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The Witt group of real surfaces

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Abstract. Let V be an algebraic variety defined over $\mathbb{R}$, and V_{top} the space of its complex points. We compare the algebraic Witt group $W(V)$ of symmetric bilinear forms on vector bundles over V, with the topological Witt group $WR(V_{\text{top}})$ of symmetric forms on Real vector bundles over V_{top} in the sense of Atiyah, especially when V is 2-dimensional. To do so, we develop topological tools to calculate $WR(V_{\text{top}})$, and to measure the difference between $W(V)$ and $WR(V_{\text{top}})$.

If V is a smooth algebraic surface defined over the real numbers $\mathbb{R}$, the Witt group $W(V)$ is a finitely generated abelian group, whose rank equals the number ν of components of the 2-manifold $V(\mathbb{R})$ [**30**]. Sujatha has given formulas for the torsion subgroup in [**48**]; although the formulas she gives are not always easy to compute, they involve étale cohomology and are thus mostly topological in nature. Indeed, Cox' theorem [**14**] states that $H^n_{\text{et}}(V,\mathbb{Z}/2) \cong \mathbb{H}^n_G(V_{\text{top}},\mathbb{Z}/2)$. Here V_{top} is the 4-manifold underlying the complex variety $V\times_{\mathbb{R}}\mathbb{C}$, G is the cyclic group $G = \text{Gal}(\mathbb{C}/\mathbb{R})$ of order 2, acting on V_{top} by complex conjugation, and $\mathbb{H}^*_G$ is Borel cohomology.

Together with Schlichting, we introduced the topological Witt group $WR(X)$ of a G-space X in [**23**] (see Definition 2.1 below), and showed that the natural map $W(V) \to WR(V_{\text{top}})$ is always an isomorphism modulo bounded 2-primary torsion; for curves it is an isomorphism; for surfaces, the kernel and cokernel of the map $W(V) \to WR(V_{\text{top}})$ have exponent 2 by [**23**, Thm. 8.7].

One of the main goals of this paper is to develop topological tools to compute $WR(X)$ when X has dimension ≤ 4, since if V is an algebraic surface then $WR(V_{\text{top}})$ is very close to $W(V)$. The second goal is to more precisely measure the difference between these two invariants in various examples.

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Here are three important invariants of W that have analogues for WR. One is the classical signature on $W(V)$, which factors through the topological signature on $WR(V_{\rm top})$, which is an invariant defined on the 1–skeleton of X for any G-space X (see Lemma 2.5). Another is the algebraic discriminant; its analogue defined on $WR(X)$ takes values in the group $\mathrm{Pic}_G(X)$ of equivariant real line bundles; see Definition 4.1. The third is the Hasse invariant (see Theorem 6.5.)

For any smooth projective surface V, $WR(V_{\rm top})$ is a birational invariant of V (see Theorem 5.1). If V is either a complex projective surface, so that $V_{\rm top} = G\times V(\mathbb{C})$, or a surface defined over $\mathbb{R}$ with no real points, (so that $V_{\rm top}$ is connected) the main theorems of [**26**] (see 7.1 and 7.2 below; cf. [**55**]) state that $W(V)\to WR(V_{\rm top})$ is a split surjection, and an isomorphism if and only if the geometric genus $p_g = \dim H^0_{\rm an}(X,\Omega^2_X)$ is 0. In particular, when V is a rational surface with G acting freely then $W(V)\cong WR(V_{\rm top})$. To compute this group it suffices to compute $WR(X)$ when $X = S^2\times S^2$. This is done in Theorem 7.4; there are 3 cases, corresponding to the 3 possible actions of G on $H^2(X,\mathbb{Z})$.

When G does not act freely, we determine the kernel and cokernel of $W(V)\to WR(V_{\rm top})$ in Theorem 8.7; the kernel is 0 if and only if $p_g(V)=0$. We also show in Theorem 8.6 that when $H^3_G(X;\mathbb{Z}(1))$ has no 2-torsion and X^G has $\nu>0$ connected components then

$$WR(X)\cong \mathbb{Z}^\nu \oplus H^1(X/G,\mathbb{Z}/2).$$

The tools used to compute $WR(X)$ are markedly different from the tools used to compute $W(V)$, because WR is related to Atiyah's Real K-theory and to the equivariant K-group KO_G. Both KR and KO_G satisfy Bredon's axioms [**7**] for a G-equivariant cohomology theory: homotopy invariance, excision, and long exact sequences. In the Appendix, we recall the facts we need about the Bredon spectral sequence converging to equivariant cohomology.

Our paper is organized as follows. In Section 1, we review Sujatha's results on the Witt group of a surface, as a reference for our results. In Section 2, we define $WR(X)$ and review the construction of the signature map. In Section 3, we compute $WR(X)$ when $\dim(X)\le 2$. We define the topological determinant in Section 4, relating it to the group $\mathrm{Pic}_G(X)$ of symmetric forms on Real line bundles.

In Section 5, we show that WR is a birational invariant of projective surfaces over $\mathbb{R}$. In Section 6, we use the Stiefel–Whitney classes w_1 and w_2 to define maps on $WR(X)$ compatible with the discriminant and Hasse invariant on $W(V)$.

In Section 7, we recall and illustrate the calculations of [**26**] for $WR(X)$ when V has no real points. In Section 8 we study the map $W(V) \to WR(X)$ when V has real points.

In Section 9, we say a few things about 3–folds. The group $WR(V_{\text{top}})$ is easy to describe for complex 3–folds, and not much harder for 3–folds over $\mathbb{R}$ which have no real points. We did not look very hard at 3-folds with real points, because of Parimala's result in [**37**], showing that for a smooth algebraic 3-fold that $W(V)$ is finitely generated if and only if the mod-2 Chow group $CH^2(V)/2$ is finitely generated. (We give the short proof in Theorem 9.1 below.) Totaro has shown in [**50**] that $CH^2(V)/2$ is in fact not finitely generated for a very general abelian 3-fold, making computations of $W(V)$ for 3-folds problematic.

The Fundamental Theorem for Witt theory says that $W(V \times \mathbb{G}_m) \cong W(V) \oplus W(V)$ for a smooth variety V. This is true more generally when $K_{-1}(V) = 0$ (which is always true for smooth V); see [**24**]. In Section 10 of the present paper, we show that if $KR_{-1}(X) = 0$ then $WR(X \times S^{1,1}) \cong WR(X) \oplus WR(X)$ where $S^{1,1}$ is the circle $(\mathbb{G}_m)_{\text{top}}$.

Notation. Recall that G denotes $\mathbb{Z}/2$. If X is a space with involution, we write $\mathbb{H}^*_G(X, A)$ for the Borel cohomology of X with coefficients in a G-module A. Similarly, we write $H^*_G(X, h)$ for the Bredon cohomology of X, where h is a coefficient system; see the Appendix. We will occasionally write $H_{\text{et}}(V)$ for $H_{\text{et}}(V, \mathbb{Z}/2)$.

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1. A filtration on $W(V)$

Let k be a field containing 1/2. A *symmetric form* (E, θ) on an algebraic variety V over k is an algebraic vector bundle E on V, with an isomorphism θ from E to its dual E^* such that $\theta = \theta^*$. Asssociated to any bundle E, there is a hyperbolic form $H(E) = (E \oplus E^*, \text{ev})$. The Grothendieck-Witt group $GW(V)$ is the Grothendick group of the category of symmetric forms on V, modulo the relation that $(E, \theta) = h(L)$ if E has a Lagrangian L (a subobject such that $L = L^\perp$). There is a hyperbolic map $H : K_0(V) \to GW(V)$, and the Witt group $W(V)$ is defined to be the cokernel of H. When V is connected, there is a canonical surjection $W(V) \to \mathbb{Z}/2$, and we define $I(V)$ to be its kernel.

When $\dim(V) \leq 3$ and V is smooth, $W(V)$ injects into $W(F)$, where $F = k(V)$ is the function field of V over k, and there is a well known exact sequence

$$0 \to W(V) \to W(F) \to \oplus_x W(k(x))$$

where x runs over all points of codimension 1. (See [**36**] and [**5**], for example.) There is a natural filtration on $W(F)$ by the powers $I(F)^n$ of the maximal ideal $I(F)$, and $I(F)^n/I(F)^{n+1} \cong H^2_{\mathrm{et}}(F, \mathbb{Z}/2)$ by [**35**].

Still assuming that $\dim(V) \le 3$, we define $I_n = I_n(V)$ to be the ideal $W(V) \cap I(F)^n$ of $W(V)$; $I(V) = I_1(V)$. Parimala showed in [**37**] that above sequence restricts to an exact sequence

$$0 \to I_n(V) \to I^n(F) \to \oplus_x\ I^{n-1}(k(x)) \tag{1.1}$$

which maps to the Bloch-Ogus sequence [**6**]

$$0 \to H^0(V, \mathcal{H}^n) \to H^n_{\mathrm{et}}(F, \mathbb{Z}/2) \to \oplus_x\ H^{n-1}_{\mathrm{et}}(k(x), \mathbb{Z}/2). \tag{1.2}$$

Here $\mathcal{H}^n$ is the Zariski sheaf associated to the presheaf $U \mapsto H^n_{\mathrm{et}}(U, \mathbb{Z}/2)$.

Recall that the discriminant of a diagonal form $(a_1, ..., a_n)$ over a field F is the class of $(-1)^{[n/2]} \prod a_i$ in $F^\times/F^{\times 2}$. As the hyperbolic form $H(1)$ has discriminant $+1$, this induces a function disc: $W(F) \to F^\times/F^{\times 2}$.

Definition 1.3. The map $I(V) \to I(F) \to I(F)/I^2(F) \cong F^\times/F^{\times 2}$ factors through the subgroup $H^0(V, \mathcal{H}^1)$ of $F^\times/F^{\times 2}$. The *algebraic discriminant* is the induced map $I(V) \to H^0(V, \mathcal{H}^1) \cong H^1_{\mathrm{et}}(V, \mathbb{Z}/2)$.

Similarly, the *Hasse invariant* $I(V) \to H^0(V, \mathcal{H}^2) \cong {}_2\mathrm{Br}(V)$ is the homomorphism induced by the morphism $I(V) \to I(F)$ followed by the Stiefel–Whitney class $w_2 : I(F) \to H^2_{\mathrm{et}}(F, \mathbb{Z}/2)$; see [**33**, 3.1].

Example 1.4. The algebraic discriminant $I(V) \to H^1_{\mathrm{et}}(V, \mathbb{Z}/2)$ is onto. To see this, recall from Kummer Theory [**32**, III.4] that there is a split exact sequence

$$0 \to \mathcal{O}^\times(V)/2 \to H^1_{\mathrm{et}}(V, \mathbb{Z}/2) \to {}_2\mathrm{Pic}(V) \to 0.$$

If a is a global unit of V and $\mathcal{O}_V$ is the trivial line bundle then $(\mathcal{O}_V, a)$ is a symmetric form, representing an element of $W(V)$ whose discriminant is the class of a in $\mathcal{O}^\times(V)/2$. If L is a line bundle with $L \otimes L \cong \mathcal{O}_X$, the isomorphism $\theta : L \cong L^*$ defines a symmetric form, and the discriminant of (L, θ) maps to the class of L in ${}_2\mathrm{Pic}(V)$.

If θ is any symmetric form on an algebraic vector bundle E, the discriminant and Hasse invariant of (E, θ) may be calculated by passing to an open subvariety U of V where $E \cong \mathcal{O}^n_U$ and θ is isomorphic to the diagonal form $(a_1, ..., a_n)$, where $a_i \in H^0(U, \mathcal{O}^\times)$; the discriminant of (E, θ) is the class of the product $(-1)^{[n/2]} \prod a_i$ in $H^0(U, \mathcal{O}^\times)/2 \subset F^\times/F^{\times 2}$, and the Hasse invariant of (E, θ) is the class of $\prod_{i<j} \{a_i, a_j\}$ in $H^2_{\mathrm{et}}(F, \mathbb{Z}/2)$.

Let $\widetilde{H}^1_{\text{et}}(V,\mathbb{Z}/2)$ denote the cokernel of $H^1_{\text{et}}(k,\mathbb{Z}/2)\to H^1_{\text{et}}(V,\mathbb{Z}/2)$.

LEMMA 1.5. *If* $\dim(V)\le 3$ *then* $I_n(V)/I_{n+1}(V)\to H^0(V,\mathcal{H}^n)$ *is an injection for all* $n\ge 0$*. For* $n=1$*, this is the discriminant isomorphism*

$$I_1/I_2 \xrightarrow{\simeq} H^0(V,\mathcal{H}^1)\cong H^1_{\text{et}}(V,\mathbb{Z}/2).$$

If -1 *is not a square in* $k(V)^\times$ *then* $W(V)/I_2(V)\cong \mathbb{Z}/4\oplus \widetilde{H}^1_{\text{et}}(V,\mathbb{Z}/2)$*.*

PROOF. This follows from a diagram chase on the map between sequences (1.1) and (1.2), using the isomorphisms $I^n(F)/I^{n+1}(F)\cong H^n_{\text{et}}(F,\mathbb{Z}/2)$ (which are known to hold for all n by [**35**]). It is also well known that $W(F)/I(F)^2$ contains $\mathbb{Z}/4$ if and only if $\{-1,-1\}\ne 0$ in $H^2_{\text{et}}(k(V),\mathbb{Z}/2)$ (see, e.g., [**33**, 3.3]). ☐

REMARK 1.5.1. The injection $I_2/I_3\to H^0(V,\mathcal{H}^2)\cong {}_2\mathrm{Br}(V)$ is called the *Hasse invariant* [**37**]; it is not known if it is an isomorphism for non-affine V over k, even if $k=\mathbb{R}$.

Connection to signature

Now suppose that V is a variety over $\mathbb{R}$, and $V(\mathbb{R})$ has $\nu>0$ connected components. The torsion subgroup of $W(V)$ is 2-primary (Pfister [**40**]), and Mahé [**30**] and Brumfiel [**8**] proved that the signature $W(V)\to\mathbb{Z}^\nu$ maps the torsionfree part of $W(V)$ isomorphically onto a subgroup of finite index in $\mathbb{Z}^\nu$. By [**23**, 2.4], the signature factors through $WR(V_{\text{top}})$.

Recall that the cup product with $\{-1\}\in H^0(V,\mathcal{H}^1)$ induces stabilization maps $H^0(V,\mathcal{H}^n)\to H^0(V,\mathcal{H}^{n+1})$; Colliot-Thélène and Parimala showed [**9**] that this map is an isomorphism for $n>d=\dim(V)$ and that this stable value is $(\mathbb{Z}/2)^\nu$, where ν is the number of real components of V. When $n\le d$, the composite

$$H^0(V,\mathcal{H}^n)\to H^0(V,\mathcal{H}^{d+1})\cong(\mathbb{Z}/2)^\nu$$

is called the *stabilization map.* Example 1.6 shows that $H^1_{\text{et}}(V,\mathbb{Z}/2)\cong H^0(V,\mathcal{H}^1)$ need not map onto $(\mathbb{Z}/2)^\nu$. In contrast, van Hamel proved in [**51**, 2.8] that the stabilization map $H^0(V,\mathcal{H}^d)\to H^0(V,\mathcal{H}^{d+1})\cong(\mathbb{Z}/2)^\nu$ is a surjection. For surfaces, this becomes a surjection ${}_2\mathrm{Br}(V)\cong H^0(V,\mathcal{H}^2)\xrightarrow{\text{onto}}(\mathbb{Z}/2)^\nu$.

EXAMPLE 1.6. A nonsingular cubic surface in $\mathbb{P}^3_{\mathbb{C}}$ is a special type of *Del Pezzo* surface, and is birationally equivalent to $\mathbb{P}^2_{\mathbb{C}}$ (see [**16**, V.4.7.1]). There are nonsingular cubic surfaces defined over $\mathbb{R}$ which are not birationally equivalent to $\mathbb{P}^2$ over $\mathbb{R}$; they have $\mathrm{Pic}(V)\cong\mathbb{Z}^3$, and the real locus $V(\mathbb{R})$ has $\nu=2$ components, homeomorphic to S^2

and $\mathbb{RP}^2$ respectively; see [**47**, V.5.4]. Since $H^1_{\text{et}}(V,\mathbb{Z}/2) \cong \mathbb{Z}/2$ by Kummer theory, it follows that $H^1_{\text{et}}(V,\mathbb{Z}/2) \to (\mathbb{Z}/2)^\nu$ is not onto.

The torsion subgroup

The torsion subgroup $I(V)_{\text{tors}}$ of $I(V)$ has an induced filtration by the ideals $I_n(V)_{\text{tors}} = I_n(V) \cap I(V)_{\text{tors}}$.

PROPOSITION 1.7. *If V is a smooth d-dimensional variety over $\mathbb{R}$, then*
(i) $I_{d+1}(V)$ is torsionfree as an abelian group; $I_{d+1}(V)_{\text{tors}} = 0$.
(ii) the torsion subgroup $I(V)_{\text{tors}}$ of $I(V)$ has exponent 2^d.
(iii) If $V(\mathbb{R}) = \emptyset$ then $W(V)$ is an algebra over $\mathbb{Z}/2^{d+1}$.

PROOF. It is classical [**15**, Thm. E] that $I^{d+1}(F)$ is a torsionfree abelian group. Hence the subgroup $I_{d+1}(V)$ is also torsionfree. Since the element '2' of $W(V)$ lies in $I(V)$ and $I^n(V) \subseteq I_n(V)$, $2^{d+1} \in 2I_d(V) \subseteq I_{d+1}(V)$. Finally, (iii) is immediate from (ii) and the fact that the torsionfree part has rank ν. □

When $\dim(V) = 2$, $I_3(V)$ is torsionfree and $I(V)_{\text{tors}}$ has exponent 4.

$$I(V)_{\text{tors}} \supseteq I_2(V)_{\text{tors}} \supseteq I_3(V)_{\text{tors}} = 0.$$

Following Sujatha [**48**], let $H^0_{\text{tors}}(V,\mathcal{H}^n)$ denote the kernel of the stable map $H^0(V,\mathcal{H}^n) \to H^0(V,\mathcal{H}^{d+1}) \cong (\mathbb{Z}/2)^\nu$. Sujatha described the torsion subgroup $I(V)_{\text{tors}}$ as follows.

PROPOSITION 1.8 (Sujatha). *When V is a smooth surface over $\mathbb{R}$, there is a short exact sequence*

$$0 \to H^0_{\text{tors}}(V,\mathcal{H}^2) \to I(V)_{\text{tors}} \to H^0_{\text{tors}}(V,\mathcal{H}^1) \to 0.$$

PROOF. Sujatha proves in [**48**, 2.1, 2.2] that the maps

$$I(V)_{\text{tors}}/I_2(V)_{\text{tors}} \to H^0_{\text{tors}}(V,\mathcal{H}^1), \quad I_2(V)_{\text{tors}}/I_3(V)_{\text{tors}} \to H^0_{\text{tors}}(V,\mathcal{H}^2)$$

are isomorphisms, and $I_3(V)_{\text{tors}} = 0$ by Proposition 1.7. □

REMARK 1.8.1. Set $j = \dim H^0_{\text{tors}}(V,\mathcal{H}^1)$ and $k = \dim H^0_{\text{tors}}(V,\mathcal{H}^2)$. It follows that (when $\nu > 0$) the torsion subgroup $W(V)_{\text{tors}}$ of $W(V)$ is a group of exponent 4 and order 2^{j+k}. (See [**48**, Lemma 3.2].) Note that $k + \nu = \dim {}_2\text{Br}(V)$ by van Hamel's result [**51**, 2.9].

Sujatha [**48**, 3.1] also defines a group N fitting into an exact sequence

$${}_2\operatorname{Pic}(V) \to {}_2\operatorname{Pic}(V_{\mathbb{C}}) \to N \to \operatorname{Pic}(V)/2 \to \operatorname{Pic}(V_{\mathbb{C}})/2.$$

Her invariant l is the dimension of N; she proves in [**48**, 3.3] that there are $k + l$ summands in $W(V)_{\text{tors}}$, and $j - l$ summands $\mathbb{Z}/4$ in $W(V)$.

It follows by a counting argument that there are $k+2l-j$ summands $\mathbb{Z}/2$ in $W(V)$. However, the numbers j,k and l are not always easy to determine; see Krasnov [**28**], [**29**].

Recall that a variety V over $\mathbb{R}$ is *geometrically connected* if $V\times_{\mathbb{R}}\mathbb{C}$ is connected, i.e., if the function field of V does not contain $\mathbb{C}$.

COROLLARY 1.9. *Let V be a geometrically connected surface over $\mathbb{R}$ with no real points. Then we have a short exact sequence*

$$0\to {}_2\mathrm{Br}(V)\to W(V)\to \mathbb{Z}/4\times \widetilde{H}^1_{\text{et}}(V,\mathbb{Z}/2)\to 0,$$

$\widetilde{H}^1_{\text{et}}(V,\mathbb{Z}/2)$ *denoting the cokernel of* $H^1_{\text{et}}(\mathrm{Spec}\ \mathbb{R},\mathbb{Z}/2)\to H^1_{\text{et}}(V,\mathbb{Z}/2)$.

