ALGEBRAIC K-THEORY OF OPERATOR RINGS

MAX KAROUBI AND MARIUSZ WODZICKI

Introduction

Let $\mathcal{K} = \mathcal{K}(H)$ be the Banach algebra of compact linear operators on an infinite-dimensional separable complex Hilbert space H. The comparison map

$$K_*(\mathcal{K}) \longrightarrow K_*^{\mathrm{top}}(\mathcal{K}), \tag{0.1}$$

is an isomorphism. This was conjectured by one of us [6] and established in [10], [11], using the fact that all C^*-algebras are H-unital [12].

The algebra of compact operators is the largest proper two-sided ideal in the algebra $\mathcal{B} = \mathcal{B}(H)$ of bounded linear operators on H. The smallest nonzero ideal, $\mathcal{F} = \mathcal{F}(H)$, consists of all finite rank operators, and $\mathcal{K}$ is the closure of $\mathcal{F}$ in the norm topology. The correspondence

$$1 \longmapsto E$$

that sends 1 to any nonzero idempotent $E \in \mathcal{F}$ of rank 1, induces an inclusion of $\mathbf{C}$-algebras,

$$\mathbf{C} \hookrightarrow \mathcal{F},$$

and the latter induces an isomorphism

$$K_*(\mathbf{C}) \xrightarrow{\simeq} K_*(\mathcal{F}) \ . \tag{0.2}$$

The isomorphism in (0.2) does not depend on the idempotent.

It follows that the algebraic K-groups $K_n(\mathcal{F})$ vanish for $n < 0$. The same correspondence induces an inclusion of Banach algebras,

$$\mathbf{C} \hookrightarrow \mathcal{K},$$

and the latter induces an isomorphism of topological K-groups

$$K_*^{\mathrm{top}}(\mathbf{C}) \xrightarrow{\simeq} K_*^{\mathrm{top}}(\mathcal{K}) \tag{0.3}$$

that, again, does not depend on the idempotent.

The pairing induced by the tensor product of operators,

$$\mathcal{K} \times \mathcal{K} \longrightarrow \mathcal{K}\big(H \otimes_{\ell^2} H\big) \ , \qquad (S, T) \longmapsto S \otimes T,$$

followed by the ring isomorphism $\mathcal{K}\big(H \otimes_{\ell^2} H\big) \simeq \mathcal{K}$, associated with a fixed Hilbert space identification

$$H \otimes_{\ell^2} H \simeq H, \tag{0.4}$$

1991 *Mathematics Subject Classification.* 19K99.

Key words and phrases. operator ideals, algebraic K-theory.

yields a pairing

$$\mathcal{K} \times \mathcal{K} \longrightarrow \mathcal{K} \ . \tag{0.5}$$

The induced biadditive pairings,

$$K_*(\mathcal{K}) \times K_*(\mathcal{K}) \longrightarrow K_*(\mathcal{K}) \quad \text{and} \quad K_*^{\mathrm{top}}(\mathcal{K}) \times K_*^{\mathrm{top}}(\mathcal{K}) \longrightarrow K_*^{\mathrm{top}}(\mathcal{K}) \ , \tag{0.6}$$

are graded-commutative, associative, do not depend on the identification chosen in (0.4). Finally, maps (0.1) and (0.3) are $\mathbf{Z}$-graded ring isomorphisms.

The topological K-theory ring $K_*^{\mathrm{top}}(\mathbf{C})$ is the ring of Laurent polynomials $\mathbf{Z}[\beta, \beta^{-1}]$ where β, a generator of $K_2^{\mathrm{top}}(\mathbf{C})$, is often referred to as the *Bott* element. Below, we shall refer to it, as the 'Bott element of degree 2', while its multiplicative inverse will be referred to as the 'Bott element of degree -2'. We shall use the same terminology and notation for the corresponding elements of $K_*^{\mathrm{top}}(\mathcal{K})$ and $K_*(\mathcal{K})$.

A subring $\mathcal{L} \subseteq \mathcal{K}$ will be said to be a *Bott$_n$ ring of compact operators* if the inclusion map induces an epimorphism

$$K_n(\mathcal{L}) \longrightarrow\!\!\!\!\!\rightarrow K_n(\mathcal{K}) \ .$$

This is of interest, of course, only for even n. If, for some identification (0.4), restriction of (0.5) to $\mathcal{L}$ yields a pairing

$$\mathcal{L} \times \mathcal{L} \longrightarrow \mathcal{L} \ ,$$

then $\mathcal{L}$ is automatically a Bott$_{m+n}$ ring whenever $\mathcal{L}$ is both a Bott$_m$ and a Bott$_n$ ring and $m, n < 0$. A subring of $\mathcal{K}$ which contains a Bott$_n$ subring is itself a Bott$_n$-subring.

The primary goal of the present article is to produce small Bott$_n$ rings, for $n < 0$. We do this in Chapter 5. As a corollary, we obtain a criterion for a ring of compact operators to be a Bott$_n$ ring, n being negative, cf. Theorem 3 and its Corollary 5.4 therein.

A second main result is presented in Chapter 3, where we prove that, for any Banach algebra A and any proper Banach operator ideal $I \subset \mathcal{B}(H)$, the algebraic K-groups in negative degrees,

$$K_n(A \widehat{\otimes}_\pi I) \qquad (n \leq 0),$$

are canonically isomorphic to the topological K-groups $K_n^{\mathrm{top}}(A)$, cf. Corollary 3.6.

Here, A can be either a complex or a real Banach algebra, and H, accordingly, a real or a complex Hilbert space. Compared to the analogous results established in [15] and [8], our method does not utilize algebraic K-groups in positive degrees, excision or H-unitality, thus it does not depend on the excision theorems of [10]–[14]. The complex case has been known since 1992, it is a corollary of the proofs by the second author of several results conjectured by the first author. Another proof appeared in [1].

The article is organized as follows. In Chapter 1, we introduce the concept of an algebra graded by a semilattice $\mathcal{S}$. There is a functorial embedding of the category of algebra extensions into the category of $\mathbf{2}$-graded algebras where $\mathbf{2}$ denotes the appropriate ordinal. The categories of semilattice-graded algebras are equipped with the exterior tensor product: given an $\mathcal{S}$-graded algebra $\mathcal{A}$ and an $\mathcal{S}'$-graded algebra $\mathcal{A}'$, the exterior tensor product $\mathcal{A} \boxtimes \mathcal{A}'$ is an $\mathcal{S} \times \mathcal{S}'$-graded algebra. This leads us to view q-tuple tensor products of algebra extensions as $\mathbf{2}^q$-graded algebras.

The concepts of a semilattice-graded Banach or C^*-algebra are defined in the same way, the respective categories are equipped with the corresponding exterior products that utilize projective tensor product $\widehat{\otimes}_\pi$, in the Banach algebra case, and maximal tensor product $\otimes_{\max}$, in the C^*-algebra case.

The framework of $\mathcal{S}$-graded algebras is subsequently used in the constructions and arguments of Chapters 3 and 5, where the main results of the present article are stated and proved.

A short Chapter 2 is meant to provide a self-contained reference for a few basic notions involving the concepts of a Banach ideal. The material of this chapter is used in Chapter 3.

In Chapter 4 we introduce the concept of a *secant extension*. These are extensions of nonunital k-algebras

$$\begin{tikzcd} A & A' \arrow[l, two heads, "\pi"'] & K \arrow[l, hook', "\iota"'] \arrow[from=1-1, to=1-2, dotted, bend right, "s"'] \end{tikzcd},$$

with a k-linear splitting s whose image generates the middle term A'. Here, k stands for any unital commutative ring, e.g., $\mathbf{Z}$. The kernel ideal of a secant extension is generated by the *multiplicative curvature* of the splitting, cf. Proposition 4.1.

With any k-linear map of k-algebras $s\colon A \longrightarrow B$, we associate a canonical algebra extension. If the original map yields a splitting of that extension, we call the map *secant*. We are particularly interested in secant dilations,

$$a \longmapsto eae \qquad (a \in A)$$

where A is a subalgebra of B and $e \in B$ is an idempotent. We offer a few topological criteria for a dilation to be secant.

An example of a secant dilation is provided by the Toeplitz map. It produces, for any k-subalgebra $\mathcal{O} \subseteq L^\infty(S^1)$, a k-algebra extension

$$\begin{tikzcd} \mathcal{O} & \mathcal{T}_{\mathcal{O}} \arrow[l, two heads] & \mathcal{K}_{\mathcal{O}} \arrow[l, hook'] \arrow[from=1-1, to=1-2, dotted, bend right, "T"'] \end{tikzcd}$$

where $\mathcal{T}_{\mathcal{O}}$ is the k-subalgebra of the algebra of bounded operators on $L^2(S^1)$, generated by the image of $\mathcal{O}$ under the Toeplitz map

$$T\colon L^\infty(S^1) \longrightarrow \mathcal{B}(L^2(S^1)).$$

The Toeplitz map, Toeplitz operators, and C^*-algebra extensions involving Toeplitz operators, since the 1960-ies have been a constant presence in various aspects of K-theory, cf., e.g., [4]. We shall use them in Chapter 5 to prove that the tensor product of the ideals,

$$\mathcal{K}_{\mathcal{O}_1} \otimes_k \cdots \otimes_k \mathcal{K}_{\mathcal{O}_q},$$

or, more precisely, its image in

$$\mathcal{K}\big(H^{\otimes_{\ell^2} q}\big),$$

is a Bott_{-2q} ring if each $\mathcal{O}_i$ is an algebra of continuous functions on S^1 containing functions

$$\theta_i(1-\theta_i) \qquad \text{and} \qquad \theta_i^2(1-\theta_i),$$

where θ_i is an arbitrary continuous function on the interval $[0, 2\pi]$ that vanishes at 0 and has value 1 at 2π.

The proof we present does not use advanced features of algebraic K-theory or homotopy theory.

