PARTIALLY ALGEBRAIC MAPS AND K-THEORY OF OPERATOR ALGEBRAS

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Abstract. We investigate partially algebraic maps between a torus and a sphere and show how this problem is related to the algebraic K-theory of real or complex operator algebras. This paper is part of the author's joint work with M. Wodzicki. Since the results are of independent topological interest, it seemed better to publish them on their own.

1. Partially algebraic maps

Let X and Z be two real algebraic varieties and Y be a topological space. A continuous map $f : X \times Y \to Z$ is "partially algebraic" with respect to X if, for each y in Y, the map $x \mapsto f(x, y)$ defines an algebraic map from X to Z. In this paper we are particularly interested in the case when $Z = S^n, X$ is the product of p-circles and Y the product of $(n - p)$-circles, as in the following definition:

Definition 1.1. A continuous map f from the n-torus T^n to the n-sphere S^n is partially algebraic of level p if f is algebraic with respect to the first p variables of the torus.

Theorem 1.2. *If n is even (resp. odd) and if $p > n/2$ (resp. $p > (n + 1)/2$), then f is homotopic to a constant map.*

We note that this theorem generalizes a well known result of Loday [9]: if $n > 1$, any algebraic map from the n-torus to the n-sphere is homotopic to a constant map.

On the positive side, we have the following theorem:

Theorem 1.3. *In n is even (resp. odd), there is a degree one partially algebraic map $f : T^n \to S^n$ of level $n/2$ (resp. $(n + 1)/2$).*

Example 1.4. There is a remarkable map from the 2-torus $S^1 \times S^1$ to the 2-sphere S^2 given by the following formula in polar coordinates, (x, y, z) being the classical cartesian coordinates on the sphere:

$$2x = 1 + \cos(\theta) + (1 - \cos(\theta))\cos(\varphi)$$

$$2y = (1 - \cos(\theta))\sin(\varphi)$$

$$z = \sin(\theta)\,|\sin(\varphi/2)|$$

This map is partially algebraic of level 1. We leave as an exercise to the reader that this map is surjective and induces an homeomorphism $S^2 \simeq S^1 \times S^1 - S^1 \vee S^1$. It is shown in [6] that the existence of such a map implies Bott periodicity of the algebraic K-theory of complex stable C^*-algebras (assuming an excision theorem proved independently by Suslin and Wodzicki [11]).

Proof of Theorem 2 (n even). Let us denote by $C_{top}(X)$ the ring of (complex) continuous functions on the space X and $C_{a\lg}(X)$ the ring of algebraic functions. We also denote by $C_{p-a\lg}(X)$ the ring of partially algebraic functions of level p (if $X = T^n$). Assume now that we have a partially algebraic map f of level $p > n/2$ and of non zero degree, that is $f : T^n = S^1 \times \ldots \times S^1 \to S^n$. We then have a commutative diagram:

$$\begin{array}{ccc} K_0(C_{p-a\lg}(T^n) & \leftarrow & K_0(C_{a\lg}(S^n)) \\ \downarrow & & \downarrow \\ K_0(C_{top}(T^n)) & \leftarrow & K_0(C_{top}(S^n) \end{array}$$

The right vertical map is surjective because the generators of the group $K_0(C_{top}(S^n))$ are given by Clifford modules according to [1] (see also Section 5). The lower horizontal map is also non zero on the part of $K_0(C_{top}(T^n))$ associated to the n-smash product of the circle with itself (which is a direct summand) because f is of non trivial degree. According to the fundamental theorem in Algebraic K-theory, the reduced K-group associated to this smash product in $K_0(C_{p-a\lg}(T^n))$ is $K_{-p}(C_{top}(T^{n-p}))$; This group is 0 if $p > n - p$ that is $p > n/2$, according to a conjecture of J. Rosenberg, proved by Cortinas and Thom [3]. This contradiction proves the theorem.

Proof of Theorem 2 (n odd). We apply the same method, replacing the functor K_0 by the functor K_1, more precisely K_1^{top} when we are dealing with continuous functions. In the last part of the previous argument the main summand in $K_1(C_{p-a\lg}(T^n))$ is the group $K_{-p+1}(C_{top}(T^{n-p}))$ and we get again a contradiction if $p - 1 > n - p$, that is $p > (n+1)/2$ as required.

The proof of Theorem 3 is more elaborate and needs some preliminary lemmas.

Lemma 1.5. *For every n there exists a degree one map $f_n : S^1 \times S^n \to S^{n+1}$ such that the cartesian coordinates of $f_n(x)$ are finite sums of products of continuous functions which are partially algebraic with respect to the sphere S^n.*

Proof. Let us write $y = (y_1, y_2)$ for the cartesian coordinates on S^1 and $X = (x_1, X_2)$ for the cartesian coordinates on S^n, viewed as a subvariety of $\mathbf{R} \times \mathbf{R}^n$. The function f is then given by the following formulas, when we write $f(y, X) = (a, b, C)$ in $\mathbf{R} \times \mathbf{R} \times \mathbf{R}^n$:

$$2a = (1 + x_1) + (1 - x_1)y_1$$

$$2b = (1 - x_1)y_1$$

$$C = X_2\sqrt{\tfrac{1-y_1}{2}}$$

One easily checks that $a^2 + b^2 + C^2 = 1$, where C^2 is the usual dot product. In particular, f_1 is the map described in Example 1.4 and is surjective. Since $\|C\|= \|X_2\|\sqrt{\frac{1-y_1}{2}}$, it follows that f_n is also surjective. Indeed, if $C \neq 0, y_1$ is determined by the equation above and then $X_2 = C/\sqrt{\frac{1-y_1}{2}}$; if $C = 0$, one can use again the surjectivity of f_1. Finally, f_n is of degree 1 because $(f_n^{-1})(m)$ is reduced to a point when m is a regular point in S^{n+1}. □

