THÈSE

Spécialité : **Mathématiques**

présentée et soutenue publiquement le 08 novembre 2010 par

Maxime ZAVIDOVIQUE

en vue d'obtenir le grade de

Docteur de l'Université de Lyon - École Normale Supérieure de Lyon
Ecole Doctorale InfoMaths
Unité de Mathématiques Pures et Appliquées

Modèle discret et intégrales premières en théorie KAM faible

Directeur de thèse : Monsieur Albert FATHI

Après avis de : Monsieur Patrick BERNARD
Monsieur John MATHER

Devant la commission d'examen formée de :

Monsieur Guy BARLES	*Membre*
Monsieur Patrick BERNARD	*Membre/Rapporteur*
Monsieur Albert FATHI	*Membre*
Monsieur John MATHER	*Membre/Rapporteur*
Monsieur Antonio SICONOLFI	*Membre*

Numéro d'ordre : 591 Numéro attribué par la bibliothèque : __ENSL591

Modèle discret et intégrales premières en théorie KAM faible

M. ZAVIDOVIQUE

8 novembre 2010

A mes grands-mères

Table des matières

Introduction

Résumé

Dans cette thèse, nous nous intéressons à deux problèmes concernant le semi-groupe de Lax-Oleinik et étudions leurs conséquences en théorie KAM faible. Le premier concerne des généralisations des théories d'Aubry-Mather et KAM faible de Fathi suite à une discrétisation en temps par rapport au modèle classique Hamiltonien. Le second problème traite, dans le cadre Hamiltonien Tonelli classique, des relations entre semi-groupes de Lax-Oleinik, théorie d'Aubry-Mather et théorie KAM faible pour des Hamiltoniens qui commutent au sens de Poisson. Commençons par décrire ces théories et les motiver. Nous présentons ensuite les articles contenant les résultats obtenus.

Chapitre 1

Point de vue dynamique

Dans la suite, M sera une variété lisse compacte sans bord, connexe. On dira qu'une fonction lisse $H : T^*M \to \mathbb{R}$ est un Hamiltonien Tonelli si elle vérifie les conditions suivantes :

1. **superlinearité :** pour tout $K > 0$, il existe $C^*(K) \in \mathbb{R}$ tel que
$$\forall (x,p) \in T^*M,\ H(x,p) \geqslant K\|p\| - C^*(K),$$
2. **stricte convexité dans les fibres :** pour tout $(x,p) \in T^*M$, la dérivée seconde $\partial^2 H/\partial p^2(x,p)$ est définie positive.

Rappelons que T^*M est naturellement muni d'une structure symplectique. Notons λ la forme de Liouville définie comme suit :
$$\forall (x,p,X,P) \in TT^*M,\ \lambda(x,p,X,P) = p(X).$$
La 2-forme $\Omega = -\,\mathrm{d}\,\lambda$ est alors une forme symplectique sur T^*M. La non-dégénérescence de Ω permet de définir le champ de vecteur Hamiltonien X_H par
$$\forall (x,p) \in T^*M,\ \Omega(X_H(x,p), .) = \mathrm{d}_{(x,p)}\, H.$$
De plus, les hypothèses sur M et H ont pour conséquence que le champ de vecteur X_H est complet. On note ϕ le flot de X_H, c'est le flot Hamiltonien.

On peut à ce stade définir un Lagrangien $L : TM \to \mathbb{R}$ comme suit
$$\forall (x,v) \in TM,\ L(x,v) = \max_{p \in T^*M} p(v) - H(x,p).$$
Des propriétés classiques d'analyse convexe affirment que L est lui aussi lisse, superlinéaire et strictement convexe dans les fibres. De plus, on a des informations sur le point où le suprémum est atteint données par les égalités suivantes :
$$L\left(x, \frac{\partial H}{\partial p}(x,p)\right) = p\left(\frac{\partial H}{\partial p}(x,p)\right) - H(x,p),$$

$$H\left(x, \frac{\partial L}{\partial v}\right) = \frac{\partial L}{\partial v}(x,v)(v) - L(x,v).$$

Ces relations motivent l'introduction de la transformée de Legendre $\mathcal{L}$: $TM \to T^*M$ définie par

$$\forall (x,v) \in TM,\ \mathcal{L}(x,v) = \left(x, \frac{\partial L}{\partial v}(x,v)\right).$$

La stricte convexité et la superlinéarité de L impliquent que la transformée de Legendre, $\mathcal{L}$, est un difféomorphisme, son inverse étant donné par la relation

$$\forall (x,p) \in T^*M,\ \mathcal{L}^{-1}(x,p) = \left(x, \frac{\partial H}{\partial p}(x,p)\right).$$

On peut alors définir le flot d'Euler-Lagrange φ comme le conjugué du flot Hamiltonien par la transformée de Legendre :

$$\forall ((x,v),t) \in TM \times \mathbb{R},\ \varphi^t(x,v) = \mathcal{L}^{-1} \circ \phi \circ \mathcal{L}(x,v).$$

Fixons momentanément $(x,v) \in TM$ et un temps t, mettons $t \geqslant 0$. On note alors

$$\forall s \in [0,t],\ \varphi^s(x,v) = (x(s), v(s)).$$

Un calcul entraîne que le flot d'Euler-Lagrange est solution du système d'EDO

$$\left\{ \begin{array}{c} v(s) = \dot{x}(s) \\ \dfrac{\mathrm{d}}{\mathrm{d}\, s}\dfrac{\partial L}{\partial v}(x(s), v(s)) = \dfrac{\partial L}{\partial x}(x(s), v(s)) \end{array} \right. \tag{EL}$$

La seconde équation est appelée équation d'Euler-Lagrange et apparaît classiquement dans un contexte de calcul des variations. Etant donné notre Lagrangien lisse strictement convexe et superlinéaire, et une paire de points $(x,y) \in M^2$, on cherche à minimiser l'action Lagrangienne :

$$\mathbb{L}(\gamma) = \int_0^t L(\gamma(s), \dot{\gamma}(s))\, \mathrm{d}\, s$$

sur les courbes γ absolument continues vérifiant $\gamma(0) = x$ et $\gamma(t) = y$. Il s'avère que les courbes réalisant le minimum existent, qu'elles sont elles aussi lisses et solutions de l'équation d'Euler-Lagrange (EL). Réciproquement, un théorème de Weierstrass affirme que les trajectoires du flot d'Euler-Lagrange réalisent localement, c'est-à-dire pour des temps petits, le minimum dans ce problème de minimisation de l'action Lagrangienne à extrémités fixées.

1.1 Systèmes intégrables et théorème KAM

Ainsi, un objectif majeur en systèmes dynamiques est de trouver des ensembles invariants par le flot Hamiltonien, voire de chercher ces ensembles sous forme de sous-variétés Lagrangiennes, c'est-à-dire des sous-variétés de T^*M sur lesquelles Ω est identiquement nulle. Ce que nous avons rappelé ci-dessus explique en quoi ce problème est intimement lié à l'étude des courbes minimisant l'action du Lagrangien associé.

Un premier cas particulier est le cas intégrable. Si M est de dimension n, on dit que H est intégrable si il existe n Hamiltoniens, $H_0, H_1, \dots, H_{n-1}$ vérifiant :

1. $H_0 = H$,
2. les Hamiltoniens sont en involution, c'est-à-dire que pour tout couple d'entiers $(i, j) \in [0, n-1]^2$ la fonction $\Omega(X_{H_i}, X_{H_j})$ est identiquement nulle,
3. les différentielles $\mathrm{d}_x H_0, \dots, \mathrm{d}_x H_{n-1}$ sont linéairement indépendantes en tout point $x \in M$.

Sous ces conditions, le surprenant théorème d'Arnol'd-Liouville ([Arn63c, Arn89]) affirme que M est en fait difféomorphe à un tore $\mathbb{T}^n$ et que de plus, le cotangent T^*M est feuilleté par des tores invariants par le flot Hamiltonien ϕ sur lesquels ce flot est linéaire.

Une question naturelle est d'étudier le comportement quand on perturbe légèrement un système intégrable. Ce problème s'avère remarquablement difficile et a donné naissance à la théorie KAM. Le résultat fondamental est que, sous certaines conditions de régularité, pour des petites perturbations d'un Hamiltonien intégrable et pour certains vecteurs de rotation ρ vérifiant une condition diophantienne, des tores invariants subsistent sur lesquels le flot Hamiltonien est conjugué à une rotation de vecteur de rotation ρ. La première version de ce théorème a été énoncée par Kolmogorov en 1954 dans un article de 4 pages [Kol54] ! L'auteur propose une preuve utilisant une sorte d'algorithme de Newton : bien évidemment, cette démonstration est incomplète. Elle a plus tard été achevée (voir [Chi08]). La première démonstration rigoureuse du théorème de Kolmogorov apparaît probablement à l'occasion des 60 ans de ce dernier dans l'article de son élève Arnol'd [Arn63a]. Arnol'd, dans le cas d'une perturbation analytique, reprend les idées de Kolmogorov, mais n'utilise pas exactement le même schéma d'approximation. Il apporte la même année une autre contribution majeure à la théorie KAM dans un papier ([Arn63b]) dont l'objectif est d'appliquer ces idées perturbatives à la mécanique céleste. Il prouve ainsi certains résultats de stabilité dans le problème des n-corps, sujet sur lequel il avait déjà soutenu sa thèse. Citons à propos de cet article un commentaire de Moser :

> It is to be hoped that this remarkable paper and exceptional work helps to arouse the interest of more mathematicians in this subject.

Il apparaît que ses voeux ont depuis été réalisés. Le M de Moser constitue la dernière lettre du sigle KAM. Il publie en 1962 un article sur l'existence de courbes périodiques et quasi-périodiques dans l'étude des "twist maps" (applications tordues ou déviant la verticale) de l'anneau. Ces twist maps peuvent-être rapprochées de l'analogue périodique en temps des système Hamiltoniens. La nouveauté des travaux de Moser est qu'il ne considère plus des déformations analytiques mais seulement de régularité finie (il énonce un théorème pour des perturbations de régularité C^{333}). Bien sûr, cette régularité n'est pas optimale, la régularité minimale a été étudiée par Herman dans [Her83, Her86], où il montre que pour les twist maps de l'anneau, la régularité critique est C^3. Il y fournit des preuves d'existence pour des perturbations $C^{3+\varepsilon}$ et C^3 et donne des contre-exemples pour des perturbations $C^{3-\varepsilon}$.

Il n'en reste pas moins que ces tores ne persistent pas pour des perturbations "grandes" et il faut donc chercher d'autres approches pour trouver des ensembles invariants dans le cas général.

1.2 Le point de vue de Mather

L'idée révolutionnaire de Mather dans le domaine ([Mat91], voir aussi [MF94]) est, en se plaçant sur TM et en adoptant le point de vue Lagrangien, d'étudier les mesures de probabilité invariantes par le flot d'Euler-Lagrange et qui minimisent l'action Lagrangienne. Un premier obstacle est que l'espace tangent TM n'est pas compact, contrairement aux conditions du théorème de Krylov-Bogoliubov prouvant l'existence de mesure de probabilité invariante pour un système dynamique sur une variété compacte. C'est la superlinéarité qui permet de réussir. Ainsi, la réunion des supports des mesures invariantes qui sont minimisantes, notée $\mathcal{M} \subset TM$ est un ensemble compact invariant appelé ensemble de Mather. Il est contenu dans un niveau d'énergie constante (Théorème de Carneiro [Car95]) du Hamiltonien, cette constante est notée $\alpha[0]$ et s'appelle constante critique de Mañé. De plus, si μ est une mesure minimisante, on a la relation suivante :

$$-\alpha[0] = \int L \, \mathrm{d}\, \mu.$$

Enfin, Mather prouve que $\mathcal{M}$ est un graphe Lipschitz au dessus de M. Mais Mather ne s'arrête pas là ! Il remarque que si ω est une 1-forme fermée et que l'on note L_ω le Lagrangien défini comme suit :

$$\forall (x,v) \in TM,\ \ L_\omega(x,v) = L(x,v) - \omega_x(v),$$

alors les flots d'Euler-Lagrange de L et de L_ω sont les mêmes et que l'action d'une mesure invariante par rapport à L_ω ne dépend que de la classe de cohomologie, $[\omega] \in H^1(M,\mathbb{R})$ de ω. Ainsi, il note $M_{[\omega]}$ la réunion des supports de mesures qui minimisent l'action pour L_ω et $\alpha[\omega]$, le niveau d'énergie dans lequel $M_{[\omega]}$ est inclus. Encore une fois, on a la relation

$$-\alpha[\omega] = \int L_\omega \,\mathrm{d}\,\mu.$$

La fonction $\alpha : H^1(M,\mathbb{R}) \to \mathbb{R}$ est convexe et superlinéaire, c'est la fonction α de Mather. Du point de vue dual, étant donnée une mesure μ, on peut lui associer un vecteur de rotation $\rho(\mu)$ défini comme étant l'unique $h \in H_1(M,\mathbb{R})$ tel que

$$\forall [\omega] \in H^1(M,\mathbb{R}),\ \int \omega \,\mathrm{d}\,\mu = \langle h, [\omega] \rangle, \tag{1.1}$$

où $\langle , \rangle$ désigne le crochet de dualité. En effet si μ est invariante par le flot d'Euler-Lagrange, on peut montrer que l'intégrale dans (1.1) ne dépend que de la classe de cohomologie de la forme fermée ω. Ainsi, étant donné $h \in H_1(M,\mathbb{R})$, Mather prouve l'existence de mesures de probabilité μ invariantes et de vecteur de rotation $\rho(\mu) = h$. Il est alors licite de minimiser sur ces mesures définissant la fonction β de Mather :

$$\forall h \in H_1(M,\mathbb{R}),\ \beta(h) = \min\left(\int L \,\mathrm{d}\,\mu,\ \rho(\mu) = h\right) \tag{1.2}$$

et on note $\mathcal{M}^h \subset TM$ la réunion des supports des mesures réalisant le minimum dans (1.2). La fonction β est elle aussi convexe et superlinéaire. En fait, il s'avère que les fonctions α et β sont très intimement liées puisqu'elles sont transformées de Legendre l'une de l'autre. Finalement, dans le cas intégrable, ou proche d'un système intégrable, Mather fait le lien entre les tores KAM et ses ensembles associés au même vecteur de rotation.

Les travaux de Mather, dont ceux mentionné ci-dessus, ont relancé la recherche en mécanique Hamiltonienne. Rares sont les travaux qui ne s'en inspirent pas dans le domaine. Ils ont aussi des répercussions en diffusion d'Arnol'd, géométrie symplectique, transport optimal... Sans eux, la théorie KAM faible n'aurait probablement jamais vu le jour.

1.3 La théorie KAM faible de Fathi

Contrairement à la théorie de Mather dédiée surtout à l'étude de mesures minimisantes, Fathi, avec sa théorie KAM faible, revient à l'étude plus classique des courbes minimisantes. Cependant, pour une vision plus globale, il fait d'une certaine manière agir ces courbes, via leur action Lagrangienne, sur des fonctions de M dans $\mathbb{R}$. Là où la théorie KAM fournit des sous-variétés Lagrangiennes analytiques, le but initial de la théorie KAM faible est de chercher des sous ensembles intéressants de T^*M. Ces ensembles, invariants par le flot Hamiltonien, apparaissent sous la forme de graphes ou de sous-graphes de différentielles de fonctions moins régulières (d'où le nom KAM faible). A cet effet, il réintroduit le semi-groupe de Lax-Oleinik $(T^{-t})_{t\geqslant 0}$ ([Fat08]) défini comme suit :

soit $u : M \to \overline{\mathbb{R}}$ une fonction et $t \geqslant 0$ un réel positif alors la fonction $T^{-t}u : M \to \overline{\mathbb{R}}$ est définie par

$$\forall x \in M,\ T^{-t}u(x) = \inf_{\gamma} u(\gamma(0)) + \int_{s=0}^{t} L\left(\gamma(s), \dot\gamma(s)\right) \mathrm{d}\, s, \qquad (1.3)$$

où l'infimum est réalisé sur toutes les courbes γ absolument continues qui vérifient $\gamma(t) = x$. Si $\alpha \in \mathbb{R}$, on dit qu'une fonction $u : M \to \mathbb{R}$ est α-dominée ou qu'elle est une α-sous-solution, noté $u \prec L + \alpha$, si pour tout réel positif t et pour toute courbe absolument continue définie de $[0, t] \to M$, on a

$$u(\gamma(t)) - u(\gamma(0)) \leqslant \int_{s=0}^{t} L\left(\gamma(s), \dot\gamma(s)\right) \mathrm{d}\, s + t\alpha.$$

On vérifie que l'assertion $u \prec L + \alpha$ est équivalente à

$$\forall t > 0,\ u \leqslant T^{-t}u + t\alpha.$$

On notera par la suite $\mathcal{H}(\alpha)$ l'ensemble des α-sous-solutions. Il découle assez facilement, comme L est minoré, que pour α assez petit, on a $\mathcal{H}(\alpha) = \varnothing$. Il s'avère à posteriori que la constante critique de Mañé, $\alpha[0]$ vérifie

$$\alpha[0] = \inf_{\alpha\in\mathbb{R}} \left\{\alpha,\ \mathcal{H}(\alpha) \neq \varnothing\right\}.$$

Enfin, Fathi prouve que les fonctions de $\mathcal{H}(\alpha[0])$ sont équi-lipschitziennes, que $\mathcal{H}(\alpha[0])$ est fermé et que pour tout $t > 0$, l'opérateur T^{-t} est faiblement contractant pour la norme ∞ et laisse $\mathcal{H}(\alpha[0])$ invariant. Une application judicieuse du théorème de point fixe de Banach donne alors le

Théorème 1 (KAM faible)**.** *Il existe une fonction $u_- : M \to \mathbb{R}$ vérifiant*

$$\forall t > 0,\ T^{-t}u_- + t\alpha[0] = u_-.$$

De plus, si pour une constante c il existe une fonction $u : M \to \mathbb{R}$ *vérifiant*

$$\forall t > 0,\ T^{-t}u + t\alpha[0] = u,$$

alors $c = \alpha[0]$.

Une fonction u_- donnée par le théorème KAM faible est appelée solution KAM faible. Soit $u \prec L + \alpha[0]$, on dit alors que u est une sous-solution critique. Si une courbe γ vérifie

$$u(\gamma(t)) - u(\gamma(0)) = \int_{s=0}^{t} L\left(\gamma(s), \dot\gamma(s)\right) \mathrm{d}\, s + t\alpha[0],$$

on dit qu'elle est calibrante pour u. Il est clair que les courbes calibrantes sont minimisantes pour l'action Lagrangienne. On peut montrer qu'une solution KAM faible u_- est exactement une fonction u_- telle que pour tout point x, il existe une courbe $\gamma_x :]-\infty, 0] \to M$ telle que $\gamma_x(0) = x$ et

$$\forall t > 0,\ u_-(x) - u_-(\gamma_x(-t)) = \int_{s=-t}^{0} L\left(\gamma_x(s), \dot\gamma_x(s)\right) \mathrm{d}\, s + t\alpha[0].$$

De plus, comme M est compact, on vérifie que si u est une fonction sur M et $\alpha \in \mathbb{R}$ vérifie

$$\forall t > 0,\ T^{-t}u + t\alpha = u,$$

alors nécessairement, $\alpha = \alpha[0]$. On note $\mathcal{S}^-$ l'ensemble des solutions KAM faibles.