1. $\mathcal{S}$-graded algebras

1.1. **Semilattices.** Let $(\mathcal{S}, \prec)$ be a $\vee$-semilattice, i.e., a partially ordered set with all nonempty finite subsets possessing a supremum. The binary operation

$$s \vee t := \sup\{s, t\} \qquad (s, t \in \mathcal{S})$$

makes S into a commutative semigroup. Note that

$$s \prec t \qquad \text{if and only if} \qquad s \vee t = t. \tag{1.1}$$

Vice-versa, if $(\mathcal{S}, \vee)$ is a commutative semigroup with all elements being idempotent, then (1.1) defines a partial order making $(\mathcal{S}, \prec)$ into a $\vee$-semilattice. In particular, one can view a $\vee$-semilattice as a semigroup or as a partially ordered set. Those two categories are isomorphic provided one declares the morphisms between lattices to be isotone maps

$$(\mathcal{S}, \prec) \xrightarrow{\phi} (\mathcal{S}', \prec')$$

that are *right-exact*, i.e., preserving the *coproduct*,

$$\phi(s \vee t) = \phi(s) \vee \phi(t) \qquad (s, t \in \mathcal{S}).$$

Accordingly, a *sublattice* consists of a subset of $\mathcal{S}' \subseteq \mathcal{S}$ closed under the $\vee$ operation in $\mathcal{S}$.

1.2. **$\mathcal{S}$-graded algebras.** A contravariant functor $\mathcal{A}$ from $\mathcal{S}$ to a category of modules over a unital commutative ring k, equipped with k-bilinear pairings

$$\mathcal{A}_{s \vee t} \xleftarrow{\mu_{st}} \mathcal{A}_s \times \mathcal{A}_t \qquad (s, t \in \mathcal{S}), \tag{1.2}$$

will be referred to as a *k-algebra graded by* $(\mathcal{S}, \prec)$. For $k = \mathbf{Z}$ we shall be speaking of $\mathcal{S}$-graded *rings*.

Functoriality of multiplications (1.2) translates into commutativity of the diagrams

$$\begin{array}{ccc} \mathcal{A}_{s \vee t} & \longleftarrow & \mathcal{A}_s \times \mathcal{A}_t \\ \downarrow & & \downarrow \\ \mathcal{A}_{s' \vee t'} & \longleftarrow & \mathcal{A}'_s \times \mathcal{A}'_t \end{array} \qquad (s \prec s', t \prec t').$$

Associativity has the obvious meaning: the diagrams

$$\begin{array}{ccc} \mathcal{A}_{s \vee t} \times \mathcal{A}_u & \longleftarrow & \mathcal{A}_s \times \mathcal{A}_t \times \mathcal{A}_u \\ \downarrow & & \downarrow \\ \mathcal{A}_{s \vee t \vee u} & \longleftarrow & \mathcal{A}_s \times \mathcal{A}_{t \vee u} \end{array} \qquad (s, t, u \in \mathcal{S})$$

are supposed to commute.

1.3. **$\mathcal{S}$-graded ideals.** Given an $\mathcal{S}$-graded k-algebra $\mathcal{A}$ and a contravariant functor $\mathcal{B}\colon \mathcal{S} \to k$-mod, a natural transformation

$$\mathcal{A} \xleftarrow{\ \iota\ } \mathcal{B} \tag{1.3}$$

equipped with k-bilinear pairings

$$\begin{tikzcd} \mathcal{A}_s \times \mathcal{B}_t \arrow[dr, "\lambda_{st}"'] & & \mathcal{B}_s \times \mathcal{A}_t \arrow[dl, "\rho_{st}"] \\ & \mathcal{B}_{s\vee t} & \end{tikzcd} \qquad (s, t \in \mathcal{S})$$

such that the diagrams

$$\begin{tikzcd} & \mathcal{B}_s \times \mathcal{B}_t \arrow[dl, "\iota_s \times \mathrm{id}"'] \arrow[dr, "\mathrm{id} \times \iota_t"] & \\ \mathcal{A}_s \times \mathcal{B}_t \arrow[dd, "\mathrm{id} \times \iota_t"'] \arrow[dr, "\lambda_{st}"'] & & \mathcal{B}_s \times \mathcal{A}_t \arrow[dl, "\rho_{st}"] \arrow[dd, "\iota_s \times \mathrm{id}"] \\ & \mathcal{B}_{s\vee t} \arrow[dd, "\iota_{s\vee t}"] & \\ \mathcal{A}_s \times \mathcal{A}_t \arrow[dr, "\mu_{st}"'] & & \mathcal{A}_s \times \mathcal{A}_t \arrow[dl, "\mu_{st}"] \\ & \mathcal{A}_{s\vee t} & \end{tikzcd} \qquad (s, t \in \mathcal{S})$$

commute, will be referred to as an $\mathcal{S}$-*graded ideal*. The structure of an ideal endows $\mathcal{B}$ with a structure of an $\mathcal{S}$-graded k-algebra making (1.3) into a morphism of $\mathcal{S}$-graded k-algebras.

We shall say that the ideal is *associative* if $\mathcal{A}$ is associative and the following diagrams

$$\begin{tikzcd} \mathcal{B}_s \times \mathcal{A}_t \times \mathcal{A}_u \arrow[r, "\rho_{st} \times \mathrm{id}"] \arrow[d, "\mathrm{id} \times \mu_{tu}"'] & \mathcal{B}_{s\vee t} \times \mathcal{A}_u \arrow[d, "\rho_{s\vee t, u}"'] & \mathcal{A}_s \times \mathcal{B}_t \times \mathcal{A}_u \arrow[l, "\lambda_{st} \times \mathrm{id}"'] \arrow[d, "\mathrm{id} \times \rho_{tu}"] \\ \mathcal{B}_s \times \mathcal{A}_{t\vee u} \arrow[r, "\rho_{s, t\vee u}"'] & \mathcal{B}_{s\vee t\vee u} & \mathcal{A}_s \times \mathcal{B}_{t\vee u} \arrow[l, "\lambda_{s, t\vee u}"] \\ & \mathcal{A}_{s\vee t} \times \mathcal{B}_u \arrow[u, "\lambda_{s\vee t, u}"] & \mathcal{A}_s \times \mathcal{A}_t \times \mathcal{B}_u \arrow[l, "\mu_{st} \times \mathrm{id}"] \arrow[u, "\mu_{st} \times \mathrm{id}"'] \end{tikzcd}$$

commute for all triples of s, t and u in $\mathcal{S}$.

The families of kernels, images and cokernels of an $\mathcal{S}$-graded ideal

$$\mathcal{K}_s := \operatorname{Ker} \iota_s \qquad \mathcal{I}_s := \operatorname{Im} \iota_s \qquad \text{and} \qquad \mathcal{C}_s := \operatorname{Coker} \iota_s, \tag{1.4}$$

not only inherit from $\mathcal{A}$ a structure of an $\mathcal{S}$-graded k-algebra but, in fact, yield related $\mathcal{S}$-graded ideals

$$\mathcal{A} \hookleftarrow \mathcal{I}, \qquad \mathcal{I} \twoheadleftarrow \mathcal{B} \qquad \text{and} \qquad 0 \longleftarrow \mathcal{K}$$

(note that $\mathcal{B} \hookleftarrow \mathcal{K}$ is an *annihilator* ideal which means that

$$\mathcal{K}_s\mathcal{B}_t = 0 = \mathcal{B}_s\mathcal{K}_t$$

for all s and t in $\mathcal{S}$).

1.4. **The associated $\mathcal{S}\times$2-graded algebra of an $\mathcal{S}$-graded ideal.** A natural transformation of functors (1.3) is the same as a contravariant functor $\mathcal{S}\times\mathbf{2} \longrightarrow k$-mod such that

$$(s,0) \longmapsto \mathcal{A}_s \qquad \text{and} \qquad (s,1) \longmapsto \mathcal{B}_s \qquad (s \in \mathcal{S}).$$

Here $\mathbf{2} = \{0,1\}$ is the two-element ordinal. The four families of k-linear pairings

$$\mu_{st}, \qquad \lambda_{st}, \qquad \rho_{st}, \qquad \text{and} \qquad \lambda_{st} \circ (\iota_s \times \mathrm{id}) = \rho_{st} \circ (\mathrm{id} \times \iota_t)$$

equip this functor with a structure of a k-algebra graded by $\mathcal{S}\times\mathbf{2}$.

Vice-versa, any $\mathcal{S}\times\mathbf{2}$-graded algebra $\mathcal{D}$ yields an $\mathcal{S}$-graded ideal by setting

$$\mathcal{A}_s := \mathcal{D}_{s0} \qquad \text{and} \qquad \mathcal{B}_s := \mathcal{D}_{s1}.$$

This natural correspondence between $\mathcal{S}$-graded ideals and $\mathcal{S}\times\mathbf{2}$-graded algebras preserves and reflects associativity: associative $\mathcal{S}\times\mathbf{2}$-graded algebras correspond to associative $\mathcal{S}$-graded ideals.

1.5. **Exterior tensor product of graded algebras.** Given q semilattices $\mathcal{S}_1,\dots,\mathcal{S}_q$ and k-algebras, $\mathcal{A}_1,\dots,\mathcal{A}_q$, each graded by the respective semilattice, the assignment

$$(s_1,\dots,s_q) \longmapsto \mathcal{A}_{1,s_1} \otimes_k \cdots \otimes_k \mathcal{A}_{q,s_q}$$

gives rise to a k-algebra graded by

$$\mathcal{S}_1 \times \cdots \times \mathcal{S}_q.$$

We shall denote it $\mathcal{A}_1 \boxtimes \cdots \boxtimes \mathcal{A}_q$ and refer to it as the *external tensor product* of the algebras involved.

1.6. **Graded Banach algebras.** Banach algebras graded by $\mathcal{S}$ are by definition contravariant functors from $\mathcal{S}$ to the category of Banach k-spaces, where k denotes $\mathbf{C}$ or $\mathbf{R}$, equipped with bounded bilinear pairings. In the definition of the exterior tensor product one employs the completed projective tensor product $\widehat{\otimes}_\pi$. The resulting graded Banach algebra will be denoted

$$\mathcal{A}_1 \widehat{\boxtimes}_\pi \cdots \widehat{\boxtimes}_\pi \mathcal{A}_q$$

1.7. **Graded C^*-algebras.** Graded C^*-algebras (complex or real) are similarly defined. In the definition of the exterior tensor product one employs the maximal tensor product $\otimes_{\max}$. The resulting graded Banach algebra will be denoted

$$\mathcal{A}_1 \boxtimes_{\max} \cdots \boxtimes_{\max} \mathcal{A}_q$$

2. The lattice of Banach ideals

2.1. **The lattice of Banach subspaces of a normed space.** We shall say that a vector subspace V of a normed vector space E is a *Banach subspace* if it is the image of a bounded linear map from a Banach space into E. This is an intrinsic property of V and the norms on V making V complete and the inclusion $V \hookrightarrow E$ bounded are all equivalent. Indeed, given two such norms, $\| \, \|'$ and $\| \, \|''$, the identity map

$$\mathrm{id}_V \colon (V, \| \, \|') \longrightarrow (V, \| \, \|'')$$

has a closed graph, hence it is bounded by the Closed Graph Theorem.