Theorem 1.6. *For every n there exists a degree one map $g_n : T^n \to S^n$ such that the cartesian coordinates of $g_n(x)$ are finite sums of products of continuous functions on each summand S^1 in T^n.*

Proof. This theorem is proved by a simple induction, starting from the previous lemma. □

The following statement is due to Loday (unpublished).

Lemma 1.7. *Let*

$$F\colon \mathbf{R}^{p+1}\times\mathbf{R}^q \to\mathbf{R}^q$$

be an "orthogonal multiplication", that is a bilinear map such that

$$\|F(x, y)\| =\|x\|.\|y\|$$

There then is an algebraic degree one map

$$\tilde{F} : S^p \times S^q \to S^{p+q}$$

Proof. Let us write $x = (x_0, .., x_p)$ and $y = (y_0, .., y_q)$ for the cartesian coordinates on the spheres S^p and S^q respectively. We also write $z = (z_0, \ldots, z_{p+q})$ for the cartesian coordinates on the sphere S^{p+q}. We then put $z = \tilde{F}(x, y)$ with the following formulas:

$$1 - z_0 = (1 - x_0)(1 - y_0)/2$$

$$z_j = x_j(1 - y_0)/2 \text{ for } j = 1, ..., p$$

$$x_{p+i} = F_i(1 - x_0, x_1, .., x_p, y_1, ..., y_q) \text{ for } i = 1, ..., q,$$

the F_i being the components of F. Finally, $\tilde{F}$ is of degree one because the counterimage of a regular point is reduced to a point. □

Corollary 1.8. *Let n be even. Then there is an algebraic map $S^1 \times S^n \to S^{n+1}$ of degree one.*

Proof. In the previous lemma we choose $q = n$ and $p = 1$ and we identify $\mathbf{R}^n$ as a complex vector space of dimension $n/2$, F being defined by the complex scalar multiplication. □

Proof of Theorem 3. The proof is by induction on n, starting from Lemma 1.5 and Corollary 1.8. For instance, if n is even and if we have constructed a partially algebraic map $T^n \to S^n$ of level $n/2$, we consider the composition $S^1 \times T^n \to S^1 \times S^n \to S^{n+1}$, where the last map is given by Corollary 1.8. This composition is partially algebraic of level $n/2 + 1 = (n + 2)/2$. The proof is similar for n odd, using Lemma 1.5 instead of Corollary 1.8.

Remark 1.9. In the previous arguments we have considered algebraic or continuous maps. We may use C^∞ maps in lieu of continuous maps, replacing absolute values in the formulas by flat functions at the origin, as described in [6] p. 128.

Remark 1.10. When S^n is viewed in $\mathbf{R}^{n+1}$, we define an involution on the sphere by changing all the coordinates, except the first one, to their opposites. This is also the involutipn induced by the antipode on $\mathbf{R}^n$ when we view S^n as the one point compactification of $\mathbf{R}^n$. The explicit formulas we defined previously show that the degree one maps $T^n \to S^n$ we consider are equivariant with respect to these involutions.

2. K-THEORY OF COMPLEX OPERATOR ALGEBRAS

In a previous paper [6] we have shown how to relate complex Bott periodicity with partially algebraic maps. We recall this relation in this section with another point of view. As we shall see in next section, this new method can be applied to the real case. We also put the emphasis here on smooth maps and general operator algebras, in contrast with continuous maps and C^*-algebras, as in [8].

We write H as a complex separable complex Hilbert space, $\mathcal{B}$ the algebra of bounded operators in H and $\mathcal{K}$ the ideal of compact operators. The idea, as expounded in [6], is to compare the ring $C_0(S^1)$ of continuous functions on the circle, vanishing at the base point $e = 1$ with the Calkin algebra $\mathcal{C} =\mathcal{B}/\mathcal{K}$. Let us consider the composite C^*-algebra map

$$C_0(S^1) \to C(S^1) \xrightarrow{\theta} \mathcal{C}$$

with $H = l^2(\mathbf{Z})$, where θ sends $z = e^{it}$ to the class of the right shift operator. The following theorem is a direct consequence of complex Bott periodicity:

Theorem 2.1. *The previous map induces an isomorphism between the associated topological K-groups.*

Proof. We prove that the map $K_i^{top}(C_0(S^1)) \to K_i^{top}(\mathcal{C})$ is an isomorphism for $i = 0$ and $i = 1$ and therefore for all i by complex Bott periodicity. For $i = 0$, both sides vanish. For $i = 1$, this is the classical isomorphism

$$K_i^{top}(\mathcal{C}) \simeq K_i^{top}(C(S^1)) \simeq K_i^{top}(C_0(S^1)) \simeq \mathbf{Z}$$

□

In the sequel, we shall identify systematically all the groups K_i^{top} with K_0 for i even and with K_1^{top} for i odd. Therefore, the groups K_i are now defined for $i \in \mathbf{Z}/2$. Let us consider the p-iterated maximal tensor product φ of the map defined at the beginning, that is

$$\Lambda^{\otimes \max p} \to \mathcal{C}^{\otimes \max p}$$

with $\Lambda = C_0(S^1)$.