PROOF. Since $\nu=0$, Proposition 1.8 implies that $I_2(V)\cong {}_2\mathrm{Br}(V)$. We conclude by Lemma 1.5. $\square$

The group extension in Corollary 1.9 need not split, as illustrated by Example 1.11.

REMARK 1.9.1. Corollary 1.9 recovers Sujatha's result [**48**, 3.4] that if V is a geometrically connected surface with $V(\mathbb{R})=\emptyset$ then $W(V)$ has order 2^{j+k+1}, because $j=\dim H^1_{\text{et}}(V,\mathbb{Z}/2)$ and $k=\dim {}_2\mathrm{Br}(V)$.

Let Q_1 denote the projective Brauer-Severi curve $x^2+y^2+z^2=0$.

LEMMA 1.10. *Suppose that V is a smooth geometrically connected surface with no real points. Then $W(V)$ is a $\mathbb{Z}/8$-algebra and surjects onto $\mathbb{Z}/4$. It is a $\mathbb{Z}/4$-algebra if and only if there is a morphism $V\to Q_1$ defined over $\mathbb{R}$.*

PROOF. Since $W(V)$ injects into $W(F)$, it suffices to study the exponent of $W(F)$. When V is geometrically connected ($\mathbb{R}(V)$ does not contain $\mathbb{C}$), then -1 is not a square in F. Hence the element '2' of $W(F)$ is nonzero, because its discriminant is -1 in $F^\times/F^{\times 2}$. The group $W(F)$ has $I^3(F)=0$ by Proposition 1.7(iii) (or [**15**, Thm. E]). Thus the ideal $I(F)$ has exponent 4, and '8' is zero in $W(F)$. Thus the image of $\mathbb{Z}$ in $W(F)$ is either $\mathbb{Z}/4$ or $\mathbb{Z}/8$.

It is well known [**33**, 3.3] that the element '4' in $W(F)$ corresponds to the symbol $\{-1,-1\}$ in $I^2(F)\cong H^2_{\text{et}}(F,\mathbb{Z}/2)$. This symbol vanishes if and only if there is a morphism $V\to Q_1$ defined over $\mathbb{R}$. $\square$

REMARK 1.10.1. Milnor showed in [**33**, 3.1, 3.3, 4.1] that $1+w_1+w_2$ is an additive isomorphism from $I(F)/I^3(F)$ to $1+H^1_{\text{et}}(F)+H^2_{\text{et}}(F)$, which is a group under $(1+a_1+a_2)(1+b_1+b_2)=1+(a_1+b_1)+(a_2+b_2+a_1\cup b_1)$, with the inverse given by s_1 and s_2. For example, the element '2' of $I(F)$ maps to $1+\{-1\}$ and '4' maps to $(1+\{-1\})^2=1+\{-1,-1\}$.

EXAMPLE 1.11. There are real surfaces V for which the image of $\mathbb{Z} \to W(V)$ is $\mathbb{Z}/8$; since $W(V) \subset W(F)$, this is equivalent to the condition that the function field F of V has level 4 (i.e., -1 is a sum of 4 squares). As pointed out by Parimala in [**39**], this is the case if V is the hypersurface in $\mathbb{P}^3_{\mathbb{R}}$ defined by $z^2 - f$, where $f(x,y)$ is the Motzkin polynomial $x^4y^2 + x^2y^4 - 3x^2y^1 + 1$ (or certain other positive polynomials).

EXAMPLE 1.12. Consider the real surface $V = E \times E$, where E is an elliptic curve with 2 real connected components; this is a real form of the abelian variety $V_{\mathbb{C}} = E_{\mathbb{C}} \times E_{\mathbb{C}}$. To compute $W(V)$ using Remark 1.8.1, we first compute $H^*_{\text{et}}(V, \mathbb{Z}/2)$, using the Leray spectral sequence

$$H^p(G, H^q_{\text{et}}(V_{\mathbb{C}}, \mathbb{Z}/2)) \Rightarrow H^{p+q}_{\text{et}}(V, \mathbb{Z}/2).$$

Because $E(\mathbb{R})$ contains the exponent 2 points of the group $E(\mathbb{C})$, G acts trivially on $H^*_{\text{et}}(E_{\mathbb{C}}, \mathbb{Z}/2)$. Dropping the coefficients $\mathbb{Z}/2$ from the notation, the Künneth formula shows that G also acts trivially on $H^1_{\text{et}}(V_{\mathbb{C}}) \cong (\mathbb{Z}/2)^4$ and $H^2_{\text{et}}(V_{\mathbb{C}}) \cong (\mathbb{Z}/2)^6$. We get an exact sequence

$$0 \to (\mathbb{Z}/2) \to H^1_{\text{et}}(V) \to H^1(V_{\mathbb{C}})^G \to 0.$$

Hence $H^1_{\text{et}}(V) \cong (\mathbb{Z}/2)^5$. In the corresponding spectral sequence for $H^*_{\text{et}}(E)$, the differential $\mathbb{Z}/2 = E_2^{0,2}(E) \to E_2^{2,1}(E) = (\mathbb{Z}/2)^2$ is an injection. Hence the differential from $E_2^{0,2}(V) = H^2_{\text{et}}(V_{\mathbb{C}})$ to $E_2^{2,1}(V) = H^1_{\text{et}}(V_{\mathbb{C}})$ has rank 2, and we deduce that $H^2_{\text{et}}(V) \cong (\mathbb{Z}/2)^7$.

Since the stabilization map $H^1_{\text{et}}(V) \to H^2_{\text{et}}(V)$ is the cup product with $[-1]$, we also see that $H^1_{\text{tors}}(V) \cong (\mathbb{Z}/2)^3$ injects into $H^2_{\text{tors}}(V) \cong (\mathbb{Z}/2)^4$, so Sujatha's invariant l vanishes, and $j = 3$, $k = 4$. It follows from Remark 1.8.1 that

$$W(E \times E) \cong \mathbb{Z}^4 \oplus (\mathbb{Z}/4)^3 \oplus \mathbb{Z}/2.$$

2. WR and the signature

In this section, we define the signature map on $WR(X)$ and relate it to the edge map η^0 in the Bredon spectral sequence for $KO_G(X)$. First we recall the definition of $WR(X)$ from [**23**].

Let X be a topological space with involution. By a *Real vector bundle* on X we mean a complex vector bundle E with an involution σ compatible with the involution on X and such that for each $x \in X$ the isomorphism $\sigma : E_x \to E_{\sigma x}$ is $\mathbb{C}$-antilinear. Following Atiyah [**2**], we write $KR(X)$ for the Grothendieck group of Real vector bundles on a compact space X.

Recall from [**23**] that the Grothendieck-Witt group $GR(X)$ is the Grothendieck group of the category of symmetric forms (E, ϕ), where

E is a Real vector bundle on X and ϕ is an isomorphism of Real vector bundles from E to its dual E^* such that $\phi = \phi^*$.

DEFINITION 2.1. The Real Witt group $WR(X)$ of a compact G-space X is the cokernel of the hyperbolic map $H : KR(X) \to GR(X)$.

By [**23**, Thm. 2.2], there is an isomorphism $KO_G(X) \cong GR(X)$ identifying the hyperbolic map with the map $KR(X) \to KO_G(X)$, sending a Real vector bundle to its underlying $\mathbb{R}$-linear bundle with involution. Thus $WR(X)$ is also the cokernel of $KR(X) \to KO_G(X)$.

EXAMPLE 2.1.1. It is easy to see that $KO_G(\mathrm{pt}) \cong RO(G)$ and the signature $WR(\mathrm{pt}) \cong \mathbb{Z}$ sends $[\mathbb{R}]$ and $[\mathbb{R}(1)]$ to 1 and -1. Suppose more generally that G acts trivially on X. Then $KO_G(X) \cong KO(X) \otimes R(G)$; we showed in [**23**, 2.4] that $KR(X) \cong KO(X)$ and $WR(X) \cong KO(X)$. In this case, the rank of a KO-bundle on each component of X defines a map $WR(X) \to \mathbb{Z}^\nu$, where $\nu = |\pi_0(X)|$.

In contrast, $WR(X)$ is a $\mathbb{Z}/2$-algebra when $X = G \times Y$. In this case, $WR(X)$ is the cokernel of the natural map $KU(Y) \to KO(Y)$ and hence a $\mathbb{Z}/2$-algebra; see [**23**, 2.4(b) and 2.6]. The following example illustrates this.

LEMMA 2.2. *When V is complex projective space $\mathbb{P}^d_{\mathbb{C}}$, $X = (\mathbb{P}^d_{\mathbb{C}})_{\mathrm{top}}$ is $G \times \mathbb{CP}^d$ and*

$$W(\mathrm{pt}) \cong W(\mathbb{P}^d_{\mathbb{C}}) \cong WR(X) \cong \mathbb{Z}/2.$$

PROOF. The isomorphism $\mathbb{Z}/2 = W(\mathrm{pt}) \cong W(\mathbb{P}^d_{\mathbb{C}})$ is due to Arason [**1**]. As noted above, $WR(X)$ is the cokernel of $KU(\mathbb{CP}^d) \to KO(\mathbb{CP}^d)$. In this case, Karoubi and Mudrinski [**22**] proved that the forgetful map $KU(\mathbb{CP}^d, \mathrm{pt}) \to KO(\mathbb{CP}^d, \mathrm{pt})$ is onto, so $WR(\mathbb{CP}^d) = \mathbb{Z}/2$ as well. $\square$

The topological signature. Now let X be any compact G-space, such that X^G has ν components. The signature of a symmetric form (E, ϕ) on X is an element of $\mathbb{Z}^\nu$; the signature on a component of X^G is the signature of the form ϕ_x on the vector space E_x, for any point x of the component (it is independent of the choice of x). This passes to a homomorphism $KO_G(X) \to \mathbb{Z}^\nu$.

Since hyperbolic forms have signature 0, it also defines a topological *signature* $\sigma : WR(X) \to \mathbb{Z}^\nu$; the signature of $w \in WR(X)$ is the signature of a quadratic form (E, ϕ) representing w. While σ may not be onto (see Remark 4.3), its image has finite 2-primary index in $\mathbb{Z}^\nu$ by [**23**, 7.4].

As observed in [**23**], there is a canonical map $W(V) \to WR(V_{\mathrm{top}})$ and the composition with the topological signature is the classical algebraic signature.

On each component, the signature of an n-dimensional symmetric form ϕ is congruent to n modulo 2. If X is connected, it follows that the signature of ϕ lies in the subgroup Γ of $\mathbb{Z}^\nu$, defined as follows.

Definition 2.3. Let Γ denote the subgroup of $\mathbb{Z}^\nu$ consisting of all $(c_1, ..., c_\nu)$ with either all c_i even or all c_i odd; Γ contains $2\mathbb{Z}^\nu$ as an index 2 subgroup. The group Γ is free abelian of rank ν, but there is no isomorphism of Γ with $\mathbb{Z}^\nu$ which is natural in X.

We now assume X is connected, and relate the signature to the canonical edge map

$$\eta^0 : KO_G(X) \to H^0_G(X; KO^0_G) \cong \mathbb{Z} \oplus \mathbb{Z}^\nu \tag{2.4}$$

in the Bredon spectral sequence (A.1); see Lemma A.5. The first coordinate of η^0 is the rank of the underlying vector bundle. To describe the other coordinates of η^0, let $\widetilde{KO}_G(X)$ denote the kernel of the rank map $KO_G(X) \to \mathbb{Z}$. For each point x of X^G, $\widetilde{KO}_G(x)$ is isomorphic to $\mathbb{Z}$ on $[\mathbb{R}] - [\mathbb{R}(1)]$. This yields a map $\widetilde{KO}_G(X) \to \widetilde{KO}_G(x) \cong \mathbb{Z}$; if E is a G-vector bundle of rank r on X, the map sends $[E] - r$ to $[E_x] - r$. Choosing a point on each of the ν components of X^G yields the edge map η^0 from $\widetilde{KO}_G(X)$ to $\mathbb{Z}^\nu$. It sends the line bundles $X \times \mathbb{R}$ and $X \times \mathbb{R}(1)$ to $(0, ..., 0)$ and $(-1, ..., -1)$.

Both $H^0_G(X; KO_G)$ and ν are determined by the 1–skeleton of X.

Lemma 2.5. *Let X be a connected finite G-CW complex whose fixed locus X^G has ν components. The signature $WR(X) \xrightarrow{\sigma} \mathbb{Z}^\nu$ is induced by*

$$KO_G(X) \xrightarrow{\eta^0} H^0_G(X; KO_G) \cong \mathbb{Z} \oplus \mathbb{Z}^\nu \xrightarrow{\rho} \Gamma \subseteq \mathbb{Z}^\nu,$$

where η^0 is the edge map in (A.1) *and ρ is given by*

$$\rho(r, a_1, ..., a_\nu) = (r + 2a_1, ..., r + 2a_\nu).$$

Proof. If pt is a single point, we choose $[\mathbb{R}]$ and $\lambda = [\mathbb{R}] - [\mathbb{R}(1)]$ as a basis of $H^0_G(\mathrm{pt}, \mathbb{Z}) \cong RO(G) \cong \mathbb{Z}^2$. The projection '+' in Example A.4(a) from $H^0_G(\mathrm{pt}, \mathbb{Z}) \cong RO(G)$ to $H^0(\mathrm{pt}/G, \mathbb{Z}) \cong \mathbb{Z}$ is split by the map sending the generator to $[\mathbb{R}]$, and the kernel is generated by λ.

The image of $KR(\mathrm{pt}) \cong \mathbb{Z}$ in $H^0_G(\mathrm{pt}, \mathbb{Z})$ is generated by $[\mathbb{R}] + [\mathbb{R}(1)] = 2[\mathbb{R}] - \lambda$; it is the kernel of the map $H^0_G(\mathrm{pt}, \mathbb{Z}) \xrightarrow{s} \mathbb{Z}$ sending $(r[\mathbb{R}] + a\lambda)$ to its signature $r + 2a$. Therefore s induces the signature $\sigma : WR(\mathrm{pt}) \xrightarrow{\cong} \mathbb{Z}$.

Now let Y denote ν discrete points. It follows that the signature $WR(Y) \xrightarrow{\cong} \mathbb{Z}^\nu$ is induced by the composition of the map

$$KO_G(Y) \xrightarrow{\cong} H^0_G(Y, \mathbb{Z}) \xrightarrow{\cong} H^0(Y^G, \mathbb{Z}) \oplus H^0(Y/G, \mathbb{Z}) \cong \mathbb{Z}^\nu \oplus \mathbb{Z}^\nu,$$

with the map $\mathbb{Z}^{2\nu} \xrightarrow{s} \mathbb{Z}^\nu$, sending $(r_1, ..., a_1, ...)$ to $(r_1+2a_1, ..., r_\nu+2a_\nu)$. The general case now follows from the following diagram, whose rows are from Lemma A.5

$$\begin{tikzcd}
0 \arrow[r] & H^0(X^G,\mathbb{Z}) \arrow[r] \arrow[d, "\cong"] & H^0_G(X;KO_G) \arrow[r, "+"] \arrow[d, "\text{into}"] & H^0(X/G,\mathbb{Z}) \arrow[r] \arrow[d, "\text{diag}"] & 0 \\
0 \arrow[r] & H^0(Y^G,\mathbb{Z}) \arrow[r, "\lambda"] & H^0_G(Y;KO_G) \arrow[r, "+"] & H^0(Y,\mathbb{Z}) \arrow[r] & 0
\end{tikzcd}$$

and whose columns are associated to the evident map $Y \to X^G \subseteq X$, obtained by choosing a basepoint for each component of X^G. By inspection, the map $\rho : H^0_G(X;KO_G) \to \Gamma$ is the restriction of the map $s : H^0_G(Y;KO_G) \to \mathbb{Z}^\nu$. It follows that $\rho\eta^0_X$ is the restriction of $s\eta^0_Y$, which is the signature. $\square$

Remark 2.5.1. The sequence $KR(X) \to \mathbb{Z}^{1+\nu} \xrightarrow{\rho} \Gamma$ of Lemma 2.5 is exact. Indeed, the kernel of ρ is the subgroup of $\mathbb{Z}^{1+\nu}$ generated by $(2, -1, ..., -1)$. This is the image of $\eta^0 \circ H : KR(X) \to \mathbb{Z}\oplus\mathbb{Z}^\nu$, because $\eta^0 \circ H$ maps the trivial Real line bundle $X \times \mathbb{C}$ to $(2, -1, ..., -1)$.

3. WR for low dimensional X

In this section, we determine $WR(X)$ when $\dim(X) \le 2$. There is no harm in assuming that X is an irreducible G-space, so that X is either connected or has the form $X = G\times Y$ with Y connected. We saw in [**26**, 7.1] that $WR(G \times Y) \cong \mathbb{Z}/2 \oplus H^1(Y,\mathbb{Z}/2)$ when $\dim(Y) \le 2$. Thus we are reduced to the case when X is connected.

In general, there is a map $w_1 : WR(X) \to \mathbb{H}^1_G(X,\mathbb{Z}/2)$, defined in 4.2 and 6.1 below. Let us write $[-1]$ for the element $w_1(X\times\mathbb{R}(1))$ of $\mathbb{H}^1_G(X,\mathbb{Z}/2)$. We will see in Remark 4.1.1 that $[-1]$ is always nonzero in $\mathbb{H}^1_G(X,\mathbb{Z}/2)$. Let $\widetilde{H}^1(X/G,\mathbb{Z}/2)$ denote the quotient of $H^1(X/G,\mathbb{Z}/2)$ by the subgroup generated by the element $[-1] = w_1(X\times\mathbb{R}(1))$.

Lemma 3.1. *When X is a connected 1-dimensional G-complex, and X^G has ν components, we have*

$$WR(X) = \begin{cases} \mathbb{Z}^\nu \oplus H^1(X/G,\mathbb{Z}/2), & \nu > 0, \\ \mathbb{Z}/4 \oplus \widetilde{H}^1(X/G,\mathbb{Z}/2), & \nu = 0. \end{cases}$$

Proof. The spectral sequences of (A.1) for KO_G and KR converge, having only two nonzero rows. By Lemma A.5, we have

$$KO_G(X) \cong \mathbb{Z}\oplus\mathbb{Z}^\nu\oplus H^1_G(X;KO_G^{-1}), \qquad KR(X) \cong \mathbb{Z}\oplus H^1_G(X;KR^{-1}),$$

and the cokernel of $H^1_G(X;KR^{-1}) \to H^1_G(X;KO_G^{-1})$ is $H^1(X/G,\mathbb{Z}/2)$. Thus it remains to consider the cokernel of the map of H^0_G-components

$\mathbb{Z} \to \mathbb{Z} \oplus \mathbb{Z}^\nu$. It is described in Remark 2.5.1, and yields the displayed calculation. $\square$

Example 3.2. There are three actions of G on the circle. Recall that $S^{p,q}$ denotes the sphere in $\mathbb{R}(1)^p \times \mathbb{R}^q$.

(i) $S^{0,2}$ is the circle with trivial G-action. Let μ denote the Möbius line bundle. Then $KO_G(S^{0,2})$ is $\mathbb{Z}^2 \oplus (\mathbb{Z}/2)^2$ with generators the line bundles 1, $X \times \mathbb{R}(1)$, μ and $\mu \otimes \mathbb{R}(1)$, while $WR(S^{0,2}) \cong KO(S^{0,2})$ is $\mathbb{Z} \oplus \mathbb{Z}/2$ by Example 2.1.1.

(ii) $S^{2,0}$ is the circle with the antipodal involution on S^1. Lemma 3.1 yields $WR(S^{2,0}) \cong \mathbb{Z}/4$.

(iii) $S^{1,1}$ is the unit circle in $\mathbb{C}$, with the induced complex conjugation. By Lemma 3.1,

$$WR(S^{1,1}) \cong \mathbb{Z}^2.$$

The Real line bundle $E = S^{1,1} \times \mathbb{C}$ carries a canonical Real symmetric form $\theta = t$, which is multiplication by t on the fiber over $t \in S^{1,1}$. By [**23**, Thm. 2.2], this corresponds to the G-subbundle of E whose fiber over t is $it \cdot \mathbb{R}$. It is a nontrivial element of $\mathrm{Pic}_G(S^{1,1}) \cong \{\pm 1\}^2$, and its discriminant $\iota = w_1(E, t)$ is nontrivial.

If $X \xrightarrow{a} S^{1,1}$ is an equivariant map, the pullback of (E, t) under the map $a^* : \mathrm{Pic}_G(S^{1,1}) \to \mathrm{Pic}_G(X)$ is the class of $(X \times \mathbb{C}, a)$ in $\mathrm{Pic}_G(X)$, and its discriminant is $w_1(a^*(E, t)) = a^*(\iota)$.