The intersection $V \cap W$ and the sum $V + W$ of Banach subspaces is again Banach. Indeed, $V + W$ is the image of the bounded linear map from

$$\big(V \oplus W, \ \| \, \|_V + \| \, \|_W\big)$$

to E where $\| \, \|_V$ and $\| \, \|_W$ are complete norms on V and W, respectively.

As to $V \cap W$, a Cauchy sequence (u_n) in $V \cap W$ with respect to the norm $\| \, \|_V + \| \, \|_W$ is a Cauchy sequence with respect to both $\| \, \|_V$ and $\| \, \|_W$. In particular, it converges to a limit v in $(V, \| \, \|_V)$ and to a limit w in $(W, \| \, \|_W)$. Since the inclusions of $(V, \| \, \|_V)$ and $(W, \| \, \|_W)$ into E are bounded, both v and w must be the limit of (u_n) in E. Thus, they are equal which shows that they belong to $V \cap W$ and that $v = w$ is the limit of (u_n) with respect to $\| \, \|_V + \| \, \|_W$. In particular, $\| \, \|_V + \| \, \|_W$ is a complete norm on $V \cap W$.

If A is a normed algebra, the intersection of the lattice of Banach subspaces of A with the lattice of ideals in A, forms the lattice whose members will be referred to as *Banach ideals* in A.

2.2. **Contractive bimodule norms.** A normed vector space M with a unitary[1] (A, B)-bimodule structure over a pair of unital normed algebras A and B admits an equivalent norm $\| \, \|'$ such that

$$\|amb\|' \leq \|a\|_A \, \|m\|' \, \|b\|_B \qquad (a \in A, \, b \in B).$$

We shall call such a norm *bicontractive*. Indeed, if $\| \, \|_M$ is a norm on M such that

$$\|am\|_M \leq K_l \|a\|_A \, \|m\|_M \qquad \text{and} \qquad \|mb\|_M \leq K_r \|m\|_M \, \|b\|_B \qquad (a \in A, \, b \in B),$$

for certain constants $K_l > 0$ and $K_r > 0$, then

$$\|m\|' := \sup_{\|a\| \leq 1, \|b\| \leq 1} \|amb\|_M$$

is bicontractive and is equivalent to $\| \, \|_M$:

$$\| \, \|_M \ \leq \ \| \, \|' \ \leq \ K_l K_r \| \, \|_M.$$

This discussion lends itself to extension to nonunitary bimodules by noting that a normed nonunitary (A, B)-bimodule is the same as a unitary bimodule over the unitalizations $\tilde{A}_k$ and $\tilde{B}_k$ where $k = \mathbf{R}$ or $\mathbf{C}$ as appropriate.

In particular, every normed ideal I in a normed algebra A admits a bicontractive norm. We shall denote it $\| \, \|_I$. If $(a', a'') \in A^2$ is pair of elements satisfying

$$a'' a' = 1_A \qquad \text{and} \qquad \|a'\|_A = \|a''\|_A = 1, \tag{2.1}$$

then

$$\|a' x\|_I \ = \ \|x\|_I \ = \ \|x a''\|_I \qquad (x \in I).$$

[1]This means that $1_A m = m = m 1_B$ for all $m \in M$.

Condition (2.1) is equivalent to the apparently weaker condition

$$a''a' = 1_A, \qquad \|a'\|_A \leq 1 \qquad \text{and} \qquad \|a''\|_A \leq 1.$$

In particular, the set $A_1^\natural$ of such pairs $(a', a'') \in A^2$ forms a monoid under the composition law

$$(a', a'') \circ (b', b'') := (a'b', b''a'')$$

and the original bicontractive norm is invariant under the action of $A_1^\natural$ on I:

$$(a', a'') \cdot x := a'xa'' \qquad (x \in I).$$

We shall refer to members of $A_1^\natural$ as *unit multipliers*.

For $A = \mathcal{B}(H)$, the algebra of bounded operators on a Hilbert space H, the correspondence

$$U \longmapsto (U, U^*)$$

identifies the monoid of partial isometries on H with a submonoid of $\mathcal{B}(H)_1^\natural$. It follows that any normed ideal of operators on a Hilbert space admits a bicontractive norm and every bicontractive norm is invariant under the action of the monoid of partial isometries. Norms of this kind are said to be *symmetric*. Any such norm is determined by its restriction to the vector subspace $\operatorname{Diag} I$ of diagonal operators in a certain basis Γ and that restriction is *rearrangement invariant*, i.e., invariant under the action of the monoid of injective selfmaps $\Gamma \rightarrowtail \Gamma$. Vice-versa, any rearrangement invariant norm on $\operatorname{Diag} I$ has a unique extension to a symmetric norm on I, cf. [2].

2.3. **The multiplier ideal of a pair of operator ideals.** Given a pair of ideals I and J in $\mathcal{B}(H)$, the set $\mathcal{M}(I, J)$ of operators $M \in \mathcal{B}(H)$, such that the operator

$$M \otimes (\ ) \colon \mathcal{B}(H) \longrightarrow \mathcal{B}(H \tilde{\otimes} H), \qquad T \longmapsto M \otimes T, \tag{2.2}$$

sends operators from $I(H)$ to operators in $J(H \tilde{\otimes} H)$ is an ideal in $\mathcal{B}(H)$. We shall refer to it as the *tensor multiplier ideal from I to J*.

The lattice of nonzero proper ideals in the algebra of bounded operators on a separable Hilbert space has the smallest element, the ideal of finite rank operators $\mathcal{F}$, and the largest element, the ideal of compact operators $\mathcal{K}$. Note that

$$\mathcal{M}(I, J) \subseteq \mathcal{M}(\mathcal{F}, J) = J \qquad \text{if} \qquad I \neq 0$$

and

$$I \subseteq J \qquad \text{if and only if} \qquad \mathcal{M}(I, J) \neq 0.$$

If $\|\ \|_J$ is a symmetric norm on J and $p \in \mathcal{F}$ is a rank one projection, then $p \otimes T$ is of the form UTU^* for a certain partial isometry U when we identify $H \tilde{\otimes} H$ with H. In particular,

$$\|p \otimes T\|_J = \|T\|_J \qquad (T \in J).$$

It follows that, given any sequence (p_n) of projections of rank 1 and a summable sequence (λ_n) of coefficients, the sequence of operators

$$\sum_{i \leq n} \lambda_i p_i \otimes T$$

is a Cauchy sequence in J. If J is complete with respect to $\|\ \|_J$, then the sequence converges and its limit equals

$$\Big(\sum \lambda_n p_n \Big) \otimes T.$$

This shows that $\mathcal{M}(J, J)$, and therefore also $\mathcal{M}(I, J)$, for any $I \subseteq J$, contain the ideal of trace class operators $\mathcal{L}_1$, if J is a Banach ideal.

The tensor multiplier ideal $\mathcal{M}(I, J)$ has been introduced by one of us in 1990 in the study leading to the calculation of the algebraic K-groups of a broad class of operator ideals on a separable complex Hilbert space. That was reported in [15], in numerous seminar and conference talks, and in detailed lecture courses, from early 1990-ties until 2004. In particular, the fact that every Banach ideal in $B(H)$ is tensorially multiplied by $\mathcal{L}_1$ played an essential role in dermination of the commutator structure of operator ideals [2].

3. Algebraic K-theory of operator ideals in degrees less or equal 0

Algebraic K-theory in nonpositive degrees possesses two essential features: it annihilates filtered unions of nilpotent rings and satisfies excision, which manifests itself in the existence of a natural long exact sequence

$$\cdots \xleftarrow{\partial'_n} K_n(A'') \xleftarrow{K_n(\pi)} K_n(A) \xleftarrow{K_n(\iota)} K_n(A') \xleftarrow{\partial'_{n+1}} K_{n+1}(A'') \xleftarrow{K_{n+1}(\pi)} \cdots, \tag{3.1}$$

in degrees $n < 0$, for any composable pair of ring homomorphisms

$$A'' \overset{\pi}{\twoheadleftarrow} A \xleftarrow{\iota} A'$$

exact in A and with $\operatorname{Ker}\iota$ nilpotent. The connecting homomorphisms in (3.1) are the unique homorphisms making the triangles

$$\begin{tikzcd} K_n(A') \arrow[dr, "\simeq"'] & & K_{n+1}(A'') \arrow[ll, dashed, "\partial'_{n+1}"'] \arrow[dl, "\partial_{n+1}"] \\ & K_n(\operatorname{Ker}\pi) & \end{tikzcd}$$

commute. The following lemma is an immediate consequence.

Lemma 3.1. *Let n be a negative integer. Given an associative $\mathcal{S}$-graded ideal* (1.3)*, if*

$$K_n(\mathcal{A}_s) = K_{n+1}(\mathcal{A}_s) = 0 \qquad (s \in \mathcal{S}),$$

then the connecting homomorphism

$$K_n(\mathcal{B}_s) \xleftarrow{\partial'_{n+1}} K_{n+1}(\mathcal{C}_s)$$

is an isomorphism. Here $\mathcal{C}$ denotes the $\mathcal{S}$-graded algebra formed by the cokernels of ι_s, cf. (1.4).

In particular, given a morphism of graded $\mathcal{S}$-ideals,

$$\begin{tikzcd} \mathcal{B}' \arrow[d, "\iota'"] & \mathcal{B} \arrow[l, dashed] \arrow[d, "\iota"] \\ \mathcal{A}' & \mathcal{A} \arrow[l, dashed] \end{tikzcd}$$

with the property that also

$$K_n(\mathcal{A}'_s) = K_{n+1}(\mathcal{A}'_s) = 0 \qquad (s \in \mathcal{S}),$$

the induced map

$$K_n(\mathcal{B}'_s) \longleftarrow K_n(\mathcal{B}_s)$$

is an isomorphism, for any particular $s \in \mathcal{S}$, if and only if the map

$$K_{n+1}(\mathcal{C}'_s) \longleftarrow K_{n+1}(\mathcal{C}_s)$$

is an isomorphism.