Theorem 2.2. *For $n \in \mathbf{Z}/2$, the previous map induces an isomorphism*

$$K_n^{top}(\Lambda^{\circledast \max p}) \simeq K_n^{top}(\mathcal{C}^{\circledast \max p})$$

Proof. If $n \neq p \bmod 2$, both groups are 0 as it can be shown by using the exactness of the maximal tensor product of C^*-algebras [4]. Otherwise, they are isomorphic to $\mathbf{Z}$ by the same argument. The generators are the products of those of K_1^{top} of Λ or $\mathcal{C}$, p times by themselves, according to Kunneth's theorem in topological K- theory. The theorem now follows since the generators are exchanged by the induced map. $\square$

In the last theorem, we would like to consider algebraic tensor products instead of topological ones. There is a commutative diagram comparing the two situations:

$$\begin{array}{ccc} \Lambda^{\otimes p} & \rightarrow & \mathcal{C}^{\otimes p} \\ \downarrow & & \downarrow \\ \Lambda^{\otimes \max p} & \rightarrow & \mathcal{C}^{\otimes \max p} \end{array}$$

Theorem 2.3. *The first vertical map induces epimorphisms*

$$K_0(\Lambda^{\otimes p}) \rightarrow K_0^{top}(\Lambda^{\otimes \max p})$$

and

$$K_1(\Lambda^{\otimes p}) \rightarrow K_1^{top}(\Lambda^{\otimes \max p})$$

Proof. By complex Bott periodicity and Kunneth formula in topological K-theory, the generators of the K-groups on the right hand side are products of elements of degree 1. Therefore, by induction on p, it is enough to prove that the product in $K_2^{top}(\Lambda^{\otimes \max 2}) \simeq K_0(\Lambda^{\otimes \max 2})$ of two elements in $K_1^{top}(\Lambda)$ is the image of an element in $K_0(\Lambda^{\otimes 2})$. The group $K_0(\Lambda^{\circledast \max 2})$, which is the reduced topological K-theory of the sphere S^2 is isomorphic to $\mathbf{Z}$, one generator being the class of the pull-back of the Hopf bundle by a degree one map from $S^1 \times S^1$ to the 2-sphere which has been described in Section 1 as the map u_1. On the other hand, the projection operator defining the Hopf bundle over S^2 as a direct summand in a trivial bundle is given by algebraic functions of the cartesian coordinates (x, y, z) on the 2-sphere (see Section 5). Therefore, the pull-back of this bundle by the map u_1, viewed as an element of $K_0(\Lambda^{\otimes 2})$, lies in the image of the comparison map $K_0(\Lambda^{\otimes 2}) \rightarrow K_0(\Lambda^{\otimes \max 2})$, thanks to Theorem 1.6. $\square$

Corollary 2.4. *Let p be a positive even integer. Then the comparison map*

$$K_{-p}(\mathcal{K}) \to K_{-p}^{top}(\mathcal{K}) \simeq \mathbf{Z}$$

is surjective.

Proof. By the standard exact sequence in algebraic and topological K-theory detailed in [8], we have the following isomorphisms (with $\mathcal{C} = \mathcal{B}/\mathcal{K}$):

$$K_0(\mathcal{C}^{\otimes p}) \simeq K_{-p}(\mathcal{K}^{\otimes p})$$

and

$$K_0(\mathcal{C}^{\otimes \max p}) \simeq K_{-p}^{top}(\mathcal{K}^{\otimes \max p})$$

We now write the commutative diagram

$$\begin{array}{ccc} K_0(\Lambda^{\otimes p}) & \to & K_0(\mathcal{C}^{\otimes p}) \simeq K_{-p}(\mathcal{K}^{\otimes p}) \\ \downarrow \theta & & \downarrow \gamma \\ K_0(\Lambda^{\otimes \max p}) & \stackrel{\varphi_*}{\to} & K_0(\mathcal{C}^{\otimes \max p}) \simeq K_{-p}(\mathcal{K}^{\otimes \max p}) \end{array}$$

Since θ and φ_* are surjective by the previous considerations, it follows that γ is also surjective. On the other hand, we also have a commutative diagram, using the fact that $H \otimes_2 \ldots \otimes_2 H$ is isomorphic to H by the usual L^2 completed tensor product:

$$\begin{array}{ccc} K_{-p}(\mathcal{K}^{\otimes p}) & \to & K_{-p}(\mathcal{K}) \\ \downarrow \gamma & & \downarrow \gamma' \\ K_{-p}^{top}(\mathcal{K}^{\otimes \max p}) & \to & K_{-p}^{top}(\mathcal{K}) \end{array}$$

From this diagram, it follows that γ' is surjective as expected. □

One would like to extend the previous considerations to subrings of $\mathcal{K}$. An important example, once a basis of H is given, is the ring of infinite matrices (a_{ij}) with rapid decay: for any $s > 0$, there is a constant C_s such that for any i and j we have $[a_{ij}] \leq C_s i^{-s} j^{-s}$. We write $\mathcal{L}$ for this ring of infinite matrices. We denote by $\mathcal{B}'$ its idealizer in the ring $\mathcal{B} = \mathcal{B}(H)$. There is a commutative diagram of rings:

$$\begin{array}{ccc} C_0^\infty(S^1) & \to & \mathcal{B}'/\mathcal{L} \\ \downarrow & & \\ C_0(S^1) & \to & \mathcal{B}/\mathcal{K} \end{array}$$

where $C_0^\infty(S^1)$ is the ring of C^∞ functions on S^1 which vanish at $e = 1$ and where the last horizontal arrow has been defined at the beginning of this section. This diagram is already detailed in [6], p. 131.