Un zeste de calcul des variations permet de mieux caractériser ces courbes. En effet, on peut montrer que si u_- est différentiable en x alors on a

$$(x, \mathrm{d}_x\, u_-) = \left(x, \frac{\partial L}{\partial v}(\gamma_x(0), \dot\gamma_x(0))\right) = \mathcal{L}(\gamma_x(0), \dot\gamma_x(0)).$$

Comme de plus une courbe calibrante d'une sous-solution critique est clairement une courbe minimisante pour l'action Lagrangienne, c'est nécessairement une trajectoire du flot d'Euler-Lagrange et on obtient donc que

$$\forall s < 0,\ (\gamma_x(s), \dot\gamma_x(s)) = \varphi^s(\gamma_x(0), \dot\gamma_x(0)) = \varphi^s(\mathcal{L}^{-1}(x, \mathrm{d}_x\, u_-)).$$

Finalement, on montre que u_- est différentiable sur la trajectoire γ_x et on obtient donc la relation suivante,

$$\forall s < 0,\ (\gamma_x(s), \mathrm{d}_{\gamma_x(s)}\, u_-) = \mathcal{L}(\gamma_x(s), \dot\gamma_x(s)) = \mathcal{L}(\varphi^s(\mathcal{L}^{-1}(x, \mathrm{d}_x\, u_-))) = \phi^s(x, \mathrm{d}_x\, u_-).$$

Ceci montre que si l'on note $\mathrm{Diff}(u_-) = \{(x, \mathrm{d}_x u_-)\} \subset T^*M$ où x parcourt les points de différentiabilité de u_-, alors on a

$$\forall s > 0,\ \phi^s(\mathrm{Diff}(u_-)) \subset \mathrm{Diff}(u_-)$$

et par continuité du flot on a aussi

$$\forall s > 0,\ \phi^s\left(\overline{\mathrm{Diff}(u_-)}\right) \subset \overline{\mathrm{Diff}(u_-)}$$

Notons que $\overline{\mathrm{Diff}(u_-)}$ est un ensemble relativement gros du fait qu'il se projette sur M tout entier. Finalement, si on note

$$\widetilde{\mathcal{A}}_{u_-} = \bigcap_{s<0} \phi^s\left(\overline{\mathrm{Diff}(u_-)}\right)$$

alors $\widetilde{\mathcal{A}}_{u_-}$ est un ensemble compact, invariant par le flot hamiltonien, de plus on peut montrer que c'est un graphe partiel bi-lipschitzien au dessus de sa projection sur M, notée $\mathcal{A}_{u_-}$. On appel $\widetilde{\mathcal{A}}_{u_-}$ l'ensemble d'Aubry relatif à u_-. L'ensemble $\mathcal{A}_{u_-}$ est en fait l'ensemble des $x \in M$ tel qu'il existe une courbe $\gamma_x :]-\infty, +\infty[\to M$ telle que $\gamma_x(0) = x$ et

$$\forall (s,t) \in \mathbb{R}^2,\ u_-(\gamma_x(t)) - u_-(\gamma_x(s)) = \int_s^t L\left(\gamma_x(\sigma), \dot\gamma_x(\sigma)\right) \mathrm{d}\,\sigma + (t-s)\alpha[0].$$

On sait dans ce cas que l'unique point au dessus de x qui est dans $\widetilde{\mathcal{A}}_{u_-}$ est

$$(x,p) = \mathcal{L}\left(x, \dot\gamma_x(0)\right) \in T_x^*M.$$

On peut alors définir l'ensemble d'Aubry, $\widetilde{\mathcal{A}} \subset T^*M$ par

$$\widetilde{\mathcal{A}} = \bigcap_{u_- \in \mathcal{S}^-} \widetilde{\mathcal{A}}_{u_-}.$$

Il découle ainsi aisément des propriétés des $\widetilde{\mathcal{A}}_{u_-}$ que $\widetilde{\mathcal{A}}$ est un ensemble compact invariant par le flot Hamiltonien ϕ. De plus, $\widetilde{\mathcal{A}}$ est un graphe partiel bi-lipschitzien au dessus de sa projection $\mathcal{A}$ sur M. Finalement, il est clair que si $x \in \mathcal{A}$ et si u_- est un solution KAM faible, alors u_- est différentiable en x et $(x, \mathrm{d}_x u_-) \in \widetilde{\mathcal{A}}$. Finalement, $\widetilde{\mathcal{A}}$ est non vide et en particulier, on peut montrer que

$$\mathcal{L}(\mathcal{M}) \subset \widetilde{\mathcal{A}}.$$

L'ensemble d'Aubry a en fait été introduit par Mather qui a montré qu'il est une sorte de généralisation des tores KAM. En particulier, dans les conditions d'application d'un tore KAM, alors l'ensemble d'Aubry et le tore KAM coïncident. Ce genre de coïncidence est abordé dans [Mat91, FGS08].

Chapitre 2

Point de vue analytique

2.1 Equation d'Hamilton-Jacobi et solutions de viscosité

Etant donné une variété lisse compacte connexe sans bord M et un Hamiltonien $H : T^*M \to \mathbb{R}$ strictement convexe et superlinéaire dans les fibres, on s'intéresse aux équations d'Hamilton-Jacobi suivantes :

l'équation d'évolution : soit $u_0 : M \to \mathbb{R}$ une fonction lipschitzienne, on veut trouver une fonction $u : \mathbb{R}_+ \times M \to \mathbb{R}$ telle que

$$\left\{ \begin{array}{rcl} \dfrac{\partial u}{\partial t} & + & H(x, \mathrm{d}_x u) \ = 0 \\ u(0,x) & = & u_0(x) \end{array} \right. \tag{HJ}$$

l'équation stationnaire : soit $c \in \mathbb{R}$ une constante, on cherche une fonction $u : M \to \mathbb{R}$ telle que

$$H(x, \mathrm{d}_x u) = c. \qquad \text{(HJS).}$$

Une première difficulté consiste à trouver une bonne notion de solution de (HJ) ou de (HJS). Une étude classique, par exemple en utilisant la méthode des caractéristiques, montre que même si u_0 est lisse, on obtiendra rarement une solution lisse de (HJ) au sens fort. Au contraire, si on demande qu'une fonction u soit solution si elle est lipschitzienne et vérifie (HJ) ou (HJS) presque partout alors la contrainte est trop faible et mène à un trop grand nombre de solutions. Par exemple si $M = S^1 = \mathbb{R}/\mathbb{Z}$ et $H(x,p) = \|p\|^2$, on voit que pour tout ensemble mesurable $A \subset [0,1]$ de mesure $1/2$, si χ_A désigne la fonction caractéristique de A et si on définit la fonction

$$\forall x \in [0,1], \ f(x) = \int_0^x (2\chi_A(t) - 1)\,\mathrm{d}\,t,$$

alors f a une dérivée presque partout égale à 1 ou -1 et définit bien une fonction sur le cercle qui est solution de (HJS) pour $c = 1$.

Une notion plus pertinente de solution est celle de solution de viscosité introduite dans [CL83, CEL84]. L'idée, comme en théorie des distributions, est d'introduire des fonctions test.

déf 2.1. Soit $u : \mathbb{R}_+ \times M \to \mathbb{R}$ et $(t_0, x_0) \in \mathbb{R}_+ \times M$, on dira qu'une fonction φ est strictement tangente supérieurement à u en (t_0, x_0) si $u(t_0, x_0) = \varphi(t_0, x_0)$ et si pour tout couple $(t, x) \neq (t_0, x_0)$ on a $u(t, x) < \varphi(t, x)$. On définit de manière analogue la notion de fonction strictement tangente inférieurement.

En reprenant les notations introduites précédemment, on dira qu'une fonction $u : \mathbb{R}_+ \times M \to \mathbb{R}$ est une sous-solution (resp. sur-solution) de viscosité de l'équation (HJ) si elle est continue, est telle que $u(0, .) = u_0$ et si pour tout couple $(t_0, x_0) \in \mathbb{R}_+ \times M$ et pour toute fonction φ de régularité C^1 strictement tangente supérieurement à u l'égalité suivante est vérifiée :

$$\frac{\partial \varphi}{\partial t} + H(x, \mathrm{d}_x \varphi) \leqslant 0.$$

$$\left(\text{resp.} \qquad \frac{\partial \varphi}{\partial t} + H(x, \mathrm{d}_x \varphi) \geqslant 0\right)$$

Finalement, on dira que u est une solution de viscosité de (HJ) si c'est à la fois une sous-solution de viscosité et une sur-solution de viscosité de (HJ).

On définit de manière similaire les sous-solutions, sur-solutions et solutions de viscosité de (HJS).

Dans le cadre particulier des Hamiltoniens strictement convexes et superlinéaires dans lequel nous nous plaçons, il existe une formule de représentation explicite des solutions de viscosité :

Théorème 2. *Soit $u_0 : M \to \mathbb{R}$ une donnée initiale lipschitzienne, alors il existe une unique solution de viscosité $u : \mathbb{R}_+ \times M \to \mathbb{R}$ à l'équation d'Hamilton-Jacobi donnée par la formule :*

$$\forall (t, x) \in \mathbb{R}_+ \times M, \;\; u(t, x) = T^{-t} u_0(x),$$

où T^- est le semi-groupe de Lax-Oleinik (1.3).

Le lien entre solutions de viscosité et théorie KAM faible est alors évident. On remarque de plus que les solutions KAM faibles sont en pratique des solutions de l'équation d'Hamilton-Jacobi stationnaire (HJS).

2.2 L'homogénéisation d'Hamilton-Jacobi

Les problèmes d'homogénéisation de l'équation d'Hamilton-Jacobi ont été initiés dans le preprint [LPV87] et suscitent encore une vive activité mathématique (voir [IM08, IMR08, FIM09b, FIM09a, DP07, DS09] pour n'en citer qu'un petit nombre parmi les plus récents). Nous n'exposerons ici que les principaux résultats de [LPV87].

On supposera que notre variété compacte est un tore : $M = \mathbb{T}^n$. Par abus de notation, on notera indifféremment par H l'Hamiltonien défini sur $T^*\mathbb{T}^n$ ou son relevé sur $\mathbb{R}^{2n}$.

On s'intéresse ici au comportement asymptotique des solutions u^ε des équations

$$\left\{\begin{array}{lcl} \dfrac{\partial u^\varepsilon}{\partial t} & + & H\left(\frac{x}{\varepsilon}, \mathrm{d}_x u^\varepsilon\right) = 0 \\ u^\varepsilon(0,x) & = & u_0(x) \end{array}\right. \tag{HJε}$$

Lions, Papanicolaou et Varadhan prouvent le résultat suivant :

Théorème 3. *Si la condition initiale $u_0 : \mathbb{R}^n \to \mathbb{R}$ est bornée et uniformément continue, alors pour tout $T > 0$ les solutions u^ε convergent uniformément sur $[0,T] \times \mathbb{R}^n$ vers une fonction U solution de l'équation*

$$\left\{\begin{array}{lcl} \dfrac{\partial U}{\partial t} & + & \overline{H}(\mathrm{d}_x U) = 0 \\ U(0,x) & = & u_0(x) \end{array}\right.$$

où pour tout $p \in \mathbb{R}^n$, $\overline{H}(p)$ est défini comme étant l'unique constante telle que l'équation stationnaire

$$H(x, p + \mathrm{d}_x u) = \overline{H}(p)$$

admette une solution.

Dans l'énoncé précédent, toutes les solutions sont à interpréter au sens des viscosités. En homogénéisation, la fonction $\overline{H}$ s'appelle "Hamiltonien effectif" et l'équation $H(x, p + \mathrm{d}_x u) = \overline{H}(p)$ s'appelle "problème de la cellule". On aura bien évidemment remarqué que la constante $\overline{H}(p)$ coïncide avec la constante critique de Mañé $\alpha[p]$ et qu'une solution au problème de la cellule est en fait une solution KAM faible pour la classe de cohomologie du vecteur p. Ainsi, l'Hamiltonien effectif n'est rien d'autre que la fonction α de Mather. Il est donc convexe et superlinéaire. On peut donc construire un Lagrangien effectif $\overline{L}$ qui est la transformée de Legendre de $\overline{H}$. Ce Lagrangien effectif est alors la fonction β de Mather.

2.3 KAM faible et le transport optimal

Le problème du transport optimal se conçoit très bien, cependant, comme souvent en mathématiques, contrairement au vieil adage, il a fallu un certain temps avant de réussir à l'énoncer clairement, au sein d'un modèle et avec un formalisme qui permette de le résoudre.

Dans son mémoire de 1781 ([Mon81]), Gaspard Monge considère un amas de matière première qu'il voudrait déplacer pour construire un édifice. Le fait de déplacer la matière a un coût, bien évidemment, on veut minimiser le coût total de l'opération. Voici comment est modélisé tout cela.

On se place sur une variété lisse M et on dispose d'un coût $c : M\times M \to \mathbb{R}$. La quantité $c(x,y)$ correspond à la dépense engendrée par le fait de déplacer une quantité de masse 1 du point x au point y. On supposera la fonction $c : M\times M \to \mathbb{R}$ continue. Le déblais de matière première initiale est modélisé par une mesure de probabilité μ et la construction finale par une mesure de probabilité ν. On cherche donc une fonction mesurable $T : M \to M$ tel que $T_\#\mu = \nu$ c'est-à-dire telle que pour tout ensemble ν-mesurable A on ait $\nu(A) = \mu(T^{-1}(A))$. Cette fonction encode le fait que la matière d'un point x est transportée au point $T(x)$. On voudrait ensuite minimiser sur de telles fonctions de transport T la quantité

$$C(T) = \int_X c(x, T(x))\,\mathrm{d}\,\mu(x),$$

qui est le coût total de l'opération. Cette approche confronte rapidement à des difficultés techniques difficilement surmontables : dans quels espaces fonctionnels chercher T, manque de linéarité, de compacité, de convexité...

Une grande avancée a été réalisée par Kantorovitch ([Kan42]). Plutôt que chercher de directement des applications de transport, il propose de considérer des plans de transport, c'est-à-dire des mesures de probabilité γ sur $M \times M$ telles que

$$(\pi_1)_\#\gamma = \mu \quad \text{et} \quad (\pi_2)_\#\gamma = \nu \tag{2.1}$$

où π_1 et π_2 sont les projections canoniques sur les premières et secondes coordonnées. L'objectif est alors de trouver de telles mesures γ minimisant le coût total

$$C(\gamma) = \iint_{X\times X} c(x,y)\,\mathrm{d}\,\gamma(x,y).$$

Pour ce problème de minimisation, l'ensemble des mesures admissibles est un convexe compact et l'existence d'un minimum est alors facile à obtenir. On peut ensuite essayer de montrer que, sous de bonnes hypothèses, les mesures minimisantes sont concentrées sur des graphes pour récupérer des fonctions de transport.

Comme souvent dans ce genre de problèmes d'optimisation linéaire, il existe un problème dual que nous exposons maintenant.

déf 2.2. *Une mesure de probabilité γ sur $M \times M$ est appelée mesure admissible si elle vérifie 2.1. On note $\mathfrak{M}(\mu,\nu)$ l'ensemble des mesures admissibles.*

Soient φ et ψ deux fonctions réelles sur M. On dit que la paire (φ,ψ) est une paire admissible si

$$\forall y \in M,\ \psi(y) = \min_{x\in M} \varphi(x) + c(x,y),$$

$$\forall x \in M,\ \varphi(x) = \max_{y\in M} \psi(y) - c(x,y).$$

On note $\mathfrak{P}(\mu,\nu)$ l'ensemble des paires admissibles.

Notons que toute paire admissible (φ,ψ) vérifie

$$\forall (x,y) \in M \times M,\ \psi(y) - \varphi(x) \leqslant c(x,y).$$

Le résultat suivant est dû à Kantorovitch :

proposition 2.1 (dualité de Monge-Kantorovitch)**.** *On a l'égalité :*

$$\min_{\gamma\in\mathfrak{M}(\mu,\nu)} C(\gamma) = \max_{(\varphi,\psi)\in\mathfrak{P}(\mu,\nu)} \int_X \psi(y)\,\mathrm{d}\,\nu(y) - \int_X \varphi(x)\,\mathrm{d}\,\mu(x). \tag{2.2}$$

De plus, si une paire admissible (φ,ψ) réalise 2.2 alors toute mesure γ optimale est concentrée sur l'ensemble

$$\left\{(x,y) \in M \times M,\ \psi(y) - \varphi(x) = c(x,y)\right\}.$$

Depuis Kantorovitch, nombre de mathématiciens se sont attaqués au problème du transport optimal et la littérature sur ce sujet est énorme. Cependant, même pour des coûts relativement simples, de type énergie cinétique plus énergie potentielle, des résultats d'existence de transport ont tardé à être exhibés. Pour plus de références et de détails on pourra consulter les livres de Cédric Villani ([Vil03, Vil09]).

On notera cependant que les inf et sup-convolution qui interviennent dans la définition des paires admissibles (2.2) ne sont pas sans rappeler le semi-groupe de Lax-Oleinik (1.3). Il n'est donc pas surprenant que des méthodes issues des théories d'Aubry-Mather et KAM faible aient joué un rôle en transport optimal. Reste que l'ampleur de ce rôle et la clarté des démonstrations finales apparaissent surprenants. Patrick Bernard et Boris Buffoni sont les premiers à avoir fait ce lien, ils décrivent l'analogie formelle dans [BB07b] puis l'utilisent pour prouver des résultats d'existence de plans de transport dans [BB07a, BB06]. Finalement, les idées de ces derniers articles sont reprises dans [FF07] où le cas d'une variété M non compacte est traitée. Les idées essentielles sont mises en exergue de manière éclairante et le théorème suivant est montré :

Théorème 4 (Bernard, Buffoni, Fathi, Figalli). *Soit M une variété lisse connexe, $c : M \times M \to \mathbb{R}$ un coût minoré, semi-continu inférieurement, μ et ν deux mesures boréliennes de probabilité sur M tel que*

$$\iint_{X\times X} c(x,y)\,\mathrm{d}\,\mu(x)\,\mathrm{d}\,\nu(y) < +\infty.$$

On suppose de plus que

- *la famille d'applications $x \mapsto c(x,y)$ est localement semi-concave en x, localement uniformément en y,*
- *le coût satisfait une condition de torsion à gauche (définie dans le chapitre 2),*
- *la mesure μ ne charge pas les ensembles de mesure de Hausdorff $(n-1)$-dimensionnelle σ-finie,*

alors il existe une application mesurable $T : M \to M$ tel que tout plan γ, optimal pour c, soit concentré sur le graphe de T.

Chapitre 3

Panorama des résultats obtenus

Dans un premier temps, nous nous sommes intéressé au cours de cette thèse au formalisme discrétisé en temps de la théorie KAM faible qui, comme nous l'avons vu, apparaît naturellement en transport optimal, mais aussi dans l'étude de lagrangiens dépendant du temps périodiques en temps. Les résultats obtenus sont exposés dans les deux premières parties.

Nous avons ensuite étudié les interactions entre les dynamiques associées à deux hamiltoniens en involution, c'est la troisième et dernière partie.

3.1 Sous-solutions strictes et potentiel de Mañé en théorie KAM faible discrète

Dans ce premier article, nous avons généralisé quelques notions et résultats de théorie d'Aubry-Mather et de théorie KAM faible dans un cadre discret. Une première remarque est que la structure lisse d'une variété n'est absolument pas nécessaire, c'est pourquoi nous proposons le cadre d'étude suivant :

déf 3.1 (espace de longueur à grande échelle)**.** Soit $K \in \mathbb{R}$, $B \geqslant 1$ des constantes, on dit qu'un espace métrique X est un B-espace de longueur à échelle K si pour toute paire $(x,y) \in X^2$, il existe une chaîne $(x = x_0, \dots, x_n = y) \in X^{n+1}$ tel que pour tout $i \leqslant n-1$, $\mathrm{d}(x_i, x_{i+1}) \leqslant K$ et, $\sum_{0 \leqslant i \leqslant n-1} \mathrm{d}(x_i, x_{i+1}) \leqslant B\,\mathrm{d}(x,y)$ où d est la fonction distance.

On suppose tout au long de ce travail que X est un B-espace de longueur à échelle K pour certaines constantes B et K. On suppose de plus que les boules fermées de X sont compactes.

On considère ensuite un coût continu $c : X \times X \to \mathbb{R}$ qui vérifie les deux propriétés suivantes :

1. **Uniforme super-linearité** : pour toute constante $k \geqslant 0$, il existe $C(k) \in \mathbb{R}$ tel que
$$\forall (x,y) \in X^2, c(x,y) \geqslant k\,\mathrm{d}(x,y) - C(k);$$
2. **Borne uniforme** : pour tout $R \in \mathbb{R}$, il existe $A(R) \in \mathbb{R}$ tel que $\mathrm{d}(x,y) \leqslant R \Rightarrow c(x,y) \leqslant A(R)$.

Par analogie au cadre lagrangien et aux paires admissibles de Kantorovitch, on dit qu'une fonction $u : X \to \mathbb{R}$ est une α-sous-solution ou qu'elle est dominée par $c+\alpha$ (abrégé $u \prec c+\alpha$) si pour tout $(x,y) \in X^2$ on a $u(x) - u(y) \leqslant c(y,x) + \alpha$. On note $\mathcal{H}(\alpha)$ l'ensemble de ces fonctions.
Finalement, les semi-groupes de Lax-Oleinik sont définis ici par :
à une fonction $u : X \to \overline{\mathbb{R}}$ on associe les fonctions
$$T_c^- u : X \to \overline{\mathbb{R}}, \quad T_c^- u(x) = \inf_{y \in X} \{u(y) + c(y,x)\},$$
$$T_c^+ u : X \to \overline{\mathbb{R}}, \quad T_c^+ u(x) = \sup_{y \in X} \{u(y) - c(x,y)\}.$$

La première étape est de vérifier que dans un tel cadre, on a encore un théorème KAM faible. La réponse à cette question est donnée par le théorème suivant :

Théorème 5 (KAM faible)**.** *Il existe une constante $\alpha[0]$ telle que l'équation $u = T_c^- u + \alpha[0]$ (resp. $u = T_c^+ u - \alpha[0]$) admet une solution et telle que $\mathcal{H}(\alpha)$ est vide pour $\alpha < \alpha[0]$.*

On a ensuite étudié les sous-solutions critiques, c'est-à-dire les fonctions $u \in \mathcal{H}(\alpha[0])$. Notre recherche s'est orientée suivant deux axes. On a commencé par remarquer que contrairement au cas lagrangien, où toutes les sous-solutions sont équi-lipschitziennes, les sous-solutions du problème discrétisé peuvent présenter des discontinuités. Un moyen simple de s'en rendre compte est donné par la proposition suivante :

lemme 1. *Soit $u \prec c + \alpha[0]$ et soit v une fonction qui vérifie les inégalités suivantes :*
$$u \leqslant v \leqslant T_c^- u + \alpha[0].$$
Alors v est elle-même une sous-solution critique : $v \prec c + \alpha[0]$.