Aided by the above lemma we shall establish the following proposition.

Proposition 3.2. *Let q be a natural number and n be an integer satisfying*

$$n + q \leq 0.$$

Suppose $\mathcal{A}$ to be an associative $\mathbf{2}^q$-graded ring such that

$$K_m(\mathcal{A}_s) = 0 \qquad (n \leq m \leq n + q)$$

for all $s \in \mathbf{2}^q$ except the maximal element $1 \cdots 1$. Then one has a natural isomorphism

$$K_n(\mathcal{A}_{1\cdots 1}) \xleftarrow{\;\simeq\;} K_{n+q}\left(\frac{\mathcal{A}_{0\cdots 0}}{\bar{\mathcal{A}}^{(1)} + \cdots + \bar{\mathcal{A}}^{(q)}}\right)$$

where $\bar{\mathcal{A}}^{(j)}$ denotes the image of

$$\mathcal{A}^{(j)} = \mathcal{A}_{0\cdots 1\cdots 0}$$

in $\mathcal{A}_{0\cdots 0}$ and $0 \cdots 1 \cdots 0$ is the q-tuple with the single 1 located at position j.

The assertion is proven by induction on q. For $q = 1$, it is an immediate corollary of the remarks opening this chapter. For a general $\mathbf{2}^q$-graded ring $\mathcal{A}$, we consider the corresponding $\mathbf{2}^{q-1}$-graded ideal

$$\mathcal{A}_0 \longleftarrow \mathcal{A}_1 \tag{3.2}$$

where the restrictions of $\mathcal{A}$ to $\mathbf{2}^{q-1} \times \{0\}$ and, respectively, to $\mathbf{2}^{q-1} \times \{1\}$,

$$(\mathcal{A}_0)_s := \mathcal{A}_{s0} \qquad \text{and} \qquad (\mathcal{A}_1)_s := \mathcal{A}_{s1}, \qquad \left(s \in \mathbf{2}^{q-1}\right),$$

are viewed as $\mathbf{2}^{q-1}$-graded rings. We apply Lemma 3.1 and note that the cokernel $\mathcal{C}$ of (3.2) satisfies the proposition hypothesis for $n + 1$ and $q - 1$. Invocation of the inductive hypothesis combined with the Noether isomorphism

$$\frac{\mathcal{C}_{0\cdots 0}}{\bar{\mathcal{C}}^{(1)} + \cdots + \bar{\mathcal{C}}^{(q-1)}} \xrightarrow{\;\simeq\;} \frac{\mathcal{A}_{0\cdots 0}}{\bar{\mathcal{A}}^{(1)} + \cdots + \bar{\mathcal{A}}^{(q)}}$$

concludes the proof.

Corollary 3.3. *Given a morphism of associative $\mathbf{2}^q$-graded rings $\mathcal{A} \to \mathcal{A}'$, satisfying*

$$K_m(\mathcal{A}'_s) = 0 \qquad (n \leq m \leq n + q,\ s \neq 1 \cdots 1),$$

the induced map

$$K_n(\mathcal{A}_{1\cdots 1}) \longrightarrow K_n(\mathcal{A}'_{1\cdots 1})$$

is an isomorphism if and only if the map

$$K_{n+q}\left(\frac{\mathcal{A}_{0\cdots 0}}{\bar{\mathcal{A}}^{(1)} + \cdots + \bar{\mathcal{A}}^{(q)}}\right) \longrightarrow K_{n+q}\left(\frac{\mathcal{A}'_{0\cdots 0}}{\bar{\mathcal{A}}'^{(1)} + \cdots + \bar{\mathcal{A}}'^{(q)}}\right)$$

is an isomorphism.

If $\mathcal{A}$ forms the octant

$$A_{p_1\cdots p_q} \quad (0 \le p_1, \ldots, p_q \le 1),$$

in the commutative $3\times\cdots\times 3$ hypercube

$$A_{p_1\cdots p_q} \quad (-1 \le p_1, \ldots, p_q \le 1),$$

formed by the extensions

$$A_{p_1\cdots\,-1\,\cdots p_q} \twoheadleftarrow A_{p_1\cdots\,0\,\cdots p_q} \longleftarrow\!\prec A_{p_1\cdots\,1\,\cdots p_q}\ , \tag{3.3}$$

then

$$\frac{\mathcal{A}_{0\cdots 0}}{\bar{\mathcal{A}}^{(1)} + \cdots + \bar{\mathcal{A}}^{(q)}} \;\simeq\; A_{-1\,\cdots\,-1}.$$

In this case, Proposition 3.2 reads

Proposition 3.4. *Let q be a natural number and n be an integer such that $n+q \le 0$. If*

$$K_m(\mathcal{A}_s) = 0 \qquad (n \le m \le n+q),$$

for all $s \in \mathbf{2}^q$ except the maximal element $1\cdots 1$, then one has a natural isomorphism

$$K_n(\mathcal{A}_{1\cdots 1}) \xleftarrow{\ \simeq\ } K_{n+q}(A_{-1\,\cdots\,-1})\ .$$

If extensions (3.3) are extensions of Banach algebras, the same argument establishes also the topological version of the above proposition.

Proposition 3.5. *Let q be a natural number and n be an integer satisfying*

$$n+q \le 0.$$

If

$$K_m^{\mathrm{top}}(\mathcal{A}_s) = 0 \qquad (n \le m \le n+q)$$

for all $s \in \mathbf{2}^q$ except the maximal element $1\cdots 1$, then one has a natural isomorphism

$$K_n^{\mathrm{top}}(\mathcal{A}_{1\cdots 1}) \xleftarrow{\ \simeq\ } K_{n+q}^{\mathrm{top}}(A_{-1\,\cdots\,-1})$$

The following scenario is of particular interest to us. Suppose that we are given Banach ideals $I_1, \ldots, I_q$, in the sense of Chapter 2, in *topologically* flabby Banach algebras $B_1, \ldots, B_q$. By *topologically flabby* we mean a property that guarantees vanishing of the algebraic K-groups of $A\widehat{\otimes}_\pi B_1, \ldots, A\widehat{\otimes}_\pi B_q$ for any *unital* Banach algebra A (cf. Chapter III of [3]).

By resorting to the unitalization extension

$$k \twoheadleftarrow \tilde{A}_k \longleftarrow\!\prec A \qquad (k = \mathbf{R} \text{ or } \mathbf{C}, \text{ as appropriate})$$

which, being split, induces split short exact sequences of K_n-groups for $n \le 0$, we deduce that the above mentioned algebraic K-groups vanish in nonpositive degrees for any nonunital algebra A.

Let us form the corresponding $\mathbf{2}$-graded Banach ideals

$$\mathcal{A}_1 = (B_1 \hookleftarrow I_1), \qquad \ldots, \qquad \mathcal{A}_q = (B_q \hookleftarrow I_q).$$

The exterior tensor product

$$A\widehat{\boxtimes}_\pi \mathcal{A}_1\widehat{\boxtimes}_\pi \cdots \widehat{\boxtimes}_\pi \mathcal{A}_q$$

is a $\mathbf{2}^q$-graded ring that satisfies the hypothesis of Proposition 3.2. In particular, one has a natural isomorphism

$$K_{-q}\big(A\widehat{\otimes}_\pi I_1\widehat{\otimes}_\pi \cdots \widehat{\otimes}_\pi I_q\big) \xleftarrow{\ \simeq\ } K_0\left(\frac{A\widehat{\otimes}_\pi B_1\widehat{\otimes}_\pi \cdots \widehat{\otimes}_\pi B_q}{\bar{I}^{(1)}+\cdots+\bar{I}^{(q)}}\right)$$

where $\bar{I}^{(j)}$ is the image of

$$A\widehat{\otimes}_\pi B_1\widehat{\otimes}_\pi \cdots \widehat{\otimes}_\pi I_j\widehat{\otimes}_\pi \cdots \widehat{\otimes}_\pi B_q$$

in

$$A\widehat{\otimes}_\pi B_1\widehat{\otimes}_\pi \cdots \widehat{\otimes}_\pi B_q. \tag{3.4}$$

The closure L of the Banach ideal $\bar{I}^{(1)}+\cdots+\bar{I}^{(q)}$ in the projective tensor product of Banach algebras (3.4) is a two-sided ideal. By a density argument established in [7], Theorem 1.12, the epimorphism

$$\frac{A\widehat{\otimes}_\pi B_1\widehat{\otimes}_\pi \cdots \widehat{\otimes}_\pi B_q}{\bar{I}^{(1)}+\cdots+\bar{I}^{(q)}} \twoheadrightarrow \frac{A\widehat{\otimes}_\pi B_1\widehat{\otimes}_\pi \cdots \widehat{\otimes}_\pi B_q}{L}$$

induces an isomorphism of K_0-groups. The same result holds if we replace ideals $I_1,\dots,I_q$ by their closures $I'_1,\dots,I'_q$ in $B_1,\dots,B_q$. The morphisms of $\mathbf{2}$-graded ideals

$$\begin{tikzcd} I'_1 & I_1 \arrow[l, tail] \arrow[d, tail] \\ B_1 \arrow[u, tail, leftarrow] \arrow[r, equal] & B_1 \end{tikzcd}, \quad \dots, \quad \begin{tikzcd} I'_q & I_q \arrow[l, tail] \arrow[d, tail] \\ B_q \arrow[u, tail, leftarrow] \arrow[r, equal] & B_q \end{tikzcd},$$

induce a homomorphism of the corresponding exterior tensor products. In the commutative triangle

$$\begin{tikzcd} K_0\left(\frac{A\widehat{\otimes}_\pi B_1\widehat{\otimes}_\pi \cdots \widehat{\otimes}_\pi B_q}{\bar{I}'^{(1)}+\cdots+\bar{I}'^{(q)}}\right) \arrow[dr, "\simeq"'] & & K_0\left(\frac{A\widehat{\otimes}_\pi B_1\widehat{\otimes}_\pi \cdots \widehat{\otimes}_\pi B_q}{\bar{I}^{(1)}+\cdots+\bar{I}^{(q)}}\right) \arrow[ll] \arrow[dl, "\simeq"] \\ & K_0\left(\frac{A\widehat{\otimes}_\pi B_1\widehat{\otimes}_\pi \cdots \widehat{\otimes}_\pi B_q}{L}\right) & \end{tikzcd}$$

two maps are isomorphisms, hence also the third one. By invoking Corollary 3.3, we deduce that the comparison map

$$K_{-q}\big(A\widehat{\otimes}_\pi I'_1\widehat{\otimes}_\pi \cdots \widehat{\otimes}_\pi I'_q\big) \longleftarrow K_{-q}\big(A\widehat{\otimes}_\pi I_1\widehat{\otimes}_\pi \cdots \widehat{\otimes}_\pi I_q\big) \tag{3.5}$$

is an isomorphism.