The topological K-theory of $C_0^\infty(S^1)$ is defined in many ways (see [6] p. 117 for instance). As a main difference with $C_0(S^1)$, we should take the projective tensor product $\otimes\pi$ of Fréchet spaces instead of the maximal tensor product of C^*-algebras. For instance, by taking a p -fold projective tensor product, we get an isomorphism $C_0^\infty(S^1)^{\otimes_\pi p} \simeq C_0^\infty(S^p)$.

Theorem 2.5. *Let Γ be the Fréchet algebra $C_0^\infty(S^1)$ and let φ be the p-fold projective tensor product of the map $C_0^\infty(S^1) \to \mathcal{C} = \mathcal{B}/\mathcal{K}$ by itself, more precisely*

$$\varphi : \Gamma \otimes_\pi \ldots \otimes_\pi \Gamma \to \mathcal{C} \otimes_\pi \ldots \otimes_\pi \mathcal{C} \to \mathcal{C} \otimes_{\max} \ldots \otimes_{\max} \mathcal{C}$$

Then φ induces an isomorphism between the associated topological K-groups

$$\varphi_* : K_n^{top}(\Gamma \otimes_\pi \ldots \otimes_\pi \Gamma) \simeq K_n^{top}(\mathcal{C} \otimes_{\max} \ldots \otimes_{\max} \mathcal{C})$$

Proof. According to Theorem 2.2, we only have to show that the topological K-theory of the ring $\Lambda \otimes_{\max} \ldots \otimes_{\max} \Lambda$ is isomorphic to the topological K-theory of the subring $\Gamma \otimes_\pi .. \otimes \pi \Gamma$. If we interpret geometrically these groups as K-groups of vector bundles over a p-dimensional sphere, this is a consequence of a classical fact: on a manifold the classification of topological vector bundles is the same as the classification of C^∞ vector bundles [10]. □

The next theorem requires more care.

Theorem 2.6. *The completion map*

$$\Gamma \otimes \ldots \otimes \Gamma \to \Gamma \otimes_\pi \ldots \otimes_\pi \Gamma$$

induces epimorphisms

$$K_0(\Gamma \otimes \ldots \otimes \Gamma) \to K_0(\Gamma \otimes_\pi \ldots \otimes_\pi \Gamma)$$

$$K_1(\Gamma \otimes \ldots \otimes \Gamma) \to K_1^{top}(\Gamma \otimes_\pi \ldots \otimes_\pi \Gamma)$$

Proof. We slightly modify the proof of Theorem 2.3 in the C^∞ case, following [6] p. 128. More precisely, we write

$$2x = 1 + \cos\theta + (1 - \cos\theta)\cos e(\varphi)$$

$$2y = (1 - \cos\theta)\sin e(\varphi)$$

$$z = \sin\theta\, |\sin e(\varphi)/2|$$

where $e(\varphi)$ is a C^∞ modification of the function φ "flattened" at $\varphi = 0$, which is strictly increasing so that $|\sin(e(\varphi)/2|$ is a C^∞ function. □

Theorem 2.7. *For any positive even integer p, the comparison map*

$$K_{-p}(\mathcal{L}) \to K_{-p}^{top}(\mathcal{K}) \simeq \mathbf{Z}$$

is surjective.

Proof. As in Theorem 2.4, the previous theorem implies the surjectivity of the map

$$\sigma' : K_{-p}(\mathcal{L} \otimes \ldots \otimes \mathcal{L}) \to K_{-p}^{top}(\mathcal{K} \otimes_{\max} \ldots \otimes_{\max} \mathcal{K})$$

In order to deal with $\mathcal{L}$ instead of $\mathcal{L} \otimes \ldots \otimes \mathcal{L}$, we consider the commutative diagram:

$$\begin{array}{ccc} K_{-p}(\mathcal{L} \otimes \ldots \otimes \mathcal{L}) & \to & K_{-p}(\mathcal{L}) \\ \downarrow \sigma' & & \downarrow \\ K_{-p}^{top}(\mathcal{K} \otimes_{\max} \ldots \otimes_{\max} \mathcal{K}) & \overset{\simeq}{\to} & K_{-p}^{top}(\mathcal{K}) \end{array}$$

Here we use the fact that the tensor product of infinite matrices with rapid decay is of the same kind (with a suitable order of the basis of $H \otimes \ldots \otimes H$: see [6] p. 115). From the surjectivity of $\sigma,'$ we then deduce the surjectivity of σ. □

3. K-theory of real operator algebras

Let us now look at the real situation. As in previous Section, we shall describe the continuous case first and the C^∞ case afterwards.