Le second axe a été de généraliser le potentiel de Mañé, aussi appelé semi-distance de viscosité que nous avons défini comme suit :

déf 3.2. On définit le potentiel de Mañé par la formule

$$\varphi(x,y) = \sup_{u \prec c+\alpha[0]} u(y) - u(x),$$

où le supremum est pris sur les sous-solutions critiques (pas nécessairement continues).

Les propriétés principales de ce potentiel sont résumées ci-dessous :

proposition 3.1. *Le potentiel vérifie les propriétés suivantes :*

1. *Pour tout $(x,y) \in X^2$ on a $\varphi(x,y) \leqslant c(x,y) + \alpha[0]$. En particulier, le potentiel est partout fini.*
2. *Pour tout $x \in X$, le potentiel vérifie $\varphi(x,x) = 0$.*
3. *une fonction u est une sous-solution critique si et seulement si pour tout (x,y) dans X^2 on a $u(y) - u(x) \leqslant \varphi(x,y)$.*
4. *La fonction φ vérifie une inégalité triangulaire : pour tout x, y, z dans X on a $\varphi(x,y) + \varphi(y,z) \geqslant \varphi(x,z)$.*
5. *Pour tout $x \in X$, la fonction $\varphi_x = \varphi(x,.)$ est une sous-solution critique.*
6. *Soit $x \in X$, alors pour tout $y \neq x$ on a*

$$\varphi_x(y) = T_c^- \varphi_x(y) + \alpha[0],$$

en particulier, la fonction φ_x est semi-continue inférieurement et continue sur $X \setminus \{x\}$.

Ainsi, le potentiel de Mañé donne pour tout $x \in X$ la sous-solution maximale qui s'annule en x.

Il s'est avéré que le potentiel de Mañé permet aussi de quantifier les discontinuités des sous-solutions critiques comme le montre le résultat suivant.

lemme 2. *Pour tout $x \in X$, on définit la quantité $F(x) = \sup_{u \prec c+\alpha[0]} T_c^- u(x) + \alpha[0] - u(x)$, où le suprémum est pris sur l'ensemble des sous-solutions critiques. On a alors l'égalité $F(x) = T_c^- \varphi_x(x) + \alpha[0]$.*
De plus, pour tout point x non isolé, la fonction F vérifie

$$F(x) = \lim_{\substack{y \to x \\ y \neq x}} \varphi_x(y).$$

On montre également que dans la définition du potentiel de Mañé, le suprémum peut en fait être pris sur des fonctions continues.

On a aussi voulu construire dans ce cadre, à l'instar des résultats de [FS04], des sous-solutions strictes sur un ensemble maximal. Pour le faire on doit introduire quelques notions.

déf 3.3. Soit $x_0 \in X$ et $u \prec c + \alpha[0]$ une sous-solution critique. on dit que u est stricte en $(x, y) \in X^2$ si et seulement si

$$u(x) - u(y) < c(y, x) + \alpha[0].$$

On dit que u est stricte en $x \in X$ si

$$\forall y \in X, u(y) - u(x) < c(x, y) + \alpha[0] \text{ et } u(x) - u(y) < c(y, x) + \alpha[0].$$

On dit qu'une chaîne $(x_i)_{0 \leqslant i \leqslant n}$ de points de X est $(u, c, \alpha[0])$-calibrée si

$$u(x_n) = u(x_0) + c(x_0, x_1) + \cdots + c(x_{n-1}, x_n) + n\alpha[0].$$

On notera qu'une sous-chaîne formée d'éléments consécutifs d'une chaîne calibrée est encore calibrée puisque $u \prec c + \alpha[0]$.

Finalement, d'après Bernard et Buffoni [BB07b] on appelle ensemble d'Aubry de u, l'ensemble $\widetilde{\mathcal{A}}_u$ de $X^{\mathbb{Z}}$ constitué des suites dont les sous-chaînes finies sont $(u, c, \alpha[0])$-calibrées. L'ensemble d'Aubry projeté de u est

$$\mathcal{A}_u = \{x \in X, \exists (x_n)_{n\in\mathbb{Z}}, (u, c, \alpha[0])\text{-calibrée avec } x_0 = x\}.$$

L'ensemble d'Aubry est

$$\widetilde{\mathcal{A}} = \bigcap_{u \prec c+\alpha[0]} \widetilde{\mathcal{A}}_u.$$

L'ensemble d'Aubry projeté est

$$\mathcal{A} = \bigcap_{u \prec c+\alpha[0]} \mathcal{A}_u,$$

où dans les deux cas, les intersections sont prises sur toutes les sous-solutions critiques.

On montre alors le théorème suivant :

Théorème 6. *Il existe une fonction continue $u_1 : X \to \mathbb{R}$ qui est une $\alpha[0]$-sous-solution et qui est stricte en $x \in X$ dès que x n'est pas dans l'ensemble d'Aubry projeté $\mathcal{A}$.*

On montre enfin que le potentiel de Mañé fait le lien entre solutions KAM faibles, ensemble d'Aubry et continuité des sous-solutions. En effet, on obtient qu'un point $x \in X$ est dans l'ensemble d'Aubry projeté si et seulement si la fonction φ_x est une solution KAM faible. Si le point x n'est pas isolé, on a aussi que $x \in X$ est dans l'ensemble d'Aubry projeté si et seulement si la fonction φ_x est continue en x ce qui est aussi équivalent au fait que toute sous-solution critique est continue en x.

3.2 Existence de sous-solutions $C^{1,1}$ en théorie KAM faible discrète

Contrairement à l'approche qui précède, où on a voulu mettre le moins de structure sur la base, on réintroduit ici un peu de différentiabilité pour, à l'instar des travaux [FS04, Ber07, FFR09] dans le cas lagrangien, montrer l'existence de sous-solutions critiques $C^{1,1}$. On suppose donc dorénavant que l'espace métrique X est en fait une variété connexe lisse que l'on note M. On suppose que le coût c est de surcroît localement semi-concave (c'est-à-dire, dans des cartes, que c'est la somme d'une fonction concave et d'une fonction lisse) et qu'il vérifie des conditions de torsion :

déf 3.4. On dit que c vérifie la *condition de torsion à droite* si pour tout $x \in M$, la fonction $y \mapsto \partial c/\partial x(x,y)$ est injective sur son domaine de définition.

Similairement, on dit que c vérifie la *condition de torsion à gauche* si pour tout $y \in M$, la fonction $x \mapsto \partial c/\partial y(x,y)$ est injective sur son domaine de définition.

Finalement, on dit que c vérifie la *condition de torsion* si c vérifie les conditions de torsion à droite et à gauche.

Sous ces hypothèses, on obtient le résultat suivant :

Théorème 7. *Si u est une sous-solution critique alors il existe une sous-solution critique $C^{1,1}$ notée u' tel que u et u' coïncident sur $\mathcal{A}_u$ et u' est stricte en dehors de $\mathcal{A}_u$.*
Il existe une sous-solution critique $C^{1,1}$ qui est stricte en dehors de $\mathcal{A}$.

Rappelons brièvement les étapes de la preuve. On commence par étudier les propriétés régularisantes des semi-groupes de Lax-Oleinik :

proposition 3.2. *Soit $u \prec c + \alpha[0]$ alors $T_c^- u$ est localement semi-concave et $T_c^+ u$ est localement semi-convexe.*

On modifie ensuite la fonction u pour obtenir une sous-solution stricte en dehors de son ensemble d'Aubry :

Théorème 8. *Soit $u \prec c + \alpha[0]$ une sous-solution critique. Il existe une sous-solution continue u' qui est stricte en dehors de $\mathcal{A}_u$ et qui est égale à u sur $\mathcal{A}_u$. En particulier, on a*

$$\mathcal{A}_u = \mathcal{A}'_u.$$

Il existe une sous-solution continue u_0 qui est stricte en dehors de $\mathcal{A}$. En particulier on a

$$\mathcal{A} = \mathcal{A}_{u_0}.$$

Et on vérifie que l'action des semi-groupes de Lax-Oleinik ne perturbe pas les ensembles d'Aubry :

proposition 3.3. *Si* $u \prec c + \alpha[0]$ *est une sous-solution critique continue qui est stricte en dehors de* $\mathcal{A}_u$ *alors* $T_c^- u$ *et* $T_c^+ u$ *sont aussi des sous-solutions strictes en dehors de* $\mathcal{A}_{T_c^- u} = \mathcal{A}_u = \mathcal{A}_{T_c^+ u}$.

On se réfère alors à l'idée de Patrick Bernard dans [Ber07] qui montre qu'en appliquant successivement le semi-groupe négatif puis le semi-groupe positif pour des petits temps, on obtient une fonction $C^{1,1}$. Le problème est que notre discrétisation en temps ne nous autorise pas de petits temps. Comme suggéré par Pierre Cardaliaguet, on va pallier cette difficulté en utilisant le lemme d'Ilmanen ([Ilm93, Ber09a, FZ09]) :

Théorème 9. *Soit* $f : M \mapsto \mathbb{R}$ *une fonction localement semi-concave et* $g : M \mapsto \mathbb{R}$ *une fonction localement semi-convexe tel que* $f \geqslant g$. *Il existe une fonction* $C^{1,1}$, $h : M \mapsto \mathbb{R}$ *telle que* $f \geqslant h \geqslant g$. *De plus,* h *peut être construite de telle sorte qu'elle vérifie* $h(x) = g(x)$ *seulement si* $f(x) = g(x)$.

On a de plus le lemme suivant :

lemme 3. *Soit* u *une sous-solution stricte en dehors de* $\mathcal{A}_u$, *et* v *tel que*

$$u \leqslant v \leqslant T_c^- u + \alpha[0].$$

supposons de plus que $u(x) = v(x)$ *si et seulement si* $x \in \mathcal{A}_u$ *alors* v *et* u *coïncident sur* $\mathcal{A}_u = \mathcal{A}_v$ *et* v *est stricte en dehors de* $\mathcal{A}_v$.

Si u est une sous-solution critique et u' est donnée conformément au théorème 8 il suffit d'appliquer le lemme d'Ilmanen à $T_c^- u'$ et à $T_c^+ T_c^- u'$ pour obtenir le théorème 7.

Dans cet article, on montre également comment on peut utiliser la condition de torsion pour introduire une dynamique partielle des configurations extrémales et minimisantes sur $M \times M$ qui correspond au flot d'Euler-Lagrange dans le cas lagrangien.

Enfin, suivant les idées de [FM07], on étudie des coûts symétriques. On applique ensuite les résultats obtenus au cas d'un coût défini sur le revêtement maximal abélien de M et on définit par la même occasion la fonction α de Mather sur la cohomologie dans ce cadre là.

3.3 KAM faible pour des hamiltoniens en involution

On s'est efforcé dans cet article d'établir des liens entre les objets des théories d'Aubry-Mather et KAM faible associés à deux hamiltoniens H et G

en involution. Ce type d'approche a été utilisé dans [Sor09] afin d'affaiblir les hypothèses d'involution dans le théorème d'Arnol'd-Liouville. Mentionnons aussi le preprint [CL09] dans lequel des résultats très semblables sont énoncés.

On rappelle que deux hamiltoniens Tonelli H et G sont dits en involution si la fonction $\{G, H\} = \Omega(X_G, X_H)$ est identiquement nulle. De la définition de champ de vecteur hamiltoniens, on obtient immédiatement que

$$\Omega(X_G, X_H) = \mathrm{d}\, H \cdot X_G = \mathrm{d}\, G \cdot X_H = 0$$

ce qui montre que G et H commutent si et seulement si G est constant sur les trajectoires du flot hamiltonien de H, si et seulement si H est constant sur les trajectoires du flot hamiltonien de G. On dit aussi que H est une intégrale première de G ou que G est une intégrale première de H. Finalement, on peut démontrer la formule suivante :

$$[X_G, X_H] = X_{\{G,H\}} = 0$$

ce qui montre que si G et H commutent, il en est de même pour leurs champs de vecteurs hamiltoniens.

L'étude des hamiltoniens en involution apparaît en EDP pour l'étude de l'équation d'Hamilton-Jacobi multitemps. Etant donné une fonction $u_0 : M \to \mathbb{R}$ lipschitzienne, on cherche une fonction $u : \mathbb{R}^2 \times M \to \mathbb{R}$ notée $u(s, t, x)$ tel que le système suivant soit vérifié au sens des viscosités :

$$\begin{array}{rcl} \forall x \in M,\ u(0,0,x) & = & u_0(x), \\ \dfrac{\partial u}{\partial s} + G(x, \mathrm{d}_x u) & = & 0, \\ \dfrac{\partial u}{\partial t} + H(x, \mathrm{d}_x u) & = & 0. \end{array}$$

D'après le lien que nous avons déjà souligné entre solutions de l'équation d'Hamilton-Jacobi et semi-groupe de Lax-Oleinik, l'existence d'une telle solution équivaut à la commutation des semi-groupes associés, $(T_G^{-s})_{s>0}$ et $(T_H^{-t})_{t>0}$. A notre connaissance, les premiers résultats dans ce sens se trouvent dans [LR86] où il est démontré que si $M = \mathbb{R}^n$ et si les hamiltoniens strictement convexes ne dépendent que de la seconde variable p alors l'équation multitemps a toujours une solution.

Dans le cas général, on ne peut pas espérer obtenir un résultat aussi fort, il faut imposer en plus que G et H commutent. Ainsi, sous-cette hypothèse et quelques conditions de croissance, Barles et Tourin ([BT01]) montrent l'existence de solutions pour des hamiltoniens Tonelli (ils ne demandent en fait qu'une régularité C^1) en commutation définis sur $\mathbb{R}^n\times\mathbb{R}^n$. Notons que des Hamiltoniens ne dépendant pas de la première variable x sont automatiquement en involution, ainsi le résultat de Barles et Tourin contient celui de Lions et Rochet. On remarquera que, contrairement au premier article, Barles et Tourin n'utilisent que des techniques de solutions de viscosité et jamais la formule explicite du semi-groupe de Lax-Oleinik. Enfin, les hypothèses de croissance de [BT01] sont essentiellement utilisées pour assurer l'unicité d'une éventuelle solution de l'équation multitemps et sont automatiquement vérifiées en cas d'Hamiltoniens périodiques définis sur $\mathbb{T}^n\times\mathbb{R}^n$. Ont ensuite suivi des travaux de Rampazzo et Motta ([MR06]) qui montrent des résultats semblables pour des hamiltoniens lipschitziens et enfin Cardin et Viterbo, dans une tonalité plus symplectique, ont étudié le cas d'hamiltoniens non convexes ([CV08]).

Revenons maintenant au cas d'une variété compacte M et supposons donc dorénavant que G et H sont deux hamiltoniens Tonelli définis sur T^*M. La première partie de notre travail a été de donner une démonstration plus géométrique du théorème de Barles et Tourin en utilisant, dans ce cadre, les formules de Lax-Oleinik.

Nous avons ainsi obtenu le résultat suivant :

Théorème 10 (G. Barles-A. Tourin)**.** *Les semi-groupes de Lax-Oleinik commutent, c'est-à-dire que si $u : M \to \mathbb{R}\cup\{-\infty,+\infty\}$ est une fonction et s,t sont deux nombres réels positifs alors on a*

$$T_G^{-s}T_H^{-t}u = T_H^{-t}T_G^{-s}u.$$

On s'est ensuite penché sur les solutions KAM faible respectives de H et de G. On commence par montrer l'existence d'une solution commune à l'aide d'un théorème de DeMarr sur l'existence de points fixes communs à une famille d'applications contractantes qui commutent dans un espace de Banach :

Théorème 11. *Il existe une fonction $u_0 : M \to \mathbb{R}$ qui est une solution KAM faible à la fois pour G et pour H.*

Ces deux théorèmes sont les ingrédients fondamentaux pour prouver les résultats qui suivent. On a commencé par en déduire que les solutions KAM faibles de G et de H sont en fait les mêmes :

Théorème 12. *Soit $u_- : M \to \mathbb{R}$ une solution KAM faible pour G alors c'est aussi une solution KAM faible pour H. En résumé on a l'égalité suivante :*

$$\mathcal{S}_G^- = \mathcal{S}_H^-.$$

Grâce au résultat ci-dessus, on obtient gratuitement l'égalité de plusieurs objets cruciaux en théorie comme les ensembles d'Aubry :

$$\widetilde{\mathcal{A}}_G = \widetilde{\mathcal{A}}_H,$$

mais aussi les ensembles de Mañé :

$$\widetilde{\mathcal{N}}_G = \widetilde{\mathcal{N}}_H,$$

et les barrières de Peierl :

$$h_G = h_H.$$

Ces deux derniers objets seront introduits plus tard.

La convexité et stricte convexité de la fonction α joue un rôle important dans l'étude de dynamiques hamiltoniennes. On commence par remarquer que si G et H sont en involution et si ω est une 1-forme fermée sur M alors les hamiltoniens

$$H_\omega(x,p) = H(x, p + \omega_x) \quad \text{et} \quad G_\omega(x,p) = G(x, p + \omega_x)$$

commutent encore. Cela a donc un sens de comparer les comportements des fonctions Finalement, on montre le résultat suivant :

Théorème 13. *Soit $C \subset H^1(M, \mathbb{R})$ un ensemble convexe. Si la restriction à C de la fonction α_G associée au hamiltonien G est affine alors il en est de même de la restriction à C de la fonction α_H.*

Remerciements

Albert Fathi, tu m'as tendu la perche,
Que j'ai saisie pour faire de la recherche,
Tu m'as su guider, sans cesse à l'écoute,
De mes rares idées, chassant mes doutes,
Je te remercie

Papa, Maman, tous les Zavidovique,
Vous m'avez inculqué l'esprit critique,
En mathématiques un indispensable,
A jamais je vous en suis redevable,
Je vous remercie

Messieurs Randé, Gianella et Sanchez,
Quand j'étais au fond assis sur ma chaise,
Vous m'avez insufflé avec passion,
L'amour des mots rigueur et réflexion,
Je vous remercie

Tous mes co-bureaux, anciens et présents,
Les autres thésards côté du levant,
Votre labeur ne fut point facilité
Par mon incongru désir de parler
Je vous remercie

Magalie, Virginia, anges gardiennes,
Du laboratoire vous tenez les rennes,
Vous gérez mes déboires sans panique,
L'été venu, nous parlons botanique
Je vous remercie

L'UMPA dans ton ensemble général,
Ton unité rarement mise à mal,
Sait accueillir, protéger sous ses ailes,
Les jeunes recrues, naïves et frêles,
Je te remercie

Ceux qui partagent mes soirées de ouf,
Entre autres Gwen Frero Momo Pignouf,
Avec vous, sans pareil, point de dilemme,
Juste parfois au réveil, une flemme !
Je vous remercie

Toi qui n'est pas remercié nommément,
Je t'apprécie quand même énormément,
Dans ces strophes emplies de maladresses,
Il en est bien une que je t'adresse
Je te remercie

Je finirai par la plus importante,
Par ton soutien tu es toujours présente,
Pour toi je romps la versification,
Comme Eluard, j'écris ici ton nom,
Anne je te remercie.

MERCI A TOUS !

Part I

Strict sub-solutions and Mañe potential in discrete weak KAM theory [1]

1. accepté pour publication dans Commentarii Mathematici Helvetici

Introduction

In the past twenty years, new techniques have been developed in order to study time-periodic or autonomous Lagrangian dynamical systems. Among them, Aubry-Mather theory (for an introduction see [Ban88] for the annulus case and [Mat93], [MF94] for the compact, time periodic case) and Albert Fathi's weak KAM theory (see [Fat08] for the compact case and [FM07] for the non-compact case) have appeared to be very fruitful. More recently, a discretization of weak KAM theory applied to optimal transportation has allowed to obtain deep results of existence of optimal transport maps (see for example [BB07b],[BB07b], [BB07a], [FF07]). A quite similar formalism was also used in the study of time periodic Lagrangians, for example in ([CISM00] or [Mas03]). In this paper, we give analogue results in this discrete setting of those already obtained in the continuous one. In particular, our phase space X will be required to have very little regularity (for example a length space with compact closed balls will do) and no global compactness assumption.