Lemma 3.3. *Let X be a closed connected 2-manifold with involution, such that X^G is a union of ν circles. Then*

$$WR(X) = \begin{cases} \mathbb{Z}^\nu \oplus (\mathbb{Z}/2)^h, & \nu > 0 \\ \mathbb{Z}/4 \times (\mathbb{Z}/2)^{h-1}, & \nu = 0, \end{cases} \quad \textit{where } h = \dim H^1(X/G, \mathbb{Z}/2).$$

Proof. If $\nu = 0$, this was proven in [**26**, 8.1], so suppose $\nu > 0$. Since X/G is a 2-manifold with boundary, $H^2(X/G, \mathbb{Z}/2) = 0$. (See the argument of [**41**, 1.9].) As $H^2(X^G, \mathbb{Z}/2) = 0$, Lemma A.5 yields $H^2_G(X; KO_G^{-2}) = 0$ and $H^1_G(X; KO_G^{-1}) = (\mathbb{Z}/2)^{h+\nu}$. By (A.1), we have $KO_G(X) = \mathbb{Z}^{\nu+1} \oplus (\mathbb{Z}/2)^{h+\nu}$. Now use the argument of Lemma 3.1. $\square$

If C is a smooth projective curve over $\mathbb{R}$, C_{top} is a 2-manifold whose genus g is h when $\nu > 0$, and $h - 1$ when $\nu = 0$. Thus we recover the following calculation of $WR(C_{\mathrm{top}})$, which was derived in [**23**, Thms. 4.6 and 4.7], using $H^2_{\mathrm{et}}(C, \mathbb{Z}/2) \cong (\mathbb{Z}/2)^{2\nu}$ and $\mathrm{Br}(C) \cong (\mathbb{Z}/2)^\nu$. (The fact that $W \cong WR$ for curves is also in *loc. cit.*)

COROLLARY 3.4. *If C is a smooth geometrically connected projective curve of genus g over $\mathbb{R}$, with ν real components, then*

$$W(C) \cong WR(C_{\mathrm{top}}) \cong \begin{cases} \mathbb{Z}^\nu \oplus (\mathbb{Z}/2)^g, & \nu > 0, \\ \mathbb{Z}/4 \oplus (\mathbb{Z}/2)^g, & \nu = 0. \end{cases}$$

REMARK 3.4.1. When $C(\mathbb{R}) = \emptyset$, and $U \subset C$ is the complement of $n > 0$ (complex) points, then $W(U) \cong W(C) \oplus (\mathbb{Z}/2)^{n-1}$. This follows from Corollary 3.4, (1.1), and the fact that $\mathrm{Pic}(C)/2 \cong \mathbb{Z}/2$. In addition, $W(U) \cong WR(U)$ by [**23**, 4.1]. We leave the details to the reader.

EXAMPLE 3.5. We can now calculate $WR(X)$ of a 2-sphere with involution. Writing $S^{p,q}$ for the sphere in $\mathbb{R}(1)^p \times \mathbb{R}^q$, we have:

$$WR(S^{0,3}) \cong \mathbb{Z} \oplus \mathbb{Z}/2; \quad WR(S^{1,2}) \cong \mathbb{Z}; \quad WR(S^{2,1}) \cong \mathbb{Z}^2 \oplus \mathbb{Z}/2;$$

and $WR(S^{3,0}) \cong \mathbb{Z}/4$. The calculation $WR(S^{0,3}) \cong KO(S^2) \cong \mathbb{Z} \oplus \mathbb{Z}/2$ is from Example 2.1.1. The formulas for the genus 0 curves $S^{1,2} = \mathbb{CP}^1$ and $S^{3,0} = Q_1$ are from Lemma 3.3.

For $X = S^{2,1}$, we note that X^G is 2 points and $X/G \simeq S^2$. Therefore $H^1_G(S^{2,1}, \mathbb{Z}/2) = 0$ and $H^2_G(X; KO^{-2}) \cong \mathbb{Z}/2$, using Lemma A.5. Thus $KO_G(X) \cong \mathbb{Z}^3 \oplus \mathbb{Z}/2$. We also have $\mathbb{H}^1_G(S^{2,1}, \mathbb{Z}/2) \cong \mathbb{Z}/2$. As in the proof of Lemma 3.3, we have $KR(X) = \mathbb{Z}$, and hence $WR(S^{2,1}) \cong \mathbb{Z}^2 \oplus \mathbb{Z}/2$.

More generally, if $q > 0$ then $KR^n(S^{p,q}, \mathrm{pt}) \cong KO^{n+p-q+1}(\mathrm{pt})$. This follows easily from Atiyah's sequence [**2**, (2.2)] for $(X, Y) = (B^{p,q}, S^{p,q})$. In addition, because $S^{p,q}$ is the 1–point compactification of $\mathbb{R}(1)^p \times \mathbb{R}^{q-1}$, $KO_G(S^{p,q}, \mathrm{pt})$ is isomorphic to the K-theory with compact supports $KO^{1-q}_G(\mathbb{R}(1)^p)$, a group isomorphic to the topological K-theory $K^{1-q}_{\mathrm{top}}(C^{0,p+1})$ of the Clifford algebra $C^{0,p+1}$; see [**19**].

If G acts trivially on Y then $KO_G(Y \times S^{p,q}, Y) \cong K^{1-q}_{\mathrm{top}}(C^{0,p+1}(Y))$, where $C^{0,p+1}(Y)$ is the Banach algebra of continuous functions from Y to $C^{0,p+1}$.

For completeness, we note that when $q = 0$ and $p > 0$ the group $KO_G(Y \times S^{p,0})$ is $KO(Y \times \mathbb{RP}^{p-1})$. This group was computed by Adams (when Y is a point) and in general by the first author in [**19**].

For later use in Example 8.9, we record that

$$KO_G(S^{1,3}, \mathrm{pt}) \cong \mathbb{Z}/2,\ KO_G(S^{2,2}, \mathrm{pt}) = 0 \text{ and } KO_G(S^{2,3}, \mathrm{pt}) = \mathbb{Z}.$$

Using this, we can check that $WR(S^{1,3}) \cong WR(S^{2,2}) \cong WR(S^{2,3}) \cong \mathbb{Z}$. (An alternative calculation of $WR(S^{2,2})$ was given in Example A.9.)

EXAMPLE 3.6. Let T be the torus $S^1 \times S^{1,1}$, where G acts trivially on S^1. Then T is a closed connected 2-manifold with involution, T^G

is a union of 2 circles, and $T/G \cong S^1 \times [0,1]$. From Lemma 3.3 and Example 3.2(i), we see that $WR(S^1) \cong \mathbb{Z} \oplus \mathbb{Z}/2$ and

$$WR(T) \cong \mathbb{Z}^2 \oplus \mathbb{Z}/2 \not\cong WR(S^1) \oplus WR(S^1).$$

If C is the affine circle $x^2 + y^2 = 1$ over $\mathbb{R}$ and $V = C \times \mathbb{G}_m$, then $W(C) \cong \mathbb{Z} \oplus \mathbb{Z}/2$ and $W(V) \cong \mathbb{Z}^2 \oplus (\mathbb{Z}/2)^2$ by the Fundamental Theorem for $W(C)$ [**20**, pp. 138–9]. Since C_{top} and V_{top} are equivariantly homotopic to S^1 and T, respectively, it follows that $W(V) \not\cong WR(V_{\text{top}})$.

EXAMPLE 3.7. The projective plane $\mathbb{P}^2$ over $\mathbb{R}$ has $\mathbb{P}^2_{\text{top}} = \mathbb{CP}^2$ and

$$W(\mathbb{P}^2) \cong WR(\mathbb{CP}^2) = \mathbb{Z}.$$

Because W and WR are birational invariants of a surface (see Theorem 5.1), this also implies that $W(\mathbb{P}^1 \times \mathbb{P}^1) \cong WR(\mathbb{CP}^1 \times \mathbb{CP}^1) \cong \mathbb{Z}$.

Arason proved that $W(\mathbb{P}^d) = \mathbb{Z}$ for all d; see [**1**]. The G-space $X = \mathbb{P}^2_{\text{top}}$ is $\mathbb{CP}^2$ with $X^G = \mathbb{RP}^2$. The quotient X/G has $H^*(X/G, \mathbb{Z}) \cong H^*(S^4, \mathbb{Z})$. In fact, X/G is homotopy equivalent to S^4; writing $\mathbb{CP}^2$ as the cone of the Hopf map, one sees that $X/G \simeq \Sigma(S^3/G) \simeq S^4$. By Lemma A.5,

$$E_2^{p,q} = H^p_G(X; KO^q_G) = H^p(\mathbb{RP}^2, KO^q) \oplus H^p(X/G, KO^q),$$

and the second factor vanishes unless $p = 0, 4$. Thus

$$H^p_G(X; KO^{-p}_G) = \begin{cases} \mathbb{Z}^2, & p = 0; \\ \mathbb{Z}/2, & p = 1, 2; \\ \mathbb{Z}, & p = 4; \\ 0, & \text{otherwise.} \end{cases}$$

Since $KO^n_G(X)$ maps onto $KO^n_G(\text{pt})$ and $KO_G(X)$ maps to $KO(\mathbb{RP}^2) = \mathbb{Z} \oplus \mathbb{Z}/4$, the spectral sequence (A.1) degenerates to show that $KO_G(X)$ is the extension $\mathbb{Z}^3 \oplus \mathbb{Z}/4$ of $\mathbb{Z}^3 \oplus \mathbb{Z}/2$ by $\mathbb{Z}/2$. By Lemma A.5, Remark A.5.1, Lemma A.8 and Remark 2.5.1, $KR(X)$ maps onto the torsion, the final $\mathbb{Z}$, and the subgroup of $\mathbb{Z}^2 \oplus \mathbb{Z}/2 \oplus \mathbb{Z}$ generated by $(2, -1, [-1], 0)$. It follows that $WR(X) \cong \mathbb{Z}$.

REMARK 3.7.1. Atiyah proved in [**2**, 2.4] that $KR(\mathbb{CP}^2) \cong \mathbb{Z}[t]/t^3$. In the filtration arising from (A.1), t is detected by $H^1_G(X, KR^{-1}) \cong \mathbb{Z}/2$; t^2 and $2t$ are in filtration level 2.

4. The equivariant Picard group

In this section X will be a connected G-space, and ν will denote the number of components of X^G.

DEFINITION 4.1. Let $\mathrm{Pic}_G(X)$ denote the group of rank 1 real G-vector bundles on X with product $\otimes$; its unit '1' is the trivial line bundle $X \times \mathbb{R}$. If L is such a bundle, $L \otimes L$ is trivial, and the involution on the fibers of L over fixed points is multiplication by ± 1. This involution is locally constant, so if X^G has ν components it defines a natural map $\mathrm{Pic}_G(X) \to \{\pm 1\}^\nu$, called the *sign map*.

Let $\mathrm{Pic}^0_G(X)$ denote the kernel of the sign map. There is a natural isomorphism $\mathrm{Pic}^0_G(X) \stackrel{\cong}{\to} H^1(X/G, \mathbb{Z}/2)$, defined as follows. If L is a G-line bundle on X with trivial sign (so G acts trivially on L_x for all $x \in X^G$, then the identifications $L_x \cong L_{\sigma x}$ imply that L descends to a line bundle L/G on X/G. Since real line bundles on X/G are classified by the group $H^1(X/G, \mathbb{Z}/2)$, this defines a map $\mathrm{Pic}^0_G(X) \to H^1(X/G, \mathbb{Z}/2)$. Conversely, the pullback of such a line bundle along $X \to X/G$ is an equivariant line bundle on X with trivial sign, i.e., an element of $\mathrm{Pic}^0_G(X)$.

A natural isomorphism $\mathrm{Pic}_G(X) \stackrel{\omega}{\to} \mathbb{H}^1_G(X, \mathbb{Z}/2)$, was constructed in [**26**, 3.2]. It is compatible with the exact sequence

$$0 \to H^1(X/G, \mathbb{Z}/2) \to \mathbb{H}^1_G(X, \mathbb{Z}/2) \xrightarrow{\text{sign}} H^0(X^G, \mathbb{Z}/2).$$

of Example A.3, as we see from the map $\mathbb{H}^1_G(X, \mathbb{Z}/2) \to \mathbb{H}^1_G(X^G, \mathbb{Z}/2)$. This exact sequence, combined with either Example 1.6 or Example 4.4, shows that the sign map is not always onto when $\nu \geq 2$.

REMARK 4.1.1. The G-line bundle $X \times \mathbb{R}(1)$ is nontrivial. This is clear if $X^G \neq \emptyset$, as $\mathbb{R}(1)$ has a nontrivial sign. When $X^G = \emptyset$, it is a nonzero element of $\mathrm{Pic}^0_G(X) \cong H^1(X/G, \mathbb{Z}/2)$. This is because $X \to X/G$ is a nontrivial covering space, covering spaces with group G are classified by elements of $H^1(X/G, \mathbb{Z}/2)$, and we showed in Example 2.7 of [**23**] that $\omega(X \times \mathbb{R}(1))$ is the element classifying this particular cover.

REMARK 4.1.2. If V is a geometrically ireducible variety over $\mathbb{R}$, and $X = V_{\text{top}}$, Cox's Theorem [**14**] states that

$$\mathrm{Pic}_G(X) \cong \mathbb{H}^1_G(X, \mathbb{Z}/2) \cong H^1_{\text{et}}(V, \mathbb{Z}/2) \cong \mathcal{O}^\times(V)/\mathcal{O}^{\times 2}(V) \oplus {}_2\,\mathrm{Pic}(V).$$

If V is projective then $\mathcal{O}^\times(V) = \mathbb{R}^\times$ and $\mathrm{Pic}_G(X) \cong \mathbb{Z}/2 \oplus {}_2\,\mathrm{Pic}(V)$.

DEFINITION 4.2. There is a determinant map $KO_G(X) \stackrel{\det}{\to} \mathrm{Pic}_G(X)$ satisfying $\det(E \oplus F) = \det(E) \otimes \det(F)$; $\det(L) = [L]$ if $\mathrm{rank}(L) = 1$.

Composing det with the isomorphism of Definition 4.1 defines a surjection $w_1 : KO_G(X) \to \mathbb{H}^1_G(X, \mathbb{Z}/2)$, which we call the *discriminant*; we will relate the discriminant to the Stiefel–Whitney class w_1 in Section 6 below.

As in the algebraic setting, the discriminant does not factor through $WR(X)$, because the discriminant of $H(1)$ is -1. Instead, the discriminant factors through the subgroup $I(X)$ of forms in $WR(X)$ of even degree, as we shall see in Theorem 6.2 below.

REMARK 4.3. If E is a rank r G-bundle on X, then $e = [E] - r$ has rank 0 in $KO_G(X)$; if E has signature a_i on the i^{th} component of X^G then the edge map η^0 of (2.4) sends e to $\eta^0(e) = (0, a_1 - r, ..., a_\nu - r)$, which is an element of $0 \times (2\mathbb{Z})^\nu$. Let $\widetilde{KO}_G(X)$ denote the kernel of $KO_G(X) \xrightarrow{\text{rank}} \mathbb{Z}$; since $\det(e) = \det(E)$ restricts to a line bundle with sign $(-1)^{(a_i - r)/2}$ on the i^{th} component of X^G, we have a commutative diagram and a map from the kernel of $KO_G(X) \xrightarrow{\eta^0} \mathbb{Z}^{1+\nu}$ onto $\mathrm{Pic}^0_G(X)$.

$$\begin{array}{ccccc} \ker(\eta^0) & \longrightarrow & \widetilde{KO}_G(X) & \xrightarrow{\eta^0} & (2\mathbb{Z})^\nu \\ \big\downarrow{\scriptstyle \text{onto}} & & \big\downarrow{\scriptstyle \det} & & \big\downarrow{\scriptstyle \bmod 2} \\ \mathrm{Pic}^0_G(X) & \longrightarrow & \mathrm{Pic}_G(X) & \xrightarrow{\text{sign}} & \{\pm 1\}^\nu. \end{array}$$

As noted in Example 4.4, the sign map $\mathrm{Pic}_G(X) \to \{\pm 1\}^\nu$ need not be onto; it follows that the signature $KO_G(X) \to \Gamma \subseteq \mathbb{Z}^\nu$ of 2.5 need not be onto either.

EXAMPLE 4.4. The 2-sphere $X = S^{2,1}$ has $\nu = 2$ as X^G is two points, and we saw in Example 3.5 that $KO_G(S^{2,1}) \cong \mathbb{Z}^2 \oplus (\mathbb{Z}/2)$. It follows that $\mathrm{Pic}_G(X) \cong \mathbb{H}^1_G(X, \mathbb{Z}/2) = \mathbb{Z}/2$. From Remark 4.3, we see that the signature $KO_G(X) \to \Gamma \subseteq \mathbb{Z}^2$ is not onto.

Given an algebraic surface V defined over $\mathbb{R}$, such that $V(\mathbb{R})$ has ν real components, let S_a and S_t denote the images of the algebraic signature $W(V) \to \mathbb{Z}^\nu$ and the topological signature $WR(V_{\text{top}}) \to \mathbb{Z}^\nu$, respectively. Since the algebraic signature factors through the topological one, $S_a \subseteq S_t$. We do not know any examples where $S_a \neq S_t$.

As we noted in Lemma 2.5, $S_t \subseteq \Gamma$. Colliot-Thélène and Sansuc proved in [**10**, 4.1] that $I_3(V)$ maps onto $8\mathbb{Z}^\nu$, i.e, that S_a contains $8\mathbb{Z}^\nu$. (See also [**9**, proof of Thm. 2.3.2].) Therefore we have

$$8\mathbb{Z}^\nu \subseteq S_a \subseteq S_t \subseteq \Gamma \subseteq \mathbb{Z}^\nu.$$

We also know from [**23**, 8.7] that $2S_t \subseteq S_a$.

PROPOSITION 4.5. *The images of the algebraic and topological signatures in $\mathbb{Z}^\nu/4\mathbb{Z}^\nu$ are the same.*

PROOF. Set $X = V_{\mathrm{top}}$; we may suppose X connected. Since both $W(V)/I(V)$ and $WR(X)/I(X)$ map isomorphically to $\Gamma/2\mathbb{Z}^\nu \cong \mathbb{Z}/2$, the signature maps $I(V)$ and $I(X)$ to $2\mathbb{Z}^\nu$. We need to consider their images modulo $4\mathbb{Z}^\nu$.

Let us write $\{\pm 1\}^\nu$ for $(2\mathbb{Z}/4\mathbb{Z})^\nu$. By Remark 4.3, the reduction modulo 2 of the topological signature, i.e., $I(X) \to (2\mathbb{Z})^\nu \to \{\pm 1\}^\nu$, factors as the discriminant $I(X) \to \mathrm{Pic}_G(X)$, which is onto, followed by the sign map $\mathrm{Pic}_G(X) \to \{\pm 1\}^\nu$. The same is true of the algebraic signature. Hence we have a commutative diagram:

$$\begin{array}{ccccc} I(V) & \xrightarrow{\text{onto}} & H^1_{\mathrm{et}}(V, \mathbb{Z}/2) & \xrightarrow{\text{sign}} & \{\pm 1\}^\nu \\ \downarrow & & \downarrow \cong & & \downarrow = \\ I(X) & \xrightarrow{\text{onto}} & \mathbb{H}^1_G(X, \mathbb{Z}/2) & \xrightarrow{\text{sign}} & \{\pm 1\}^\nu. \end{array}$$

By inspection, the images of $I(V)$ and $I(X)$ in $\{\pm 1\}^\nu$ agree. □

REMARK 4.5.1. Here are two consequences of Proposition 4.5:
i) If the signature $WR(X) \to \Gamma$ is onto, so is the signature $W(V) \to \Gamma$.
ii) If $4\mathbb{Z}^\nu \subseteq S_a$ then $S_a = S_t$.

5. WR is a birational invariant

Let V, V' be smooth projective surfaces defined over $\mathbb{R}$. We say that V and V' are *(real) birationally equivalent* if there exists a chain of blow-ups at smooth closed points (and their inverses, blow-downs) connecting them. This is equivalent to the assertion that their function fields are isomorphic (see [**47**, II.6.4]). By a *birational invariant* of smooth projective surfaces over $\mathbb{R}$ we mean a function that sends real birationally equivalent surfaces to isomorphic objects. For example, the number of real components is a birational invariant.

THEOREM 5.1. *WR is a birational invariant of smooth projective surfaces over $\mathbb{R}$.*

PROOF. Since every birational equivalence between smooth projective surfaces is a composition of blow-ups and blow-downs at smooth closed points, the result is immediate from Theorem 5.2 below. □

REMARK 5.1.1. We recall how to topologically construct the blow-up $V' \to V$ of a complex algebraic surface V at a smooth point x of V. Let D^4 denote a small 4-disk about x; the inverse image of D^4 in V'_{top} is a tubular neighborhood T of the exceptional fiber $\mathbb{P}^1_{\mathrm{top}} \cong S^2$, and the boundary of T is the same as the boundary S^3 of D^4. (T has Euler class 1.) Thus $V'_{\rm top}$ is the union of $V_{\rm top} - D^4$ and T along S^3.