In the real framework, we replace the ring $C(S^1)$, NOT by the ring of real continuous functions on S^1, but by the ring of complex continuous functions $f : S^1 \to \mathbf{C}$ such that $\overline{f(z)} = f(\overline{z})$. The Fourier series associated to f is $\sum a_n z^n$, where $a_n \in \mathbf{R}$. We shall denote this ring by $CR(S^1)$: this is a particular case of rings considered by Atiyah in [2]. We also consider the subring $CR_0(S^1)$ of functions which vanish at the base point $e = 1$. On the other hand, let $H_{\mathbf{R}}$ be the real Hilbert space spanned by the functions (z^n) and let $\mathcal{B}_{\mathbf{R}}$ (resp. $\mathcal{K}_{\mathbf{R}}$) be the ring of real bounded operators (resp. compact operators) on $H_{\mathbf{R}}$. Finally, let $\mathcal{C}_{\mathbf{R}} = \mathcal{B}_{\mathbf{R}}/\mathcal{K}_{\mathbf{R}}$ be the real Calkin algebra. We have an algebra map

$$CR_0(S^1) \to \mathcal{C}_{\mathbf{R}}$$

which associates to a function f its Fourier series desplayed as the class of an infinite matrix (which is real). This is the restriction to $CR_0(S^1)$ of the map $C_0(S^1) \to \mathcal{C}$ defined in Section 2.

Theorem 3.1. *The previous algebra map induces an isomorphism*

$$K_n^{top}(CR_0(S^1)) \simeq K_n^{top}(\mathcal{C}_{\mathbf{R}})$$

Proof. We apply a general theorem on real Banach algebras proved in [7]. If a real Banach algebra map $S \to T$ induces an isomorphism $K_n^{top}(S \otimes_{\mathbf{R}} \mathbf{C}) \simeq K_n^{top}(T \otimes_{\mathbf{R}} \mathbf{C})$, it also induces an isomorphism $K_n^{top}(S) \simeq K_n^{top}(T)$. If we apply this general result to our situation, the theorem becomes a consequence of Theorem 2.1 since $CR_0(S^1) \otimes_{\mathbf{R}} \mathbf{C} \simeq C(S^1)$ and $\mathcal{C}_{\mathbf{R}} \otimes_{\mathbf{R}} \mathbf{C} \simeq \mathcal{C}$. □

Theorem 3.2. *The maximal tensor product of the previous algebra map by itself induces an isomorphism*

$$K_n^{top}(CR_0(S^1) \otimes_{\max} \ldots \otimes_{\max} CR_0(S^1)) \simeq K_n^{top}(\mathbf{C}_{\mathbf{R}} \otimes_{\max} \ldots \otimes_{\max} \mathcal{C}_{\mathbf{R}})$$

Proof. We apply the same argument as above, starting from Theorem 2.2. □

We now introduce the C^∞ analog of $CR_0(S^1)$ which is the ring of C^∞ complex functions on the circle, which vanish at the base point, such that $\overline{f(z)} = f(\overline{z})$. We denote this ring by $CR_0^\infty(S^1)$.

Theorem 3.3. *The composition*

$$K_n^{top}(CR_0^\infty(S^1) \otimes_\pi \ldots \otimes_\pi CR_0^\infty(S^1))$$
$$\to K_n^{top}(CR_O(S^1) \otimes_{\max} \ldots \otimes_{\max} CR_0(S^1))$$
$$\to K_n^{top}(\mathcal{C}_{\mathbf{R}} \otimes_{\max} \ldots \otimes_{\max} \mathcal{C}_{\mathbf{R}})$$

is an isomorphism.

Proof. We just proved that the second map is an isomorphism. Therefore, it is enough to show the same property for the first one. This follows from a classical topology statement, already used in the complex situation: the classification of topological real vector bundles over a manifold coincides with the C^∞ classification [10]. □

The next theorem uses in an essential way Theorem 1.6 of the first section.

Theorem 3.4. *Let p be an arbitrary integer. Then the comparison map between the K-theory of algebraic and projective p-tensor products*

$$\varphi_* : K_0(CR_0^\infty(S^1) \otimes \ldots \otimes CR_0^\infty(S^1)) \\ \to K_0(CR_0^\infty(S^1) \otimes_\pi \ldots \otimes_\pi CR_0^\infty(S^1))$$

is surjective.

Proof. The algebra $CR_0^\infty(S^1) \otimes_\pi \ldots \otimes_\pi CR_0^\infty(S^1))$ may be identified[1] with $CR_0^\infty(S^p)$. According to Atiyah [2], the K_0-group of this algebra is generated by Clifford modules. More precisely, the associated Real bundles over S^p are images of idempotents in a trivial Real bundle, which coefficients are algebraic functions of the coordinates (see Section 5 for more details). Therefore, the pull-back of these bundles by the specific degree one map defined in Theorem 1.6, belongs to the image of φ_* as expected. □

Theorem 3.5. *Let $\mathcal{L}_{\mathbf{R}}$ be the ring of infinite real matrices with rapid decay and let p be a positive integer. Then the comparison map*

$$K_{-p}(\mathcal{L}_{\mathbf{R}}) \to K_{-p}(\mathcal{K}_{\mathbf{R}}) \simeq K_{-p}^{top}(\mathcal{K}_{\mathbf{R}}) \simeq K_{-p}(\mathbf{R})$$

is surjective. Note that the target $K_{-p}(\mathbf{R})$ is $\mathbf{Z}$ if $p = 0$ mod 4, $\mathbf{Z}/2$ if $p = 6$ or 7 mod 8 and 0 otherwise.