Theorem 1. *Given Banach ideals $I_1, \ldots, I_q$ in topologically flabby Banach algebras $B_1, \ldots, B_q$, the comparison map* (3.5) *is an isomorphism for any Banach algebra A.* □

By applying this to $B_1 = \cdots = B_q$ being the algebra of bounded operators on a separable Hilbert space, we deduce that the comparison maps

$$K_{-q}(A\widehat{\otimes}_\pi \mathcal{K}\widehat{\otimes}_\pi \cdots \widehat{\otimes}_\pi \mathcal{K}) \longleftarrow K_{-q}(A\widehat{\otimes}_\pi I_1\widehat{\otimes}_\pi \cdots \widehat{\otimes}_\pi I_q)$$

are isomorphisms whenever $I_1, \cdots, I_q$ are arbitrary Banach ideals in $\mathcal{B}(H)$. Recalling that

$$K^{\mathrm{top}}_{-q}(A) \xleftarrow{\simeq} K_{-q}(A\widehat{\otimes}_\pi \mathcal{K}\widehat{\otimes}_\pi \cdots \widehat{\otimes}_\pi \mathcal{K})$$

(the complex case was established in 1991 by one of us, cf. [15], the real case was established more recently in [8]), we obtain the following result.

Theorem 2. *For any complex or real Banach algebra A and arbitrary proper Banach ideals $I_1, \ldots, I_q$ in $\mathcal{B}(H)$, the canonical comparison maps*

$$K^{\mathrm{top}}_{-q}(A) \xleftarrow{\simeq} K_{-q}(A\widehat{\otimes}_\pi I_1\widehat{\otimes}_\pi \cdots \widehat{\otimes}_\pi I_q) \qquad (q \geq 0)$$

are isomorphisms. □

In the terminology introduced in Chapter 1 of [8], Theorem 2 asserts that the Banach algebra

$$A\widehat{\otimes}_\pi I_1\widehat{\otimes}_\pi \cdots \widehat{\otimes}_\pi I_q$$

is K_{-q}-stable, for $q \geq 0$. Let us recall from [8], *ibidem*, that a *stable retract* of a K_n-stable Banach algebra is K_n-stable. In Section 2.3, we observed that the ideal of trace class operators $\mathcal{L}_1$ is contained in the tensor multiplier $\mathcal{M}(I, I)$. Noting that the composite inclusion

$$I \xrightarrow{\simeq} \mathbf{K}^{\otimes_{\mathbf{K}}(q-1)} \otimes_{\mathbf{K}} I \hookrightarrow \mathcal{F}^{\otimes_{\mathbf{K}}(q-1)} \otimes_{\mathbf{K}} I \hookrightarrow \mathcal{L}_1^{\widehat{\otimes}_\pi (q-1)}\widehat{\otimes}_\pi I \longrightarrow I\big(H^{\tilde{\otimes} q}\big),$$

is isomorphic to the stabilization inclusion

$$I \hookrightarrow M_2(I),$$

shows that I is a stable retract of the Banach algebra

$$\mathcal{L}_1^{\widehat{\otimes}_\pi (q-1)}\widehat{\otimes}_\pi I.$$

Here $\mathbf{K}$ denotes the ground field, $\mathbf{C}$ or $\mathbf{R}$. In particular, $A\widehat{\otimes}_\pi I$ is a stable retract of

$$A\widehat{\otimes}_\pi \mathcal{L}_1^{\widehat{\otimes}_\pi (q-1)}\widehat{\otimes}_\pi I$$

and we arrive at the following important corollary of Theorem 2.

Corollary 3.6. *For any complex or real Banach algebra A and an arbitrary proper Banach ideal I in $\mathcal{B}(H)$, the canonical comparison maps*

$$K^{\mathrm{top}}_{-q}(A) \xleftarrow{\simeq} K_{-q}(A\widehat{\otimes}_\pi I) \qquad (q \geq 0)$$

are isomorphisms. □

4. Secant extensions

Let k be a unital commutative ring. An extension of nonunital k-algebras, equipped with a k-module splitting,

$$\mathcal{E}: \qquad A \underset{s}{\overset{\pi}{\twoheadleftarrow}} B \overset{\iota}{\hookleftarrow} K\,, \tag{4.1}$$

is said to be *secant* if B is the smallest k-algebra of B which contains the image of s.

Secant extensions of a k-algebra A form a category, $\operatorname{Sec} A$, with morphisms from $\mathcal{E}$ to another secant extension,

$$\mathcal{E}': \qquad A \underset{s'}{\overset{\pi'}{\twoheadleftarrow}} B' \overset{\iota'}{\hookleftarrow} K\,,$$

given by k-algebra homomorphisms $\alpha\colon B \longrightarrow B'$, such that

$$\pi' \circ \alpha = \pi \qquad \text{and} \qquad s' = \alpha \circ s.$$

The category of secant extension of A is skeletal, i.e., there is no more than a single morphism from an object to an object. The extension

$$\mathcal{E}_{\text{ter}}: \qquad A \underset{\text{id}}{\overset{\text{id}}{\longleftarrow}} A \longleftarrow 0\,,$$

is a terminal object. The extension

$$\mathcal{E}_{\text{in}}: \qquad A \underset{i}{\overset{\mu}{\twoheadleftarrow}} F \overset{\kappa}{\hookleftarrow} R\,, \tag{4.2}$$

where

$$F := \bigoplus_{n>0} A^{\otimes_k n}$$

is the nonunital tensor algebra of the k-module A and

$$a_1 \cdots a_n \overset{\mu}{\longleftarrow\!\shortmid} a_1 \otimes \cdots \otimes a_n$$

denotes the multiplication map, is an initial object. The k-linear splitting identifies A with the degree 1 component of F and κ is the kernel of μ.

Note that

$$R = R_2 + R_3 + \cdots$$

where R_n is the k-submodule of F, additively spanned by terms

$$\rho(a_1, \ldots, a_n) := a_1 \cdots a_n - a_1 \otimes \cdots \otimes a_n.$$

Proposition 4.1. *The left and the right ideals in F, generated by the k-submodule R_2, coincide with R*

$$R_2 + FR_2 = R = R_2 + R_2 F.$$

Proof. The identities

$$\begin{aligned} \rho(a_1, \ldots, a_n) &= \rho(a_1, a_2 \cdots a_n) + a_1 \otimes \rho(a_2, \ldots, a_n) \\ &= \rho(a_1 \cdots a_{n-1}, a_n) + \rho(a_1, \ldots, a_{n-1}) \otimes a_n \end{aligned}$$

demonstrate that

$$R_n \subset (R_2 + A \otimes R_{n-1}) \cap (R_2 + R_{n-1} \otimes A) \qquad (n > 2).$$

$\square$

The unique morphism $\mathcal{E}_{\mathrm{in}} \longrightarrow \mathcal{E}$ yields the commutative diagram

$$\begin{tikzcd}
 & R_s \arrow[r, equal] \arrow[d, tail] & R_s \arrow[d, tail] \\
A \arrow[d, equal] & F \arrow[l, two heads, "\mu"'] \arrow[d, two heads, "\tilde{s}"] & R \arrow[l, tail, "\kappa"'] \arrow[d, two heads] \\
A \arrow[u, phantom] & B \arrow[l, two heads, "\pi"'] & K \arrow[l, tail, "\iota"']
\arrow[from=2-1, to=2-2, dotted, bend right, "i"']
\arrow[from=3-1, to=3-2, dotted, bend right, "s"']
\end{tikzcd}$$

where the k-algebra homomorphism $\tilde{s}$ is induced by the splitting of $\mathcal{E}$ and R_s denotes the kernel of $\tilde{s}$. The correspondence

$$\mathcal{E} \leadsto R_s$$

gives rise to a functor from $\operatorname{Sec} A$ to the lattice Λ_A of two-sided ideals in F, contained in the ideal of A-relations R. This is an equivalence of categories with a quasi-inverse provided by the functor

$$J \leadsto \mathcal{E}_J\,, \qquad (J \in \Lambda_A)$$

that associates, with any member of Λ_A, the *quotient* of $\mathcal{E}_{\mathrm{in}}$ by J

$$\mathcal{E}_J: \qquad \begin{tikzcd} A & F/J \arrow[l, two heads] & R/J \arrow[l, tail] \arrow[from=1-1, to=1-2, dotted, bend right] \end{tikzcd} .$$

4.1. **An extension associated with a k-linear map.** A k-module map $s\colon A \longrightarrow B$ between nonunital k-algebras gives rise to the commutative square of k-algebra extensions

(4.3)
$$\begin{tikzcd}
I \arrow[d, dotted, tail] & R_s \arrow[l, dotted, two heads] \arrow[d, tail] & R_s \cap R \arrow[l, dotted, tail] \arrow[d, tail] \\
A \arrow[d, dotted, two heads] \arrow[dr, dotted, "s"'] & F \arrow[l, two heads, "\mu"'] \arrow[d, two heads, "\tilde{s}"] & R \arrow[l, tail, "\kappa"'] \arrow[d, two heads] \\
A_s/K_s & A_s \arrow[l, two heads] & K_s \arrow[l, tail]
\arrow[from=2-1, to=2-2, dotted, bend right, "i"']
\end{tikzcd}$$

with A_s being the subring of B generated by the image of s and $\tilde{s} \circ i = s$.

Note that $I = 0$, i.e., s yields the splitting of a secant extension of A, precisely when the kernel R_s of $\tilde{s}$ is contained in the ideal of relations $R \subseteq F$. Equivalently stated, equality

(4.4)
$$\sum s(a_{j1}) \cdots s(a_{jn_j}) = 0 \qquad \text{in} \quad B$$

implies

(4.5)
$$\sum a_{j1} \cdots a_{jn_j} = 0 \qquad \text{in} \quad A.$$

We shall call such s a *secant map* from a k-algebra A to a k-algebra B.