In a first part we introduce the Lax-Oleinik semi-groups T_c^- and T_c^+ and study its sub-solutions. We start with a cost $c : X^2 \to \mathbb{R}$ continuous which verifies:

1. **Uniform super-linearity**: for every $k \geqslant 0$, there exists $C(k) \in \mathbb{R}$ such that
$$\forall (x,y) \in X^2, c(x,y) \geqslant k\,\mathrm{d}(x,y) - C(k);$$
2. **Uniform boundedness**: for every $R \in \mathbb{R}$, there exists $A(R) \in \mathbb{R}$ such that $\mathrm{d}(x,y) \leqslant R \Rightarrow c(x,y) \leqslant A(R)$.

A function u is an α-sub-solution for c if

$$\forall (x,y) \in X^2, u(y) - u(x) \leqslant c(x,y) + \alpha. \tag{1}$$

The critical constant $\alpha[0]$ is the smallest constant α such that there are α-sub-solutions. In the first part we prove, as in [FS04], the existence of critical sub-solutions which are strict on a maximal set:

Theorem 1. *There is a continuous function $u_1 : X \to \mathbb{R}$ which is an $\alpha[0]$-sub-solution such that for every $(x, y) \in X^2$, if there exists an $\alpha[0]$-sub-solution, u such that*

$$u(y) - u(x) < c(x, y) + \alpha[0],$$

then we also have

$$u_1(y) - u_1(x) < c(x, y) + \alpha[0].$$

The proof is done using the Lax-Oleinik semi-groups T_c^- and T_c^+ and the notion of Aubry set as introduced in [BB07b].

The second part is devoted to the study of the continuity of sub-solutions and of an analogue of Mañe's potential. Those two problems are closely related. As a matter of fact, in the Lagrangian continuous case, all critical sub-solutions are equi-Lipschitz maps and the Aubry set may be defined as the set of points $x \in X$ such that any sub-solution is differentiable at x. Moreover, this information is encrypted in the Mañe potential $\phi : X^2 \to \mathbb{R}$. More precisely, Fathi and Siconolfi ([FS04]) proved that a point x is in the projected Aubry set if and only if the function $\phi_x : y \mapsto \phi(x, y)$ is differentiable at x. In the discrete case, we will see that sub-solutions are not necessarily continuous. However, analogously to the continuous case, the projected Aubry set is the set of points where all sub-solutions are continuous. Moreover, our Mañe potential will verify the following:

Theorem 2. *There in a function $\varphi : X^2 \to \mathbb{R}$ which satisfies the following:*

(1) for any $x \in X$, $\varphi(x, x) = 0$;

(2) a function u is a critical sub-solution if and only if

$$\forall (x, y) \in X^2, u(y) - u(x) \leqslant \varphi(x, y);$$

(3) for any $x \in X$, the function $\varphi_x : y \mapsto \varphi(x, y)$ is a critical sub-solution;

(4) a non isolated point $x \in X$ is in the Aubry set if and only if the function $\varphi_x : y \mapsto \varphi(x, y)$ is continuous at x;

(5) if $x \in X$ is non isolated, the function φ_x is continuous at x if and only if it is a negative weak KAM solution, that is a fixed point of $T_c^- + \alpha[0]$.

For the definition of the semi-group T_c^- see section 1.

Acknowledgment

I would like to thank Albert Fathi without whom this article would never have been written. His remarks and comments were of invaluable help. This paper was partially elaborated during a stay at the Sapienza University in Rome. I wish to thank Antonio Siconolfi, Andrea Davini and the Dipartimento di Matematica "Guido Castelnuovo" for their hospitality while I was there. I also would like to thank Explora'doc which partially supported me during this stay. Finally, I would like to thank the ANR KAM faible (Project BLANC07-3_187245, Hamilton-Jacobi and Weak KAM Theory) for its support during my research.

Chapter 1

On critical sub-solutions

In this section we will fix a metric space X which is a B-length space at scale K for some constants B and K (see 3.1 for the exact definition) with compact closed balls and let $c : X \times X \to \mathbb{R}$ be a continuous function which is uniformly super-linear and uniformly bounded that is which verifies condition 1 and 2 of the introduction.

def 1.1. If $\alpha \in \mathbb{R}$ and $u : X \to \mathbb{R}$ is a (not necessarily continuous) function, we will say that u is α-dominated (in short $u \prec c + \alpha$) if

$$\forall (x, y) \in X^2, u(y) - u(x) \leqslant c(x, y) + \alpha.$$

We will denote by $\mathcal{H}(\alpha)$ the set of α-dominated functions.

Following Albert Fathi's weak KAM theory we introduce the Lax-Oleinik semi-groups:

$$T_c^- u(x) = \inf_{y \in X} u(y) + c(y, x);$$

$$T_c^+ u(x) = \sup_{y \in X} u(y) - c(x, y).$$

Theorem 3 (weak KAM)**.** *There is a constant* $\alpha[0]$ *such that the equation* $u = T_c^- u + \alpha[0]$ *(resp.* $u = T_c^+ u - \alpha[0]$*) admits a continuous solution and such that* $\mathcal{H}(\alpha)$ *is empty for* $\alpha < \alpha[0]$*.*

Proof. see the end of the appendix (3). □

We say that a function u is critically dominated or that it is a critical sub-solution if it is $\alpha[0]$-dominated. Finally, we call negative (resp. positive) weak KAM solution a fixed point of the operator $T_c^- + \alpha[0]$ (resp. $T_c^+ - \alpha[0]$). Let us state that weak KAM solutions exist by (3). The following proposition is a direct consequence of the definitions:

proposition 1.1. *A function u is a critical sub-solution if and only if it verifies one of the following properties:*

(i) $\forall (x,y) \in X^2, u(x) - u(y) \leqslant c(y,x) + \alpha[0]$ *(or* $u \prec c + \alpha[0]$*);*

(ii) $u \leqslant T_c^- u + \alpha[0]$*;*

(iii) $u \geqslant T_c^+ u - \alpha[0]$.

The more analytical denomination of sub-solution is useful because it allows to introduce the notion of being strict at some point:

def 1.2. Consider $x_0 \in X$ and $u \prec c + \alpha[0]$ a critical sub-solution. We will say that u is strict at $(x,y) \in X^2$ if and only if

$$u(x) - u(y) < c(y,x) + \alpha[0].$$

We will say that u is strict at $x \in X$ if

$$\forall y \in X, u(y) - u(x) < c(x,y) + \alpha[0] \text{ and } u(x) - u(y) < c(y,x) + \alpha[0].$$

We first give a characterization of continuous strict sub-solutions.

proposition 1.2. *A **continuous** sub-solution u is strict at x if and only if* $u(x) < T_c^- u(x) + \alpha[0]$ *and* $u(x) > T_c^+ u(x) - \alpha[0]$.

Proof. By definition, if u is strict at x then

$$\forall y \in X, u(x) - u(y) < c(y,x) + \alpha[0].$$

In the appendix (3.2 and 17), it is shown that the function $y \mapsto c(y,x) + \alpha[0] - u(y) + u(x)$ tends to $+\infty$ when $\mathrm{d}(x,y)$ tends to $+\infty$. Since closed balls are compact, by continuity of u, the infimum in the definition of T_c^- is achieved. Therefore we must have

$$u(x) < T_c^- u(x) + \alpha[0].$$

Similarly, if for every $y \in X$, $u(y) - u(x) < c(x,y) + \alpha[0]$ then

$$u(x) > \sup_{y \in X} u(y) - c(x,y) - \alpha[0] = T_c^+ u(x) - \alpha[0].$$

The converse is clear. □

Before going any further, let us give some definitions:

def 1.3. Let $u : X \to \mathbb{R}$ verify $u \prec c + \alpha[0]$. We will say that a chain $(x_i)_{0 \leqslant i \leqslant n}$ of points in X is $(u, c, \alpha[0])$-calibrated if

$$u(x_n) = u(x_0) + c(x_0, x_1) + \cdots + c(x_{n-1}, x_n) + n\alpha[0].$$

Notice that a sub-chain formed by consecutive elements of a calibrated chain is again calibrated since $u \prec c + \alpha[0]$.

Following Bernard and Buffoni [BB07b] we will call Aubry set of u, the subset $\widetilde{\mathcal{A}}_u$ of $X^{\mathbb{Z}}$ consisting of the sequences whose finite sub-chains are $(u, c, \alpha[0])$-calibrated. The projected Aubry set of u is

$$\mathcal{A}_u = \{x \in X, \exists (x_n)_{n \in \mathbb{Z}}, (u, c, \alpha[0])\text{-calibrated with } x_0 = x\}.$$

The Aubry set is

$$\widetilde{\mathcal{A}} = \bigcap_{u \prec c + \alpha[0]} \widetilde{\mathcal{A}}_u.$$

The projected Aubry set is

$$\mathcal{A} = \bigcap_{u \prec c + \alpha[0]} \mathcal{A}_u,$$

where in both cases, the intersection is taken over all critically dominated functions.

We begin by a very simple lemma that will be of great use:

lemma 1. *Let $u \prec c + \alpha[0]$ be a critically dominated function and $(x, y) \in X^2$. If the following identity is verified:*

$$u(x) - u(y) = c(y, x) + \alpha[0],$$

then $u(x) = T_c^- u(x) + \alpha[0]$. If the following identity is verified

$$T_c^- u(x) - T_c^- u(y) = c(y, x) + \alpha[0],$$

then $u(y) = T_c^- u(y) + \alpha[0]$ and $T_c^- u(x) = u(y) + c(y, x)$.

Proof. The first part is straightforward from the definitions. For the second point write

$$T_c^- u(x) = T_c^- u(y) + c(y, x) + \alpha[0] \geqslant u(y) + c(y, x) \geqslant T_c^- u(x)$$

therefore, all inequalities must be equalities which proves the lemma. □

The following lemma, along with the fact that the image by the Lax-Oleinik semi-group of a dominated function is continuous (cf. 3.2), show that all the intersections in the definitions of the Aubry sets and projected Aubry sets may be taken on continuous functions.

proposition 1.3. *Let $u \prec c+\alpha[0]$ be a dominated function, then $\widetilde{\mathcal{A}}_u = \widetilde{\mathcal{A}}_{T_c^- u}$. In particular, we also have $\mathcal{A}_u = \mathcal{A}_{T_c^- u}$.*

Proof. First we prove the inclusion $\widetilde{\mathcal{A}}_u \subset \widetilde{\mathcal{A}}_{T_c^- u}$. Let us consider the sequence $(x_n)_{n\in\mathbb{Z}} \in \widetilde{\mathcal{A}}_u$. Since u is dominated and the sequence $(x_n)_{n\in\mathbb{Z}} \in \widetilde{\mathcal{A}}_u$ is $(u, c, \alpha[0])$-calibrated we have for all $k \in \mathbb{Z}$

$$u(x_{k+1}) = u(x_k) + c(x_k, x_{k+1}) + \alpha[0],$$

therefore lemma 1 yields

$$\forall k \in \mathbb{Z}, T_c^- u(x_{k+1}) + \alpha[0] = u(x_{k+1}).$$

Therefore, the sequence $(x_n)_{n\in\mathbb{Z}}$ is $(T_c^- u, c, \alpha[0])$-calibrated and belongs to $\widetilde{\mathcal{A}}_{T_c^- u}$.

We now prove the reverse inclusion $\widetilde{\mathcal{A}}_{T_c^- u} \subset \widetilde{\mathcal{A}}_u$. Let $(x_n)_{n\in\mathbb{Z}} \in \widetilde{\mathcal{A}}_{T_c^- u}$. We have that for any $k \in \mathbb{Z}$

$$T_c^- u(x_{k+1}) = T_c^- u(x_k) + c(x_k, x_{k+1}) + \alpha[0],$$

therefore using the second part of 1

$$\forall k \in \mathbb{Z}, u(x_k) = T_c^- u(x_k) + \alpha[0],$$

and the sequence $(x_n)_{n\in\mathbb{Z}}$ is $(u, c, \alpha[0])$-calibrated. □

Here is a lemma that will be useful in the sequel:

lemma 2. *There is a continuous function $u \prec c + \alpha[0]$ such that $\widetilde{\mathcal{A}}_u = \widetilde{\mathcal{A}}$.*

Proof. Let us consider the set $\mathcal{S} = \{u \in C^0(X, \mathbb{R}), u \prec c + \alpha[0]\}$ of continuous dominated functions . This set is separable for the compact open topology so let $(u_n)_{n\in\mathbb{N}^*}$ be a sequence dense in $\mathcal{S}$. Consider now $(a_n)_{n\in\mathbb{N}^*}$ a sequence of positive real numbers such that $\sum a_n = 1$ and $u = \sum a_n u_n$ converges uniformly on each compact subset of X. To construct such a sequence, one can for example fix an $x_0 \in X$ and for any $n > 1$, take $a_n = \min\{2^{-n}, 1/(2^n \|u_n\|_{\infty,\overline{B(x_0,n)}})\}$ then take $a_1 = 1 - \sum_{n>1} a_n > 0$. The function u is clearly continuous and since u is a convex sum of elements of $\mathcal{S}$, one can easily verify that $u \in \mathcal{S}$. Moreover, since each u_n is dominated, if a chain is $(u, c, \alpha[0])$-calibrated then it is $(u_n, c, \alpha[0])$-calibrated for every $n \in \mathbb{N}^*$. As a matter of fact, if

$$u(x_{n'}) - u(x_n) = \sum_{k\in\mathbb{N}^*} a_k(u_k(x_{n'}) - u_k(x_n)) = (n'-n)\alpha[0] + \sum_{i=n}^{n'-1} c(x_i, x_{i+1}),$$

since for each k the following inequality holds

$$\forall k \in \mathbb{N}^*, u_k(x_{n'}) - u_k(x_n) \leqslant (n'-n)\alpha[0] + \sum_{i=n}^{n'-1} c(x_i, x_{i+1}),$$

and considering that $\sum a_n = 1$ and $a_n > 0$, the inequalities above must be equalities

$$\forall k \in \mathbb{N}^*, u_k(x_{n'}) - u_k(x_n) = (n'-n)\alpha[0] + \sum_{i=n}^{n'-1} c(x_i, x_{i+1}).$$

Finally, since the u_k are dense in $\mathcal{S}$ we obtain

$$\forall u' \in \mathcal{S}, u'(x_{n'}) - u'(x_n) = (n'-n)\alpha[0] + \sum_{i=n}^{n'-1} c(x_i, x_{i+1}).$$

Hence such a calibrated chain is calibrated by every element of $\mathcal{S}$. In particular, for every $u' \in \mathcal{S}$, we have $\widetilde{\mathcal{A}}_u \subset \widetilde{\mathcal{A}}_{u'}$ therefore $\widetilde{\mathcal{A}}_u \subset \widetilde{\mathcal{A}}$. The reverse inclusion follows from the definition of $\widetilde{\mathcal{A}}$. Similarly, projecting on X, we get that $\mathcal{A}_u = \mathcal{A}$. □

As an immediate consequence we get the following:

corollary 1.4. *The following equality holds:*

$$\mathcal{A} = p(\widetilde{\mathcal{A}}),$$

where p denotes the canonical projection from $X^{\mathbb{Z}}$ to X.

The following lemma is useful:

lemma 3. *If $u \prec c + \alpha[0]$ and $x \in X$ then x is in $\mathcal{A}_u$ implies*

$$\forall p \in \mathbb{N}, (T_c^-)^p u(x) + p\alpha[0] = u(x) = (T_c^+)^p u(x) - p\alpha[0].$$

Moreover, if u is continuous then the converse is true, that is if

$$\forall p \in \mathbb{N}, (T_c^-)^p u(x) + p\alpha[0] = u(x) = (T_c^+)^p u(x) - p\alpha[0],$$

then $x \in \mathcal{A}_u$.

Proof. If $(x_n)_{n\in\mathbb{Z}} \in X^{\mathbb{Z}}$ is calibrating for u then for every integer p

$$u(x_p) - u(x_0) = p\alpha[0] + \sum_{i=0}^{p-1} c(x_i, x_{i+1}).$$

Therefore, the domination hypothesis gives us that

$$\forall p \in \mathbb{N}^*, (T_c^-)^p u(x_0) + p\alpha[0] = u(x_0),$$

and

$$\forall p \in \mathbb{N}^*, (T_c^+)^p u(x_0) - p\alpha[0] = u(x_0).$$

Conversely, let us assume that for every $p \in \mathbb{N}$,

$$(T_c^-)^p u(x) + p\alpha[0] = u(x) = (T_c^+)^p u(x) - p\alpha[0].$$

then by successive applications of point (iv) of proposition 3.2 we can find chains $(x_{-p}^p, \dots x_{-1}^p, x_0^p = x, x_1^p, \dots, x_p^p)$ such that

$$\forall p \in \mathbb{N}, (T_c^-)^p u(x) = u(x_{-p}^p) + \sum_{i=-p}^{-1} c(x_i^p, x_{i+1}^p),$$

and

$$\forall p \in \mathbb{N}, (T_c^+)^p u(x) = u(x_p^p) - \sum_{i=0}^{p-1} c(x_i^p, x_{i+1}^p).$$

Using the assumption we made, we obtain that

$$\forall p \in \mathbb{N}, u(x) - u(x_{-p}^p) = \sum_{i=-p}^{-1} c(x_i^p, x_{i+1}^p) + p\alpha[0],$$

and

$$\forall p \in \mathbb{N}, u(x_p^p) - u(x) = \sum_{i=0}^{p-1} c(x_i^p, x_{i+1}^p) + p\alpha[0].$$

Summing these two last equalities we get

$$\forall p \in \mathbb{N}, u(x_p^p) - u(x_{-p}^p) = \sum_{i=-p}^{p-1} c(x_i^p, x_{i+1}^p) + 2p\alpha[0],$$

which proves that the chains $(x_{-p}^p, \dots x_{-1}^p, x_0^p = x, x_1^p, \dots, x_p^p)$ are calibrating for u.

By 17, for every integer $n \in \mathbb{Z}$, the sequence $(x_n^p), p \geqslant |n|$ is bounded hence, by a diagonal extraction ($p_l \to +\infty$ as $l \to +\infty$) we can assume each $(x_n^{p_l}), p_l \geqslant |n|$ converges to a $x_n \in X$. Let us now fix two integers m and n such that $m \leqslant n$. If $p_l \geqslant |m|, |n|$ we have

$$u(x_n^{p_l}) - u(x_m^{p_l}) = \sum_{i=m}^{n-1} c(x_i^{p_l}, x_{i+1}^{p_l}) + (n-m)\alpha[0],$$

letting p_l go to $+\infty$, using the continuity of u, the following holds

$$u(x_n) - u(x_m) = \sum_{i=m}^{n-1} c(x_i, x_{i+1}) + (n-m)\alpha[0].$$

Since m and n were taken arbitrarily, this proves that the sequence $(x_k)_{k\in\mathbb{Z}}$ is calibrating for u and therefore is the bi-infinite chain that we are looking for. □

Let us define yet another Aubry set :

def 1.4. *Let S from $X^{\mathbb{Z}}$ to $X^{\mathbb{Z}}$ be the shift operator. We define*

$$\widehat{\mathcal{A}}_u = \{(x,y) \in X^2, \exists z \in \widetilde{\mathcal{A}}_u, x = p(z) \text{ and } y = p \circ S(z)\},$$

and

$$\widehat{\mathcal{A}} = \{(x,y) \in X^2, \exists z \in \widetilde{\mathcal{A}}, x = p(z) \text{ and } y = p \circ S(z)\}.$$

We are now ready to prove the following theorem, which in particular is stronger than theorem 1. The proof is inspired from the unpublished manuscript [FS03].