Proof. As in the complex case, we denote by $\mathcal{B}'_{\mathbf{R}}$ the idealizer of $\mathcal{L}_{\mathbf{R}}$ in $\mathcal{B}_{\mathbf{R}}$. We denote by $\mathcal{C}_{\mathbf{R}}$ the real Calkin algebra $\mathcal{B}_{\mathbf{R}}/\mathcal{K}_{\mathbf{R}}$ and by $\mathcal{C}'_{\mathbf{R}}$ the quotient ring $\mathcal{B}'_{\mathbf{R}}/\mathcal{L}_{\mathbf{R}}$. We then have a commutative diagram

$$\begin{array}{ccc} K_0((CR_0^\infty(S^1)^{\otimes p}) & \to & K_0((\mathcal{C}'_{\mathbf{R}})^{\otimes p}) \simeq K_{-p}((\mathcal{L}_{\mathbf{R}})^{\otimes p}) \\ \downarrow \theta & & \downarrow \gamma \\ K_0(CR_0(S^1)^{\otimes p}) & \overset{\varphi_*}{\to} & K_0((\mathcal{C}_{\mathbf{R}})^{\otimes p}) \simeq K_{-p}((\mathcal{K}_{\mathbf{R}})^{\otimes p}) \end{array}$$

Since θ and φ_* are surjective, it follows that γ is also surjective. We finish the proof as for Theorem 2.7, but with real infinite matrices with rapid decay instead of complex ones. □

[1]where $S^p = S^1 \Lambda \ldots \Lambda\, S^1$ is the p sphere with the induced involution. Note that S^p is the one-point compactification of $\mathbf{R}^p = \mathbf{R} \times .. \times \mathbf{R}$.

4. Periodicity of algebraic K-groups

The construction of "Bott elements" in K-theory is one of the possible strategies to prove Bott periodicity as shown in previous publications, for instance [6] and [8]. By "Bott elements" we mean elements of algebraic K-theory in appropriate degrees which map onto generators of topological K-theory. In Sections 3 and 4, we proved their existence in degree -2 in the complex case and -8 in the real case, for a large class of operator rings containing the ring of infinite matrices with rapid decay. Typical examples of such rings are Schatten classes L^p $(p \geq 1)$ in a real or complex Hilbert spaces. An element of L^p is a compact operator k such that the eigenvalues λ_i of k^*k satisfy the condition

$$(\sum(\lambda_i)^{p/2})^{1/p} < \infty$$

The ring of matrices with rapid decay is clearly a subring of L^p. From now on, we shall reserve the notation L^p for operators in a complex Hilbert spaces and the notation $L^p_{\mathbf{R}}$ in the real framework.

Our goal in this section is to prove Bott periodicity for the algebraic K-theory of this type of rings. Note that the complex case is already considered in previous publications, e.g. [6] and [8]. Therefore, we shall be mainly interested in the real case. Our first result is about K-theory with finite coefficients $\mathbf{Z}/n$.

Theorem 4.1. *For any r in $\mathbf{Z}$ the "cup-product" with the Bott element in $K_{-8}(L_{\mathbf{R}})$ induces an isomorphism*

$$K_r(L^p_{\mathbf{R}}; \mathbf{Z}/n) \simeq K_{r-8}(L^p_{\mathbf{R}}; \mathbf{Z}/n)$$

Proof. As detailed in [6] and [8], the proof of this theorem is purely formal if we show the existence of a' "positive" Bott element in $K_8(L^p_{\mathbf{R}}; \mathbf{Z}/n)$, taking into account two basic facts:

1) The existence of excision in K-theory with finite coefficients

2) The fact that $L^p_{\mathbf{R}} \otimes L^p_{\mathbf{R}}$ is contained in $L^p_{\mathbf{R}}$ when we use the identification $H \otimes_2 H \simeq H$. This justifies the existence of a "cup-product".

The crucial point is of course the construction of this "positive" Bott element in $K_8(L^p_{\mathbf{R}}; \mathbf{Z}/n)$. For this, we rely on a theorem of Suslin who constructed an element in $K_8(\mathbf{R}; \mathbf{Z}/n)$ which maps isomorphically to its topological counterpart. We use the canonical inclusion $\mathbf{R} \subset L^p_{\mathbf{R}}$ to conclude. $\square$

The periodicity of the groups $K_r(L^p_{\mathbf{R}})$ for all values of r is much harder to study (and is false in general). We already know the existence of a Bott element δ in $K_{-8}(L^p_{\mathbf{R}})$ we just used above. On the other hand, we have constructed in [6] p. 142 a Bott element β in the relative group $K_2(\mathcal{B}, L^{\mathbf{p}})$ for $p > 1$. Note that we cannot write this last group $K_2(L^p)$ since algebraic K-theory does not satisfy excision in general. However, we can embed L^p in the union L^∞ of all the L^p; The advantage is that L^∞ is H-unital as proved in [13], and we may view β as an element of $K_2(\mathcal{B}, L^\infty)$, which we can write now $K_2(L^\infty)$, since the ring L^∞ satisfies excision. Taking the cup-product of β four times by itself leads to a Bott element β^4 in $K_8(L^\infty)$.