THEOREM 5.2. *Let V be a smooth projective variety defined over $\mathbb{R}$, and $V' \to V$ the blowup at a smooth closed point p. Then $WR(V_{\rm top}) \to WR(V'_{\rm top})$ is an isomorphism.*

PROOF. Set $X = V_{\rm top}$, $X' = V'_{\rm top}$. There are two cases; we may blow up either a real point of V (a fixed point x of X), or a complex point of V (a conjugate pair of points on X).

Fixed points. Consider first the case where V' is the blow-up of V at an $\mathbb{R}$-point. That is, X' is obtained by removing an equivariant disk D^4 in X about the point and replacing it with a tubular neighborhood T of S^2. For any equivariant cohomology theory H^*, the excision $H^*(X, D^4) \cong H^*(X', T)$ yields a Mayer-Vietoris sequence.

Writing $\mathbb{P}$ for the G-space $\mathbb{CP}^1$, we have a commutative diagram, whose rows are Mayer–Vietoris sequences and whose columns are the exact Bott sequences (see [**44**, Thm. 6.1] and [**23**, (1.5)]).

(5.2.1)

$$\begin{tikzcd}
KR_1(\mathbb{P},\mathrm{pt}) \arrow[r] \arrow[d, "\cong"] & KR(X) \arrow[r] \arrow[d, "H"] & KR(X') \arrow[r, "\text{onto}"] \arrow[d, "H"] & KR(\mathbb{P},\mathrm{pt}) = \mathbb{Z} \arrow[d, "\text{onto}"] \\
GR_1(\mathbb{P},\mathrm{pt}) \arrow[r] \arrow[d, "0"] & GR(X) \arrow[r] \arrow[d, "\partial"] & GR(X') \arrow[r] \arrow[d, "\partial"] & GR(\mathbb{P},\mathrm{pt}) = \mathbb{Z}/2 \arrow[d] \\
GR_0^{[-1]}(\mathbb{P},\mathrm{pt}) \arrow[r, "\sigma"] \arrow[d, shift left, "2"] & GR_{-1}^{[-1]}(X) \arrow[r, "u"] \arrow[d, shift left, "F"] & GR_{-1}^{[-1]}(X') \arrow[r] \arrow[d, shift left, "F"] & GR_{-1}^{[-1]}(\mathbb{P},\mathrm{pt}) = 0 \arrow[d] \\
KR(\mathbb{P},\mathrm{pt}) \arrow[r, "0"] \arrow[u, shift left, "\tau"] & KR_{-1}(X) \arrow[r] \arrow[u, shift left, "\tau"] & KR_{-1}(X') \arrow[r] \arrow[u, shift left, "\tau"] & KR_{-1}(\mathbb{P},\mathrm{pt}) = 0
\end{tikzcd}$$

The Bott sequences in the first and last columns are well known, with $KR_1(\mathbb{P},\mathrm{pt}) \cong \mathbb{Z}/2$ mapping onto $GR_1(\mathbb{P},\mathrm{pt}) \cong \mathbb{Z}/2$, $KR(\mathbb{P},\mathrm{pt}) \cong \mathbb{Z}$ mapping onto $GR_1(\mathbb{P},\mathrm{pt}) \cong \mathbb{Z}/2$, and $GR_0^{[-1]}(\mathbb{P},\mathrm{pt}) = \mathbb{Z}$ mapping to $KR(\mathbb{P},\mathrm{pt}) = \mathbb{Z}$ by multiplication by 2.

Since the canonical bundle on $\mathbb{P}$ lifts to the Real bundle on X' associated to the invertible sheaf $\mathcal{O}(1)$ on V' [**16**, II.7], the top right horizontal map is onto, and the bottom left horizontal map is zero. It follows from a diagram chase that the map $WR(X) \to WR(X')$ is onto. To see that it is an injection, we will show that the map σ is zero, and hence that the map u is an injection; this suffices, since $WR(X)$ and $WR(X')$ are the images of the vertical maps ∂ in $GR_{-1}^{[-1]}(X)$ and $GR_{-1}^{[-1]}(X')$.

The map $\tau : KR_n(X) \to GR_n^{[-1]}(X)$ in (5.2.1) is defined as the composition of the map $KR_n(X) \to {}_-GR_n^{[1]}(X)$ in the Bott sequence with the isomorphism ${}_-GR_n^{[1]} \cong GR_n^{[-1]}$ of the Fundamental Theorem; see [**44**, 6.2]. Identifying $GR_0^{[-1]}(\mathbb{P}, \mathrm{pt})$ with the relative group associated to the hyperbolic functor, τ sends the class of a Real bundle E to the hyperbolic bundle $E \oplus E^*$, equipped with its two Lagrangians $E \oplus 0$ and $0 \oplus E^*$. The composition $F \circ \tau$ sends $[E]$ to $[E] - [E^*]$. Since $[E^*] = -[E]$ in $KR(\mathbb{P}, \mathrm{pt})$, the composition $F \circ \tau$ on the lower left group $KR(\mathbb{P}, \mathrm{pt}) \cong \mathbb{Z}$ is multiplication by 2; it follows that $\tau : KR(\mathbb{P}, \mathrm{pt}) \to GR_0^{[-1]}(\mathbb{P})$ is an isomorphism.

Conjugate points. We now consider the case where X' is the blow-up of X at a conjugate pair of points. In this case, we have a diagram like (5.2.1), with $(\mathbb{P}, \mathrm{pt})$ replaced by $(G \times \mathbb{P}, G \times \mathrm{pt})$. As in [**23**, 2.4(b) and 2.6] (cf. Lemma 2.2 above), $GR_0^{[-1]}(G \times \mathbb{P}, G) \cong KO^{-2}(S^2, \mathrm{pt})$ and $KR(G \times \mathbb{P}, G) \cong KU(S^2, \mathrm{pt})$. Thus the bottom left and upper right vertical maps are the familiar maps:

$$\mathbb{Z} = KO^{-2}(S^2, \mathrm{pt}) \xrightarrow{2} KU(S^2, \mathrm{pt}) = \mathbb{Z}, \text{ and}$$
$$\mathbb{Z} = KU(S^2, \mathrm{pt}) \xrightarrow{\text{onto}} KO(S^2, \mathrm{pt}) = \mathbb{Z}/2.$$

Again, the Real bundle on X' associated to $\mathcal{O}(1)$ maps to the canonical complex bundle generating $KU(S^2, \mathrm{pt})$ so, as in (5.2.1), the upper right horizontal map is onto and the lower left horizontal map is zero. Again, as in diagram (5.2.1), this implies that the map σ is zero and hence that u is an injection. (In fact, u is an isomorphism as $KO^1(S^2, \mathrm{pt}) = 0$.) As before, this suffices to show that $WR(X) \to WR(X')$ is an isomorphism. $\square$

6. Stiefel–Whitney classes on KO_G

If X is a compact G-space, let us write X_G for $X \times_G EG$, so that the (Borel) equivariant cohomology group $H^n_G(X, \mathbb{Z}/2)$ is defined to be $H^n(X_G, \mathbb{Z}/2)$. Following [**3**], we can identify vector bundles on X_G with equivariant vector bundles on $X \times EG$; the pullback of a bundle on X_G is an equivariant vector bundle on $X \times EG$. As in [**3**], we define $KU(X_G)$ and $KO(X_G)$ to be representable K-theory: $KU(X_G) = [X_G, KU]$ and $KO(X_G) = [X_G, KO]$.

The map $X \times EG \to X$ induces natural maps $KU_G(X) \to KU(X_G)$ and $KO_G(X) \to KO(X_G)$, called *Atiyah-Segal maps*, since they were first studied by Atiyah and Segal in [**3**]. Atiyah and Segal showed in [**3**] that $KO(X_G)$ is the completion of $KO_G(X)$ (for finite CW complexes X) with respect to the augmentation ideal of $KO_G(\mathrm{pt}) \cong RO(G)$.

DEFINITION 6.1. The maps $w_n : KO_G(X) \to \mathbb{H}^n_G(X, \mathbb{Z}/2)$, obtained by composing the Atiyah–Segal map with the usual Stiefel–Whitney classes on X_G, are called the equivariant Stiefel–Whitney classes.

The equivariant Stiefel–Whitney class w_1 agrees on $KO_G(X)$ with the discriminant w_1 defined in Definition 4.2. To see this, note that $\mathrm{Pic}_G(X) \cong \mathrm{Pic}_G(X \times EG)$, and elements of $\mathrm{Pic}_G(X \times EG)$ correspond to rank 1 $\mathbb{R}$-linear vector bundles on X_G, which are classified by the group $H^1(X_G, \mathbb{Z}/2)$; see [**53**, I.4.11].

The equivariant Stiefel–Whitney class w_1 does not factor through $WR(X)$, as the example $X = \mathrm{pt}$ shows (see Example 2.1.1). However, we showed in [**26**, 5.2] that the restriction of w_1 to $I(X)$, the kernel of $KO_G(X) \to \mathbb{H}^0_G(X, \mathbb{Z}/2)$, does induce a map $I(X) \to \mathbb{H}^1_G(X, \mathbb{Z}/2)$, which we shall call w_1 by abuse of notation. The following result was proven in [**26**, Thm. 5.3].

THEOREM 6.2. *The algebraic discriminant of a smooth variety V factors as*

$$I(V) \to I(V_{\mathrm{top}}) \xrightarrow{w_1} \mathbb{H}^1_G(V_{\mathrm{top}}, \mathbb{Z}/2) \cong H^1_{\mathrm{et}}(V, \mathbb{Z}/2).$$

Write $I_2(X)$ for the kernel of $w_1 : I(X) \to \mathbb{H}^1_G(X, \mathbb{Z}/2)$.

COROLLARY 6.3. $W(V)/I_2(V) \overset{\cong}{\to} WR(X)/I_2(X)$, *where* $X = V_{\mathrm{top}}$.

PROOF. By Definition 4.2 and Theorem 6.2, the isomorphism of Lemma 1.5 is the composition

$$I(V)/I_2(V) \to I(X)/I_2(X) \xrightarrow{\cong} \mathbb{H}^1_G(X, \mathbb{Z}/2).$$

The extension from $I(V) \cong I(X)$ to $W(V) \cong W(X)$ is routine. $\square$

As noted in Definition 4.2, the rank and w_1 give a ring homomorphism $KO_G(X) \to \mathbb{Z} \times \mathbb{H}^1_G(X, \mathbb{Z}/2)$. It is onto, because $\mathrm{Pic}^0_G(X)$ maps onto $\mathbb{H}^1_G(X, \mathbb{Z}/2) \cong H^1(X/G, \mathbb{Z}/2)$ (see 4.1).

LEMMA 6.4. *Suppose that G acts freely on a connected space X. Then the rank and w_1 induce a non-trivial extension*

$$0 \to H^1(X/G, \mathbb{Z}/2) \to WR(X)/I_2(X) \xrightarrow{\mathrm{rank}} \mathbb{Z}/2 \to 0.$$

This is the extension such that $2 \cdot 1$ is $[-1]$, where '1' is the image of the unit of the ring $WR(X)$, and $[-1] \in H^1(X/G, \mathbb{Z}/2) \cong \mathrm{Pic}^0_G(X)$ is the class of the line bundle $X \times \mathbb{R}(1)$.

PROOF. The map $H : KR(X) \to KO_G(X)$ is compatible with the filtration F_* associated to the Bredon spectral sequence (A.1), and we have $KO_G(X)/F_2 = \mathbb{Z} \times \mathbb{H}^1_G(X, \mathbb{Z}/2)$. Since $H^1_G(X; KR^{-1}) = 0$ we have $KR(X)/F_2 \cong \mathbb{Z}$. By Remark 2.5.1, $(2, [-1])$ is the image under $w_1 \circ H$ of the trivial Real line bundle in $\mathbb{Z} \times \mathbb{H}^1_G(X, \mathbb{Z}/2)$. Since $[-1] \neq 0$ by Remark 4.1.1, the result follows. □

The Stiefel–Whitney class $w_2 : KO_G(X) \to \mathbb{H}^2_G(X, \mathbb{Z}/2)$ does not factor through $WR(X)$ either, because w_2 need not vanish on the image of $H : KR(X) \to KO_G(X)$. Consider the composition

$$\bar{w}_2 : WR(X) \to {}_2\mathbb{H}^3_G(X, \mathbb{Z}(1)).$$

of the map $w_2 : KO_G(X) \to \mathbb{H}^2_G(X, \mathbb{Z}/2)$, followed by the Bockstein. We proved the following result in [**26**, 5.6 and 5.8].

THEOREM 6.5. *If V is a smooth variety, then the composition of*

$$GW(V) \to GR(V_{\mathrm{top}}) \cong KO_G(V_{\mathrm{top}}) \to WR(V_{\mathrm{top}})$$

with $\bar{w}_2$ agrees with the algebraic Hasse invariant 1.3, followed by the Bockstein:

$$GW(V) \xrightarrow{\text{Hasse}} H^2_{\mathrm{et}}(V, \mathbb{Z}/2)/(\operatorname{Pic} V/2) \xrightarrow{\beta} {}_2\mathbb{H}^3_G(V_{\mathrm{top}}, \mathbb{Z}(1)).$$

That is, $\beta\,\mathrm{Hasse}(\theta) = \bar{w}_2(\theta_{\mathrm{top}})$ for all $\theta \in GW(V)$.

We end this section with a discussion of equivariant Chern classes.

DEFINITION 6.6. In [**17**], Bruno Kahn defined equivariant Chern classes $c_n : KR(X) \to \mathbb{H}^{2n}_G(X, \mathbb{Z}(n))$ for Real vector bundles, with the first Chern class c_1 inducing an isomorphism between the group of rank 1 Real vector bundles on X and $\mathbb{H}^2_G(X, \mathbb{Z}(1))$, where $\mathbb{Z}(1)$ is the sign representation. In particular, $c_1 : KR(X) \to \mathbb{H}^2_G(X, \mathbb{Z}(1))$ is a surjection.

The following calculation is taken from [**17**, Th. 1]. Recall from Definition 6.6 that the first Chern class $c_1(E)$ of a Real vector bundle E on X is an element of $H^2_G(X, \mathbb{Z}(1))$. If E has rank $d+1$ and $\mathbb{P}(E)$ is the associated projective bundle, then the fiber of $\mathbb{P}(E)$ over a point $x \in X$ is the copy of $\mathbb{CP}^d$ corresponding to $\pi^{-1}(x)$.

THEOREM 6.7 (Kahn). *Let $E \xrightarrow{\pi} X$ be a rank 2 Real bundle on a G-space X, and let $\mathbb{P}(E)$ denote the associated projective space. If $\xi = c_1(E) \in \mathbb{H}^2_G(X, \mathbb{Z}(1))$ is the first Chern class of E then for any coefficient system A*

$$\mathbb{H}^n_G(\mathbb{P}(E), A) = \mathbb{H}^n_G(X, A) \oplus \mathbb{H}^{n-2}_G(X, A(1)) \cdot \xi.$$

In particular, $\mathbb{H}^1_G(\mathbb{P}(E),\mathbb{Z}/2)=\mathbb{H}^1_G(X,\mathbb{Z}/2)$ *and*

$$\mathbb{H}^2_G(\mathbb{P}(E),\mathbb{Z}(1))=\mathbb{H}^2_G(X,\mathbb{Z}(1))\oplus\mathbb{H}^0_G(X,\mathbb{Z}).$$

Remark 6.7.1. Kahn's formula is more general. In particular, if X is a point and $E=\mathcal{O}^2_X$ so that $\mathbb{P}(E)=\mathbb{P}^2$, it yields

$$\mathbb{H}^n_G(\mathbb{P}^2,A)=\mathbb{H}^n_G(\mathrm{pt},A)\oplus\mathbb{H}^{n-2}_G(\mathrm{pt},A(1))\cdot\xi\oplus\mathbb{H}^{n-4}_G(\mathrm{pt},A)\cdot\xi^2.$$

7. Algebraic surfaces with no $\mathbb{R}$-points

Let V be an ireducible variety defined over $\mathbb{R}$. Recall [**16**, Ex. II.3.15] that V is said to be *geometrically connected* (or *geometrically integral*) if $V\times_{\mathbb{R}}\mathbb{C}$ is connected, i.e., the function field of V does not contain $\mathbb{C}$. In this case, V_{top} is connected.

If V is an irreducible surface defined over $\mathbb{R}$, and V_{top} is not connected, then V is also defined over $\mathbb{C}$, i.e., V is a complex variety. In this case, the G-space $V_{\mathrm{top}}=\mathrm{Hom}_{\mathbb{R}}(\mathrm{Spec}(\mathbb{C}),V)$ is $G\times V(\mathbb{C})$, where $V(\mathbb{C})=\mathrm{Hom}_{\mathbb{C}}(\mathrm{Spec}(\mathbb{C}),V)$ is the usual space of complex points.

We proved the following theorem in [**26**, Thm. 7.4]; it is a refinement of a theorem of Zibrowius [**55**, 5.12]. The invariant ρ is the rank of the cokernel of the first Chern class $c_1: NS(V)\to H^2(V(\mathbb{C}),\mathbb{Z})$, and p_g is the geometric genus; the inequality $\rho\ge 2\,p_g$ in Theorem 7.1 follows from Hodge theory.

Theorem 7.1. *Suppose that V is a smooth projective surface over $\mathbb{C}$. Then there is a split exact sequence*

$$0\to(\mathbb{Z}/2)^\rho\to W(V)\to WR(V_{\mathrm{top}})\to 0,$$

where $\rho\ge 2\,p_g$, and $\rho=0$ when $p_g=0$. Thus $W(V)\to WR(V_{\mathrm{top}})$ is an isomorphism if and only if $p_g=0$.

Surfaces with $p_g=0$ include the projective plane $\mathbb{P}^2$, rational surfaces, ruled surfaces, K3 surfaces, and Enriques surfaces (see [**16**]). It also includes some surfaces of general type, such as Godeaux surfaces, Burniat surfaces and Mumford's fake projective plane.

When V is geometrically connected, i.e., V_{top} is connected, the following result was proven in [**26**, Thm. 8.5]. The invariant ρ_0 is the rank of $c_1:\mathrm{Pic}(V)\to\mathbb{H}^2_G(X,\mathbb{Z}(1))$, and the relation $\rho_0\ge p_g$ follows from Hodge theory.

THEOREM 7.2. *Let V be a smooth geometrically connected projective surface over $\mathbb{R}$ with no real points. Then there is a split exact sequence*

$$0 \to (\mathbb{Z}/2)^{\rho_0} \to W(V) \to WR(V_{\text{top}}) \to 0,$$

where $\rho_0 \geq p_g$, and $W(V) \to WR(V_{\text{top}})$ is an isomorphism if and only if V has geometric genus $p_g = 0$.

The following calculation was given in [**26**, 8.1]. As in Lemma 3.1, $\widetilde{H}^1(X/G, \mathbb{Z}/2)$ denotes the quotient of $H^1(X/G, \mathbb{Z}/2)$ by the subgroup generated by $[-1] = w_1(X \times \mathbb{R}(1))$.

THEOREM 7.3. *Let X be a connected 4-dimensional G-CW complex. If G acts freely on X, then $WR(X)$ is a $\mathbb{Z}/8$-algebra, and there is an extension:*

$$0 \to {}_2\mathbb{H}^3_G(X, \mathbb{Z}(1)) \to WR(X) \ \to \mathbb{Z}/4 \times \widetilde{H}^1(X/G, \mathbb{Z}/2) \to 0;$$

In the rest of this section, we describe $WR(V_{\text{top}})$ for several surfaces with $p_g = 0$ and with no real points.

Forms of $\mathbb{P}^1 \times \mathbb{P}^1$. A variety V is a *form* of $\mathbb{P}^1 \times \mathbb{P}^1$ if $V \otimes_{\mathbb{R}} \mathbb{C} \cong \mathbb{P}^1 \times \mathbb{P}^1$. In this case, $p_g = 0$ and V_{top} is $S^2 \times S^2$, so $W(V) \cong WR(V_{\text{top}})$ by Theorem 7.2. In this case, the calculation of WR is dictated by the structure of $H^2(V_{\text{top}}, \mathbb{Z})$ as a G-module.

THEOREM 7.4. *Suppose that G acts freely on $X = S^2 \times S^2$.*
(i) If G acts as -1 on $H^2(X, \mathbb{Z})$ then $WR(X) \cong \mathbb{Z}/4$.
(ii) If $H^2(X, \mathbb{Z}) \cong \mathbb{Z}[G]$ as a G-module then $WR(X) \cong \mathbb{Z}/4 \oplus \mathbb{Z}/2$.
(iii) If $H^2(X, \mathbb{Z}) \cong \mathbb{Z} \oplus \mathbb{Z}(1)$ as a G-module then $WR(X) \cong \mathbb{Z}/4 \oplus \mathbb{Z}/2$.