Theorem 4. *For every sub-solution u there is a continuous sub-solution u' which is strict at every $(x,y) \in X^2 - \widehat{\mathcal{A}}_u$ and such that $u = u'$ on $\mathcal{A}_u$. There is a continuous sub-solution which is strict at every $(x,y) \in X^2 - \widehat{\mathcal{A}}$.*

Proof. Replacing u by $T_c^- u$ (which does not change the Aubry set by 1.3) we can assume that u is continuous. Consider the function

$$u' = \sum_{n\in\mathbb{N}} a_n (T_c^-)^n u + \sum_{n\in\mathbb{N}^*} b_n (T_c^+)^n u,$$

where the a_n and the b_n are chosen as in the proof of lemma 2, positive, such that the sums above are convergent for the compact open topology and $\sum a_n + \sum b_n = 1$. For the same reasons as in the proof of 2, u' is a continuous and critically dominated function. Let $(x,y) \in X^2$ verify $u'(x) - u'(y) =$

$c(y,x)+\alpha[0]$. Since this equality implies the following ones (cf. the proof of 2) for all integers n

$$\begin{aligned}(T_c^-)^n(u)(x)-(T_c^-)^n u(y)=c(y,x)+\alpha[0],\\ (T_c^+)^n(u)(x)-(T_c^+)^n u(y)=c(y,x)+\alpha[0].\end{aligned}$$

By domination of u, we therefore have for every n

$$\begin{aligned}(T_c^-)^{(n+1)}u(x)+\alpha[0] &= (T_c^-)^n u(y)+c(y,x)+\alpha[0]\\ &= (T_c^-)^n u(x), \end{aligned}\tag{1.1}$$

and

$$\begin{aligned}(T_c^+)^{(n+1)}u(y)-\alpha[0] &= (T_c^+)^n u(x)-c(y,x)-\alpha[0]\\ &= (T_c^+)^n u(y). \end{aligned}\tag{1.2}$$

Using the same argument as in the previous lemma (3), by successive applications of 3.2 we can find chains $(x_{-n}^n,\dots x_{-1}^n=y,x_0^n=x)$ such that

$$\forall n\in\mathbb{N}, (T_c^-)^n u(x)=u(x_{-n}^n)+\sum_{i=-n}^{-1}c(x_i^n,x_{i+1}^n),$$

and chains $(x_{-1}^n=y,x_0^n=x,\dots,x_n^n)$ such that

$$\forall n\in\mathbb{N}, (T_c^+)^n u(x)=u(x_n^n)-\sum_{i=0}^{n-1}c(x_i^n,x_{i+1}^n).$$

Using 1.1 and 1.2, we get that

$$\forall n\in\mathbb{N}, u(x)-u(x_{-n}^n)=\sum_{i=-n}^{-1}c(x_i^n,x_{i+1}^n)+n\alpha[0],$$

and

$$\forall n\in\mathbb{N}, u(x_n^n)-u(x)=\sum_{i=0}^{n-1}c(x_i^n,x_{i+1}^n)+n\alpha[0].$$

Summing these two last equalities we get

$$\forall n\in\mathbb{N}, u(x_n^n)-u(x_{-n}^n)=\sum_{i=-n}^{n-1}c(x_i^n,x_{i+1}^n)+2n\alpha[0],$$

which proves that the chains $(x_{-n}^n,\dots x_{-1}^n=y,x_0^n=x,x_1^n,\dots,x_n^n)$ are calibrating for u.

By 17, for every integer $k \in \mathbb{Z}$, the sequence $(x_k^n), n \geqslant |k|$ is bounded hence, by a diagonal extraction ($n_l \to +\infty$ as $l \to +\infty$) we can assume each $(x_k^{n_l}), n_l \geqslant |k|$ converges to a $x_k \in X$. Let us now fix two integers m and m' such that $m \leqslant m'$. If $n_l \geqslant |m|, |m'|$ we have

$$u(x_{m'}^{n_l}) - u(x_m^{n_l}) = \sum_{i=m}^{m'-1} c(x_i^{n_l}, x_{i+1}^{n_l}) + (m'-m)\alpha[0],$$

letting n go to $+\infty$, using the continuity of u, the following holds

$$u(x_{m'}) - u(x_m) = \sum_{i=m}^{m'-1} c(x_i, x_{i+1}) + (m'-m)\alpha[0].$$

Since m and m' were taken arbitrarily, this proves that the sequence $(x_k)_{k\in\mathbb{Z}}$ is calibrating for u and therefore that $(x, y) \in \widehat{\mathcal{A}}_u$. Therefore, u' is a sub-solution strict at $X^2 - \widehat{\mathcal{A}}_u$. Moreover, by 3 and since $\sum a_n + \sum b_n = 1$, u and u' coincide on $\mathcal{A}$ which finishes to prove the first part of the theorem.

To prove the second part, pick u such that $\widehat{\mathcal{A}_u} = \mathcal{A}$ which is possible according to 2. The function u' is strict outside of $\widehat{\mathcal{A}}$. □

Chapter 2

Towards a discrete analog of Mañe's potential

In the study of globally minimizing curves in Lagrangian dynamics, two functions appear naturally. The first one is used to study infinite orbits of the Euler-Lagrange flow and is Mather's Peierls' barrier which was introduced in the Lagrangian setting in [Mat93]. This barrier was studied in the discrete case in [BB07b]. The other function is Mañe's potential and was introduced in [Mañ97]. As it is proved in [FS04], Mañe's potential gives nice characterizations of the projected Aubry set in terms of differentiability and weak KAM solutions (see Theorems 4.3 and 5.3 in [FS04]). However, in the discrete setting, this notion seems less natural.

In this section, we propose two versions of Mañe's potential. It appears that they are closely related. Moreover, by analogy with Fathi and Siconolfi's results, we characterize the Aubry set in terms of continuity of the potential. In order to stay consistent with the rest of the text, we will only consider the critical case. However, all the results of this section hold in the supercritical case (that is to consider the cost $c + \alpha$, $\alpha > \alpha[0]$). Moreover, in this section, let us stress the fact that X and c only need to satisfy the hypothesis of the beginning of the article being that X is a B-length space at scale K for some constants B and K (see 3.1 for the exact definition) with compact closed balls and c is continuous, super-linear and uniformly bounded (see conditions 1 and 2 in the introduction).

The following construction is inspired from Perron's method to construct viscosity solutions in PDE. It is also reminiscent of ideas of Gabriel Paternain and results obtained in [FS04].

def 2.1. We define the potential

$$\varphi(x, y) = \sup_{u \prec c+\alpha[0]} u(y) - u(x),$$

where the supremum is taken over all critical sub-solutions (not necessarily continuous).

We begin with some properties.

proposition 2.1. *The potential satisfies the following properties:*

1. *For all* $(x, y) \in X^2$ *we have* $\varphi(x, y) \leqslant c(x, y) + \alpha[0]$*. In particular, the potential is everywhere finite.*
2. *For all* $x \in X$*, the potential verifies* $\varphi(x, x) = 0$*.*
3. *A function* u *is critically dominated if and only if for all* (x, y) *in* X^2 *we have* $u(y) - u(x) \leqslant \varphi(x, y)$*.*
4. *The function* φ *verifies the triangular inequality that is for all* x, y, z *in* X *we have* $\varphi(x, y) + \varphi(y, z) \geqslant \varphi(x, z)$*.*

In particular, this proves points (1) and (2) of theorem 2.

Proof. Items 1. and 2. are clear. The third one comes from the fact that for any dominated function u we clearly have that

$$\forall (x, y) \in X^2, u(y) - u(x) \leqslant \varphi(x, y).$$

For the reverse implication, since by the first point of the proposition, we have $\varphi(x, y) \leqslant c(x, y) + \alpha[0]$, any function which satisfies

$$\forall (x, y) \in X^2, u(y) - u(x) \leqslant \varphi(x, y),$$

is necessarily critically dominated. The fourth point is clear from the definition. □

Before going any further, let us state two simple lemmas that we will use throughout this section. The first one helps to understand how to construct sub-solutions:

lemma 4. *Let* $u \prec c + \alpha[0]$ *and let a function* v *that verifies the following inequalities*

$$u \leqslant v \leqslant T_c^- u + \alpha[0].$$

Then v *itself is a critical sub-solution:* $v \prec c + \alpha[0]$*.*

Proof. The proof is merely based on the monotonicity of the Lax-Oleinik semi-group

$$u \leqslant v \leqslant T_c^- u + \alpha[0] \leqslant T_c^- v + \alpha[0],$$

which proves that v is itself critically dominated. □

lemma 5. *Let u be any critical sub-solution and $x \in \mathcal{A}_u$, then u is continuous at x.*

Proof. The following inequalities are true

$$T_c^+ u - \alpha[0] \leqslant u \leqslant T_c^- u + \alpha[0]$$

and are equalities at x. Therefore, the conclusion is a direct consequence of the fact that both $T_c^- u + \alpha[0]$ and $T_c^+ u - \alpha[0]$ are continuous (cf. 3.2). □

The reason why we are interested in this potential is that it generates the greatest possible sub-solutions.

proposition 2.2. *The potential verifies the following properties:*

1. *for all $x \in X$, the function $\varphi_x = \varphi(x, .)$ is a critical sub-solution.*
2. *Let $x \in X$, then for any $y \neq x$ we have*

$$\varphi_x(y) = T_c^- \varphi_x(y) + \alpha[0],$$

   therefore, the function φ_x is lower semi-continuous, and continuous on $X \setminus \{x\}$.
3. *A point $x \in X$ is in the projected Aubry set if and only if the function φ_x is a weak KAM solution.*
4. *If the point $x \in X$ is not isolated, the function φ_x is continuous at x if and only if $x \in \mathcal{A}$.*

In particular, this ends the proof of theorem 2.

Proof. The first part is a direct consequence of part 4 and part 3 of the previous proposition (2.1).

Let us consider the function ψ_x defined as follows:
- $\psi_x(x) = \varphi_x(x) = 0$,
- $\psi_x(y) = T_c^- \varphi_x(y) + \alpha[0]$ if $y \neq x$.

The function ψ_x is lower semi-continuous. As a matter of fact, it is continuous outside of x and at x it verifies

$$\liminf_{y \to x} \psi_x(y) = \liminf_{y \to x} T_c^- \varphi_x(y) + \alpha[0] \geqslant \liminf_{y \to x} \varphi_x(y) \geqslant 0 = \psi_x(x),$$

where the last inequality follows from the existence of a continuous critical sub-solution u which implies

$$\liminf_{y \to x} \varphi_x(y) \geqslant \liminf_{y \to x} u(y) - u(x) = 0.$$

Note that $\varphi_x \leqslant \psi_x \leqslant T_c^- \varphi_x + \alpha[0]$ therefore using the "in between" lemma (4), we obtain at once that the function ψ_x is critically dominated and greater or equal to φ_x by definition. Since, by definition of φ we also have

$$\forall y \in X, \varphi_x(y) \geqslant \psi_x(y) - \psi_x(x) = \psi_x(y),$$

we obtain in fact that $\varphi_x = \psi_x$. In particular, $\varphi_x = T_c^- \varphi_x + \alpha[0]$ on $X \setminus \{x\}$. This finishes the proof of point (2).

To prove 3, note that if $x \in \mathcal{A}$, then for any sub-solution u, the following equality holds by 3

$$T_c^- u(x) + \alpha[0] = u(x).$$

In particular, $\varphi_x(x) = T_c^- \varphi_x(x) + \alpha[0]$, and by the previous point, those functions also coincide on $X \setminus \{x\}$.
To prove the converse, assume $x \notin \mathcal{A}$ and pick a sub-solution u which is strict at x (such a function exists by 4). Without loss of generality, we can assume that $u(x) = 0$. In particular, the following holds

$$T_c^- u(x) + \alpha[0] > u(x) = 0.$$

We already know that

$$\forall y \in X, u(y) \leqslant \varphi_x(y).$$

By the monotonicity of the Lax-Oleinik semi-group, we obtain that

$$\forall y \in X, T_c^- u(y) + \alpha[0] \leqslant T_c^- \varphi_x(y) + \alpha[0].$$

Taking $y = x$, we obtain that

$$\varphi_x(x) = 0 < T_c^- u(x) + \alpha[0] \leqslant T_c^- \varphi_x(y) + \alpha[0].$$

Finally, let us assume $x \in X$ is not isolated. We prove that φ_x is continuous at x if and only if $x \in \mathcal{A}$. Assume first that $x \notin \mathcal{A}$. Pick $u \prec c + \alpha[0]$ such that u is strict at x and that u is continuous and vanishes at x. We can find an open neighborhood V of x and an $\varepsilon > 0$ such that on V, $u + \varepsilon \leqslant T_c^- u + \alpha[0]$ and $|u| \leqslant \frac{\varepsilon}{2}$. Now the function $v = u + \varepsilon \chi_{V \setminus \{x\}}$ verifies $v(x) = 0$. Again it is dominated by 4. Therefore we have that if $y \in V \setminus \{x\}$ (which is not empty because x is not isolated),

$$\varphi_x(y) \geqslant v(y) = u(y) + \varepsilon \geqslant \frac{\varepsilon}{2},$$

which proves that φ_x is not continuous at x. The other implication is clear since we know that any sub-solution is continuous at x as soon as $x \in \mathcal{A}$. $\square$

Part 2 of proposition 2.2 shows that when $x \notin \mathcal{A}$, the function φ_x has a lower jump at x. Here is a property of this "jump". It is a direct consequence of the previous proposition:

lemma 6. *For any $x \in X$, the quantity $F(x) = \sup_{u \prec c+\alpha[0]} T_c^- u(x) + \alpha[0] - u(x)$, where this supremum is taken on the set of all sub-solutions, exists and is equal to $T_c^- \varphi_x(x) + \alpha[0]$.*
Moreover, for any non isolated point x, the function F verifies

$$F(x) = \lim_{\substack{y \to x \\ y \neq x}} \varphi_x(y).$$

Proof. For the first equality, let u be any critically dominated function and let $x \in X$. We already know that

$$\forall y \in X, u(y) - u(x) \leqslant \varphi_x(y).$$

By the monotonicity of the Lax-Oleinik semi-group, we obtain that

$$\forall y \in X, T_c^- u(y) - u(x) + \alpha[0] \leqslant T_c^- \varphi_x(y) + \alpha[0].$$

Taking $y = x$, we obtain that

$$T_c^- u(x) + \alpha[0] - u(x) \leqslant T_c^- \varphi_x(x) + \alpha[0].$$

Therefore, the supremum in the definition of $F(x)$ is reached by the sub-solution φ_x:

$$F(x) = T_c^- \varphi_x(x) + \alpha[0] - \varphi_x(x),$$

since $\varphi_x(x) = 0$.
Now, The continuity of the function $T_c^- \varphi_x + \alpha[0]$ at x together with the equality $\varphi_x = T_c^- \varphi_x + \alpha[0]$ on $X \setminus \{x\}$ imply the second equality. □

Let us now "reverse time" and look what happens when we consider the reversed Lax-Oleinik semi-group:

$$T_c^+ u(x) = \sup_{y \in X} u(y) - c(x, y).$$

This semi group may also be interpreted as a negative Lax-Oleinik semi-group for the symmetric cost $\overline{c}(x, y) = c(y, x)$ by the following relation:

$$T_c^+ u = -T_{\overline{c}}^-(-u).$$

Let us stress the fact that the critical value is unchanged when considering the positive semi-group T_c^+. As a matter of fact, the critical value is the smallest α such that there exists $u \prec c + \alpha$. But $u \prec c + \alpha$ if and only if $-u \prec \overline{c} + \alpha$. Hence the critical values are the same.
Therefore, the same properties, with the same proofs, hold. Let us simply state the results.

lemma 7. *Let $u \prec c+\alpha[0]$ and let v be a function that verifies the following inequalities:*

$$u \geqslant v \geqslant T_c^+ u - \alpha[0],$$

then v itself is a sub-solution: $v \prec c+\alpha[0]$.

proposition 2.3. *The function φ verifies the following properties:*

1. *for all $x \in X$, the function $\varphi^x = -\varphi(.,x)$ is a critical sub-solution.*
2. *Let $x \in X$, then for any $y \neq x$ the function φ^x verifies*

   $$\varphi^x(y) = T_c^+\varphi^x(y) - \alpha[0]$$

   therefore, it is upper semi-continuous, and continuous on $X \setminus \{x\}$.
3. *A point $x \in X$ is in the projected Aubry set if and only if the function φ^x is a positive weak KAM solution.*
4. *If x is not isolated, the function φ^x is continuous at x if and only if $x \in \mathcal{A}$.*

lemma 8. *For any $x \in X$, the quantity $f(x) = \inf_{u \prec c+\alpha[0]} T_c^+u(x) - \alpha[0] - u(x)$ exists and is equal to $T_c^+\varphi^x(x) - \alpha[0]$.*
Moreover, whenever x is not isolated, the function f verifies

$$\forall x \in X, f(x) = \lim_{\substack{y \to x \\ y \neq x}} \varphi^x(y).$$

Until now, we mostly considered general sub-solutions. However, it is much easier to deal with semi-continuous or even continuous functions. We have already noticed that the functions φ_x are lower semi-continuous and therefore that in the definition of φ we can restrict the supremum to lower semi-continuous functions. The following theorem strengthens the result.

Theorem 5. *Let $x \in X$. The function φ_x is a simple limit of continuous critical sub-solutions. Moreover, the limit may be chosen to be uniform outside of any given neighborhood of x.*

Proof. If $x \in \mathcal{A}$, the function φ_x is a weak KAM solution and is therefore continuous. If $x \notin \mathcal{A}$, then $T_c^-\varphi_x(x) + \alpha[0] > 0$. Let $\varepsilon \in]0,1[$ be such that $\varepsilon < T_c^-\varphi_x(x) + \alpha[0]$. We will see in the appendix (3.2 and 17) that any sub-solution has a growth that is at most linear (and which can be bounded independently from the sub-solution) while c is super-linear. Therefore, we can find a real number $1 < R$ such that whenever $y \in B(x,1)$ and $\mathrm{d}(x,z) > R$ then for any critical sub-solution u,

$$u(y) - u(z) < c(z,y) + \alpha[0] - 2(T_c^-\varphi_x(x) + \alpha[0]) \tag{2.1}$$

and

$$u(z) - u(y) < c(y,z) + \alpha[0] - 2(T_c^- \varphi_x(x) + \alpha[0]). \tag{2.2}$$

Using the continuity of c and the compactness of the ball $B(x,R)$, we can find a neighborhood $V \subset B(x,1)$ of x verifying:

– if $y,z,t,u \in V$ then $|c(y,z) - c(t,u)| < \frac{\varepsilon}{2}$,

– if $z \in B(x,R)$ and $y,t \in V$ then

$$|c(z,y) - c(z,t)| < \varepsilon$$

and

$$|c(y,z) - c(t,z)| < \varepsilon,$$

Cutting down V, by continuity of $T_c^- \varphi_x$ we can assume

– if $y \in V \setminus \{x\}$ then $\varphi_x(y) = T_c^- \varphi_x(y) + \alpha[0] > \varepsilon$,

– if $y,t \in V$ then $|T_c^- \varphi_x(y) - T_c^- \varphi_x(t)| < \frac{\varepsilon}{2}$,

Note that from the last condition it follows that for $(y,t) \in V \setminus \{x\}$ we have

$$|\varphi_x(y) - \varphi_x(t)| = |T_c^- \varphi_x(y) - T_c^- \varphi_x(t)| < \frac{\varepsilon}{2}.$$

Let us now consider the function φ_ε defined as follows. Let $\theta : X \to [0,1]$ be a Urysohn function equal to 1 on $X \setminus V$, which vanishes at x and define

$$\forall z \in X, \varphi_\varepsilon(z) = \theta(z)\left(T_c^- \varphi_x(z) - \varepsilon\right) = \theta(z)\left(\varphi_x(z) - \varepsilon\right).$$

The function φ_ε verifies the following properties:

– on $X \setminus V$, $\varphi_\varepsilon(y) = \varphi_x(y) - \varepsilon$,

– on V, φ_ε is non-negative, vanishes at x and verifies

$$\forall y \in V \setminus \{x\}, \varphi_\varepsilon(y) \leqslant \varphi_x(y) - \varepsilon.$$

Now let us check that the function φ_ε is critically dominated. It is enough to separately consider several cases. If both $y,z \notin V$ then

$$\varphi_\varepsilon(y) - \varphi_\varepsilon(z) = \varphi_x(y) - \varphi_x(z) \leqslant c(z,y) + \alpha[0].$$

If $y \in V$ and $z \notin V$, we distinguish between cases. First, let us notice that if $z \notin B(x,R)$ then, since φ_ε is non negative on V, taking into consideration the fact that

$$T_c^- \varphi_x(x) - \varphi_x(y) + \alpha[0] + \frac{\varepsilon}{2} \geqslant 0,$$

which is clear for $y = x$, since $T_c^- \varphi_x(x) + \alpha[0] \geqslant \varphi_x(x) = 0$, and for $y \neq x$, follows from

$$|T_c^- \varphi_x(x) - T_c^- \varphi_x(y)| < \frac{\varepsilon}{2} \text{ and } \varphi_x(y) = T_c^- \varphi_x(y) + \alpha[0],$$

and the fact (using (2.2)) that

$$\varphi_x(z) - \varphi_x(y) \leqslant c(y,z) + \alpha[0] - 2(T_c^- \varphi_x(x) + \alpha[0]),$$

we obtain that

$$\begin{aligned}
\varphi_\varepsilon(z) - \varphi_\varepsilon(y) &\leqslant \varphi_x(z) - \varepsilon \\
&\leqslant \varphi_x(z) - \varepsilon - \varphi_x(y) + T_c^- \varphi_x(x) + \alpha[0] + \frac{\varepsilon}{2} \\
&\leqslant c(y,z) + \alpha[0] - 2(T_c^- \varphi_x(x) + \alpha[0]) + T_c^- \varphi_x(x) + \alpha[0] - \frac{\varepsilon}{2} \\
&< c(y,z) + \alpha[0],
\end{aligned}$$

because $T_c^- \varphi_x(x) + \alpha[0] \geqslant \varphi_x(x) = 0$.
If $z \in B(x,R)$ then using $\varphi_x(x) = 0$ and $\varphi_\varepsilon(y) \geqslant 0$, we obtain

$$\varphi_\varepsilon(z) - \varphi_\varepsilon(y) \leqslant \varphi_x(z) - \varepsilon - \varphi_x(x) \leqslant c(x,z) + \alpha[0] - \varepsilon \leqslant c(y,z) + \alpha[0].$$