The previous considerations can be rephrased in the real framework, except the existence of β for which we need a complex structure. However, the proof given in [8] p. 15 for the ring of compact operators $\mathcal{K}$ goes through for the ring L^∞ as well. In other words, β^4 is the image of a Bott element γ in $K_8(L^\infty_{\mathbf{R}})$ by the complexification homomorphism

$$K_8(L^\infty_{\mathbf{R}}) \to K_8(L^\infty)$$

With the other Bott element δ in $K_{-8}(L^\infty_{\mathbf{R}})$, we can conclude this discussion by the following theorem:

Theorem 4.2. *For any r in $\mathbf{Z}$, the cup-product with the element γ above defines a periodicity isomorphism*

$$K_r(L^\infty_{\mathbf{R}}) \simeq K_{r+8}(L^\infty_{\mathbf{R}})$$

Remark 4.3. The same argument shows that the groups $K_r(L^\infty_{\mathbf{R}} \otimes A)$ are 8-periodic for any unital ring A, since $L^\infty_{\mathbf{R}} \otimes A$ is also H-unital.

A priori, the previous theorem does not enable us to compute the groups $K_r(L^p_{\mathbf{R}})$ for a specific p. However, $L^\infty_{\mathbf{R}}/L^p_{\mathbf{R}}$ is an union of nilpotent rings and it is well known that algebraic K-theory in non positive degrees is invariant under nilpotent extensions. Therefore, we get at least half of the periodicity statement for negative values of r.

Theorem 4.4. *The cup-product with the element δ above defines a periodicity isomorphism*

$$K_r(L^p_{\mathbf{R}}) \simeq K_{r-8}(L^p_{\mathbf{R}})$$

for $r \leq 0$ and $p \geq 1$.

Remark 4.5. The full algebraic K-theory of Schatten classes L^p and $L^p_{\mathbf{R}}$ is quite subtle. For instance, the link with classical regulators in number theory is made in [6] p. 150 and also in [13].

Remark 4.6. A number of questions remain open. For instance, we dont know if the algebraic K-groups $K_r(L^p_{\mathbf{R}})$ coincide with their topological counterparts $K^{top}_r(L^p_{\mathbf{R}})$, even for $r < 0$. We dont know neither if the groups $K_r(\mathcal{L}_{\mathbf{R}})$ are periodic for $r < 0$.

5. Appendix: Clifford modules

This appendix contains nothing new. It is essentially a recollection of results in [1], [2] and [5], presented in the format we need. Following these references, we denote by $C^{p,q}$ the Clifford algebra of $\mathbf{R}^{p+q}$ provided with the quadratic form $-(x_1)^2 - \ldots - (x_p)^2 + (x_{p+1})^2 + \ldots + (x_{p+q})^2$. Similarly, we denote by $\mathcal{E}^{p,q}(X)$ the category of vector bundles (real or complex) which are also $C^{p,q}$-modules. As proved in [5], the topological K-theory with compact support of $X \times \mathbf{R}^n$ (for X compact) is isomorphic to the Grothendieck group of the forgetful functor $\varphi^{0,n} : \mathcal{E}^{0,n+1}(X) \to \mathcal{E}^{0,n}(X)$. In other words we have the formula

$$K(X \times \mathbf{R}^n) \simeq K(\varphi^{0,n})$$

Similarly, using the suspension isomorphism, we have the formula[2]

$$K(X \times \mathbf{R}^n) \simeq K^1(X \times \mathbf{R}^{n+1}) \simeq K^1(\varphi^{0,n+1})$$

We are particularly interested in the case when X is reduced to a point, i.e. in the computation of $K(\mathbf{R}^n)$, which is the reduced K-theory of the sphere S^n. This is of course linked with Bott periodicity and was the main historical motivation in [1].

Theorem 5.1. *(cf. [1]) The reduced K-theory of the sphere S^n is isomorphic to the cokernel of the map*

$$K(C^{0,n+2}) \to K(C^{0,n+1})$$

Proof. The group $K^1(\varphi^{0,n+1})$ is inserted in an exact sequence of K-groups of Clifford algebras:

$$K(C^{0,n+2}) \to K(C^{0,n+1}) \to K^1(\varphi^{0,n+1}) \to K^1(C^{0,n+2}) = 0$$

and we can apply the isomorphism above. Note that the Clifford algebra is understood over the basic field $\mathbf{R}$ or $\mathbf{C}$, according to the type of K-theory, real or complex, we are considering. □

In the basic reference [1], an explicit map from $K(C^{0,n+1})$ to the topological K-theory of S^n is given: if we start with a $C^{0,n+1}$ -module

[2]We use indifferently the notation K^n or K_{-n} .

E, we consider the involution J on the trivial bundle $S^n \times E$ defined over a point $x = (x_0, x_1, .., x_n)$ of the sphere by the formula

$$J = x_0 e_0 + x_1 e_1 + .. + x_n e_n$$

where $e_0, e_1, \ldots, e_n$ are the classical generators of the Clifford algebra $C^{0,n+1}$ acting on E. The bundle over the sphere associated to E is then the kernel of the projection operator $(1 - J)/2$.