EXAMPLES 7.4.1. Cases (i)–(iii) are the only possibilities, because the only G-module structures on $\mathbb{Z}^2$ are: $\mathbb{Z}^2$, $\mathbb{Z}(1)^2$, $\mathbb{Z}[G]$ and $\mathbb{Z} \oplus \mathbb{Z}(1)$ (see [**42**, 2.1]), and G cannot act trivially on $H^2(X, \mathbb{Z})$ (by the Lefschetz Fixed Point Theorem).

Let X be the space $S^2 \times S^2$, where $S^2 = \mathbb{C} \cup \{\infty\}$. The involutions sending (z_1, z_2) to $(-\bar{z}_1^{-1}, -\bar{z}_2^{-1})$, $(\bar{z}_2^{-1}, -\bar{z}_1^{-1})$ and $(-\bar{z}_1^{-1}, z_2)$ have no fixed points, and yield the G-space structures $\mathbb{Z}(1)^2$, $\mathbb{Z}[G]$ and $\mathbb{Z} \oplus \mathbb{Z}(1)$. These illustrate cases (i), (ii) and (iii) of Theorem 7.4.

PROOF OF THEOREM 7.4. Since $\pi_1(X) = 0$, we have $\pi_1(X/G) = \mathbb{Z}/2$. Hence $H^1(X/G, \mathbb{Z}/2)$ is $\mathbb{Z}/2$. It follows from Theorem 7.3 that $WR(X)$ is an extension of $\mathbb{Z}/4$ by ${}_2\mathbb{H}^3_G(X, \mathbb{Z}(1))$.

To determine ${}_2\mathbb{H}^3_G(X, \mathbb{Z}(1))$, we recall from A.3 that

$$\mathbb{H}^*_G(X, \mathbb{Z}) \cong H^*(X/G, \mathbb{Z}) \quad \text{and} \quad \mathbb{H}^*_G(X, \mathbb{Z}[G]) \cong H^*(X, \mathbb{Z}).$$

Hence the long exact cohomology sequence associated to the exact sequence $0 \to \mathbb{Z} \to \mathbb{Z}[G] \to \mathbb{Z}(1) \to 0$ of G-modules becomes:

$$0 = H^3(X,\mathbb{Z}) \to \mathbb{H}^3_G(X,\mathbb{Z}(1)) \to H^4(X/G,\mathbb{Z}) \to H^4(X,\mathbb{Z}). \tag{7.4}$$

Now consider the spectral sequence $H^p(G, H^q(X,\mathbb{Z})) \Rightarrow H^{p+q}(X/G,\mathbb{Z})$ associated to $X_G \to BG$. For convenience, we omit the coefficient when it is $\mathbb{Z}$. If t is the nonzero element of $E_\infty^{2,0} = E_2^{2,0} = H^2(BG) \cong \mathbb{Z}/2$ then except for $p = 0$, cup product with t is an isomorphism $E_2^{p,q} \cong E_2^{p+2,q}$, commuting with the differentials.

We now consider the possible G-module structures on $H^2(X) \cong \mathbb{Z}^2$.

(i) If $H^2(X) \cong \mathbb{Z}(1)^2$ then $H^4(X) \cong \mathbb{Z}$ as a G-module. For even $p > 4$, the sequence

$$0 \to \mathbb{Z}/2 = E_3^{p-6,4} \xrightarrow{d_3} E_3^{p-3,2} = (\mathbb{Z}/2)^2 \xrightarrow{d_3} E_3^{p,0} = \mathbb{Z}/2 \to 0$$

must be exact. By t–periodicity, the map $d_3^{1,2} : E_3^{1,2} \to E_3^{4,0}$ must be onto. It follows that $H^4(X/G) \to H^4(X)$ is an injection (with cokernel $\mathbb{Z}/2$). The exact sequence (7.4) yields $\mathbb{H}^3_G(X,\mathbb{Z}(1)) = 0$.

(ii) If $H^2(X)$ is $\mathbb{Z}[G]$, we have $\mathbb{Z}/2 \cong H^4(BG) \xrightarrow{\cong} H^4(X/G)$. This is because $H^4(X) \cong \mathbb{Z}(1)$ so $H^4(X)^G = 0$, and $H^p(G, H^2(X)) = 0$ for $p > 0$. The result follows from (7.4).

(iii) If $H^2(X,\mathbb{Z}) \cong \mathbb{Z} \oplus \mathbb{Z}(1)$ then $H^4(X) \cong \mathbb{Z}(1)$ as a G-module. For even $p \geq 6$, the sequences

$$0 = E_3^{p-6,4} \xrightarrow{d_3} E_3^{p-3,2} = \mathbb{Z}/2 \xrightarrow{d_3} E_3^{p,0} = \mathbb{Z}/2 \to 0,$$
$$0 \to \mathbb{Z}/2 = E_3^{p-5,4} \xrightarrow{d_3} E_3^{p-2,2} = \mathbb{Z}/2 \xrightarrow{d_3} E_3^{p+1,0} = 0$$

must be exact. It follows that $H^4(X/G,\mathbb{Z}) \cong \mathbb{Z}/2$. □

The anisotropic quadric. Let Q_2 be the anisotropic quadric surface $x^2+y^2+z^2+w^3 = 0$. It is well known that the topological G-space X underlying Q_2 is $S^{3,0} \times S^{3,0}$, where $S^{3,0}$ is the 2-sphere with the antipodal involution. (See [**47**, VI.5] or [**11**, Proof of 2.4] for example.)

Corollary 7.5. $W(Q_2) \cong WR(S^{3,0} \times S^{3,0}) \cong \mathbb{Z}/4$.

Proof. This follows from Theorem 7.4(i) and Theorem 7.2. □

Remark 7.5.1. The fact that $W(Q_2) \cong \mathbb{Z}/4$ is due to Parimala [**38**, p. 92]. See Theorem 7.6 for a short algebraic proof.

Rational surfaces. An algebraic surface defined over $\mathbb{R}$ is called a *real rational surface* if $V_{\mathbb{C}}$ is birational to the projective plane over $\mathbb{C}$. For example, Q_2 is a rational surface.

If V is a real rational surface with no real points, then $p_g = 0$, and $W(V) \cong WR(V_{\mathrm{top}})$ by Theorem 7.2.

THEOREM 7.6. *Let V be a real rational surface with no real points. Then $W(V) \xrightarrow{\cong} WR(V_{\mathrm{top}}) \cong \mathbb{Z}/4$.*

PROOF. We calculate $W(V)$ and $WR(V_{\mathrm{top}})$ separately, for comparison purposes. We begin with the algebraic calculation of $W(V)$.

Sujatha proved in her thesis that $W(V) \cong \mathbb{Z}/4$. Unfortunately, the published version [**48**, 4.2] asserts that $W(V) = \mathbb{Z}/4 \oplus \mathbb{Z}/2$, due to a typo in the proof (on top of p. 100 in *loc. cit.*): the incorrect formula $\mathrm{Pic}(V_{\mathbb{C}})^G \cong \mathrm{Pic}(V) \oplus \mathbb{Z}/2$, should say that the torsionfree group $\mathrm{Pic}(V_{\mathbb{C}})^G$ is a nontrivial extension of $\mathrm{Br}(\mathbb{R}) \cong \mathbb{Z}/2$ by $\mathrm{Pic}(V)$:

$$0 \to \mathrm{Pic}(V) \to \mathrm{Pic}(V_{\mathbb{C}})^G \xrightarrow{d_3} \mathrm{Br}(\mathbb{R}) \to 0.$$

With this correction, Sujatha's proof of [**48**, 4.2] yields

$$\dim H^2(V, \mathbb{Z}/2) = \dim(\mathrm{Pic}\, V)/2 \quad \text{and} \quad {}_2\mathrm{Br}(V) = 0$$

(so $k = 0$ in the notation of [**48**]), and hence $W(V) \cong \mathbb{Z}/4$.

We turn to the topological calculation, setting $X = V_{\mathrm{top}}$. In this case, $H^1(X, \mathbb{Z}) = H^3(X, \mathbb{Z}) = 0$, and $H^2(X, \mathbb{Z})$ is given by Lemma 7.7 below. The spectral sequence $E_2^{p,q} = H^p(G, H^q(X, \mathbb{Z})) \Rightarrow \mathbb{H}^*_G(X, \mathbb{Z})$ agrees with the spectral sequence in the proof of Theorem 7.4(i), except for the $E_2^{0,2}$ term. As this term played no role in the proof of *loc. cit.*, the same argument (using 7.2) applies to yield $WR(X) \cong \mathbb{Z}/4$. $\square$

LEMMA 7.7. *Let V be a real rational surface with no real points. Then there is an integer c such that $H^2(V_{\mathrm{top}}, \mathbb{Z}) \cong \mathbb{Z}(1)^2 \oplus \mathbb{Z}[G]^c$.*

PROOF. It is well known that $\mathrm{Pic}(V_{\mathbb{C}}) \cong \mathbb{Z}^2 \oplus \mathbb{Z}[G]^c$ as a G-module; see [**39**] or [**47**, p. 46]. Since $H^1_{\mathrm{an}}(X, \mathcal{O}) = H^2_{\mathrm{an}}(X, \mathcal{O}) = 0$, the equivariant exponential sequence $0 \to \mathbb{Z}(1) \to \mathcal{O}_{\mathrm{an}} \to \mathcal{O}^\times_{\mathrm{an}} \to 0$ on X implies that as a G-module:

$$\mathrm{Pic}(V_{\mathbb{C}}) \cong H^1_{\mathrm{an}}(X, \mathcal{O}^\times_{\mathrm{an}}) \cong H^2_{\mathrm{an}}(X, \mathbb{Z}(1)) \cong H^2(X, \mathbb{Z}) \otimes \mathbb{Z}(1). \qquad \square$$

REMARK 7.7.1. Every finitely generated torsionfree G-module A is isomorphic to $\mathbb{Z}^a \oplus \mathbb{Z}(1)^b \oplus \mathbb{Z}[G]^c$ for some integers a, b, c. The integer c is called the *Comesssatti characteristic* of A, since Comessatti used it to study abelian varieties over $\mathbb{R}$ in the 1920's; see [**12, 13**]. The integer c in Lemma 7.7 is called the *Comessatti characteristic* of $H^2(X)$. The Comessatti character of $H^d(V_{\mathrm{top}}, \mathbb{Z})$, $d = \dim V$, has played a major role in computations by Silhol [**47**, IV.1], Parimala–Sujatha [**39**] and the authors [**41**, 1.1], [**23**, 4.2].

Ruled surfaces. Recall that a *ruled surface* over a curve C is a variety of the form $V = \mathbb{P}(E)$, where E is a locally free sheaf of rank 2 over C. For example, if E is free then $V = C \times \mathbb{P}^1$. Ruled surfaces have $p_g = 0$, so Theorem 7.2 yields $W(V) \cong WR(V_{\text{top}})$ if C has no $\mathbb{R}$-points, since in that case V has no $\mathbb{R}$-points either.

THEOREM 7.8. *Let C be a smooth, geometrically connected projective curve over $\mathbb{R}$ of genus g. If C has no real points and V is a ruled surface over C, then*

$$W(V) \cong WR(V_{\text{top}}) \cong WR(C_{\text{top}}) \cong \mathbb{Z}/4 \oplus (\mathbb{Z}/2)^g.$$

PROOF. It suffices to show that $WR(V_{\text{top}}) \cong WR(C_{\text{top}})$, because $W(C) \cong WR(C_{\text{top}}) \cong \mathbb{Z}/4 \oplus (\mathbb{Z}/2)^g$ by Corollary 3.4. Because $\mathbb{H}^3_G(C_{\text{top}}, \mathbb{Z}(1)) = 0$, we see from the Projective Bundle Theorem 6.7 that

$$\mathbb{H}^3_G(V_{\text{top}}, \mathbb{Z}(1)) \cong \mathbb{H}^1_G(C_{\text{top}}, \mathbb{Z}) \cong H^1(C_{\text{top}}/G, \mathbb{Z}),$$

which is torsionfree. Theorem 7.3 yields $WR(V_{\text{top}}) \cong WR(C_{\text{top}})$. $\square$

ALGEBRAIC PROOF. It suffices to show that $W(V) \cong W(C)$. By the algebraic Projective Bundle Theorem (which follows from 6.7 and [**14**]), $H^1_{\text{et}}(V, \mathbb{Z}/2) = H^1_{\text{et}}(C, \mathbb{Z}/2)$ and $H^2_{\text{et}}(V, \mathbb{Z}/2) = H^2_{\text{et}}(C, \mathbb{Z}/2) \oplus \mathbb{Z}/2$. In addition, $\text{Pic}(V) \cong \text{Pic}(C) \oplus \mathbb{Z}$. Hence $\text{Br}(V) = \text{Br}(C) = 0$, and $W(V) \cong W(C)$ by Corollary 1.9. $\square$

REMARK 7.8.1. If C' is the (affine) curve obtained by removing $n \geq 1$ (complex) points from the projective curve C, and $V' = C' \times \mathbb{P}^1$, then we still have $W(V') \cong WR(V') \cong W(C')$, but now (using Remark 3.4.1) $W(V') \cong W(V) \oplus (\mathbb{Z}/2)^{n-1}$. This follows from Corollary 7.2, since $\text{Br}(V') = \text{Br}(C') = 0$ (see [**41**, 3.6]) and $H^1_{\text{et}}(V', \mathbb{Z}/2) \cong H^1_{\text{et}}(C', \mathbb{Z}/2)$.

Let Q_1 denote the projective curve $x^2 + y^2 + z^2 = 0$, which is a real form of $\mathbb{CP}^1$; its underlying G-space $Q_{1,\text{top}}$ is $S^{3,0}$ (S^2 with the antipodal involution), and $Q_{1,\text{top}}/G$ is $\mathbb{RP}^2$. Thus $H^1(Q_1, \mathbb{Z}/2) \cong \mathbb{Z}/2$.

Let C be a smooth (geometrically connected) projective curve over $\mathbb{R}$, and consider the real form $V = C \times Q_1$ of the ruled surface $C_{\mathbb{C}} \times \mathbb{P}^1_{\mathbb{C}}$.

PROPOSITION 7.9. *When $V = C \times Q_1$, and $C(\mathbb{R})$ has ν components,*

$$W(V) \xrightarrow{\cong} WR(V_{\text{top}}) \cong \begin{cases} \mathbb{Z}/4 \oplus (\mathbb{Z}/2)^g, & \nu = 0; \\ \mathbb{Z}/4 \oplus (\mathbb{Z}/2)^g \oplus (\mathbb{Z}/4)^{\nu-1}, & \nu > 0. \end{cases}$$

PROOF. Since $p_g = 0$, this follows from Theorem 7.2. $\square$

EXAMPLES 7.10. Both $Q_1 \times Q_1$ and the ruled surface $Q_1 \times \mathbb{P}^1$ are surfaces with no real points, and $p_g = 0$. Their underlying G-space is $X = S^2 \times S^2$. By Theorem 7.2, $W(V) \cong WR(V_{\mathrm{top}}) \cong \mathbb{Z}/4$.

8. Real points

If $X^G \neq \emptyset$ and $\dim(X^G) \leq 3$ then $WR(X)$ is the sum of $\mathbb{Z}^\nu$ (given by the signature, as described in Lemma 2.5) and a finite 2–group. This follows from [**23**, 7.4], since $KO(X^G)$ has this structure. In particular, if V is a real algebraic surface and $V(\mathbb{R})$ has $\nu > 0$ components, then both $W(V)$ and $WR(V_{\mathrm{top}})$ are the sum of $\mathbb{Z}^\nu$ and a finite 2–group.

To describe the torsion, we use the cellular filtrations on KR and KO_G from the Bredon spectral sequence (A.1), defining $F_pWR(X)$ as the image of $F_pKO_G(X) \to WR(X)$. Thus $F_1WR(X)$ is the torsion subgroup and $WR(X)/F_1 \cong \mathbb{Z}^\nu$.

LEMMA 8.1. *Suppose that* $\dim(X) < 8$ *and* $\dim(X^G) \leq 3$. *Then* $F_3WR(X) = 0$.

PROOF. The groups $E_2^{p,-p}(KO_G)$ are zero if $p = 3, 5, 6, 7$. The map $E_2^{4,-4}(KR) \to E_2^{4,-4}(KO_G)$ is onto by Lemma A.8. Since $\dim X < 8$, $E_\infty^{4,-4}$ is a quotient of $E_2^{4,-4}$ for both the KR and KO_G cases, $E_\infty^{4,-4}(KR)$ maps onto $E_\infty^{4,-4}(KO_G)$, and hence $F_3WR(X) = 0$. $\square$

Since $F_1/F_2 \cong E_\infty^{1,-1}$ for both KO_G and KR, we have an exact sequence

$$E_\infty^{1,-1}(KR) \xrightarrow{H_\infty} E_\infty^{1,-1}(KO_G) \to F_1WR(X)/F_2WR(X) \to 0.$$

When $\dim(X) \leq 4$ and $\dim(X^G) \leq 2$, this fits into a diagram with exact rows, in which d_2 and $\bar{d}_2$ are the E_2 differentials:

$$\begin{tikzcd}
0 \to E_\infty^{1,-1}(KR) \arrow[r] \arrow[d, "H_\infty"] & H^1(X^G, \mathbb{Z}/2) \arrow[r, "d_2"] \arrow[d, "\text{into}"] & {}_2H^3_G(X, \mathbb{Z}(1)) \arrow[d, "\text{mod } 2"] \\
0 \to E_\infty^{1,-1}(KO_G) \arrow[r] \arrow[d, "\text{onto}"] & H^1_G(X; KO_G^{-1}) \arrow[r, "\bar{d}_2"] \arrow[d, "\text{onto}"] & H^3(X/G, \mathbb{Z}/2) \\
F_1WR(X)/F_2 \arrow[r, "\text{onto}"] & \mathrm{Pic}^0_G(X). &
\end{tikzcd} \tag{8.2}$$

The bottom horizontal arrow in (8.2) is onto because, as pointed out in 4.1, $F_1WR(X)$ maps onto $\mathrm{Pic}^0_G(X)$. The top middle vertical is an injection with cokernel $H^1(X/G, \mathbb{Z}/2) \cong \mathrm{Pic}^0_G(X)$ by Lemma A.5 and Definition 4.1. It follows that H_∞ is also an injection.

By Example A.2, $H^3_G(X, \mathbb{Z}/2) \cong H^3(X/G, \mathbb{Z}/2)$; the right vertical map is reduction mod 2.

Definition 8.3. Let Δ denote the kernel of the induced surjection $F_1WR(X)/F_2 \to \mathrm{Pic}^0_G(X)$ in the bottom row of (8.2).

By the snake lemma applied to the columns of (8.2), Δ is the image in $H^3(X,\mathbb{Z}(1))$ of the group of all a in $H^1_G(X^G,\mathbb{Z}/2)$ which vanish in $H^3(X/G,\mathbb{Z}/2)$ under the map $\overline{d}_2$ of (8.2), i.e., Δ is

$$\{a \in \mathrm{image}\big[H^1(X^G,\mathbb{Z}/2) \stackrel{d_2}{\to} H^3_G(X,\mathbb{Z}(1))\big] : \bar{a} = 0 \text{ in } H^3(X/G,\mathbb{Z}/2)\}.$$

Remark 8.3.1. Since the image of d_2 has exponent 2, Δ is also the intersection of image(d_2) and $2 \cdot H^3_G(X,\mathbb{Z}(1))$.

Clearly, $\Delta = 0$ if $H^3_G(X,\mathbb{Z}(1))$ has no 2-torsion. We also have $\Delta = 0$ if the even Tate cohomology of $\mathbb{Z}/2$ acting on $H^3_G(X,\mathbb{Z}(1))$ is trivial.

Lemma 8.4. *When* $\dim(X) \le 4$ *and* $\dim(X^G) \le 2$,

$$F_2WR(X) \cong F_2KO_G(X)/H(F_2KR(X)) \cong E^{2,-2}_\infty(KR)/H(E^{2,-2}_\infty(KR)).$$

Proof. Because H_∞ is an injection in diagram (8.2), the intersection $H(F_1KR(X)) \cap F_2KO_G(X)$ is the image of $F_2KR(X)$. $\square$

Lemma 8.5. *If* $\dim(X) \le 4$ *and* X^G *has* $\nu > 0$ *components, of dimension* ≤ 2*, then there is an isomorphism* ${}_2H^3_G(X;\mathbb{Z}(1)) \stackrel{\cong}{\to} F_2WR(X)$.

Proof. By Lemma A.5 and Corollary A.7, and Lemma 8.4, we also have a commutative diagram with exact columns:

$$\begin{tikzcd}
(\mathbb{Z}/2)^\nu \arrow[r, "d_2"] \arrow[d] & H^2_G(X;\mathbb{Z}(1)) \oplus H^2(X^G,\mathbb{Z}/2) \arrow[r] \arrow[d] & E^{2,-2}_\infty(KR) \to 0 \arrow[d, "H^{2,-2}_\infty"] \\
(\mathbb{Z}/2)^{1+\nu} \arrow[r, "d_2"] \arrow[d, "\text{onto}"] & H^2(X/G,\mathbb{Z}/2) \oplus H^2(X^G,\mathbb{Z}/2) \arrow[r] \arrow[d, "\text{onto}"] & E^{2,-2}_\infty(KO_G) \to 0 \arrow[d, "\text{onto}"] \\
(\mathbb{Z}/2) \arrow[r] & {}_2H^3_G(X;\mathbb{Z}(1)) \arrow[r] & F_2WR(X) \to 0.
\end{tikzcd}$$

A comparison with the corresponding diagram when X is a point shows that the bottom left horizontal map is zero and hence that we have ${}_2H^3_G(X;\mathbb{Z}(1)) \cong F_2WR(X)$. $\square$

Recall that the group Δ is defined in 8.3.