A simple criterion for s to be secant is provided by the following corollary of Proposition 4.1

Proposition 4.2. *If $L \subset B$ is a left or right ideal such that*

$$\rho_s(a_1, a_2) := s(a_1 a_2) - s(a_1)s(a_2) \in L \qquad (a_1, a_2 \in A)$$

and

$$s(a) \notin L \qquad (a \neq 0),$$

then s is secant. □

4.2. **Example: Secant evaluation exensions.** Let θ be a continuous function on the interval $[0, 2\pi]$ such that

$$\theta(0) = 0 \qquad \text{and} \qquad \theta(2\pi) = 1.$$

Given a unital subring $k \subseteq \mathbf{C}$ of the field of complex numbers, let us consider the k-linear map

$$s\colon k \longrightarrow \mathcal{C}', \qquad 1 \longmapsto \theta, \tag{4.6}$$

from k to the algebra $\mathcal{C}' := C_0[0, 2\pi]$ of continuous complex valued functions on the interval $[0, 2\pi]$ that vanish at the left end.

Let us denote by $\mathcal{O}'_\theta$ the k-subalgebra of $\mathcal{C}'$ generated by the image of s. Evaluation at the right end yields the diagram of extensions

$$\begin{tikzcd} k & \mathcal{O}'_\theta \arrow[l, two heads] \arrow[d, tail] & \mathcal{O}_\theta \arrow[l, tail] \arrow[d, tail] \\ \mathbf{C} \arrow[from=u, tail] & \mathcal{C}' \arrow[l, two heads] & \mathcal{C} \arrow[l, tail] \end{tikzcd} \tag{4.7}$$

where

$$\mathcal{C} := C_0[0, 2\pi]_0$$

denotes the ideal of functions vanishing at both ends, and $\mathcal{O}_\theta = \mathcal{O}'_\theta \cap \mathcal{C}$.

Since the image of s, the rank 1 k-module $k\theta$, has zero intersection with $\mathcal{C}$, the map given by (4.6) satisfies the hypothesis of Proposition 4.2. Accordingly, the top row of (4.7) is a secant extension.

As a k-module, $\mathcal{O}_\theta$ is freely generated by the functions

$$\vartheta_l := \theta^l(1 - \theta) \qquad (l = 1, 2, \dots).$$

Since

$$\vartheta_l \vartheta_m = \vartheta_{l+m} - \vartheta_{l+m+1},$$

the nonunital k-algebra $\mathcal{O}_\theta$ is generated by ϑ_1 and ϑ_2. It is isomorphic to the principal ideal in $k[t]$ generated by $t(1-t)$. We leave an elementary proof to the reader. The latter algebra is isomorphic to the maximal ideal $\mathfrak{m}_{(0,0)}$ of the singular point of the cubic curve

$$\operatorname{Spec} k[x, y]/(x^3 - xy + y^2).$$

4.3. **Secant dilations.** Let A be a subalgebra of B and $e \in B$ be an idempotent. Consider the associated *dilation* map

$$s_e\colon A \longrightarrow B, \qquad a \longmapsto eae \qquad (a \in A). \tag{4.8}$$

The subalgebra A_e of B, generated by the k-submodule eAe, is additively spanned by products

$$ea_1 e \cdots ea_n e.$$

Assuming that $B \ni 1$, the monoid of *multipliers*

$$B^\natural := \{(b', b'') \in B^2 \mid b''b' = 1\}$$

acts by endomorphisms on B

$$x \longmapsto b'xb'' \qquad (x \in B).$$

If either b' or b'' centralizes A, then

$$b'(ea_1ea_2e \cdots ea_ne)b'' = e'a_1e' \cdots e'a_ne'$$

where $e' = b'eb''$. We shall say, in this case, that the multiplier is *A-central*. An A-central multiplier (b', b'') induces a homomorphism of algebras

$$A_e \longrightarrow A_{e'}.$$

Assume that B is a topological algebra and (u_λ, v_λ) is a net of A-central multipliers such that

$$u_\lambda e v_\lambda \longrightarrow 1. \tag{4.9}$$

Then

$$u_\lambda(ea_1ea_2e \cdots ea_ne)v_\lambda \longrightarrow a_1 \cdots a_n. \tag{4.10}$$

In particular, s_e satisfies Condition (4.4)–(4.5) and we arrive at the following criterion for a dilation to be secant.

Proposition 4.3. *Let A be a subalgebra of a unital topological algebra B and e be an idempotent in B, such that there exists a net (u_λ, v_λ) of A-central multipliers in B satisfying condition* (4.9). *Then the dilation s_e associated with e is secant.* □

In the resulting secant extension

$$\begin{tikzcd} A \arrow[r, dotted, bend right, "s_e"'] & A_e \arrow[l, two heads, "\pi"'] & K_e \arrow[l, tail, "\iota"'] \end{tikzcd}, \tag{4.11}$$

one has

$$\pi(b) = \lim u_\lambda b v_\lambda \qquad (b \in A_e). \tag{4.12}$$

4.4. **Secant dilations of Banach algebras.** Multiplication on A may not be jointly continuous in the topology for which (4.9) holds, which does not make (4.10) a consequence of (4.9). This occurs, for example, for the weak topology on a Banach algebra, and for the strong topology on a Banach algebra of bounded operators on a Banach space. In either case, however, (4.10) holds and, moreover, the correspondence

$$b \longmapsto \pi(b) = \lim u_\lambda b v_\lambda \qquad (b \in A_e) \tag{4.13}$$

is *bounded*, if the net (u_λ, v_λ) is *bounded*, i.e., if

$$\sup_\lambda \|u_\lambda\| + \|v_\lambda\| < \infty.$$

Proposition 4.4. *Let A be a subalgebra of a unital Banach algebra B and e be an idempotent in B with the property that there exists a bounded net (u_λ, v_λ) of A-central multipliers in B such that condition* (4.10) *holds for a certain topology compatible with the structure of a vector space. Then dilation* (4.8) *is secant.*

If A is a closed subalgebra and correspondence (4.13) *is bounded, then dilation* (4.8) *becomes a bounded linear splitting of the Banach algebra extension*

$$A \underset{s_e}{\overset{\hat{\pi}_e}{\twoheadleftarrow}} \hat{A}_e \longleftarrow \hat{K}_e \tag{4.14}$$

where $\hat{\pi}_e$ denotes the extension by continuity of π_e to the norm-closure $\hat{A}_e$ of A_e and $\hat{K}_e$ denotes the kernel of $\hat{\pi}_e$. □

As mentioned above, the hypothesis of Proposition 4.4 is satisfied when condition (4.9) holds in the weak or in the strong topology and the net (u_λ, v_λ) is bounded.

The Banach algebra extension (4.14) is *topologically secant*, in the sense that s is a continuous linear splitting whose image generates a dense subalgebra in $\hat{A}_e$.

4.5. **Example: Toeplitz extensions.** Multiplication of complex valued functions on S^1 makes the Hilbert space $L^2(S^1)$ a module over the algebra $L^\infty(S^1)$. The correspondence

$$M\colon f \longmapsto M_f \qquad (f \in L^\infty(S^1)),$$

that associates with f the operator M_f of multiplication by f, embeds $L^\infty(S^1)$ isometrically onto the C^*-subalgebra $M(L^\infty(S^1))$ of $\mathcal{B}(L^2(S^1))$.

Let $P \in \mathcal{B}(L^2(S^1)))$ be the orthogonal projection onto the Hardy space H^2, the latter being spanned by the restrictions to S^1 of functions

$$1,\, z,\, z^2,\, \dots\,. \tag{4.15}$$

Note that in the strong operator topology one has

$$M_{\bar{z}^n} P M_{z^n} \longrightarrow \mathrm{id}_{L^2(S^1)}$$

and the strong limit correspondence

$$\pi\colon T \longmapsto s\text{-}\lim M_{\bar{z}^n} T M_{z^n} \qquad (T \in M(L^\infty(S^1))_P)$$

is a contractive map. In particular, the map

$$f \longmapsto PM_fP \qquad (f \in L^\infty(S^1))$$

is, according to Proposition 4.4, secant and π_P extends by continuity to the closure of the $*$-algebra $M(L^\infty(S^1))_P$ in $\mathcal{B}(L^2(S^1))$. By construction, the closure,

$$M(L^\infty(S^1))\widehat{\ }_P\,, \tag{4.16}$$

is a C^*-subalgebra of

$$P\mathcal{B}(L^2(S^1))P\,,$$

and the latter is naturally identified with the algebra of bounded operators on the Hardy space $\mathcal{B}(H^2)$. Viewed as an operator on H^2, the operator PM_fP is denoted T_f and is referred to as the *Toeplitz operator* associated with $f \in L^\infty(S^1)$. We shall refer to (4.16), viewed as a subalgebra of $\mathcal{B}(H^2)$, as the C^*-algebra of *Toeplitz operators*, shall denote it $\hat{\mathfrak{T}}$, and will refer to the topologically secant extension

$$L^\infty(S^1) \overset{\hat{\pi}}{\twoheadleftarrow} \hat{\mathfrak{T}} \longleftarrow \hat{\mathcal{K}} \tag{4.17}$$

as the *Toeplitz C^*-extension*. Epimorphism $\hat{\pi}$ is called the *symbol* map.

Given any subalgebra $\mathcal{O} \subseteq L^\infty(S^1)$, the restriction of the dilation s_P to $\mathcal{O}$ defines a subextension of (4.17)

$$\mathcal{O} \overset{\pi_{\mathcal{O}}}{\twoheadleftarrow} \mathcal{T}_{\mathcal{O}} \leftarrowtail \mathcal{K}_{\mathcal{O}} \tag{4.18}$$

and, for a Banach subalgebra of $L^\infty(S^1)$, the corresponding Banach algebra subextension

$$\mathcal{O} \overset{\hat{\pi}_{\mathcal{O}}}{\twoheadleftarrow} \hat{\mathcal{T}}_{\mathcal{O}} \leftarrowtail \hat{\mathcal{K}}_{\mathcal{O}} \tag{4.19}$$

For any $f, g \in C(S^1) + H^\infty$, the operator

$$T_{fg} - T_f T_g$$

is compact, where H^∞ denotes the subalgebra in $L^\infty(S^1)$ of functions extending to a holomorphic function on the open disk of radius one. Since T_f is not compact for $f \neq 0$, with the aid of Proposition 4.2 we obtain another proof that the Toeplitz correspondence

$$f \longmapsto T_f$$

is secant when restricted to any subalgebra of $C(S^1) + H^\infty$. In this case, the secant extension ideal is contained in the ideal of compact operators on H^2,

$$\mathcal{K}_{\mathcal{O}} \subseteq \mathcal{K}(H^2).$$

Incidentally, the fact that $C(S^1) + H^\infty$ is closed under multiplication was first pointed out by Sarason, cf. the last paragraph of Section 6 in [9]. It is a consequence of $\mathbf{C}[z, z^{-1}] + H^\infty$ being closed under multiplication, of $\mathbf{C}[z, z^{-1}]$ being dense in $C(S^1)$, and of

$$C(S^1) + A(D)^{**} = C(S^1) + H^\infty$$

being closed in $C(S^1)^{**} = L^\infty(S^1)$. Here

$$A(D) = C(S^1) \cap H^\infty$$

denotes the disk algebra.