Example 5.2. The simplest example is the (complex) Hopf bundle over S^2. It is associated to the Clifford module $E = \mathbf{C}^2$ with $e_0 = ie_1e_2$. Here e_1 and e_2 are the so called "Pauli matrices":

$$e_1 = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, \; e_2 = \begin{bmatrix} 0 & i \\ -i & 0 \end{bmatrix}$$

Example 5.3. An example in real K-theory is $E = \mathbf{R}^{16}$. Indeed, since the Clifford algebra $C^{0,8}$ is isomorphic to the matrix algebra $M_{16}(\mathbf{R})$ (see [1]), we may choose E as an irreducible $C^{0,8}$-module. If $e_1, \ldots, e_8$ are the usual generators of $C^{0,8}$, we put $e_0 = e_1 e_2 \ldots e_8$, as in previous example, so that E is now an irreducible $C^{0,9}$ module. As proved in [1], the real vector bundle associated to it generates the reduced real K-theory of the sphere S^8, traditionnally written $\tilde{KO}(S^8) \simeq \mathbf{Z}$.

For the next step, let us follow Atiyah's notation $\mathbf{R}^{p,q}$ for the Euclidean space $\mathbf{R}^p \times \mathbf{R}^q$ with the involution $(x, y) \mapsto (-x, y)$. The previous results can be generalized through Thom's isomorphism in KR-theory:

$$KR(X \times \mathbf{R}^{p,q}) \simeq K(\varphi^{p,q}),$$

where $\varphi^{p,q} : \mathcal{E}^{p,q+1}(X) \to \mathcal{E}^{p,q}(X)$ is the forgetful functor between Real vector bundles on X provided with suitable Clifford structures. For X reduced to a point, we may write as before

$$KR(\mathbf{R}^{p,q}) \simeq KR^1(\mathbf{R}^{p,q+1}) \simeq K^1(\varphi^{p,q+1})$$

where the group $K^1(\varphi^{p,q+1})$ is inserted in an exact sequence generalizing the one described above (for $p = 0$):

$$K(C^{p,q+2}) \to K(C^{p,q+1}) \to K^1(\varphi^{p,q+1}) \to K^1(C^{p,q+2}) = 0$$

By taking now $q = 0$. we end up with the following theorem, where the sphere S^p is provided with the involution defined at the end of previous section. The notation $\tilde{KR}$ in the statement stands for reduced KR -theory.

Theorem 5.4. *(cf. [2]) The group $\tilde{KR}(S^p)$ fits in the exact sequence*

$$K(C^{p,2}) \to K(C^{p,1}) \to \tilde{KR}(S^p) \to 0$$

In fact, Atiyah's theorem is again more precise: if we start with a $C^{p,1}$-module E and if E' denotes its complexification, the Real vector bundle associated to E is $Ker(1-J)$, where J is the involution on the trivial Real bundle $S^p \times E'$ defined over the point $x = (x_0, x_1, \ldots, x_p)$ by the formula

$$J = x_0 e_0 + i x_1 e_1 + \ldots + i x_p e_p$$

Here $e_0, e_1, .., e_p$ denote the classical generators of the Clifford algebra $C^{p,1}$ with the relations

$$(e_0)^2 = 1, (e_\alpha)^2 = -1 \text{ for } \alpha = 1, .., p$$

$$e_\beta e_\gamma + e_\gamma e_\beta = 0 \text{ for } \beta \neq \gamma$$

REFERENCES

[1] M.F. Atiyah, R. Bott and A. Shapiro. Clifford modules. Topology 3 (1964), 3-38.

[2] M.F. Atiyah. K-theory and reality. Q.J. Math. Oxford (2) (1966), 367-386.

[3] G. Cortinas and A. Thom. Comparison between algebraic and topological K-theory of locally convex algebras. Adv. Math 218 (2008), 266-307.

[4] T. Fack. Exactness of the maximal tensor products for real and complex C^*-algebras. Q.J. Math 64 (2013), 941-945 (special edition in honor of D. Quillen).

[5] M. Karoubi. Algèbres de Clifford et K-théorie. Ann. Sci. Ec. Norm Sup. (4) (1968), 161-270.

[6] M. Karoubi. Homologie de groupes discrets associés à des algèbres d'opérateurs. Journal of Operator Théory 15 (1986), 109-161.

[7] M. Karoubi. A descent theorem in topological K-theory. K-theory 24 (2001), 109-114.

[8] M. Karoubi and M. Wodzicki. Algebraic and Hermitian K-theory of $\mathcal{K}$-rings. Q.J. Math 64 (2013), 903-940 (special edition in honor of D. Quillen).

[9] J.-L. Loday.Applications algébriques du tore dans la sphère. C.R. Acad. Sci. Paris (1971), 578-581.

[10] N. Steenrod. The topology of fibre bundles. Reprint of the 1957 edition. Princeton University Press 1999.

[11] A. Suslin and M. Wodzicki. Excision in Algebraic K-theory. Ann. of Math. 136 (1992), 51-122.

[12] A. Suslin. On the K-theory of local fields. Journal of Pure and Applied Algebra 34 (1984), 301-318.

[13] M. Wodzicki. Algebraic K-theory and functional analysis. First European Congress Math, Vol. II (Paris 1992), 485-496, Progr. Math. 120, Birkhauser !1994).