Theorem 8.6. *If* X *is 4-dimensional and* X^G *has* $\nu > 0$ *components, of dimension* ≤ 2*, the image* S_t *of the signature* $WR(X) \to \mathbb{Z}^\nu$ *is a subgroup of 2-primary index, and there is an exact sequence*

$$0 \to {}_2H^3_G(X;\mathbb{Z}(1)) \to WR(X) \to S_t \oplus H^1(X/G,\mathbb{Z}/2) \oplus \Delta \to 0.$$

In particular, if $H^3_G(X;\mathbb{Z}(1))$ *has no 2-torsion, then*

$$WR(X) \cong S_t \oplus H^1(X/G,\mathbb{Z}/2).$$

PROOF. Given Lemma 8.5, we need to determine the kernel of the surjection $F_1/F_2WR(X) \to \mathrm{Pic}^0_G(X)$ in (8.2). Since $H^1_G(X;KO^{-1}_G)$ has exponent 2, the subgroup $E^{1,-1}_\infty(KO_G)$ and its quotient $F_1/F_2WR(X)$ also have exponent 2. This implies that the extension has a (non-canonical) splitting: $F_1/F_2WR(X) \cong H^1(X/G,\mathbb{Z}/2)\oplus\Delta$.

Finally, if $H^3_G(X;\mathbb{Z}(1))$ has no 2-torsion, then $\Delta = 0$. $\square$

When V is a smooth projective variety over $\mathbb{R}$ and $X = V_{\mathrm{top}}$, we can use the isomorphisms $H^*_{\mathrm{et}}(V,\mathbb{Z}/2)\cong \mathbb{H}^*_G(X,\mathbb{Z}/2)$ to compare $W(V)$ and $WR(X)$. Recall from Section 4 that the image S_a of the algebraic signature $W(V)\to\mathbb{Z}^\nu$ is a subgroup of the image S_t of the topological signature $WR(X)\to\mathbb{Z}^\nu$.

Recall that ρ_0 denotes the rank of $c_1 : \mathrm{Pic}(V)\otimes\mathbb{Q}\to\mathbb{H}^2_G(X,\mathbb{Q}(1))$; we have $\rho_0 \ge p_g$ by Hodge theory, and if $p_g(V) = 0$ then $\rho_0 = 0$. The group Δ is defined in 8.3. (See also Remark 8.3.1.)

THEOREM 8.7. *Let V be a smooth geometrically connected projective surface defined over $\mathbb{R}$, with $V(\mathbb{R})\neq\emptyset$. Then there is an exact sequence*

$$0\to(\mathbb{Z}/2)^{\rho_0}\to W(V)\to WR(V_{\mathrm{top}})\to\Delta\oplus(S_t/S_a)\to 0.$$

PROOF. As the signature on $W(V)$ factors through the signature on $WR(V_{\mathrm{top}})$, we are reduced to a comparison of their torsion subgroups $I(V)_{\mathrm{tors}}$ and $F_1WR(V_{\mathrm{top}})$. By Proposition 1.8, which is due to Sujatha,

$$I(V)_{\mathrm{tors}}/I_2(V)_{\mathrm{tors}}\cong H^0_{\mathrm{tors}}(V,\mathcal{H}^1) \text{ and } I_2(V)_{\mathrm{tors}}\cong H^0_{\mathrm{tors}}(V,\mathcal{H}^2).$$

By definition, $H^0_{\mathrm{tors}}(V,\mathcal{H}^1)$ and $H^0_{\mathrm{tors}}(V,\mathcal{H}^2)$ are the kernels of the stabilization maps from $H^0(V,\mathcal{H}^1)\cong H^1_{\mathrm{et}}(V,\mathbb{Z}/2)$ and $H^0(V,\mathcal{H}^2)\cong {}_2\mathrm{Br}(V)$ to $H^3_{\mathrm{et}}(V,\mathbb{Z}/2)\cong(\mathbb{Z}/2)^\nu$.

Recall from [**26**, 6.3] that when V is a smooth geometrically connected projective variety over $\mathbb{R}$ then there is a split exact sequence

$$(8.8)\qquad 0\to(\mathbb{Q}/\mathbb{Z})^{\rho_0}\to\mathrm{Br}(V)\xrightarrow{\beta}{}_{\mathrm{tors}}\mathbb{H}^3_G(X,\mathbb{Z}(1))\to 0,$$

where β is induced from the Bockstein on $\mathbb{H}^2_G(X,\mathbb{Z}/2)$, and $X = V_{\mathrm{top}}$. Hence $(\mathbb{Z}/2)^{\rho_0}$ is the kernel of $\beta : {}_2\mathrm{Br}(V)\to {}_2\mathbb{H}^3_G(X,\mathbb{Z}(1))$; by Theorem 6.5, $(\mathbb{Z}/2)^{\rho_0}$ is the kernel of $I_2(V)_{\mathrm{tors}}\to F_2WR(X)$.

It remains to show that the induced map from $I(V)_{\mathrm{tors}}/I_2(V)_{\mathrm{tors}}\cong H^0_{\mathrm{tors}}(V,\mathcal{H}^1)$ to $F_1WR(X)/F_2WR(X)$ is an injection with cokernel Δ. As we saw in Section 4, it follows from Definition 4.1 that

$$H^0_{\mathrm{tors}}(V,\mathcal{H}^1)\cong\mathrm{Pic}^0_G(X)\xrightarrow{\cong}H^1(X/G,\mathbb{Z}/2),$$

where the second map is w_1. By Theorem 6.2 and Corollary 6.3, this is the same as the discriminant on $I(V)$. It follows from Theorem 8.6 that the cokernel of $H^0_{\text{tors}}(V,\mathcal{H}^1) \to WR(X)/F_2WR(X)$ is Δ. □

EXAMPLE 8.9. Consider the real surface $V = E \times E$, where E is an elliptic curve with 2 real connected components. We saw in Example 1.12 that $W(V) \cong \mathbb{Z}^4 \oplus (\mathbb{Z}/4)^3 \oplus \mathbb{Z}/2$. We claim that $WR(V_{\text{top}}) \cong \mathbb{Z}^4 \oplus (\mathbb{Z}/2)^3$ so that $W(V) \to WR(V_{\text{top}})$ is a surjection with kernel $(\mathbb{Z}/2)^4$.

To compute $WR(V_{\text{top}})$ we choose a basepoint fixed by the involution, set $Y = E_{\text{top}}$ and observe that since $V_{\text{top}} = Y \times Y$ we have a split exact sequence

$$0 \to KR(Y \wedge Y) \to KR(Y \times Y) \to KR(Y, \text{pt}) \oplus KR(Y, \text{pt}),$$

and similarly for $KO_G(Y \times Y)$. Thus we have

$$WR(V_{\text{top}}) \cong WR(Y \wedge Y) \oplus WR(Y, \text{pt}) \oplus WR(Y, \text{pt})$$

Finally, we use the identity $WR(Y, \text{pt}) \cong \mathbb{Z} \oplus (\mathbb{Z}/2)$ of Corollary 3.4.

In the same way, $Y = S^1 \times S^{1,1}$, where $S^{1,1}$ is the unit circle in $\mathbb{C}$. Therefore $WR(Y \wedge Y)$ decomposes as the sum of $WR(S^2) = \mathbb{Z} \oplus (\mathbb{Z}/2)$ and the groups $WR(S, \text{pt})$ for S on the ordered list

$$S^{2,3}, S^{2,2}, S^{2,2}, S^{1,3}, S^{1,3}, S^{1,2}, S^{1,2}, S^{2,1}.$$

Since $S^{1,2} \cong \mathbb{CP}^1$, we see from Corollary 3.4 that $WR(S^{1,2}) = \mathbb{Z}$. We also have $WR(S^{2,1}) = \mathbb{Z}^2$ by Example 3.5. A similar detailed computation, mentioned in Example 3.5, shows that

$$WR(S^{2,3}) = WR(S^{2,2}) = WR(S^{1,3}) = \mathbb{Z}.$$

It follows that $WR(Y \times Y) \cong \mathbb{Z}^4 \oplus (\mathbb{Z}/2)^3$. Thus $W(V) \to WR(V_{\text{top}})$ is a surjection with kernel $(\mathbb{Z}/2)^4$.

Ruled Surfaces

We return to *ruled surfaces* over a curve C, i.e., varieties of the form $\mathbb{P} = \mathbb{P}(E)$, where E is a rank 2 algebraic line bundle over C. Theorem 7.8 describes the case when $C(\mathbb{R}) = \emptyset$.

THEOREM 8.10. *Let C be a curve defined over $\mathbb{R}$ with $\nu > 0$ real components, and genus g. If $\mathbb{P} = \mathbb{P}(E)$ is a ruled surface over C, then*

$$WR(\mathbb{P}_{\text{top}}) \xleftarrow{\cong} WR(C_{\text{top}}) \cong \mathbb{Z}^\nu \oplus H^1(C_{\text{top}}/G, \mathbb{Z}/2).$$

If C is smooth projective, then we also have $W(\mathbb{P}) \xrightarrow{\cong} WR(\mathbb{P}_{\text{top}})$.

PROOF. Let X and Y denote the G-spaces underlying $\mathbb{P} = \mathbb{P}(E)$ and C, respectively. Theorem 8.6 yields $WR(Y) \cong \mathbb{Z}^\nu \oplus H^1(Y/G, \mathbb{Z}/2)$. By the Projective Bundle Theorem 6.7, $\mathbb{H}^3_G(X, \mathbb{Z}(1)) \cong \mathbb{H}^1_G(Y, \mathbb{Z})$. This group is defined to be $H^1(Y_G, \mathbb{Z})$, which is torsionfree, so by Theorem 8.6 we have $WR(X) \cong \mathbb{Z}^\nu \oplus H^1(X/G, \mathbb{Z}/2)$. Since $H^1_{\text{et}}(\mathbb{P}, \mathbb{Z}/2) \cong H^1_{\text{et}}(C, \mathbb{Z}/2)$, and $H^0(X^G) \cong H^0(Y^G)$, the exact sequence in Example A.3 implies that $H^1(X/G, \mathbb{Z}/2) \cong H^1(Y/G, \mathbb{Z}/2)$. Hence $WR(X) \cong WR(Y)$.

When C is smooth projective, $W(C) \cong WR(C_{\text{top}})$ by Corollary 3.4 and $W(\mathbb{P}) \cong W(C)$ by Lemma 8.11. This gives the second assertion. □

LEMMA 8.11. *Let C be a smooth projective curve over $\mathbb{R}$, with $\nu > 0$ real components, and genus g. If $\mathbb{P} = \mathbb{P}(E)$ is a ruled surface over C, then $W(\mathbb{P}) \cong W(C) \cong \mathbb{Z}^\nu \oplus (\mathbb{Z}/2)^g$.*

PROOF. Now suppose that C is smooth and projective; we need to show that $W(C) \to W(\mathbb{P})$ is an isomorphism. We have $H^2_{\text{et}}(\mathbb{P}, \mathbb{Z}/2) \cong H^2_{\text{et}}(C, \mathbb{Z}/2) \oplus \mathbb{Z}/2$ by the Projective Bundle Theorem [**32**, VI.10.1]. Since $\text{Pic}(\mathbb{P}) \cong \text{Pic}(C) \oplus \mathbb{Z}$, we have ${}_2\,\text{Br}(\mathbb{P}) = {}_2\text{Br}(C) = \mathbb{Z}/2^\nu$. As in Corollary 3.4, $I_2(C) = \mathbb{Z}/2^\nu$; since $I_2(\mathbb{P}) \subseteq {}_2\,\text{Br}(\mathbb{P})$, we see that $I_2(\mathbb{P})/I_2(C)$ is 2-torsionfree.

By Sujatha's formulas listed in Remark 1.8.1, $k = \dim H^0_{\text{tors}}(-, \mathcal{H}^2)$ is the same for C and $\mathbb{P}$. Similarly, $H^1_{\text{et}}(\mathbb{P}, \mathbb{Z}/2) \cong H^1_{\text{et}}(C, \mathbb{Z}/2)$, and $j = \dim H^0_{\text{tors}}(-, \mathcal{H}^1)$ is the same for C and $\mathbb{P}$. Thus $W(C) = W(\mathbb{P})$, since both groups have order 2^{j+k} by Remark 1.8.1. □

REMARK 8.11.1. Charles Walter (unpublished [**52**]) proved that for Y smooth of dimension ≤ 2, and any projective bundle $\mathbb{P}$ over Y, we have $W(Y) \cong W(\mathbb{P})$.

9. 3-folds

In this section, we collect information about smooth 3-folds that follows from the techniques in this paper. We begin by giving a short proof of a result of Parimala [**37**]. Recall from (1.2) that $\mathcal{H}^n$ is the Zariski sheaf associated to the presheaf $U \mapsto H^n_{\text{et}}(U, \mathbb{Z}/2)$.

THEOREM 9.1 (Parimala). *Let V be a smooth 3-dimensional algebraic variety over $\mathbb{R}$ or $\mathbb{C}$. Then the group $W(V)$ is finitely generated if and only if $CH^2(V)/2$ is finitely generated.*

PROOF. By Cox' theorem, both $H^1_{\text{et}}(V, \mathbb{Z}/2)$ and $H^2_{\text{et}}(V, \mathbb{Z}/2)$ are finite. Hence Lemma 1.5 and Remark 1.5.1 show that $W(V)/I_3(V)$ is finite. In addition, $W(V)/\text{torsion}$ is $\mathbb{Z}^\nu$, where $V(\mathbb{R})$ has ν connected components, and $I_4(V)$ is torsionfree (by 1.7). Hence $W(V)$ is a finitely generated group if and only if $I_3(V)/I_4(V) \cong H^0(V,\mathcal{H}^3)$ is finite. Since $H^2(V,\mathcal{H}^2) \cong CH^2(V)/2$, and $H^p(V,\mathcal{H}^q) = 0$ for $p > q$ [**6**], the coniveau spectral sequence $H^p(V,\mathcal{H}^q) \Rightarrow H^{p+q}_{\text{et}}(V,\mathbb{Z}/2)$ yields the exact sequence

$$H^3_{\text{et}}(V,\mathbb{Z}/2) \to H^0(V,\mathcal{H}^3) \xrightarrow{d_2} CH^2(V)/2 \to H^4_{\text{et}}(V,\mathbb{Z}/2).$$

As the outer terms are finite, it follows that $H^0(V,\mathcal{H}^3)$ is finitely generated if and only if $CH^2(V)/2$ is finite. □

Remark 9.1.1. If V is a 3-fold defined over $\mathbb{C}$, the Witt group $W(V)$ is an algebra over $W(\mathbb{C}) = \mathbb{Z}/2$, and no group extension issues arise.

For $WR(V_{\text{top}})$ of 3-folds, we need a new ingredient: the cohomology operation $\tilde\beta Sq^2$ from $H^2(X,\mathbb{Z}/2)$ to $H^5(X,\mathbb{Z})$, where $\tilde\beta$ is the integral Bockstein. By [**26**, 5.9], the kernel of $\tilde\beta Sq^2$ is the set of all Stiefel–Whitney classes $u = w_2(E) \in H^2(Y,\mathbb{Z}/2)$.

Since $H^5(\mathbb{CP}^\infty,\mathbb{Z}) = 0$ and $K(\mathbb{Z},2) = \mathbb{CP}^\infty$, $\tilde\beta Sq^2$ vanishes on the mod 2 reduction of $H^2(X,\mathbb{Z})$. Hence the following definition makes sense.

Definition 9.2. We define $\bar H^2(Y)$ to be the kernel of the cohomology operation ${}_2H^3(Y,\mathbb{Z}) \to H^5(Y,\mathbb{Z})$ induced by $\tilde\beta Sq^2$.

The following result and its proof is a modification of the argument we gave in [**26**, 7.1] for 4-dimensional spaces.

Theorem 9.3. *If Y is a connected 6-dimensional CW complex,*

$$WR(G \times Y) \cong \mathbb{Z}/2 \times H^1(Y,\mathbb{Z}/2) \times \bar H^2(Y).$$

Proof. Recall that $WR(G\times Y)$ is the cokernel of $KU(Y) \to KO(Y)$. We compare the filtrations on $KU(Y)$ and $KO(Y)$ associated to their Atiyah-Hirzebruch spectral sequences. Since the quotient $KU(Y)/F_2KU(Y) = \mathbb{Z}$ injects into

$$KO(Y)/F_2KO(Y) = \mathbb{Z} \times H^1(Y,\mathbb{Z}/2),$$

we see that the cokernel of $F_2KU(Y) \to F_2KO(Y)$ is the kernel of the surjection $WR(G\times Y) \to \mathbb{Z}/2 \times H^1(Y,\mathbb{Z}/2)$. Since $E_2^{p,-p}(KO) = 0$ for $p = 3$ and $p > 4$, and the map

$$E_\infty^{4,-4}(KU) \cong H^4(Y,\mathbb{Z}) \to H^4(Y,\mathbb{Z}) \cong E_\infty^{4,-4}(KO)$$

is onto (because $KU^{-4} \to KO^{-4}$ is onto), we are left to consider $F_2/F_3 = E_\infty^{2,-2}$ for $KU(Y)$ and $KO(Y)$.

The differential $d_3 : E_2^{2,-2}(KU) \to E_2^{5,-4}(KU)$ is a cohomology operation $H^2(Y,\mathbb{Z}) \to H^5(Y,\mathbb{Z})$. As noted above, it must be zero, so $E_\infty^{2,-2}(KU) \cong H^2(Y,\mathbb{Z})$. Similarly, the differential $d_3 : H^2(Y,\mathbb{Z}/2) \to H^5(Y,\mathbb{Z})$ is the cohomology operation $\tilde\beta Sq^2$, so $E_\infty^{2,-2}(KO)$ is the subgroup of Stiefel–Whitney classes $w_2(E)$ in $H^2(Y,\mathbb{Z}/2)$, and the cokernel of $E_\infty^{2,-2}(KU) \to E_\infty^{2,-2}(KO_G)$ is the group $\bar H^2(Y)$ of Definition 9.2. $\square$

Corollary 9.4. *Let V be a smooth 3-fold over $\mathbb{C}$, and set $Y = V(\mathbb{C})$. If $H^5(Y,\mathbb{Z})$ is a torsionfree group, then*

$$WR(V_{\mathrm{top}}) \cong \mathbb{Z}/2 \times H^1(Y,\mathbb{Z}/2) \times {}_2H^3(Y,\mathbb{Z}).$$

Proof. The image of $\tilde\beta Sq^2 : H^2(Y,\mathbb{Z}/2) \to H^5(Y,\mathbb{Z})$ has exponent 2. It must be zero when $H^5(Y,\mathbb{Z})$ is torsionfree. Now apply 9.3 to $G \times Y$. $\square$

Example 9.5. In particular, if V is an abelian 3-fold over $\mathbb{C}$, then $H^1(V(\mathbb{C}),\mathbb{Z}/2) \cong (\mathbb{Z}/2)^6$ and $WR(V_{\mathrm{top}}) \cong (\mathbb{Z}/2)^7$. This contrasts with Totaro's result [**50**] that $W(V)$ can be infinite for very general V.

We now consider a connnected 6-dimensional complex over $\mathbb{R}$ on which G acts freely. We need the following modification of Definition 9.2. We will see in the proof of Theorem 9.7 below that the composition of $\mathbb{H}^2_G(X,\mathbb{Z}(1)) \cong H^2(X/G,\mathbb{Z}(1)) \to H^2(X/G,\mathbb{Z}/2)$ with $\tilde\beta Sq^2$ is zero.

Definition 9.6. When G acts freely on X and $Y = X/G$, we define $\bar H^2_G(X)$ to be the kernel of the cohomology operation ${}_2H^3(Y,\mathbb{Z}(1)) \to H^5(Y,\mathbb{Z})$ induced by $\tilde\beta Sq^2$.

Here is the analogue of Theorem 7.2 for 3-folds over $\mathbb{R}$. It is a modification of the argument we gave in [**26**, 7.1] for 4-dimensional spaces; see Theorem 8.1 of [**26**]. As in Lemma 3.1, $\widetilde H^1(X/G,\mathbb{Z}/2)$ denotes the quotient of $H^1(X/G,\mathbb{Z}/2)$ by the subgroup generated by $[-1] = w_1(X \times \mathbb{R}(1))$.