5. Bott$_{-2q}$ rings of compact operators ($q > 0$)

Consider the *evaluation-at-the-right-end* extension

$$\mathbf{C} \twoheadleftarrow \mathcal{C}' \leftarrowtail \mathcal{C} \tag{5.1}$$

where

$$\mathcal{C}' := C_0[0, 2\pi]$$

denotes the algebra of continuous complex valued functions on the interval $[0, 2\pi]$, vanishing at the left end, and

$$\mathcal{C} := C_0[0, 2\pi]_0$$

denotes the ideal of functions vanishing at both ends.

Let $k \subseteq \mathbf{C}$ denote a unital subring of the field of complex numbers. Given q subextensions of k-algebras of (5.1)

$$\begin{tikzcd} k \arrow[d, tail] & \mathcal{O}'_i \arrow[l, two heads] \arrow[d, tail] & \mathcal{O}_i \arrow[l, tail] \arrow[d, tail] \\ \mathbf{C} & \mathcal{C}' \arrow[l, two heads] & \mathcal{C} \arrow[l, tail] \end{tikzcd}$$

consider the induced morphisms of the corresponding $\mathbf{2}^q$-graded ideals

$$
\begin{array}{c}
(\,\mathcal{O}_1' \longleftarrowtail \mathcal{O}_1\,) \boxtimes_k \cdots \boxtimes_k (\,\mathcal{O}_q' \longleftarrowtail \mathcal{O}_q\,) \\
\big\downarrow \\
(\,\mathcal{C}' \longleftarrowtail \mathcal{C}\,)^{\boxtimes_k q} \\
\big\downarrow \\
(\,\mathcal{C}' \longleftarrowtail \mathcal{C}\,)^{\boxtimes_{\mathbf{C}} q} \\
\big\downarrow \\
(\,\mathcal{C}' \longleftarrowtail \mathcal{C}\,)^{\widehat{\boxtimes}_\pi q} \\
\big\downarrow \\
(\,\mathcal{C}' \longleftarrowtail \mathcal{C}\,)^{\boxtimes_{\max} q}
\end{array}
\tag{5.2}
$$

Tensor product of $\mathbf{C}$-algebras is replaced in the bottom rows by projective tensor product of the same algebras viewed as Banach algebras, and by maximal tensor product when they are viewed as C^*-algebras. Since $\otimes_{\max}$ preserves exactness of extensions of C^*-algebras, the bottom row is the $\mathbf{2}^q$ octant in a $3\times\cdots\times 3$ hypercube of extensions of C^*-algebras. The row above it is, likewise, the $\mathbf{2}^q$ octant in a $3\times\cdots\times 3$ hypercube of extensions of Banach algebras in view of (5.1) being an extension split in the category of Banach spaces.

Recall that the C^*-algebra $\mathcal{C}'$ is *contractible*. That means that the identity map on $\mathcal{C}'$ is continuously homotopic to the zero map via C^*-algebra endomorphisms,

$$f \longmapsto t^*f \qquad \text{where} \qquad t^*f(x) := f(2\pi t x) \qquad (f \in \mathcal{C}').$$

Accordingly, the maximal tensor product with any C^*-algebra algebra A,

$$\mathcal{C}' \widehat{\otimes}_\pi A,$$

is contractible as a C^*-algebra, while the projective tensor product with any Banach algebra B,

$$\mathcal{C}' \widehat{\otimes}_\pi B,$$

is a contractible Banach algebra. Taking this into account, we infer that the bottom two rows of diagram (5.2) satisfy the hypothesis of Proposition 3.5. The resulting commutative diagram

$$
\begin{tikzcd}
K_{-q}(\mathcal{O}_1 \otimes_k \cdots \otimes_k \mathcal{O}_q) \arrow[dd] & K_0(k^{\otimes_k q}) \arrow[l] \arrow[dd, two heads] & \\
 & & K_0(\mathbf{Z}^{\otimes_{\mathbf{Z}} q}) \arrow[ul] \arrow[dl, "\simeq"] \\
K^{\mathrm{top}}_{-q}(\mathcal{C}^{\widehat{\otimes}_\pi q}) \arrow[d] & K_0(\mathbf{C}^{\otimes_{\mathbf{C}} q}) \arrow[l, "\simeq"'] \arrow[d, equal] & \\
K^{\mathrm{top}}_{-q}(\mathcal{C}^{\otimes_{\max} q}) & K_0(\mathbf{C}^{\otimes_{\mathbf{C}} q}) \arrow[l, "\simeq"'] &
\end{tikzcd}
\tag{5.3}
$$

yields the following proposition.

Proposition 5.1. *One has*

$$\mathbf{Z} \simeq K^{\mathrm{top}}_{-q}(\mathcal{C}^{\otimes_{\max} q}) \xleftarrow{\simeq} K^{\mathrm{top}}_{-q}(\mathcal{C}^{\widehat{\otimes}_\pi q}) \twoheadleftarrow K_{-q}(\mathcal{O}_1 \otimes_k \cdots \otimes_k \mathcal{O}_q) \ ,$$

the epimorphism being split. □

We identify $\mathcal{C}$ with the algebra of continuous complex valued functions on S^1, vanishing at 1, and $\mathcal{C}^{\otimes_{\max} q}$ with the algebra of continuous complex valued functions on the q-dimensional sphere $S^q = S^1 \wedge \cdots \wedge S^1$, vanishing at the distinguished point.

Consider now the sequence of morphisms of the 2^q-graded ideals

$$
\begin{tikzcd}
(\,\mathcal{T}_{\mathcal{O}_1} \leftarrowtail \mathcal{K}_{O_1}\,) \boxtimes_k \cdots \boxtimes_k (\,\mathcal{T}_{O_q} \leftarrowtail \mathcal{K}_{\mathcal{O}_q}\,) \arrow[d] \\
(\,\mathcal{T}_{\mathcal{C}} \leftarrowtail \mathcal{K}_{\mathcal{C}}\,)^{\boxtimes_k q} \arrow[d] \\
(\,\mathcal{T}_{\mathcal{C}} \leftarrowtail \mathcal{K}_{\mathcal{C}}\,)^{\boxtimes_{\mathbf{C}} q} \arrow[d] \\
(\,\hat{\mathcal{T}}_{\mathcal{C}} \leftarrowtail \hat{\mathcal{K}}_{\mathcal{C}}\,)^{\widehat{\boxtimes}_\pi q} \arrow[d] \\
(\,\hat{\mathcal{T}}_{\mathcal{C}} \leftarrowtail \hat{\mathcal{K}}_{\mathcal{C}}\,)^{\boxtimes_{\max} q} \arrow[d] \\
(\,\mathcal{B}(H^2) \leftarrowtail \mathcal{K}(H^2)\,)^{\boxtimes_{\max} q}
\end{tikzcd}
\tag{5.4}
$$

associated with the Toeplitz extensions of the k-algebras $\mathcal{O}_1, \dots, \mathcal{O}_q$, $\mathcal{C}$, and the C^* completion of the Toeplitz extension of $\mathcal{C}$. In the bottom row, we have the exterior power of the **2**-ideal of compact operators in the algebra of bounded operators on the Hardy space $H^2(S^1)$.

In orthogonal basis (4.15), the operator

$$T_{z^m}[T_{\bar{z}}, T_z]T_{\bar{z}^n} \tag{5.5}$$

is represented by the elementary matrix E_{mn}. Observing that (5.5) equals

$$[T_{1-\bar{z}}, T_{1-z}] - T_{1-z^m}[T_{1-\bar{z}}, T_{1-z}] - [T_{1-\bar{z}}, T_{1-z}]T_{1-\bar{z}^n} + T_{1-z^m}[T_{1-\bar{z}}, T_{1-z}]T_{1-\bar{z}^n},$$

we infer that $\mathcal{T}_{\mathcal{O}}$ contains an operator whose matrix is E_{mn} if $\mathcal{O}$ contains

$$1 - z, \quad ,1 - \bar{z}, \quad 1 - z^m \quad \text{and} \quad 1 - \bar{z}^n.$$

If so, then $\mathcal{T}_{\mathcal{C}}$ is dense in $\mathcal{K}(H^2)$ and, accordingly,

$$\hat{\mathcal{T}}_{\mathcal{C}} = \mathcal{K}(H^2).$$

Proposition 5.2. *One has*

$$K_*^{\mathrm{top}}\big(\hat{\mathcal{T}}_{\mathcal{C}}^{\widehat{\otimes}_\pi l}\widehat{\otimes}_\pi \mathcal{C}^{\widehat{\otimes}_\pi m}\big) = 0 \qquad (l > 0, m \geq 0) \tag{5.6}$$

and, similarly,

$$K_*^{\mathrm{top}}\big(\hat{\mathcal{T}}_{\mathcal{C}}^{\otimes_{\max} l}\otimes_{\max} \mathcal{C}^{\otimes_{\max} m}\big) = 0 \qquad (l > 0, m \geq 0). \tag{5.7}$$

Proof. Let us view (5.6) as a double sequence of statements S_{lm}. Contractibility of the Banach algebra $\mathcal{C}'\widehat{\otimes}_\pi A$ combined with the long exact sequence of topological K-groups associated with the Banach algebra extension

$$A \twoheadleftarrow \mathcal{C}'\widehat{\otimes}_\pi A \leftarrowtail \mathcal{C}\widehat{\otimes}_\pi A$$

yields the sequence of implications

$$S_{l,0} \implies S_{l,1} \implies S_{l,2} \implies \dots .$$

The long exact sequence associated with the Banach algebra extension

$$\hat{\mathcal{T}}_{\mathcal{C}}^{\otimes_{\max} l}\otimes_{\max}\mathcal{C} \twoheadleftarrow \hat{\mathcal{T}}_{\mathcal{C}}^{\otimes_{\max}(l+1)} \leftarrowtail \hat{\mathcal{T}}_{\mathcal{C}}^{\otimes_{\max} l}\otimes_{\max}\mathcal{K}(H^2)$$

yields the implication

$$S_{l,0} \wedge S_{l,1} \implies S_{l+1,0}.$$

It suffices, therefore, to establish $S_{1,0}$.