In both cases, the following inequalities hold

$$\varphi_\varepsilon(y) - \varphi_\varepsilon(z) \leqslant \varphi_x(y) - \varepsilon - (\varphi_x(z) - \varepsilon) = \varphi_x(y) - \varphi_x(z) \leqslant c(z,y) + \alpha[0].$$

Finally, if $y, z \in V$ then since $\varphi_x(x) = 0$ and $\varphi_\varepsilon(z) \geqslant 0$,

$$\varphi_\varepsilon(y) - \varphi_\varepsilon(z) \leqslant \varphi_x(y) - \varphi_x(x) - \varepsilon \leqslant c(x,y) + \alpha[0] - \varepsilon \leqslant c(z,y) + \alpha[0].$$

□

We now propose another version of a discrete Mañe's potential. We will show that it is very much related to φ. We begin with a definition

def 2.2. Let us define the family of functions, for all $n \in \mathbb{N}^*, (x,y) \in X^2$,

$$c_n(x,y) = \inf_{(x_1,\dots,x_{n-1}) \in X^{n-1}} \{c(x,x_1) + c(x_1,x_2) + \cdots + c(x_{n-1},y)\}.$$

proposition 2.4. *For any $n > 0$, the function c_n is continuous.*

Proof. Let n be a positive integer and let us consider a pair of points $(x^0, y^0) \in X^2$. First, let us notice that for all $(x,y) \in K = \overline{B(x^0,1) \times B(y^0,1)}$, using the uniform boundedness of c (condition 2), the following inequality holds:

$$c_n(x,y) \leqslant (n-1)c(x,x) + c(x,y) \leqslant nA(\mathrm{d}(x^0,y^0) + 2). \tag{2.3}$$

Moreover, using the super-linearity of c (condition 1), for any chain of points $(x_1,\dots,x_{n-1}) \in X^{n-1}$, we have, setting $x_0 = x$ and $x_n = y$:

$$\sum_{i=0}^{n-1} c(x_i,x_{i+1}) \geqslant -nC(1) + \sum_{i=0}^{n-1} \mathrm{d}(x_i,x_{i+1}). \tag{2.4}$$

Finally, if the chain verifies that $\sum_{i=0}^{n-1} c(x_i, x_{i+1}) \leqslant c_n(x,y)+1$, using (2.3) and (2.4), we obtain that

$$\sum_{i=0}^{n-1} \mathrm{d}(x_i, x_{i+1}) \leqslant c_n(x,y) + nC(1) + 1 \leqslant n(A(\mathrm{d}(x^0, y^0) + 2) + C(1)) + 1 = R.$$

In particular,

$$\begin{aligned}\forall i \in [0,n], \mathrm{d}(x_0, x_i) &\leqslant \sum_{j=0}^{i-1} \mathrm{d}(x_j, x_{j+1}) \\ &\leqslant \sum_{j=0}^{n-1} \mathrm{d}(x_j, x_{j+1}) \\ &\leqslant R.\end{aligned}$$

We have just proven that restricted to K, in the definition of c_n, we can take the infimum on chains of points which belong to $B(x,R)^{n-1}$ which is relatively compact. Therefore, by Heine's theorem, the restriction of c_n to K is a finite infimum of equi-continuous functions and is therefore itself continuous.

□

remark 1. In the case where X is compact, one can show that the family of functions $(c_n)_{n\in\mathbb{N}^*}$ is uniformly equi-continuous, however, in the non compact case, it is not clear whether this fact remains true.

Let us now introduce another family of functions:

def 2.3. For any $n \in \mathbb{N}^*$ and $(x,y) \in X^2$ let

$$\varphi_n(x,y) = \inf_{k\geqslant n} c_k(x,y) + k\alpha[0].$$

This quantity is always greater or equal to $\varphi(x,y)$ by the triangular inequality. Moreover, the functions φ_n are clearly increasing with n.

proposition 2.5. *For any $n \in \mathbb{N}^*$, the function φ_n is upper semi-continuous. Moreover, for any x, the function $\varphi_{n,x} = \varphi_n(x,.)$ is critically dominated. Finally, $T_c^- \varphi_{n,x} + \alpha[0] = \varphi_{n+1,x}$.*

Proof. The upper semi-continuity comes from the fact that φ_n is an infimum of continuous functions. The domination of $\varphi_{n,x}$ is consequence of the definitions. In fact, let y, z be in X, then

$$\begin{aligned}\varphi_{n,x}(y) + c(y,z) + \alpha[0] &= \inf_{k\geqslant n} c_k(x,y) + k\alpha[0] + c(y,z) + \alpha[0] \\ &\geqslant \inf_{k\geqslant n+1} c_k(x,z) + k\alpha[0] \\ &= \varphi_{n+1,x}(z) \\ &\geqslant \varphi_{n,x}(z).\end{aligned}$$

To prove the last point, just write that

$$\begin{aligned} T_c^-\varphi_{n,x}(z)+\alpha[0] &= \inf_y \varphi_{n,x}(y)+c(y,z)+\alpha[0] \\ &= \inf_{y\in X}\inf_{k\geqslant n} c_k(x,y)+k\alpha[0]+c(y,z)+\alpha[0] \\ &= \varphi_{n+1,x}(z). \end{aligned}$$

□

We now link both versions of the potential:

proposition 2.6. *On $X^2\setminus\Delta X$, $\varphi=\varphi_1$. Moreover, for any $x\in X$,*

$$\varphi_1(x,x)\geqslant\varphi(x,x)=0.$$

Proof. By definition of $\varphi_{1,x}$, if $u\prec c+\alpha[0]$,

$$\forall y\in X, u(y)-u(x)\leqslant\varphi_{1,x}(y),$$

therefore, $\varphi_x\leqslant\varphi_{1,x}$.
We then notice that $T_c^+\varphi_{1,x}(x)-\alpha[0]\leqslant 0$. As a matter of fact, for any $x_1\in X$ we have

$$\varphi_{1,x}(x_1)-c(x,x_1)-\alpha[0]\leqslant 0,$$

by definition of the function φ_1. Taking the supremum on x_1, we get the result.

Let us define the function ψ by

– $\psi(y)=\varphi_{1,x}(y)$ if $y\neq x$,
– $\psi(x)=0$.

Since $\varphi_{1,x}\geqslant\psi\geqslant T_c^+\varphi_{1,x}-\alpha[0]$ the "in-between" lemma (7) gives that the function ψ is a critical sub-solution. But ψ vanishes at x and is greater than φ_x, therefore $\psi=\varphi_x$. □

As a corollary of the previous proof we also obtain the following:

corollary 2.7. *The following equality holds:*

$$\forall x\in X, T_c^+\varphi_{1,x}(x)-\alpha[0]=0.$$

Proof. Let us fix an $x\in X$. We just saw that $T_c^+\varphi_{1,x}(x)-\alpha[0]\leqslant 0$. Assume now by contradiction that we can find an $\varepsilon>0$ such that

$$T_c^+\varphi_{1,x}(x)-\alpha[0]\leqslant-\varepsilon<0=\varphi_x(x)\leqslant\varphi_{1,x}(x).$$

By analogy with the previous proof, let us define the function ψ by

– $\psi(y) = \varphi_{1,x}(y)$ if $y \neq x$,
– $\psi(x) = -\varepsilon$

Since $\varphi_{1,x} \geqslant \psi \geqslant T_c^+\varphi_{1,x} - \alpha[0]$ the "in between" lemma (7) gives that the function ψ is a critical sub-solution. But if $y \neq x$ then $\psi(y) - \psi(x) > \varphi_x(y)$ which is in contradiction with the definition of φ. □

In the following, we will use this lemma:

lemma 9. *Let $u : X \to \mathbb{R}$ be a function and $n \in \mathbb{N}$, then*

$$(T_c^-)^n(T_c^+)^n u \geqslant u \quad \text{and} \quad (T_c^+)^n(T_c^-)^n u \leqslant u.$$

Moreover, if u is a negative (resp. positive) weak KAM solution then

$$(T_c^-)^n(T_c^+)^n u = u \quad (\text{resp.} \quad (T_c^+)^n(T_c^-)^n u = u).$$

Finally, the operators $T_c^- \circ T_c^+$ and $T_c^- \circ T_c^+$ are idempotent.

Proof. By symmetry, we will only prove one half of the lemma. By definition, for a given $x \in X$ we have

$$T_c^-T_c^+u(x) = \inf_z \sup_y u(y) - c(z,y) + c(z,x),$$

and this quantity is greater than $u(x)$ (take $y = x$). Now the first part of the proposition is obtained by induction or by applying the argument to c_n instead of c.
If u is a negative weak KAM solution, we have that $u \geqslant T_c^+u - \alpha[0]$ (this is always true for a dominated function) and therefore

$$u = T_c^-u + \alpha[0] \geqslant T_c^-T_c^+u.$$

Hence we have in fact an equality. Once again, the general result follows by induction or by using c_n instead of c.

Finally, we have already seen that $(T_c^- \circ T_c^+)^2 \geqslant T_c^- \circ T_c^+$. For the reversed inequality, note that since similarly, $T_c^+ \circ T_c^- \leqslant Id$,

$$T_c^- \circ (T_c^+ \circ T_c^-) \circ T_c^+ \leqslant T_c^- \circ T_c^+.$$

□

proposition 2.8. *Let $x \in X$ be any point, then the following inequality holds: $\varphi_{1,x}(x) \leqslant T_c^-\varphi_x(x) + \alpha[0]$. In particular, the function $\varphi_{1,x}$ is continuous. Moreover, if the point x is not isolated, we have in fact an equality: $\varphi_{1,x}(x) = T_c^-\varphi_x(x) + \alpha[0]$.*

Proof. We have already seen (2.7) that $T_c^+\varphi_{1,x}(x) - \alpha[0] = 0$. Therefore, the following inequality is true:

$$T_c^+\varphi_{1,x} - \alpha[0] \leqslant \varphi_x.$$

As a matter of fact, it is true at x, and at other points y, it is a consequence of the equality $\varphi_{1,x}(y) = \varphi_x(y)$ (2.6) and of the fact that since $\varphi_{1,x}$ is a critical sub-solution, we have $T_c^+\varphi_{1,x} - \alpha[0] \leqslant \varphi_{1,x}$. By the monotonicity of the Lax-Oleinik semi-group, the following holds

$$T_c^- T_c^+ \varphi_{1,x} \leqslant T_c^- \varphi_x + \alpha[0],$$

which by (9) gives us

$$\varphi_{1,x} \leqslant T_c^- T_c^+ \varphi_{1,x} \leqslant T_c^- \varphi_x + \alpha[0].$$

By (2.2) and (2.6) these inequalities are in fact equalities, except possibly at x. Since by (3.2) the function $T_c^-\varphi_x + \alpha[0]$ is continuous it is clear that $\varphi_{1,x}$ is lower semi-continuous and therefore continuous by (2.5).
Finally, the equality $\varphi_{1,x}(x) = T_c^-\varphi_x(x) + \alpha[0]$ whenever x is not isolated is a straight consequence of the continuity of the functions $\varphi_{1,x}$ and $T_c^-\varphi_x + \alpha[0]$ and of the fact that they coincide on $X \setminus \{x\}$. □

Actually, the last equality of the previous proposition (2.8) holds even when x is isolated, as shown below:

proposition 2.9. *For any $x \in X$, the following holds*

$$\forall y \in X, \varphi_{1,x}(y) = T_c^-\varphi_x(y) + \alpha[0].$$

Proof. We have already proven the result when $y \neq x$ and we proved above (2.8) that

$$\varphi_{1,x}(x) \leqslant T_c^-\varphi_x(x) + \alpha[0].$$

Let us prove the reverse inequality. By definition and monotonicity of the Lax-Oleinik semi-group, since $\varphi_{1,x} \geqslant \varphi_x$ the following holds

$$\begin{aligned} \forall x \in X, T_c^-\varphi_x(x) + \alpha[0] &= \inf_{y\in X} \varphi_x(y) + c(y,x) + \alpha[0] \\ &\leqslant \inf_{y\in X} \varphi_{1,x}(y) + c(y,x) + \alpha[0] \\ &= T_c^-\varphi_{1,x}(x) + \alpha[0] = \varphi_{2,x}(x), \end{aligned}$$

where we used the last part of 2.5 for the last equality. Taking $y = x$ in infimum of the Lax-Oleinik we also have

$$T_c^-\varphi_x(x) + \alpha[0] \leqslant c(x,x) + \alpha[0].$$

Since $\varphi_{1,x}(x) = \min(c(x,x) + \alpha[0], \varphi_{2,x}(x))$, this finishes the proof of the proposition. □

Obviously, similar results hold when considering the positive time Lax-Oleinik semi-group T_c^+ therefore, we obtain the following:

proposition 2.10. *For any $n \in \mathbb{N}^*$, and any x, the function $\varphi^{n,x} = -\varphi_n(.,x)$ is critically dominated. Finally, $T_c^+\varphi^{n,x} - \alpha[0] = \varphi^{n+1,x}$.*

lemma 10. *The following equality holds:*

$$\forall x \in X, T_c^-\varphi^{1,x}(x) + \alpha[0] = 0.$$

proposition 2.11. *Let $x \in X$ be any point, then the following equality holds: $\varphi^{1,x}(x) = T_c^+\varphi^x(x) - \alpha[0]$. In particular, the function $\varphi^{1,x}$ is continuous.*

We are now able to prove the following theorem:

Theorem 6. *The family of functions $\varphi_n, n \in \mathbb{N}$ is locally equi-continuous on X^2. In particular, φ_1 is a continuous extension of φ restricted to $X^2 \setminus \Delta X$.*

Proof. We first prove the continuity of φ_1. Let $(x,y) \in X^2$ By (3.2) we know that images of critically dominated functions by the Lax-Oleinik semi-groups are locally equi-continuous. Therefore, let us consider relatively compact neighborhoods V and V' of respectively x and y and let ω be a modulus of continuity for images of critically dominated functions by the Lax-Oleinik semi-groups restricted to V and V'. Let now $(x',y') \in V \times V'$. Using (2.9) and (2.11) we obtain

$$\begin{aligned}
|\varphi_1(x,y) - \varphi_1(x',y')| &\leqslant |\varphi_1(x,y) - \varphi_1(x,y')| + |\varphi_1(x,y') - \varphi_1(x',y')| \\
&\leqslant |T_c^-\varphi_x(y) - T_c^-\varphi_x(y')| + |T_c^+\varphi^{y'}(x) - T_c^+\varphi y'(x')| \\
&\leqslant \omega(\mathrm{d}(y,y')) + \omega(\mathrm{d}(x,x')).
\end{aligned}$$

This proves the continuity of φ_1. Similarly, if $n \geqslant 2$ we have

$$\begin{aligned}
|\varphi_n(x,y) - \varphi_n(x',y')| &\leqslant |\varphi_n(x,y) - \varphi_n(x,y')| + |\varphi_n(x,y') - \varphi_n(x',y')| \\
&\leqslant |T_c^-\varphi_{n-1,x}(y) - T_c^-\varphi_{n-1,x}(y')| \\
&+ |T_c^+\varphi^{n-1,y'}(x) - T_c^+\varphi^{n-1,y'}(x')| \\
&\leqslant \omega(\mathrm{d}(y,y')) + \omega(\mathrm{d}(x,x')).
\end{aligned}$$

This proves the local equi-continuity. □

remark 2. It is clear that whenever a point $x \in X$ is not isolated, the continuous extension of the potential φ is unique at (x,x).

In what follows, we will need this definition:

def 2.4. Let us define the Peierls barrier

$$h(x,y) = \liminf_{n\to+\infty} c_n(x,y) + n\alpha[0] = \lim_{n\to+\infty} \varphi_n(x,y).$$

lemma 11. *The following inequality is verified: $\varphi \leqslant h$.*

Proof. This point comes from the fact that by definition,

$$h(x,y) = \liminf_{n\to+\infty} c_n(x,y) + n\alpha[0]$$

while by the triangular inequality we have

$$\varphi(x,y) \leqslant \inf_{n\to+\infty} c_n(x,y) + n\alpha[0].$$

□

In Mather's original work ([Mat93]) , the projected Aubry set is not defined the way we did, however, we will now prove that our definition is equivalent to the one using the Peierls barrier. Note that the Peierls barrier h takes its values in $\mathbb{R}\cup\{+\infty\}$ and that it is continuous whenever it is finite by equicontinuity of the φ_n (6). Furthermore, since the functions (φ_n) are critically dominated, it follows that family of functions $(\varphi_n)_{n\in\mathbb{N}}$ is equi-Lipschitz in the large (16). Therefore, the Peierls barrier is either finite everywhere or $+\infty$ everywhere. First, let us give some properties of h which are proved in the compact case in [BB07b] and in the continuous case in [FS04]. The proof carries on similarly in the general case with the use of 17:

proposition 2.12. *For each $n, m \in \mathbb{N}$, $x, y, z \in X$, we have*

$$\varphi_{n+m}(x,z) \leqslant \varphi_n(x,y) + c_m(y,z) + m\alpha[0],$$

$$h(x,z) \leqslant h(x,y) + c_m(y,z) + m\alpha[0],$$
$$h(x,z) \leqslant c_m(x,y) + h(y,z) + m\alpha[0].$$

This gives another proof that the function h is either everywhere finite or identically $+\infty$. Moreover, when h is finite, by 6, it is continuous.
For each $l, m, n \in \mathbb{N}$ such that $n \leqslant l+m$, for each $x, y, z \in X$ we have

$$\varphi_n(x,z) \leqslant \varphi_m(x,y) + \varphi_l(y,z),$$

$$h(x,z) \leqslant h(x,y) + \varphi_n(y,z),$$
$$h(x,z) \leqslant h(x,y) + h(y,z).$$

Theorem 7. *If $x \in X$, and the Peierls barrier is finite, let us define the functions $h_x = h(x,.)$ and $h^x = -h(.,x)$. Then h^x, h_x are respectively a positive and a negative weak KAM solution.*

Proof. We only prove the theorem for the functions h_x, the rest is similar. Recall that h_x is the limit of the $\varphi_{n,x}$ and is therefore critically dominated. Moreover, by Dini's theorem, since the sequence of functions $\varphi_{n,x}$ is increasing, its convergence is uniform on compact subsets. Therefore, by the continuity property of T_c^- (3.2) the following holds

$$\begin{aligned} T_c^- h_x + \alpha[0] &= T_c^- \left(\lim_{n\to+\infty} \varphi_{n,x} + \alpha[0] \right) \\ &= \lim_{n\to+\infty} T_c^- \varphi_{n,x} + \alpha[0] \\ &= \lim_{n\to+\infty} \varphi_{n+1,x} + \alpha[0] \\ &= h_x. \end{aligned}$$

□

corollary 2.13. *For each $n \in \mathbb{N}$, $x, y \in X$ we have*

$$h(x,y) = \min_{z\in X} h(x,z) + c_n(z,y) + n\alpha[0] = \min_{z\in X} c_n(x,z) + n\alpha[0] + h(z,y).$$

Proof. It is a straight consequence of (7) and of point (iv) of (3.2). □

We will now prove a characterization of the Aubry set:

Theorem 8. *The projected Aubry set $\mathcal{A}$ coincides with the set*

$$\mathcal{A} = \{x, h(x,x) = 0\}.$$

Before proving 8, we need some results about what happens when h is finite. They are very closely related to results in the compact case.