Theorem 9.7. *Let X be a connected 6-dimensional G-CW complex. If G acts freely on X, then $WR(X)$ is an extension:*

$$0 \to \bar H^2_G(X) \to WR(X) \to \mathbb{Z}/4 \times \widetilde H^1_G(X/G,\mathbb{Z}/2) \to 0,$$

In particular, $WR(X)$ is a $\mathbb{Z}/8$-algebra.

Proof. Since X is a free G-space, $KO_G(X) = KO(X/G)$. To determine $KR(X)$, it is convenient to write Y for X/G. As in the proof of Theorem 9.3, we work with the filtration coming from the Bredon spectral sequence. By Lemma A.5, $E_2^{1,-1}(KR) \cong H^1(X^G,\mathbb{Z}/2) = 0$. Thus $KR(X)/F_2KR(X) = \mathbb{Z}$ injects into

$$KO_G(X)/F_2KO_G \cong KO(Y)/F_2KO(Y) = \mathbb{Z} \times H^1(Y,\mathbb{Z}/2),$$

and the kernel of the surjection $WR(X) \to \mathbb{Z}/4 \times \widetilde{H}^1(Y, \mathbb{Z}/2)$ is the cokernel of $F_2KR(X) \to F_2KO_G(X) \cong F_2KO(Y)$.

Since $E_2^{p,-p}(KO) = 0$ for $p > 4$, $F_5KO(Y) = 0$ and $F_4KO(Y) \cong E_\infty^{4,-4}$. For both KR and KO_G, $E_3^{3,-3} = 0$ and the group $E_\infty^{4,-4}$ is a quotient of $H^4(Y, \mathbb{Z})$. In addition, the map

$$E_2^{4,-4}(KR) \cong H^4(Y, \mathbb{Z}) \to H^4(Y, \mathbb{Z}) \cong E_2^{4,-4}(KO_G)$$

is an isomorphism (because $KU^{-4} \to KO^{-4}$ is). Thus $F_3KR(X)$ maps onto $F_3KO(Y)$, i.e., $F_3WR(X) = 0$. We are left to consider $F_2/F_3 = E_\infty^{2,-2}$ for $KR(X)$ and $KO(Y)$. We have a commutative diagram

$$\begin{tikzcd} 0 \to E_\infty^{2,-2}(KR) \arrow[r] \arrow[d] & H^2(Y, \mathbb{Z}(1)) \arrow[r, "d_3"] \arrow[d] & H^5(Y, \mathbb{Z}) \arrow[d, "\cong"] \\ 0 \to E_\infty^{2,-2}(KO_G) \arrow[r] & H^2(Y, \mathbb{Z}/2) \arrow[r, "d_3"] & H^5(Y, \mathbb{Z}). \end{tikzcd} \tag{9.8}$$

As in the proof of Theorem 9.3, the differential $d_3 : H^2(Y, \mathbb{Z}/2) \to H^5(Y, \mathbb{Z})$ is the cohomology operation $\tilde{\beta} Sq^2(u)$, so its kernel $E_\infty^{2,-2}(KO)$ is the subgroup $\{u \in H^2(Y, \mathbb{Z}/2) \mid \tilde{\beta} Sq^2(u) = 0\}$.

We claim that the differential $d_3 : H^2(Y, \mathbb{Z}(1)) \to H^5(Y, \mathbb{Z})$ is zero in the top row in (9.8), so that $E_\infty^{2,-2}(KR) \cong E_2^{2,-2}(KR) = H^2(Y, \mathbb{Z}(1))$. This will imply that the cokernel of

$$E_\infty^{2,-2}KR(X) \to E_\infty^{2,-2}KO_G(X)$$

is the subgroup $\bar{H}_G^2(Y)$ of Definition 9.6, finishing the proof.

The Real Chern class c_1 of Definition 6.6 is the composition of the map $F_2KR(X) \to E_\infty^{2,-2} \subseteq H_G^2(X, \mathbb{Z}(1)) \cong H^2(Y, \mathbb{Z}(1))$ with the isomorphism $H_G^2(X, \mathbb{Z}(1)) \cong \mathbb{H}_G^2(X, \mathbb{Z}(1))$ of A.4(c). Since c_1 is onto, the differential $d_3 : H^2(Y, \mathbb{Z}(1)) \to H^5(Y, \mathbb{Z})$ is zero, as claimed, $\square$

Remark 9.8.1. The fact that $WR(X)$ is a $\mathbb{Z}/8$–algebra can also be deduced from the considerations of Section 7 in [**23**], which are valid for any CW-complex with a free G-action, the torsion being a specific function of the dimension.

Now suppose that V is a 3-fold over $\mathbb{R}$ with no real points.

Corollary 9.9. *If V is a geometrically connected 3-fold over $\mathbb{R}$, with no $\mathbb{R}$-points, then $WR(V_{\text{top}})$ is an extension of $\mathbb{Z}/4 \times \widetilde{H}^1_{\text{et}}(V, \mathbb{Z}/2)$ by the subquotient $\bar{H}_G^2$ of $H^2_{\text{et}}(V, \mathbb{Z}/2)$ defined in 9.6.*

The kernel and cokernel of $W(V) \to WR(V_{\text{top}})$ have exponent 2, and are the same as the kernel and cokernel of $I_2(V) \to \bar{H}_G^2$.

Proof. Setting $X = V_{\mathrm{top}}$, the first assertion is Theorem 9.7. Combining this with Corollary 6.3, we get a commutative diagram with exact rows:

$$\begin{array}{ccccc}
0 \to I_2(V) & \longrightarrow & W(V) & \longrightarrow & \mathbb{Z}/4 \times \widetilde{H}^1_{\mathrm{et}}(V,\mathbb{Z}/2) \to 0 \\
\downarrow & & \downarrow & & \downarrow \cong \\
0 \to \bar{H}^2_G & \longrightarrow & WR(X) & \longrightarrow & \mathbb{Z}/4 \times \widetilde{H}^1_G(X,\mathbb{Z}/2) \to 0.
\end{array}$$

The result follows from the snake lemma. □

We now turn our attention to the anisotropic quadric 3-fold Q_3 defined by $\sum_{i=0}^4 x_i^2 = 0$. For notational simplicity, let X denote the G-manifold $Q_{3,\mathrm{top}}$. Colliot-Thélène and Sujatha [**11**, 3.4] proved that $W(Q_3) \cong \mathbb{Z}/8$, with $I_2(Q_3) \cong \mathrm{Br}(Q_3) \cong \mathbb{Z}/2$; we will show that $WR(X) \cong \mathbb{Z}/8$ as well.

Lemma 9.10. *$\mathbb{Z}/2 \cong \mathrm{Br}(Q_3) \to {}_2\mathbb{H}^3_G(X,\mathbb{Z}(1))$ is an isomorphism.*

Proof. Colliot-Thélène and Sujatha observed in [**11**, p. 9] that $\mathrm{Pic}(Q_3) = \mathbb{Z}$ and $\mathrm{Br}(Q_3) \cong \mathrm{Br}(\mathbb{R}) \cong \mathbb{Z}/2$, and observed that G acts trivially on $\mathrm{Pic}(Q_3)$. Set $m = 2^k$. From Kummer Theory, we deduce that $H^p_{\mathrm{et}}(Q_3,\mu_m)$ is $\mathbb{Z}/2$ for $p = 0, 1$, and $\mathbb{Z}/m \oplus \mathbb{Z}/2$ for $p = 2$.

By Cox' theorem, we can identify $H^p_{\mathrm{et}}(Q_3,\mu_m)$ with $\mathbb{H}^p_G(X,\mathbb{Z}(1)/m)$. Moreover, $\mathbb{H}^0_G(X,\mathbb{Z}(1)) = 0$, and the groups $\mathbb{H}^p_G(X,\mathbb{Z}(1))$ are finitely generated. Using the exact sequences

$$0 \to \mathbb{H}^p_G(X,\mathbb{Z}(1))/m \to H^p_{\mathrm{et}}(Q_3,\mu_m) \to {}_m\mathbb{H}^{p+1}_G(X,\mathbb{Z}(1)) \to 0,$$

we see that $\mathbb{H}^1_G(X,\mathbb{Z}(1)) = \mathbb{Z}/2$ and $H^2_G(X,\mathbb{Z}(1)) = \mathbb{Z}$. From

$$0 \to \mathbb{H}^2_G(X,\mathbb{Z}(1))/2 \to \mathbb{H}^2_G(X,\mathbb{Z}/2) \xrightarrow{\partial} {}_2\mathbb{H}^3_G(X,\mathbb{Z}(1)) \to 0$$

we see that ${}_2\mathbb{H}^3_G(X,\mathbb{Z}(1)) = \mathbb{Z}/2$. Since ∂ factors through the surjection $H^2_{\mathrm{et}}(Q_3,\mathbb{Z}/2) \to {}_2\,\mathrm{Br}(Q_3)$, we see that ${}_2\,\mathrm{Br}(Q_3) \cong {}_2\mathbb{H}^3_G(X,\mathbb{Z}(1))$ □

Combining Lemma 9.10 with Corollary 9.9, we conclude:

Corollary 9.11. *$W(Q_3) \cong WR(Q_{3,\mathrm{top}}) \cong \mathbb{Z}/8$.*

When V is a geometrically connected 3-fold over $\mathbb{R}$, and $V(\mathbb{R}) \neq \emptyset$, we saw in Proposition 1.7 that the torsion subgroup of $I(V)$ has exponent 8. We have not yet thought about this case.

10. Fundamental Theorem for Witt groups

The Fundamental Theorem for Witt groups concerns Laurent polynomial extensions. When $V = \mathrm{Spec}(A)$, and A is a regular $\mathbb{R}$-algebra, the formula $W(A[t,1/t]) \cong W(A) \oplus W(A)$ goes back to the work of the first author [**20**, pp. 138–9] and Ranicki [**43**, 4.6] in the 1970s. More recently, Balmer and Gille proved in [**4**, 9.13] that for smooth schemes V we have $W(V \times \mathrm{Spec}(\mathbb{R}[t,1/t])) \cong W(V) \oplus W(V)$.

Since the G-space underlying $\mathbb{G}_m$ is equivariantly homotopy equivalent to $S^{1,1}$, we consider $WR(X \times S^{1,1})$. Since $WR(S^{1,1}) \cong \mathbb{Z} \oplus \mathbb{Z}$ (by Lemma 3.1) we have a canonical external product map

$$(10.1)\quad WR(X) \oplus WR(X) \cong WR(X) \otimes WR(S^{1,1}) \xrightarrow{\mu} WR(X \times S^{1,1}).$$

When X is a finite G-set, it follows from Lemma 3.1 that the product (10.1) is an isomorphism. However, it is not true in general that $WR(X \times S^{1,1}) \cong WR(X) \oplus WR(X)$, even if $X = V_{\mathrm{top}}$, as Example 3.6 shows.

Here is a sufficient condition. To state it, recall that sending a Real vector bundle E to its dual E^* defines an action of $\mathbb{Z}/2$ on $KR(X)$, and hence on $KR_n(X) = KR^{-n}(X)$ for all n.[1] Let $kr_n(X)$ (resp. $kr'_n(X)$) denote the even (resp. odd) Tate cohomology of $\mathbb{Z}/2$ acting on the group $KR_n(X)$. That is,

$$kr_n(X) = H^2(\mathbb{Z}/2, KR_n(X)), \quad kr'_n(X) = H^1(\mathbb{Z}/2, KR_n(X)).$$

THEOREM 10.2. *If $kr_{-1}(X) = kr'_{-1}(X) = 0$, (which is true in particular when $KR_{-1}(X) = 0$), then the product* (10.1) *is an isomorphism:*

$$W(X) \oplus WR(X) \xrightarrow{\mu} WR(X \times S^{1,1}).$$

More generally, the kernel of μ is a quotient of $kr_{-1}(X)$ and the cokernel of μ is a subgroup of $kr'_{-1}(X)$.

We will prove Theorem 10.2 later in this section. We first record:

LEMMA 10.3. $KR_n(X \times S^{1,1}) \cong KR_n(X) \oplus KR_{n-1}(X)$.

PROOF. Since $B^{1,1}$ retracts equivariantly to a point in $S^{1,1}$, we have a (split) short exact sequence

$$0 \to KR_n(X \times B^{1,1}) \to KR_n(X \times S^{1,1}) \to KR_{n-1}(X \times B^{1,1}, X \times S^{1,1}) \to 0.$$

Since the Thom Isomorphism theorem [**2**, 2.4], identifies the final term with $KR_{n-1}(X^E, \mathrm{pt}) \cong KR_{n-1}(X)$, $E = X \times \mathbb{R}^{1,1}$, the result follows. $\square$

[1] The superscript notation KR^* is used by Atiyah [**2**], but the subscript notation is more appropriate for our uses.

EXAMPLE 10.3.1. When $X = S^{1,1}$, Lemma 10.3 yields $KR_{-1}(X) = KR_{-1}(\mathrm{pt}) \oplus KR_{-2}(\mathrm{pt}) = 0$ and hence $WR(X \times S^{1,1}) \cong WR(X)^2 \cong \mathbb{Z}^4$. We also have $W(\mathbb{G}_m \times \mathbb{G}_m) \cong WR(S^{1,1} \times S^{1,1})$ in this case.

Since Real vector bundles on X form a Hermitian category, there is a classical Bott exact sequence (see [**44**, Thm. 6.1]):

$$(10.4) \quad \to GR_n^{[i-1]}(X) \xrightarrow{F} KR_n(X) \xrightarrow{H} GR_n^{[i]}(X) \to GR_{n-1}^{[i-1]}(X) \xrightarrow{F} .$$

The groups $GR_0(X) = GR_n^{[0]}(X)$ are the usual Grothendieck–Witt groups of symmetric forms on Real vector bundles (see [**44**, 1.39]), and the groups $GR_n^{[2]}(X)$ are the Grothendieck–Witt groups of skew-symmetric forms.

For the rest of this section, we shall adopt the following terminology. If F is a functor on G-spaces, we'll write $F(\mathbb{R})$ and $F(X \times \mathbb{R})$ for $F(S^{1,1}, \mathrm{pt})$ and $F(X \times S^{1,1}, X)$. Thus $KR_n(X \times \mathbb{R}) \cong KR_{n-1}(X)$ by 10.3, and Theorem 10.2 asserts that $WR(X \times \mathbb{R}) \cong WR(X)$.

We will use the following description of the group $GR_1^{[1]}(X)$, viewed as the Grothendieck group of the forgetful functor F (see [**18**, II.2.13]). An element is determined by a triple (E, g_1, g_2), up to a suitable homotopy equivalence. Here E is a Real vector bundle on X and the g_i are symmetric bilinear forms on E.

Taking $i = n = 1$ in (10.4), and using Lemma 10.3, we get an exact sequence

$$(10.5)$$
$$GR_1(X \times \mathbb{R}) \xrightarrow{F} KR_0(X) \xrightarrow{H} GR_1^{[1]}(X \times \mathbb{R}) \xrightarrow{\partial} GR(X \times \mathbb{R}) \to KR_{-1}(X).$$

EXAMPLE 10.6. When X is a point, the group $GR_1^{[1]}(\mathbb{R})$ has two special elements: $u = (E, \langle 1 \rangle, \langle t \rangle)$ and $u^- = (E, \langle -1 \rangle, \langle -t \rangle)$, where E is the bundle $S^{1,1} \times \mathbb{C}$ and $\langle \pm t \rangle$ (resp., $\langle \pm 1 \rangle$) is the quadratic form $x \mapsto \pm t x^2$ (resp., $x \mapsto \pm x^2$). The boundary map ∂ in (10.5) sends u to the generator $(E, \langle t \rangle)$ of $GR(\mathbb{R}) \cong \mathbb{Z}$, and $\partial(u^-) = -\partial(u)$, while the map H in (10.5) sends the generator of $KR_0(\mathrm{pt}) \cong \mathbb{Z}$ to $u + u^-$. Since $GR_1(\mathbb{R}) \cong \mathbb{Z}/2$, the map $F : GR_1(\mathbb{R}) \to KR_0(\mathrm{pt})$ is zero. Since $KR_{-1}(\mathrm{pt}) = 0$, (10.5) reduces to:

$$0 \to KR_0(\mathrm{pt}) \xrightarrow{H} GR_1^{[1]}(\mathbb{R}) \xrightarrow{\partial} GR(\mathbb{R}) \to 0.$$

It follows that $GR_1^{[1]}(\mathbb{R})$ is isomorphic to $\mathbb{Z}^2$, with basis $\{u, u^-\}$.

Similarly, $GR(\mathrm{pt}) \cong RO(G) \cong \mathbb{Z}^2$; a basis is given by $e = (\mathbb{C}, \langle 1 \rangle)$ and $e^- = (\mathbb{C}, \langle -1 \rangle)$. Since the cup product $GR(\mathrm{pt}) \xrightarrow{\cup u} GR_1^{[1]}(\mathbb{R})$ sends e to u and e^- to u^-, it is an isomorphism.

LEMMA 10.7. *Let $\varphi_X : KO(X) \to H(X)$ be a natural transformation of functors from finite CW complexes to abelian groups such that each $H(X)$ is naturally a $KO(X)$-module and each φ_X is a $KO(X)$-module map. Then φ_X is the cup product with $t = \varphi_{\mathrm{pt}}(1)$.*

If in addition $\varphi_{\mathrm{pt}} : KO(\mathrm{pt}) \to H(\mathrm{pt})$ is an isomorphism, and $H(X) \cong KO(X)$ for all X, then $\varphi : KO^* \to H^*$ is an isomorphism of cohomology theories.

PROOF. If $x \in KO(X)$ then $\varphi_X(x) = \varphi_X(x \cdot 1) = \varphi_X(x) \cdot t$. This proves the first assertion. Now assume that $H(X) \cong KO(X)$ for all X, and that φ_{pt} is an isomorphism; as $KO(\mathrm{pt}) \cong \mathbb{Z}$, $t = \pm 1$. To see that φ_X is an isomorphism, we may assume that X is connected. In this case, $KO(X) \cong KO(\mathrm{pt}) \oplus \widetilde{KO}(X)$, and the second factor is a nilpotent ideal in $KO(X)$. Therefore $t = \varphi_X(1)$ is a unit of $KO(X)$, and hence $\varphi_X : KO(X) \to KO(X)$ is an isomorphism. Taking products with spheres and using Bott periodicity, it follows that $\varphi_X : KO^*(X) \cong H^*(X)$ for all X, i.e., φ is an isomorphism of cohomology theories. $\square$

REMARK 10.7.1. If G acts trivially on X, so that $KO_G(X) \cong KO(X)[y]/(y^2 - 1)$ and $GR^{[1]}(X \times \mathbb{R})$ is a $GR(X)$-module, the lemma holds for $KO_G(X) \to GR^{[1]}(X \times \mathbb{R})$.

THEOREM 10.8. *The cup product with u defines isomorphisms*

$$GR_n(X) \xrightarrow{\cup u} GR^{[1]}_{n+1}(X \times \mathbb{R}).$$

COROLLARY 10.9. *(See [**23**, C.6]) We have an exact sequence*

$$\to GR_1(X \times \mathbb{R}) \to KR(X) \xrightarrow{H} GR(X) \to GR(X \times \mathbb{R}) \to,$$

where the map $GR(X) \to GR(X \times \mathbb{R})$ is the cup product with $\partial(u)$.

PROOF. Use Theorem 10.8 to replace the middle term in (10.5) with $GR(X)$. $\square$

PROOF OF THEOREM 10.8. We first consider two extreme cases, and then the general case.

Case 1) When $X = X^G$, the sequence of $GR(X)$-modules (10.5) reduces to

$$0 \to KO(X) \xrightarrow{H} GR^{[1]}_1(X \times \mathbb{R}) \xrightarrow{\partial} KO(X) \to 0.$$

We see from Example 10.6 and Lemma 10.7 that the map H is the cup product with $u + u^-$, and that the map ∂ is split by the cup product $KO(X) \to GR^{[1]}_1(X)$ with u. Hence $GR^{[1]}(X \times \mathbb{R}) \cong GR(X) \cong KO(X) \oplus KO(X)$ as a $KO(X)$-module.

We claim that $GR_1^{[1]}(X\times\mathbb{R}) \cong GR(X)$ as a $GR(X)$-module. Now $GR(X) \cong KO(X)\otimes RO(G)$, and $RO(G) = \mathbb{Z}[s]/(s^2-1)$, it suffices to check the action of the sign representation s. By inspection, $s = 1$ on $KO(X)$ and $s(u) = u^-$. Therefore H and ∂ are $RO(G)$-maps, establishing the claim.

By Remark 10.7.1, establishes Theorem 10.8 when $X = X^G$.