Taking into account Bott periodicity of the complex topological K-theory and the fact that

$$K_0^{\mathrm{top}}(\mathcal{C}) = K_1^{\mathrm{top}}(\mathcal{K}(H^2)) = 0,$$

the long exact sequence associated with the Toeplitz C^*-algebra extension

$$\mathcal{C} \twoheadleftarrow \hat{\mathcal{T}}_{\mathcal{C}} \leftarrowtail \mathcal{K}(H^2)$$

reduces to the following 4-term exact sequence

$$0 \longleftarrow K_0^{\mathrm{top}}(\hat{\mathcal{T}}_{\mathcal{C}}) \longleftarrow K_0^{\mathrm{top}}(\mathcal{K}(H^2)) \xleftarrow{\partial_1} K_1^{\mathrm{top}}(\mathcal{C}) \longleftarrow K_1^{\mathrm{top}}(\hat{\mathcal{T}}_{\mathcal{C}}) \longleftarrow 0 \tag{5.8}$$

The connecting homomorphism in the long exact sequence associated with the Toeplitz extension of the unitalization of $\mathcal{C}$ is the index map, hence we obtain the following commutative diagram

$$\begin{tikzcd}
K_0^{\mathrm{top}}(\mathcal{K}(H^2)) \arrow[d, equal] & K_1^{\mathrm{top}}(\mathcal{C}) \arrow[l, "\partial_1"'] \arrow[d, tail] \\
K_0^{\mathrm{top}}(\mathcal{K}(H^2)) & K_1^{\mathrm{top}}(C(S^1)) \arrow[l, two heads, "\mathrm{ind}"'] \arrow[d, two heads] \\
& K_1^{\mathrm{top}}(\mathbf{C})
\end{tikzcd}$$

with $K_1^{\mathrm{top}}(\mathbf{C}) = 0$. It follows that ∂_1 is an epimorphism from $K_1^{\mathrm{top}}(\mathcal{C}) \simeq \mathbf{Z}$ onto $K_0^{\mathrm{top}}(\mathcal{K}(H^2)) \simeq \mathbf{Z}$, hence an isomorphism. The 4-term exact sequence (5.8) now yields the vanishing of the topological K-groups of $\hat{\mathfrak{T}}_{\mathcal{C}}$

$$K_*^{\mathrm{top}}(\hat{\mathfrak{T}}_{\mathcal{C}}) = 0.$$

By replacing in the above arguments $\widehat{\otimes}_\pi$ by $\otimes_{\max}$, we obtain (5.7). $\square$

We deduce from Proposition (5.2) that the second and the third rows from the bottom in diagram (5.4) satisfy the hypothesis of Proposition 3.5. By combinining this with Proposition 5.1, we obtain the following result.

Proposition 5.3. *One has the commutative diagram*

$$\begin{tikzcd}
K_{-2q}(\mathcal{K}_{\mathcal{O}_1} \otimes_k \cdots \otimes_k \mathcal{K}_{\mathcal{O}_q}) \arrow[d] & K_{-q}(\mathcal{O}_1 \otimes_k \cdots \otimes_k \mathcal{O}_q) \arrow[l] \arrow[d, two heads] \\
K_{-2q}^{\mathrm{top}}(\hat{\mathcal{K}}_{\mathcal{C}}^{\widehat{\otimes}_\pi q}) \arrow[d, "\simeq"] & K_{-q}^{\mathrm{top}}(\mathcal{C}^{\widehat{\otimes}_\pi q}) \arrow[l, "\simeq"'] \arrow[d] \\
K_{-2q}^{\mathrm{top}}(\hat{\mathcal{K}}_{\mathcal{C}}^{\otimes_{\max} q}) \arrow[d, equal] & K_{-q}^{\mathrm{top}}(\mathcal{C}^{\otimes_{\max} q}) \arrow[l, "\simeq"'] \arrow[d] \\
K_{-2q}^{\mathrm{top}}(\mathcal{K}(H^2)^{\otimes_{\max} q}) & K_{-q}^{\mathrm{top}}((\mathcal{B}(H^2)/\mathcal{K}(H^2))^{\otimes_{\max} q}) \arrow[l, "\simeq"']
\end{tikzcd} \tag{5.9}$$

$\square$

A similar argument also yields the following commutative diagram of isomorphisms

$$\begin{tikzcd}
K^{\mathrm{top}}_{-2q}\big(\hat{\mathcal{K}}_{\mathcal{C}}^{\widehat{\otimes}_\pi q}\big) \arrow[d, equal] & K_{-q}\big(\mathcal{C}^{\widehat{\otimes}_\pi q}\big) \arrow[l, "\simeq"'] \arrow[d] \\
K^{\mathrm{top}}_{-2q}\big(\mathcal{K}(H^2)^{\widehat{\otimes}_\pi q}\big) \arrow[d, "\simeq"] & K^{\mathrm{top}}_{-q}\big((\mathcal{B}(H^2)/\mathcal{K}(H^2))^{\widehat{\otimes}_\pi q}\big) \arrow[l, "\simeq"'] \arrow[d] \\
K^{\mathrm{top}}_{-2q}\big(\mathcal{K}(H^2)^{\widehat{\otimes}_\pi q}\big) & K^{\mathrm{top}}_{-q}\big((\mathcal{B}(H^2)/\mathcal{K}(H^2))^{\otimes_{\max} q}\big) \arrow[l, "\simeq"']
\end{tikzcd}$$

We arrive at the following theorem.

Theorem 3. *A k-subalgebra*

$$\mathcal{L} \subseteq \mathcal{K}^{\otimes_{\max} q}$$

is a Bott$_{-2q}$ ring if it contains the image of

$$\mathcal{K}_{\mathcal{O}_1} \otimes_k \cdots \otimes_k \mathcal{K}_{\mathcal{O}_q}$$

for some $\mathcal{O}_1, \ldots, \mathcal{O}_q$ appearing as the kernel ideals in function algebra subextensions of (5.1).

Note that each k-algebra $\mathcal{O}_i$ in Theorem 3 contains $\mathcal{O}_\theta$ for every $\theta \in \mathcal{O}_i'$ such that $\theta(2\pi) = 1$, cf. Section 4.2. The following special case is, therefore, an equivalent form of Theorem 3.

Corollary 5.4. *A k-subalgebra $\mathcal{L} \subseteq \mathcal{K}^{\otimes_{\max} q}$ is a Bott$_{-2q}$ ring if it contains the image of*

$$\mathcal{K}_{\mathcal{O}_{\theta_1}} \otimes_k \cdots \otimes_k \mathcal{K}_{\mathcal{O}_{\theta_q}}$$

for some continuous functions $\theta_1, \ldots, \theta_q$ on the interval $[0, 2\pi]$ that vanish at 0 and have value 1 at 2π.

The proof of Theorem 3 uses only elementary facts about algebraic K-theory. It is independent of the fact that algebraic and topological K-theories are equipped with associative multiplicative structures making the comparison map between the two a ring homomorphism. Using that fact one can derive the assertion of the theorem from its special case $q = 1$.

References

[1] G. Cortiñas, A. Thom. Comparison between algebraic and topological K-thoery of locally convex algebras. Adv. Math. **218**, 266-307 (2008).

[2] K. Dykema, T. Figiel, G. Weiss and M. Wodzicki. Commutator structure of operator ideals. Adv. Math. **185**, 1-79 (2004).

[3] M. Karoubi. *Foncteurs dérivés et K-théorie*. Séminaire Heidelberg-Saarbrucken-Strasbourg (1967/68), Springer Lecture Notes **136**, 107–186 (1970).

[4] M. Karoubi. *La périodicité de Bott en K-théorie générale*. Ann. Sci. École Normale supérieure, 4ième série, **4**, 63-95 (1971).

[5] M. Karoubi. *A descent theorem in topological K-theory*. K-theory 24, 109-114 (2001).

[6] M. Karoubi. *K-théorie algébrique de certaines algèbres d'opérateurs*. Springer Lecture Notes **725**, 254–290 (1978).

[7] M. Karoubi. *Homologie de groupes discrets associès à des algèbres d'opèrateurs.* Journal of Operator Theory **15**, 109–161 (1986).

[8] M. Karoubi and M. Wodzicki. Algebraic and Hermitian K-theory of $\mathcal{K}$-rings. The Quarterly Journal of Mathematics **64** (2013), 903—940 (Daniel Quillen's memorial volume).
[9] D. Sarason. Generalized interpolation in H^∞. Trans. Amer. Math. Soc. **127** (1967), 179–203.
[10] A. Suslin and M. Wodzicki. *Excision in algebraic K-theory and Karoubi's conjecture.* Proc. NAS USA **87**, 9582–9584 (1990).
[11] A. Suslin and M. Wodzicki. *Excision in algebraic K-theory.* Ann. of Math. **136**, 51–122 (1992).
[12] M. Wodzicki. *The long exact sequence in cyclic homology associated with an extension of algebras.* C. R. Acad. Sci. Paris Sér. I Math. **306**, 399–403 (1988).
[13] M. Wodzicki. *Excision in cyclic homology and in rational algebraic K-theory.* Ann. of Math. **129**, 591–639 (1989).
[14] M. Wodzicki. *Homological properties of rings of functional-analytic type.* Proc. NAS USA **87**, 4910–491 (1990).
[15] M. Wodzicki. *Algebraic K-theory and functional analysis.* First European Congress, Vol. II (Paris, 1992), 485–496, Progr. Math. **120**, Birkhauser (1994).

Université Denis Diderot. UFR de Mathématiques Paris and University of California, Department of Mathematics. Berkeley CA

Email address: `max.karoubi@gmail.com and wodzicki@math.berkeley.edu`