Theorem 9. *Let $u \prec c + \alpha[0]$, then for all $n, m \in \mathbb{N}$, and for every $x, y \in X$ we have*

$$\forall (x,y) \in X^2, h(x,y) \geqslant (T_c^-)^n u(y) - (T_c^+)^m u(x) + (n+m)\alpha[0].$$

Proof. Let $n, m \in \mathbb{N}$ and let $x_{-n}, \dots, x_m$ verify $x_{-n} = x$ and $x_m = y$. By definition of the Lax-Oleinik semi-group, we have

$$(T_c^-)^m u(y) \leqslant u(x_0) + \sum_{i=0}^{m-1} c(x_i, x_{i+1}),$$

and similarly,

$$(T_c^+)^n u(x) \geqslant u(x_0) - \sum_{i=-n}^{-1} c(x_i, x_{i+1}).$$

Putting these two inequalities together, we find that

$$(T_c^-)^m u(y) - (T_c^+)^n u(x) \leqslant \sum_{i=-n}^{m-1} c(x_i, x_{i+1}).$$

Since the chain between x and y was taken arbitrarily, we obtain

$$(T_c^-)^m u(y) - (T_c^+)^n u(x) \leqslant c_{n+m}(x, y).$$

If $n' > n$, since $u \prec c + \alpha[0]$ we have that

$$(T_c^+)^n u - n\alpha[0] \geqslant (T_c^+)^{n'} u - n'\alpha[0].$$

Therefore the following hold

$$(T_c^-)^m u(y) - (T_c^+)^n u(x) \leqslant (T_c^-)^m u(y) - (T_c^+)^{n'} u(x).$$

Therefore,

$$(T_c^-)^m u(y) - (T_c^+)^n u(x) + (m+n)\alpha[0] \leqslant c_{n'+m}(x, y) + (m+n')\alpha[0].$$

Finally, letting n' go to infinity and taking the liminf, we obtain

$$\begin{aligned}(T_c^-)^m u(y) - (T_c^+)^n u(x) + (m+n)\alpha[0] &\leqslant \liminf_{n' \to +\infty} c_{n'+m}(x, y) + (m+n')\alpha[0] \\ &\leqslant h(x, y).\end{aligned}$$

□

An easy consequence of the previous theorem is that whenever the function h is finite, then if u is a critically dominated function, the sequences $(T_c^-)^n u + n\alpha[0]$ and $(T_c^+)^n u - n\alpha[0]$ are both simply bounded since they are respectively non decreasing and non increasing and therefore converge to respectively u_- and u_+. Moreover, by equi-continuity (3.2), the convergences are uniform on compact subsets. Therefore, by continuity of the semi-groups for the compact open topology (see 3.2), u_- is a negative weak KAM solution and u_+ is a positive weak KAM solution. Let us state a well known and useful lemma (cf. [Con01]):

lemma 12. *Let* $(u_\alpha)_{\alpha\in A}$ *be a family of critically dominated functions. Let*

$$u = \inf_{\alpha\in A} u_\alpha,$$

this function is either identically $-\infty$ *either it is finite everywhere. Moreover if* u *is finite, the following relation holds:*

$$T_c^- \inf_{\alpha\in A} u_\alpha = \inf_{\alpha\in A} T_c^- u_\alpha.$$

If furthermore the u_α *are weak KAM solutions and if the function* u *is not identically* $-\infty$ *then it is a weak KAM solution.*

Proof. The fact that u is either identically $-\infty$ or everywhere finite comes from the fact that the domination hypothesis is stable by taking an infimum, therefore,

$$\forall (x,y) \in X, u(y) \leqslant u(x) + c(x,y) + \alpha[0].$$

Assume now that u is finite. The following holds:

$$\begin{aligned} T_c^- u(x) &= \inf_{y\in X} u(y) + c(y,x) \\ &= \inf_{y\in X} \inf_{\alpha\in A} u_\alpha(y) + c(y,x) \\ &= \inf_{\alpha\in A} \inf_{y\in X} u_\alpha(y) + c(y,x) \\ &= \inf_{\alpha\in A} T_c^- u_\alpha(x). \end{aligned}$$

If moreover the u_α are weak KAM solutions, the following holds:

$$\begin{aligned} T_c^- u(x) + \alpha[0] &= \inf_{\alpha\in A} \inf_{y\in X} u_\alpha(y) + c(y,x) + \alpha[0] \\ &= \inf_{\alpha\in A} T_c^- u_\alpha(x) + \alpha[0] \\ &= \inf_{\alpha\in A} u_\alpha(x) = u(x). \end{aligned}$$

□

As a consequence, still in the case when h is finite, we have the following theorem which first part was already established.

Theorem 10. *Assume* h *is finite. Let* $u \prec c+\alpha[0]$ *be a dominated function, then the sequences* $(T_c^-)^n u + n\alpha[0]$ *and* $(T_c^+)^n u - n\alpha[0]$ *converge respectively to* u_- *and* u_+*, a negative weak KAM solution and a positive weak KAM solution. Moreover, the functions* u_+ *and* u_- *verify the following properties:*

$$u_- = \inf_{w_- \geqslant u} w_-$$

where the infimum is taken over negative weak KAM solutions.

$$u_+ = \sup_{w_+ \leqslant u} w_+$$

where the supremum is taken over positive weak KAM solutions.

Proof. Let us consider the function u' defined by

$$u' = \inf_{w_- \geqslant u} w_-.$$

First notice that the set $\{w_-, w_- \geqslant u\}$ such that w_- is a weak KAM solution is not empty because u_- belongs to it. The previous lemma shows that u' is a negative weak KAM solution. Moreover, we have the following inequality:

$$(T_c^-)^n u + n\alpha[0] \leqslant (T_c^-)^n u' + n\alpha[0] = u'.$$

Since the sequence $(T_c^-)^n u + n\alpha[0]$ converges to the weak KAM solution u_- which is smaller or equal to u', we have in fact $u_- = u'$. The proof for the time positive case is the same. □

We now give a representation formula for the function h:

Theorem 11. *The Peierls barrier satisfies*

$$\forall x, y \in X, h(x,y) = \sup_{\substack{u \prec c+\alpha[0] \\ n,m \in \mathbb{N}}} (T_c^-)^n u(y) - (T_c^+)^m u(x) + (n+m)\alpha[0].$$

Proof. One inequality has been proved in 9, therefore, we only have to find a dominated function which realizes the case of equality. We have already seen (2.7) that

$$T_c^+ \varphi_{1,x}(x) - \alpha[0] = 0. \tag{2.5}$$

Now using the fact that the sequence of functions

$$(T_c^-)^n \varphi_{1,x} + n\alpha[0] = \varphi_{n+1,x}$$

converge to h_x we obtain that

$$\lim_{n\to+\infty} (T_c^-)^n \varphi_{1,x} - T_c^+ \varphi_{1,x}(x) + (n+1)\alpha[0] = h(x,y). \tag{2.6}$$

This ends the proof. □

corollary 2.14. *For all positive integer m we have that*

$$(T_c^+)^m \varphi_{1,x}(x) - m\alpha[0] = 0.$$

For all integer m we have $(T_c^+)^m \varphi_x(x) - m\alpha[0] = 0$. Moreover, the following hold

$$\lim_{n\to+\infty} (T_c^-)^n \varphi_{1,x}(y) - T_c^+ \varphi_{1,x}(x) + (n+1)\alpha[0] = h(x,y),$$

$$\lim_{n\to+\infty} (T_c^-)^n \varphi_x(y) - \varphi_x(x) + n\alpha[0] = h(x,y).$$

Proof. Using 2.5, and the fact that $\varphi_{1,x}$ is a critical sub-solution, we get the following generalization of 2.6:

$$\forall m \in \mathbb{N}^*, \lim_{n\to+\infty} (T_c^-)^n \varphi_{1,x}(y) - (T_c^+)^m \varphi_{1,x}(x) + (m+n+1)\alpha[0] \geqslant h(x,y).$$

Once again, this inequality is in fact an equality (by 9). Now using again the fact that the sequence of functions

$$(T_c^-)^n \varphi_{1,x} + n\alpha[0] = \varphi_{n+1,x}$$

converge to h_x we obtain that $(T_c^+)^m \varphi_{1,x}(x) - m\alpha[0] = 0$.

To prove the second point, notice that by 2.9 and $\varphi_{1,x} \geqslant \varphi_x$ we get that for all $m > 0$ and $n \in \mathbb{N}$,

$$(T_c^-)^n \varphi_x(y) - (T_c^+)^m \varphi_x(x) + \alpha[0] \geqslant \varphi_{n-1,x}(y) - (T_c^+)^m \varphi_{1,x}(x).$$

Therefore we have

$$\lim_{n\to+\infty} (T_c^-)^n \varphi_x(y) - (T_c^+)^m \varphi_x(x) + (m+n+1)\alpha[0] \geqslant h(x,y).$$

By 9, these inequalities are in fact equalities which implies that for all integer m we have $(T_c^+)^m \varphi_x(x) - m\alpha[0] = 0$. □

We are now able to give the proof of 8:

Proof of 8. We know that if u is a critically dominated function and $(x_n)_{n\in\mathbb{Z}}$ is a calibrated sequence for u, then for all $n \in \mathbb{N}$, we have (3)

$$(T_c^-)^n u(x_0) + n\alpha[0] = (T_c^+)^n u(x_0) - n\alpha[0] = u(x_0).$$

Therefore if h is identically $+\infty$, then there are no calibrated bi-infinite chains for the critically dominated function $\varphi_{1,x}$ where x is any point of X (the sequence $(T_c^-)^n \varphi_{1,x}(x_0) + n\alpha[0]$ goes to $+\infty$ and therefore may not be constant) which proves that in this case, $\widetilde{\mathcal{A}} = \varnothing$ and at the same time that $\mathcal{A} = \varnothing$.

When h is finite, by 11 and 3, $h(x,x) = 0$ if and only if for any critically dominated function u, the sequences

$$(T_c^-)^n u(x) + n\alpha[0] \text{ and } (T_c^+)^m u(x) - m\alpha[0]$$

are constantly equal to $u(x)$. Assume now that u is the function given by 2. Applying, 3 we obtain that $x \in \mathcal{A}_u = \mathcal{A}$. □

Let us now point out a phenomenon that is of some resemblance with paired weak KAM solutions in the compact case ([Fat08]).

proposition 2.15. *Assume that h is finite. Let u be a critically dominated function. Let u_- be the limit of the sequence of functions $(T_c^-)^n u + n\alpha[0]$ (it is a negative weak KAM solution). Let u_{-+} be the limit of the sequence of functions $(T_c^+)^n u_- - n\alpha[0]$ which is a positive weak KAM solution. Then, again let u_{-+-} be the limit of the $(T_c^-)^n u_{-+} + n\alpha[0]$ and u_{-+-+} be the limit of the $(T_c^+)^n u_{-+-} - n\alpha[0]$.*
Then, $u_{-+} = u_{-+-+}$.

Proof. We have seen that

$$\begin{aligned} u_- &= \inf_{w_- \geqslant u} w_- \\ u_{-+} &= \sup_{w_+ \leqslant u_-} w_+ \\ u_{-+-} &= \inf_{w_- \geqslant u_{-+}} w_- \\ u_{-+-+} &= \sup_{w_+ \leqslant u_{-+-}} w_+, \end{aligned}$$

where w_- and w_+ denote each time respectively negative and positive weak KAM solutions. Obviously, since $u_{-+} \leqslant u_{-+-}$, by the above formula $u_{-+} \leqslant u_{-+-+}$. We also have $u_- \geqslant u_{-+-}$. Therefore, by monotonicity of the Lax-Oleinik semi-group we obtain $u_{-+} \geqslant u_{-+-+}$ which gives the desired equality. □

remark 3. In other words, the operation which sends a sub-solution u to the weak KAM solution u_{-+} is idempotent. This is comparable to the result we obtained in 9.
The assumption that the Peierls barrier is finite is rather strong in the non compact case. To ensure that the sequence $(T_c^-)^n u + n\alpha[0]$ (resp. $(T_c^+)^n u - n\alpha[0]$) converges, it is enough to suppose that there is a negative (resp. positive) weak KAM solution that is greater (resp. smaller) than u.

We conclude by showing that the function φ may help solving the question of the finiteness of the Peierls barrier h.

proposition 2.16. *The following statements are equivalent:*

1. *the Peierls barrier is finite,*
2. *there is an $(x,y) \in X^2$ such that the sequence $(T_c^-)^n\varphi_x(y) + n\alpha[0]$ is bounded,*
3. *there is an $x \in X$ such that the sequence $(T_c^-)^n\varphi_x + n\alpha[0]$ is point-wise bounded,*
4. *for every $x \in X$, the sequence $(T_c^-)^n\varphi_x + n\alpha[0]$ is point-wise bounded,*
5. *for all u critically dominated, the sequences $((T_c^-)^n u + n\alpha[0])_{n\in\mathbb{N}}$ and $((T_c^+)^n u - n\alpha[0])_{n\in\mathbb{N}}$ converge uniformly on compact sets to respectively a negative weak KAM solution and a positive weak KAM solution.*

Proof. It suffices to notice that by (2.9) we have $\varphi_{1,x} = T_c^-\varphi_x + \alpha[0]$ for all $x \in X$. Hence applying (2.5) we obtain

$$(T_c^-)^n\varphi_x + n\alpha[0] = \varphi_{n-1,x}.$$

Therefore, this sequence of functions converges uniformly on all compact sets to h_x which is either everywhere finite or everywhere $+\infty$. The last point is a direct consequence of 9. □

Chapter 3

Appendix: Existence of weak KAM solutions

What comes in the following section is mostly adapted from [FM07]. Let us consider a metric space X such that its closed balls are compact and, which verifies the following:

def 3.1. Given constants $K \in \mathbb{R}$, $B \geqslant 1$ we will say the metric space X is a B-length space at scale K if for every $(x, y) \in X^2$, there exist $(x = x_0, \dots, x_n = y) \in X^{n+1}$ such that for all $i \leqslant n-1$, $\mathrm{d}(x_i, x_{i+1}) \leqslant K$ and, $\sum_{0 \leqslant i \leqslant n-1} \mathrm{d}(x_i, x_{i+1}) \leqslant B\,\mathrm{d}(x, y)$ where d denotes the distance function.

We start with a simple but fundamental lemma:

lemma 13. *If X is a B-length space at scale K then for every $(x, y) \in X^2$, there exist $(x = x_0, \dots, x_n = y) \in X^{n+1}$ such that for all $i \leqslant n-1$, $\mathrm{d}(x_i, x_{i+1}) \leqslant K$ and, $\sum_{0 \leqslant i \leqslant n-1} \mathrm{d}(x_i, x_{i+1}) \leqslant B\,\mathrm{d}(x, y)$ and*

$$n \leqslant \frac{2B\,\mathrm{d}(x, y)}{K} + 1.$$

Proof. Let us take a chain $(x = x_0, \dots, x_n = y)$ verifying the hypothesis of 3.1 and such that n is minimal. Necessarily,

$$\forall i \leqslant n-2, \mathrm{d}(x_i, x_{i+1}) + \mathrm{d}(x_{i+1}, x_{i+2}) \geqslant K,$$

for otherwise, the same sequence without x_{i+1} would itself verify the hypothesis of 3.1.

Therefore, if we call $m = \lfloor n/2 \rfloor$ then $n \leqslant 2m+1$ and $mK \leqslant B\,\mathrm{d}(x, y)$. □

exemple I.1. 1. A metric compact space C is a 1-length space at scale $\mathrm{diam}(C)$,

2. a length space is a 1-length space at scale K for every $K > 0$,
3. a graph endowed with its graph metric is a 1-length space at scale 1,
4. a grid $G_\varepsilon = \varepsilon\mathbb{Z}^n \subset \mathbb{R}^n$ endowed with the metric induced by the inclusion in $\mathbb{R}^n$ is a $\sqrt{n}$-length space at scale ε,
5. if a metric space, (X, d), whose closed balls are compact is a B-length space at scale K for every $K > 0$ then it is Lipschitz equivalent to a length space,
6. the set $\mathcal{P}$ of prime numbers endowed with the distance $\mathrm{d}(p, p') = |p - p'|$ is not a length space at any scale.

Proof. Items 1, 2,3,4 and 6 are clear. The proof of 5 uses standard ideas in topology and in the study of length spaces (see for example [Gro99], Theorem 1.8). Let $(x, y) \in X^2$ be two distinct points. We want to construct a continuous curve from x to y which metric length is less than $B\,\mathrm{d}(x, y)$. Applying that X is a B-length space at scale $1/n$ we find for any $n \geqslant 1$ a sequence of points $\big(x = x_0^n, \ldots, x_{N_n}^n = y\big) \in X^{N_n+1}$ such that for all $i \leqslant N_n - 1$ we have $\mathrm{d}(x_i^n, x_{i+1}^n) \leqslant 1/n$ and,

$$\sum_{0 \leqslant i \leqslant N_n - 1} \mathrm{d}(x_i^n, x_{i+1}^n) \leqslant B\,\mathrm{d}(x, y). \tag{3.1}$$

Moreover, it is clear that the sequence N_n goes to $+\infty$ and by 13, we can assume that for n large enough, the following holds:

$$\forall n \in \mathbb{N}^*, N_n \leqslant 2nB\,\mathrm{d}(x, y) + 1 \leqslant 3nB\,\mathrm{d}(x, y). \tag{3.2}$$

Clearly, we also have:

$$\forall n \in \mathbb{N}^*, \forall i \leqslant N_n, \mathrm{d}(x, x_i^n) \leqslant B\,\mathrm{d}(x, y). \tag{3.3}$$

We define for any integer n and $i \leqslant N_n$, $f_n(i/N_n) = x_i^n$. For any integer n large enough and any $i, j \leqslant N_n$, the following holds :

$$\mathrm{d}(f_n(i/N_n), f_n(j/N_n)) \leqslant \frac{|j - i|}{n} \leqslant 3B\,\mathrm{d}(x, y)\frac{|j - i|}{N_n}. \tag{3.4}$$

Let $(q_k), k \in \mathbb{N}$ be a dense sequence in $[0, 1]$. For any $k \in \mathbb{N}$ let us choose a sequence $(a_n^k = i_n^k/N_n), n \in \mathbb{N}$ which converges to q_k, where i_n^k is always smaller than N_n. Up to doing a diagonal extraction, using 3.3, we can assume that all the sequences $(f_n(a_n^k), n \in \mathbb{N})$ converge to an element x_k of X. Let us define

$$\forall k \in \mathbb{N}, f(q_k) = x_k.$$

By 3.4, we have for n large enough,

$$\mathrm{d}(f_n(a_n^k), f_n(a_n^{k'})) \leqslant 3B\,\mathrm{d}(x,y)|a_n^k - a_n^{k'}|,$$

therefore, letting n go to $+\infty$ we obtain

$$\forall (k,k') \in \mathbb{N}^2, \mathrm{d}(f(q_k), f(q_{k'})) \leqslant 3B\,\mathrm{d}(x,y)|q_k - q_{k'}|.$$

Since $(q_k)_{k\in\mathbb{N}}$ is dense in $[0,1]$, X is complete and by the previous inequalities f is uniformly continuous (it is in fact Lipschitz), we can extend it to a continuous function, that we will still call f, from $[0,1]$ to X. Finally, by 3.1, the length of f is smaller than $B\,\mathrm{d}(x,y)$.

Let us now denote d_l the distance on X induced by its metric length structure. More precisely, if x, y are two points, $\mathrm{d}_l(x,y)$ is nothing but the infimum of the length of a path joining x to y over all such paths (see [Gro99] (p. 2 and 3) for a more precise definition). By the above construction, the space (X, d_l) is a length space and the application identity from (X, d_l) to (X, d) is B-Lipschitz. Moreover, by definition of d_l, it's inverse from (X, d) to (X, d_l) is 1-Lipschitz. □

A complete, connected Riemannian manifold is a 1-length space at scale K for all $K > 0$ so this property will clearly hold. Assume from now on that X is a B-length space at scale K for some (B, K).

Let $c : X \times X \to \mathbb{R}$ be a continuous function which verifies the conditions of uniform super-linearity (1) and uniform boundedness (2) stated in the introduction. We recall that a function $u : X \to \mathbb{R}$ is an α-sub-solution or that it is dominated by $c + \alpha$ (in short $u \prec c + \alpha$) if for every $(x, y) \in X^2$ we have $u(x) - u(y) \leqslant c(y, x) + \alpha$ (see 1 in the introduction). We will denote by $\mathcal{H}(\alpha)$ the set of such functions.

Finally, let us state the definitions of the well known Lax-Oleinik semi-groups: for a function $u : X \to \overline{\mathbb{R}}$ we define the function

$$T_c^- u : X \to \overline{\mathbb{R}} \text{ by } T_c^- u(x) = \inf_{y\in X} \{u(y) + c(y,x)\},$$

$$T_c^+ u : X \to \overline{\mathbb{R}} \text{ by } T_c^+ u(x) = \sup_{y\in X} \{u(y) - c(x,y)\}.$$

The following lemma is not difficult to check.

lemma 14. *If $k \in \mathbb{R}$ and $u : X \to \overline{\mathbb{R}}$ then $T_c^-(u + k) = k + T_c^- u$ that is, the Lax-Oleinik semi-group commutes with the addition of constants. Moreover, if $v : X \to \overline{\mathbb{R}}$ is another function such that $u \leqslant v$ then $T_c^- u \leqslant T_c^- v$, in other words, the semi-group is monotonous.*

def 3.2. *Let $(k,b) \in \mathbb{R}^2$, we will say that $f : X \to \mathbb{R}$ is (k,b)-Lipschitz in the large or $f \in \mathrm{Lip}_{(k,b)}(X,\mathbb{R})$ if*

$$\forall (x,y) \in X^2, |f(x) - f(y)| \leqslant k\,\mathrm{d}(x,y) + b.$$

exemple I.2. *Bounded functions are Lipschitz in the large.*

Uniformly continuous functions on a length space are Lipschitz in the large.