Case 2) When $X = G\times Y$, the map $GR_n(X\times\mathbb{R}) \xrightarrow{F} KR_n(X\times\mathbb{R})$ in (10.5) is the complexification map $KO_{n-1}(Y) \to KU_{n-1}(Y)$, and (10.4) becomes

$$KO_{n+2}(Y) \xrightarrow{c} KU_n(Y) \xrightarrow{H} GR^{[1]}_{n+1}(X\times\mathbb{R}) \to KO_{n+1}(Y) \xrightarrow{c} KU_{n-1}(Y).$$

This shows that $GR^{[1]}_*(X\times\mathbb{R})$ is a cohomology theory on Y. Comparing (10.4) with the classical Bott sequence (see [**18**, III.5.18], [**2**, (3.4)])

$$KO_{n+2}(Y) \xrightarrow{c} KU_n(Y) \to KO_n(Y) \xrightarrow{\partial} KO_{n+1}(Y) \xrightarrow{c} KU_{n-1}(Y),$$

we see that $GR^{[1]}_{*+1}(X\times\mathbb{R}) \cong KO_*(Y)$ as $KO_*(Y)$-modules. Therefore it suffices to show that the cup product $\varphi : GR_0(X) \to GR_1^{[1]}(X\times\mathbb{R})$ sends 1 to a generator of the $KO(Y)$-module $GR_1^{[1]}(X) \cong KO(Y)$. By Lemma 10.7, we may assume that Y is a point. In this case, the above exact 5-term sequence reduces to

$$\mathbb{Z}/2 \xrightarrow{0} \mathbb{Z} \xrightarrow{H=2} \mathbb{Z} \xrightarrow{\partial} \mathbb{Z}/2 \to 0.$$

Since $KO(\mathrm{pt}) \cong \mathbb{Z}$ and $GR_1^{[1]}(\mathbb{R}) \cong \mathbb{Z}$, we need only show that the distinguished element $u = (\mathbb{C}, 1, \langle t\rangle)$ is a generator of $GR_1^{[1]}(\mathbb{R})$.

For this, we first note that $\partial(u)$ is nonzero and odd because, by construction, $\partial(u)$ is the class of $\langle t\rangle$ considered as a function $S^1 \to KO$, and this is the generator of $KO_1(S^1, \mathrm{pt}) \cong \mathbb{Z}/2$. Next, we compose the map $KO(Y) \to GR_1^{[1]}(X\times\mathbb{R})$ with the discriminant

$$GR_1^{[1]}(X\times\mathbb{R}) \xrightarrow{D} KR_1(X\times\mathbb{R}) \cong KU(Y).$$

Since D takes an element (E, g_1, g_2) of $GR_1^{[1]}(X\times\mathbb{R})$ to $[g_1 g_2^{-1}]$, we have $D(u) = [t]$, which is a generator of $KR(\mathbb{R}) \cong KU(\mathrm{pt}) \cong \mathbb{Z}$. Since $KU(\mathrm{pt}) \xrightarrow{H} KO(\mathrm{pt})$ is $\mathbb{Z} \xrightarrow{2} \mathbb{Z}$, we see that $|u| \le 2$. Hence u is a generator of $GR_1^{[1]}(\mathrm{pt})$, establishing Theorem 10.8 when $X = G\times Y$.

Case 3) When G acts freely on X, there is a finite cover of X by open subspaces of the form $\{G\times U_i\}$. By case 2), the cup product with u is an isomorphism for each of these opens, and for their interections. It follows from the 5–lemma that the cup product with u is an isomorphism for X.

Case 4) For general X, let T be an equivariant closed neighborhood of the subcomplex X^G, so that X^G is in the interior of T and $X^G \subset T$ is a G-homotopy equivalence. Write Y (resp., A) for the closure of $X - T$ (resp., $T \cap Y$). Then the cup product with u determines a map of Mayer–Vietoris sequences

$$\begin{tikzcd} GR_{n+1}(T \amalg Y) \arrow[r] \arrow[d, "\cong"] & GR_{n+1}(A) \arrow[r] \arrow[d, "\cong"] & GR_n(X) \arrow[r] \arrow[d, "\cup u"] & GR_n(T \amalg Y) \arrow[r] \arrow[d, "\cong"] & GR_n(A) \arrow[d, "\cong"] \\ GR^{[1]}_{n+1}(T \amalg Y) \arrow[r] & GR^{[1]}_{n+1}(A) \arrow[r] & GR^{[1]}_n(A) \arrow[r] & GR^{[1]}_n(T \amalg Y) \arrow[r] & GR^{[1]}_n(A). \end{tikzcd}$$

The general case follows from the 5–lemma. $\square$

Recall that the "co-Witt" group $WR'_0(X)$ is the kernel of the map $GR_0(X) \xrightarrow{F} KR_0(X)$ in (10.4); it is also the image of $GR^{[1]}_1(X) \to GR_0(X)$.

Proof of Theorem 10.2. The relative group $WR(X \times S^{1,1}, X) = WR(X \times \mathbb{R})$ fits into the usual 12–term sequence [**21**, p. 278], an elementary part of which is:

$$kr'_0(X \times \mathbb{R}) \xrightarrow{j} WR'_0(X \times \mathbb{R}) \xrightarrow{\beta} WR_0(X \times \mathbb{R}) \xrightarrow{d} kr_0(X \times \mathbb{R}).$$

By Lemma 10.3, $kr_0(X \times \mathbb{R}) \cong kr_{-1}(X)$ and $kr'_0(X \times \mathbb{R}) \cong kr'_{-1}(X)$. If these groups are 0, for example if $KR_{-1}(X) = 0$, then $WR'_0(X \times \mathbb{R}) \cong WR_0(X \times \mathbb{R})$. In general, the kernel of β is a quotient of $kr_{-1}(X)$ and the cokernel is a subgroup of $kr'_{-1}(X)$.

On the other hand, by definition, Theorem 10.8, and Corollary 10.9:

$$\begin{aligned} WR'_0(X \times \mathbb{R}) &= \text{im } GR^{[1]}_1(X \times \mathbb{R}) \xrightarrow{H} GR_0(X \times \mathbb{R}) \\ &= \text{im } GR(X) \xrightarrow{\cup u} GR(X \times \mathbb{R}) \\ &= GR(X)/\text{im } KR(X) \cong WR(X). \end{aligned}$$

$\square$

Appendix A. Equivariant cohomology

In this section, we recall basic facts about G-equivariant cohomology theories, and apply them to KO_G and KR. Recall from [**7**, I.2] that an *equivariant cohomology theory* is a sequence of functors h^q on pairs of G-complexes satisfying homotopy invariance, excision and the existence of long exact sequences, depending naturally on a pair of G-complex.

A *Bredon coefficient system* M for the cyclic group G of order 2 is a diagram $M(\mathrm{pt}) \xrightarrow{a} M(G)$, together with an involution σ on $M(G)$ such that $\sigma a = a$ [**7**, I.4.1]. Any coefficient system M determines an equivariant cohomology theory $H^*_G(-;M)$ [**7**, I.6.4]. Conversely, any G-equivariant cohomology theory h^* defines a family of coefficient systems $M = h^q$, $h^q(\mathrm{pt}) \to h^q(G)$, and there are canonical maps $\eta^q : h^q(X) \to H^0_G(X;h^q)$ for all $q \in \mathbb{Z}$.

If X is a finite dimensional G-CW complex, the (convergent) Bredon spectral sequence [**7**, IV.4] for an equivariant cohomology theory h^* is:

$$E_2^{p,q} = H^p_G(X;h^q) \Rightarrow h^{p+q}(X), \tag{A.1}$$

and the canonical maps η^q are the edge maps.

Any coefficient system M defines a sheaf $\mathcal{M}$ on X/G whose stalk at $\bar{x}$ is $M(\mathrm{pt})$ or $M(G)$, depending on whether the inverse image of $\bar{x}$ in X is a fixed point or isomorphic to G. The sheaf cohomology $H^p(X/G;\mathcal{M})$ agrees with $H^p_G(X;M)$; this alternative definition is due to Segal [**45**]. We remark that (A.1) agrees with Segal's spectral sequence [**45**, 5.3].

Example A.2. For any G-module M, we have the coefficient system $M(\mathrm{pt}) = M^G \to M(G) = M$, as well as a local system $\mathcal{M}$ on X/G, and $H^*_G(X;M)$ is $H^* \operatorname{Hom}_G(C_*(X), M)$, the cohomology associated to the local system $\mathcal{M}$. If A is an abelian group, regarded as a trivial G-module, we get the *constant coefficient system* $A(0)$; since $\operatorname{Hom}_G(C_*(X), A) = \operatorname{Hom}(C_*(X/G), A)$, $H^p_G(X;A(0))$ is the usual cohomology group $H^*(X/G, A)$. Note that $H^p_G(X;A(0))$ is different from the Borel cohomology $\mathbb{H}^p_G(X, A)$; see Example A.3.

Example A.3. Let A be an abelian group. The Borel cohomology groups $\mathbb{H}^p_G(X, A)$ are defined to be $H^p(X_G, A)$, where $X_G = X \times_G EG$. The $\mathbb{H}^*_G(-, A)$ form an equivariant cohomology theory; the associated coefficient system has $H^q_G(\mathrm{pt}) = H^q(BG, A)$ and $H^q_G(G) = 0$ for $q \neq 0$.

Since $C_*(X \times EG)$ is a chain complex of free $\mathbb{Z}[G]$-modules, quasi-isomorphic to $C_*(X)$, we see that $\mathbb{H}^p_G(X, A)$ is the group hypercohomology of C_*X with coefficients in A.

$$\mathbb{H}^p_G(X, A) = H^p \operatorname{Hom}(C_*(X_G), A) = H^p \operatorname{Hom}_G(C_*(X \times EG), A).$$

More generally, if M is any G-module we will write $\mathbb{H}^*_G(X, M)$ for the group hypercohomology

$$\mathbb{H}^*_G(C_*X, M) = H^p \operatorname{Hom}_G(C_*(X \times EG), M);$$

it is also an equivariant cohomology theory. For example, the reader may use the formula that $\operatorname{Hom}_{\mathbb{Z}}(A, \mathbb{Z}) \cong \operatorname{Hom}_G(A, \mathbb{Z}[G])$ for any G-module A to check that

$$\mathbb{H}^*_G(X, \mathbb{Z}[G]) \cong H^*(X, \mathbb{Z}).$$

Now suppose that $A = \mathbb{Z}/2$. The terms of the spectral sequence (A.1) are: $E_2^{p,q} = H^p(X^G, \mathbb{Z}/2)$ if $q > 0$, and $E_2^{p,0} = H^p(X/G, \mathbb{Z}/2)$. For example, if $X = X^G$ then $\mathbb{H}^n_G(X, \mathbb{Z}/2) \cong \bigoplus_{p=0}^n H^p(X, \mathbb{Z}/2)$.

The exact sequence of low degree terms is

$$0 \to H^1(X/G, \mathbb{Z}/2) \to \mathbb{H}^1_G(X, \mathbb{Z}/2) \xrightarrow{\eta^1} H^0(X^G, \mathbb{Z}/2) \xrightarrow{d_2} H^2(X/G, \mathbb{Z}/2).$$

The edge map $\mathbb{H}^1_G(X, \mathbb{Z}/2) \xrightarrow{\eta^1} H^0(X^G, \mathbb{Z}/2)$ in (A.1) need not be onto, as we see from Example 1.6.

The groups $\mathbb{H}^*_G(X, \mathbb{Z}/2)$ are graded modules over $\mathbb{H}^*_G(\mathrm{pt}, \mathbb{Z}/2) = \mathbb{Z}/2[\beta]$, natural in X. It follows that $\beta \in \mathbb{H}^1_G(\mathrm{pt}, \mathbb{Z}/2)$ acts on the Bredon spectral sequence. For $q > 0$, it sends $E_2^{p,q} \cong H^p(X^G, \mathbb{Z}/2)$ isomorphically to $E_2^{p,q+1}$. When $X = V_{\mathrm{top}}$ for a variety V, so that $H^*_{\mathrm{et}}(V, \mathbb{Z}/2) \cong \mathbb{H}^*_G(X, \mathbb{Z}/2)$, it follows from [**9**] that when $q > \dim V$ we have $E_2^{0,q} \cong H^0(V, \mathcal{H}^q) \cong (\mathbb{Z}/2)^\nu$.

EXAMPLES A.4. Let G be the group of order 2.
a) The equivariant cohomology theory KO^*_G determines coefficient systems KO^q_G:

$$KO^q_G(\mathrm{pt}) = KO^q(\mathrm{pt}) \otimes RO(G) \xrightarrow{+} KO^q_G(G) = KO^q(\mathrm{pt}).$$

Here $RO(G) \cong \mathbb{Z}^2$ is the real representation ring of G, with generators $[\mathbb{R}]$ and $[\mathbb{R}(1)]$, and '+' sends $a[\mathbb{R}] + b[\mathbb{R}(1)]$ to $a + b$.

b) Real K-theory KR^q determines the coefficient system $KO^q \to KU^q$: KR^0 is the constant system $\mathbb{Z}(0)$, KR^2 is $0 \to \mathbb{Z}(1)$, KR^4 is $\mathbb{Z} \xrightarrow{2} \mathbb{Z}$, $KR^6 = KR^{-2}$ is $\mathbb{Z}/2 \xrightarrow{0} \mathbb{Z}(1)$ and $KR^7 = KR^{-1}$ is $\mathbb{Z}/2 \to 0$. We also have $KR^q = 0$ for $q = 1, 3, 5$ and $KR^q = KR^{q+8}$.

c) The cohomology theory $h^q = \mathbb{H}^q_G(-, \mathbb{Z}(1))$ of Example A.3 has the coefficient systems $h^0 = \mathbb{Z}(1)$, $h^q = (\mathbb{Z}/2 \to 0)$ if $q > 0$ is odd, and $h^q = 0$ otherwise. In particular, there is an exact sequence

$$0 \to H^1_G(X; \mathbb{Z}(1)) \to \mathbb{H}^1_G(X, \mathbb{Z}(1)) \to (\mathbb{Z}/2)^\nu \xrightarrow{d_2}$$
$$H^2_G(X; \mathbb{Z}(1)) \to \mathbb{H}^2_G(X, \mathbb{Z}(1)) \to H^1(X^G; \mathbb{Z}/2) \xrightarrow{d_2} H^3_G(X; \mathbb{Z}(1)).$$

If G acts freely on X, then $H^n_G(X; \mathbb{Z}(1)) \cong \mathbb{H}^n_G(X, \mathbb{Z}(1))$ for all n.

Remark A.4.1. If M is a G-module, $\mathbb{H}^*_G(X, M)$ denotes the Borel cohomology with coefficients in M (see A.3), while $H^*_G(X; M)$ denotes the Bredon cohomology of the coefficient system $(M^G \to M)$ (see A.2). In particular, $H^*_G(X; M) \to \mathbb{H}^*_G(X, M)$ need not be an isomorphism unless $X^G = \emptyset$.

Lemma A.5. *For each p and q, there is a natural split exact sequence*

$$0 \to H^p(X^G, KO^q) \to H^p_G(X; KO^q_G) \xrightarrow{+} H^p(X/G, KO^q) \to 0.$$

If X/G is connected and X^G has ν components, then

$$H^0_G(X; KO_G) \cong \mathbb{Z} \oplus \mathbb{Z}^\nu.$$

Moreover, $H^1_G(X; KR^{-1}) \cong H^1(X^G, \mathbb{Z}/2)$ and

$$H^1_G(X; KO^{-1}_G) \cong H^1_G(X; KR^{-1}) \oplus H^1(X/G, \mathbb{Z}/2).$$

Proof. There is a natural surjection from the coefficient system KO^q_G to the constant coefficient system $KO^q(0) = (KO^q \to KO^q)$; the map from $KO^q_G(\mathrm{pt}) \cong KO^q \otimes R(G)$ to $KO^q(\mathrm{pt})$ is addition. It is split by the map $KO^q \to KO^q_G$ sending a to $a \otimes [\mathbb{R}]$.

Let E^q denote the kernel of this surjection; $E^q(\mathrm{pt}) = KO^q$ and $E^q(G) = 0$, so $H^p(X; E^q) \cong H^p(X^G, KO^q)$. Then $KO^q_G = E^q \oplus KO^q$ as coefficient systems. Applying $H^p_G(X; -)$ yields the exact sequence of the lemma. The last assertions follow from the isomorphisms $KR^0 \cong \mathbb{Z}(0)$ and $KR^{-1} \cong E^{-1}$ of coefficient systems. □

Remark A.5.1. The image of $H^p_G(X; KR^{-2}) \to H^p_G(X; KO^{-2}_G)$ contains the summand $H^p(X^G, \mathbb{Z}/2)$ of Lemma A.5, since $E^{-2} = (\mathbb{Z}/2 \to 0)$ is a summand of both coefficient systems KR^{-2} and KO^{-2}_G.

Lemma A.6. *Let $\mathcal{A}$ be the coefficient system $(0 \to A)$, where A is a G-module. Then for every G-CW complex X,*

$$H^*_G(X; \mathcal{A}) \cong H^*_G(X, X^G; A) \cong H^*_G(X/X^G, \mathrm{pt}; A).$$

Proof. Recall that $C^q(X;\mathcal{A}) = \mathrm{Hom}_G(C_q(X, X^G), A)$ is the group of functions f on the q-cells σ of $X - X^G$ satisfying $\overline{f(\sigma)} = f(\bar\sigma)$; see [**7**, I-14]. The usual differentials on $\mathrm{Hom}(C_*(X, X^G), A)$ make $\mathrm{Hom}_G(C_*(X, X^G), A)$ into a cochain complex, and $H^*_G(X;\mathcal{A})$ is the cohomology of this complex, i.e., the cohomology $H^*_G(X, X^G; A) \cong H^*_G(X/X^G, \mathrm{pt}; A)$ of the local system A. (See Example A.2.) □

Example A.6.1. When G acts trivially on X, $H^*_G(X;\mathcal{A}) = 0$. When G acts freely on X, $H^*_G(X;\mathcal{A}) = H^*(X/G, A)$.

Recall that the coefficient system KR^2 is $0 \to \mathbb{Z}(1)$.

Corollary A.7. *$H^q_G(X; KR^{+2}) \cong H^q_G(X;\mathbb{Z}(1))$, and*

$$H^q_G(X; KR^{-2}) \cong H^q(X^G, \mathbb{Z}/2) \oplus H^q_G(X;\mathbb{Z}(1)).$$

Proof. By construction, $C^*(X^G, \mathbb{Z}(1)) = 0$, so $H^*_G(X^G;\mathbb{Z}(1)) = 0$. It follows from this and Lemma A.6 that $H^q_G(X; KR^{+2}) \cong H^q_G(X;\mathbb{Z}(1))$.

The calculation of $H^q_G(X; KR^{-2})$ follows from this and the observation in Remark A.5.1 that the coefficient system KR^{-2} is the direct sum of $(\mathbb{Z}/2 \to 0)$ and $(0 \to \mathbb{Z}(1))$. □

Lemma A.8. *If $\dim X^G \le 3$, then*

$$h^4 : H^4_G(X; KR^{-4}) \longrightarrow H^4_G(X; KO^{-4}_G) \cong H^4(X/G, \mathbb{Z}).$$

is a surjection. If $\dim X^G \le 2$, then h^4 is an isomorphism.

Proof. The coefficient map $KO^{-4}_G \xrightarrow{+} \mathbb{Z}(0)$ of Lemma A.5 induces an isomorphism $H^4_G(X; KO^{-4}_G) \cong H^4_G(X;\mathbb{Z}(0)) \cong H^4(X/G, \mathbb{Z})$, because $\dim X^G < 4$. From the exact sequence of coefficient systems

$$0 \to KR^{-4} \to \mathbb{Z}(0) \to (\mathbb{Z}/2 \to 0) \to 0,$$

we get an exact sequence

$$H^3(X^G, \mathbb{Z}/2) \to H^4_G(X; KR^{-4}) \to H^4_G(X;\mathbb{Z}(0)) \to H^4(X^G, \mathbb{Z}/2).$$

The last term vanshes when $\dim X^G \le 3$, and the left term vanishes when $\dim X^G \le 2$, so the middle map $H^4_G(X; KR^{-4}) \to H^4_G(X;\mathbb{Z}(0))$ is onto (resp., an isomorphism) when $\dim X^G$ is at most 3 (resp., 2). As $KR^{-4} \to \mathbb{Z}(0)$ is the composite $KR^{-4} \to KO^{-4}_G \to \mathbb{Z}(0)$, the result follows. □

Example A.9. Let X be $S^{2,2}$, the 3-sphere with $X^G = S^1$. It is not hard to show that $KO_G(S^{2,2}) = \mathbb{Z}^2$. Since $KR^1(B^{2,2}, S^{2,2}) \cong KO^1 = 0$ by [**2**, 2.3], we also have $KR(S^{2,2}) \cong \mathbb{Z}$ and hence $WR(S^{2,2}) \cong \mathbb{Z}$.

In this case, the differential $d_2 : H^1(X^G, \mathbb{Z}/2) \to H^3_G(X, \mathbb{Z}/2)$ is an isomorphism in the Bredon spectral sequence (A.1) for $KO_G(X)$.

Indeed, since X/G is the suspension of $\mathbb{CP}^1$, we have $H^p(X/G, \mathbb{Z}/2) = 0$ for $p \neq 3$ and $H^3(X/G, \mathbb{Z}/2) = \mathbb{Z}/2$.

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