Although functions Lipschitz in the large are not necessarily continuous, obviously they satisfy the following lemma:

lemma 15. *A function Lipschitz in the large is bounded on any ball of finite radius.*

These functions give a nice setting to apply the Lax-Oleinik semi-groups as shown in the following propositions:

proposition 3.1. *The following properties hold:*

1. *If $k \in \mathbb{R}$ and $u : X \to \mathbb{R}$ then $u \in \mathcal{H}(\alpha)$ if and only if $u + k \in \mathcal{H}(\alpha)$.*
2. *If $u : X \to \mathbb{R}$ is (k,b)-Lipschitz in the large then $u \in \mathcal{H}(C(k) + b)$.*
3. *The subset $\mathcal{H}(\alpha)$ is convex and closed in the space $\mathcal{F}(X,\mathbb{R})$ of finite real valued functions on X endowed with the topology of point-wise convergence.*
4. *If $\alpha \leqslant \alpha'$ then $\mathcal{H}(\alpha) \subset \mathcal{H}(\alpha')$.*
5. *If $\mathcal{H}(\alpha) \neq \varnothing$ then $\alpha \geqslant \sup\{-c(x,x), x \in X\} \geqslant -A(0)$.*

Proof. Statements (1) and (4) are direct consequences of the definitions. If $u \in \mathrm{Lip}_{(k,b)}(X,\mathbb{R})$ then

$$\forall (x,y) \in X^2, u(x) - u(y) \leqslant k\,\mathrm{d}(x,y) + b \leqslant c(y,x) + C(k) + b,$$

which proves statement (2).

To prove statement (3), just notice that $\mathcal{H}(\alpha)$ is an intersection of closed half spaces for the given topology, one for each couple of points of X.

As for statement (5), observe that if $u \in \mathcal{H}(\alpha)$ and $x \in X$ then

$$0 = u(x) - u(x) \leqslant c(x,x) + \alpha,$$

which implies (5). □

In the following, we will need this lemma:

lemma 16. *Let $\alpha \in \mathbb{R}$, then there exists constants $k(\alpha)$ and $b(\alpha)$ such that for any u which is α-dominated, then u is Lipschitz in the large with constants $k(\alpha)$ and $b(\alpha)$.*

Proof. Let us consider $u \in \mathcal{H}(\alpha)$ and $x_0 \in X$. Then one has

$$\forall y \in X, u(x_0) - u(y) \leqslant c(y, x_0) + \alpha \leqslant A(\mathrm{d}(y, x_0)) + \alpha$$

where we have used first the domination of u and then the uniform boundedness of c. Moreover, using the assumption we made on the metric d and 13, the constants K, B satisfy that for any $y \in X$,

$$u(x_0) - u(y) \leqslant (A(K) + \alpha)\left(\frac{2B\,\mathrm{d}(x_0, y)}{K} + 1\right)$$

which proves that $u \in \mathrm{Lip}_{2(A(K)+\alpha)B/K, A(K)+\alpha}(X, \mathbb{R})$. □

proposition 3.2. *The following properties are verified:*

(i) Let $u : X \to \mathbb{R}$ be a function. We have $u \prec c + \alpha$ if and only if $u \leqslant T_c^- u + \alpha$.

(ii) The following holds:

$$T_c^- \left(\mathrm{Lip}_{(k,b)}(X, \mathbb{R})\right) \subset \mathcal{H}(C(k) + b) \cap C^0(M, \mathbb{R}).$$

Moreover, the set of functions $T_c^- \left(\mathrm{Lip}_{(k,b)}(X, \mathbb{R})\right)$ is locally equi-continuous. Finally the mapping T_c^- restricted to $\mathrm{Lip}_{(k,b)}(X, \mathbb{R})$ is continuous for the topology of uniform convergence on compact subsets.

(iii) The map T_c^- sends $\mathcal{H}(\alpha)$ into $\mathcal{H}(\alpha) \cap C^0(M, \mathbb{R})$ and is continuous for the topology of uniform convergence on compact subsets. Moreover, the set of functions $T_c^- (\mathcal{H}(\alpha))$ is locally equi-continuous.

(iv) If $u \in \mathrm{Lip}_{(k,b)}(X, \mathbb{R})$ is lower semi-continuous, then for every $x \in X$, there is a $y \in X$ such that $T_c^- u(x) = u(y) + c(y, x)$.

Proof. To prove (i), remark that domination of u by $c + \alpha$ is equivalent to

$$\forall (x, y) \in X^2, u(x) \leqslant u(y) + c(y, x) + \alpha,$$

which is equivalent to

$$\forall x \in X, u(x) \leqslant \inf_{y \in X} u(y) + c(y, x) + \alpha,$$

but the right hand side is precisely $T_c^- u(x) + \alpha$.
Let us prove (ii). Let $u \in \mathrm{Lip}_{(k,b)}(X, \mathbb{R})$ and let $x_0 \in X$ and $r > 0$. We know that

$$\forall y \in X, \forall x \in B(x_0, r), u(y) + c(y, x) \geqslant c(y, x) + u(x_0) - k\,\mathrm{d}(x_0, y) - b,$$

therefore, using the super-linearity of c we get that

$$\begin{aligned} u(y)+c(y,x) &\geqslant 2k\,\mathrm{d}(x,y)+C(2k)+u(x_0)-k\,\mathrm{d}(x_0,y)-b \\ &\geqslant k\,\mathrm{d}(x_0,y)-2kr+C(2k)+u(x_0)-b. \end{aligned} \tag{3.5}$$

Now, by definition of the Lax-Oleinik semi-group,

$$T_c^-u(x)=\inf_{y\in X} u(y)+c(y,x)\leqslant u(x_0)+c(x_0,x)\leqslant u(x_0)+A(r),$$

so by condition (3.5) it is not restrictive to take the infimum on points at a distance less than $D(r,k,b)=(A(r)+2kr-C(2k)+b)/k$ from x_0. Using that u (by lemma 15) and c (by continuity) are bounded below on balls of finite radius (which are compact), the infimum in the Lax-Oleinik semi-group is finite and if reached, can only be reached in $\overline{B}(x_0,D(r,k,b))$. Note that this already proves (iv) because a lower semi-continuous function achieves its minimum on a compact set. The constant $D(r,k,b)$ is independent of $x\in B(x_0,r)$ and $u\in \mathrm{Lip}_{(k,b)}(X,\mathbb{R})$. Therefore if $x_1\in\overline{B}(x_0,r)$ then in the definition of $T_c^-u(x_1)$ the infimum may be taken on points which lie in $\overline{B}(x_0,D(r,k,b))$. Since $\overline{B}(x_0,D(r,k,b))\times\overline{B}(x_0,r)$ is compact, the restriction of c to this domain is uniformly continuous, let ω be a modulus of continuity of c on that domain. One has that the restriction of T_c^-u to $\overline{B}(x_0,r)$ is a finite infimum of equi-continuous functions and is therefore itself continuous with same modulus of continuity which only depends on c, so the family $T_c^-(\mathrm{Lip}_{(k,b)}(X,\mathbb{R}))$ is in fact locally equi-continuous.
Now that we know it is finite, let us check that T_c^-u is $(C(k),+b)$-dominated. This is in fact a direct consequence of the monotonicity of the Lax-Oleinik semi-group (14). In fact, by (i), since $u\prec c+C(k)+b$ it follows that $u\leqslant T_c^-u+C(k)+b$. We therefore have that $T_c^-u\leqslant T_c^-(T_c^-u)+C(k)+b$ which proves that $T_c^-u\prec c+C(k)+b$.
It remains to prove that the restriction of this mapping to $\mathrm{Lip}_{(k,b)}(X,\mathbb{R})$ is continuous for the topology of uniform convergence on compact subsets. Let $v\in\mathrm{Lip}_{(k,b)}(X,\mathbb{R})$ be another dominated function, $x\in X$. Let $\varepsilon>0$ and $x_1\in X$ be such that

$$|T_c^-u(x)-u(x_1)-c(x_1,x)|<\varepsilon,$$

and similarly, chose x_2 such that

$$|T_c^-v(x)-v(x_2)-c(x_2,x)|<\varepsilon.$$

Note that both x_1 and x_2 are necessarily in $\overline{B}(x,D(0,k,b))$. The following inequality holds:

$$\begin{aligned} T_c^-v(x)-T_c^-u(x) &\leqslant v(x_1)+c(x_1,x)-u(x_1)-c(x_1,x)+\varepsilon \\ &\leqslant \sup_{\overline{B}(x,D(0,k,b))}|u-v|+\varepsilon. \end{aligned}$$

By a symmetrical argument, we also have

$$T_c^- u(x) - T_c^- v(x) \leqslant u(x_2) + c(x_2, x) - v(x_2) - c(x_2, x) \leqslant \sup_{\overline{B}(x, D(0,k,b))} |u - v| + \varepsilon.$$

This being true for all $\varepsilon > 0$, we have just proved that if $A \subset X$ is compact, then

$$\sup_A |T_c^- u - T_c^- v| \leqslant \sup_{A_{D(0,k,b)}} |u - v|,$$

where $A_{D(0,k,b)} = \{x \in X, \mathrm{d}(A, x) \leqslant D(0,k,b)\}$ is still compact because it is contained in a ball of finite large radius. This achieves the proof of (ii).
To prove (iii), note that by lemma 16, dominated functions are equi-Lipschitz in the large. Therefore the family of functions in $T_c^-(\mathcal{H}(\alpha))$ is locally equi-continuous. □

As an immediate consequence of the previous proof we deduce the following result:

lemma 17 (a priori compactness)**.** *Given constants k, b, $\varepsilon > 0$ and a compact set $A \subset X$ there is a compact set $A' \subset X$ such that if $v \in \mathrm{Lip}_{(k,b)}(X, \mathbb{R})$, $x \in A$ then*

$$|u(y) + c(y, x) - T_c^- u(x)| \leqslant \varepsilon \Longrightarrow y \in A'.$$

We now can prove the weak KAM theorem:

Proof of theorem 3. First, notice that saying that $\mathcal{H}(\alpha)$ is empty is equivalent to saying that $\mathcal{H}(\alpha) \cap C^0(M, \mathbb{R})$ is empty, because of part (iii) of the previous proposition (3.2). Let $\mathbb{1}$ be the constant function equal to 1 on X and let $\widehat{C^0(X, \mathbb{R})}$ be the quotient of $C^0(X, \mathbb{R})$ by the subspace of constant functions $\mathbb{R}\mathbb{1}$ and let q be the projection operator. Since the semi-group T_c^- commutes with the addition of constants, it induces a semi group on $\widehat{C^0(X, \mathbb{R})}$ that we will denote $\widehat{T}_c^-$. The topology on $\widehat{C^0(X, \mathbb{R})}$ is the quotient of the compact open topology on $C^0(X, \mathbb{R})$, which makes it a locally convex vector space.
We will call $\widehat{\mathcal{H}}(\alpha)$ the image $q(\mathcal{H}(\alpha) \cap C^0(M, \mathbb{R}))$. It is convex because so is $\mathcal{H}(\alpha) \cap C^0(M, \mathbb{R})$. Let us introduce the subset $C^0_{x_0}$ of $C^0(X, \mathbb{R})$ consisting of the functions which vanish at x_0, where x_0 is any point of X. Then, q induces a homomorphism of $C^0_{x_0}$ onto $\widehat{C^0(X, \mathbb{R})}$. Since $\mathcal{H}(\alpha) \cap C^0(M, \mathbb{R})$ is stable by addition of constants, $\widehat{\mathcal{H}}(\alpha)$ is also the image by q of the set $\mathcal{H}(\alpha) \cap C^0_{x_0} = \mathcal{H}_{x_0}(\alpha)$. Now, $\mathcal{H}_{x_0}(\alpha)$ is closed for the compact open topology, it consists of functions which all vanish at x_0. We have seen in the proof of 3.2 that $T_c^-(\mathcal{H}(\alpha))$ is a family of locally equi-continuous and equi-Lipschitz in

the large, therefore locally equi-bounded functions. By the Ascoli theorem, we deduce that $T_c^-(\mathcal{H}_{x_0}(\alpha))$ is relatively compact. Furthermore, since

$$\widehat{T}_c^-(q(u)) = q(T_c^- u - T_c^- u(x_0)\mathbb{1}),$$

we obtain that

$$\widehat{T}_c^-(\widehat{\mathcal{H}}(\alpha)) = q(T_c^-(\mathcal{H}_{x_0}(\alpha)))$$

is also relatively compact and the closed convex envelope of $\widehat{T}_c^-(\widehat{\mathcal{H}}(\alpha))$ that we will denote $H(\alpha)$ is compact. Note also that $H(\alpha) \subset \widehat{\mathcal{H}}(\alpha)$, since $\widehat{\mathcal{H}}(\alpha)$ is convex, closed for the compact open topology and it contains $T_c^-(\widehat{\mathcal{H}}(\alpha))$. As a first consequence, if

$$\alpha[0] = \inf\{\alpha \in \mathbb{R}, \mathcal{H}(\alpha) \neq \varnothing\},$$

then $\bigcap_{\alpha > \alpha[0]} H(\alpha) \neq \varnothing$ as the intersection of a decreasing family of compact nonempty sets. It follows that $\mathcal{H}(\alpha[0])$ is non empty for it contains $q^{-1}\left(\bigcap_{\alpha>\alpha[0]} H(\alpha)\right)$.
Finally, it is obvious that $\widehat{T}_c^-$ carries $H(\alpha)$ into itself. Since this last subset is a nonempty convex compact subset of a locally convex topological vector space, we can apply the Schauder-Tykhonov theorem ([Dug66] p.414, Theorem 2.2). This gives that $\widehat{T}_c^-$ has a fixed point in $H(\alpha)$ as soon as $\mathcal{H}(\alpha) \neq \varnothing$ that is for all values $\alpha \geqslant \alpha[0]$.
If we call $q(u)$ such a fixed point, with $u \in \mathcal{H}(\alpha[0])$, we see there is a constant α' such that $T_c^- u = u + \alpha'$. Obviously, $u \prec c - \alpha'$ so $-\alpha' \geqslant \alpha[0]$. Moreover since $u \in \mathcal{H}(\alpha[0])$ we must have $u \leqslant T_c^- u + \alpha[0]$ which gives $u = T_c^- u - \alpha' \leqslant T_c^- u + \alpha[0]$ and $-\alpha' \leqslant \alpha[0]$. We therefore conclude that $-\alpha' = \alpha[0]$. □

Part II

Existence of $C^{1,1}$ critical subsolutions in discrete weak KAM theory [1]

1. accepté pour publication dans Journal of Modern Dynamics

Introduction

In the past twenty years, new techniques have been developed in order to study time-periodic or autonomous Lagrangian dynamical systems. Among them, Aubry-Mather theory (for an introduction see [Ban88] for the annulus case and [Mat93], [MF94] for the compact, time periodic case) and Albert Fathi's weak KAM theory (see [Fat08] for the compact case and [FM07] for the non-compact case) have appeared to be very fruitful. More recently, a discretization of weak KAM theory applied to optimal transportation has allowed to obtain deep results of existence of optimal transport maps (see for example [BB07c],[BB07b], [BB07a],[FF07]). A quite similar formalism was also used in the study of time periodic Lagrangians, for example in ([CISM00] or [Mas07]).
In [Zav08], our goal was to study critical subsolutions and their discontinuities in a broad setting. Here, following [FS04], [Ber07] and [FFR09] we will study the existence of more regular strict subsolutions. More precisely, we start with a connected C^∞ complete Riemmanian manifold M endowed with the distance $\mathrm{d}(.,.)$ coming from the Riemmanian metric. Let $c : M \times M \to \mathbb{R}$ be a locally semi-concave cost function (in other terms, in small enough charts, c is the sum of a smooth and a concave function) which verifies:

1. **Uniform super-linearity**: for every $k \geqslant 0$, there exists $C(k) \in \mathbb{R}$ such that
$$\forall (x,y) \in M \times M,\ c(x,y) \geqslant k\,\mathrm{d}(x,y) - C(k);$$
2. **Uniform boundedness**: for every $R \in \mathbb{R}$, there exists $A(R) \in \mathbb{R}$ such that
$$\forall (x,y) \in M \times M,\ \mathrm{d}(x,y) \leqslant R \Rightarrow c(x,y) \leqslant A(R).$$

A function u is an α-subsolution for c if
$$\forall (x,y) \in M \times M,\ u(y) - u(x) \leqslant c(x,y) + \alpha. \tag{6}$$
The critical constant $\alpha[0]$ is the smallest constant α such that there exist α-subsolutions (see [Zav08]). We will moreover suppose that c verifies left

and right twist conditions (defined in section 2).
Under these hypothesis, we prove the following theorem:

Theorem 12. *There is a $C^{1,1}$ function $u_1 : M \to \mathbb{R}$ which is an $\alpha[0]$-subsolution such that for every $(x, y) \in M \times M$ and for every $\alpha[0]$-subsolution u, the following implication holds:*

$$u(y) - u(x) < c(x, y) + \alpha[0] \quad \Longrightarrow \quad u_1(y) - u_1(x) < c(x, y) + \alpha[0].$$

The proof is done, as in [Ber07], using back and forth the Lax-Oleinik semi-groups as in Lasry-Lions regularization, combined with a version of Ilmanen's insertion lemma (proved in [Ber09a, FZ09]). Let us mention that the same example as the one given in [Ber07] shows that in general, this is the best regularity one can expect.

This paper is organized as follows:

- the first two sections, 1 and 2, are devoted to recalling some results proved in [Zav08] and to introducing the notion of twist condition,
- in the third section, 3, we study the particular case of cost coming from Tonelli Lagrangians and we prove that they fit into our framework,
- in section 4 we prove the main theorem (12),
- finally in section 5 we study, following ideas of [FM07] the case of invariant cost functions and we apply this study in section 6 to symmetries coming from deck transformations of a cover. Finally, following ideas of Mather ([Mat91]), we introduce Mather's α function on the cohomology.

Acknowledgment

I first would like to thank Pierre Cardaliaguet for pointing out to me that the proof of 12 could be done using Ilmanen's lemma. I would like to thank Albert Fathi for his careful reading of the manuscript and for his comments and remarks during my research on this subject. I am particularly indebted to him regarding to sections 5 and 6 which were written after very inspiring conversations. This paper was partially elaborated during a stay at the Sapienza University in Rome. I wish to thank Antonio Siconolfi, Andrea Davini and the Dipartimento di Matematica "Guido Castelnuovo" for its hospitality while I was there. I also would like to thank Explora'doc which partially supported me during this stay. Finally, I would like to thank the ANR KAM faible (Project BLANC07-3_187245, Hamilton-Jacobi and Weak KAM Theory) for its support during my research.

First, let us recall the setting and some results proved in [Zav08].

Chapter 1

Known results

In this section we quickly survey some previously obtained results, see [Zav08]. Throughout this paper, we will assume M is a connected C^∞ complete Riemmanian manifold endowed with the distance $\mathrm{d}(.,.)$ coming from the Riemmanian metric. We will consider a cost function $c : M \times M \to \mathbb{R}$ verifying the properties 1 and 2 mentioned in the introduction. We will denote $\alpha[0]$, the Mañé critical value as defined for example in [Zav08]. We say that a function $u : M \to \mathbb{R}$ is critically dominated or that it is a critical subsolution if it is $\alpha[0]$-dominated that is if

$$\forall (x,y) \in M \times M,\ u(y) - u(x) \leqslant c(x,y) + \alpha[0].$$

Let us mention that $\alpha[0]$ is defined as being the smallest value such that there are subsolutions. More precisely, if $C \in \mathbb{R}$, we let $\mathcal{H}(C) \subset \mathbb{R}^M$ be the set of C-dominated functions, that is the set of u verifying

$$\forall (x,y) \in M \times M,\ u(y) - u(x) \leqslant c(x,y) + C.$$

Then the Mañé critical value is

$$\inf\{C \in \mathbb{R},\ \mathcal{H}(C) \neq \varnothing\}.$$

As is customary, we introduce the discrete Lax-Oleinik semi-groups:

$$T_c^- u(x) = \inf_{y \in M} u(y) + c(y,x),$$

$$T_c^+ u(x) = \sup_{y \in M} u(y) - c(x,y).$$

Finally, we call negative (resp. positive) weak KAM solution a fixed point of the operator $T_c^- + \alpha[0]$ (resp. $T_c^+ - \alpha[0]$).

proposition 1.1. *A function $u : M \mapsto \mathbb{R}$ is a critical subsolution (written $u \prec c+\alpha[0]$) if and only if it verifies one of the following equivalent properties:*

(i) $\forall (x,y) \in M \times M,\ u(x) - u(y) \leqslant c(y,x) + \alpha[0]$*;*

(ii) $u \leqslant T_c^- u + \alpha[0]$*;*

(iii) $u \geqslant T_c^+ u - \alpha[0]$*.*

def 1.1. Consider $u : M \to \mathbb{R}$ a critical subsolution ($u \prec c + \alpha[0]$). We will say that u is strict at $(x,y) \in M \times M$ if and only if

$$u(x) - u(y) < c(y,x) + \alpha[0].$$

We will say that u is strict at $x \in M$ if

$$\forall y \in M,\ u(y) - u(x) < c(x,y) + \alpha[0] \ \text{ and } \ u(x) - u(y) < c(y,x) + \alpha[0].$$

We recall a characterization of strict continuous subsolutions (see [Zav08]).

proposition 1.2. *The* ***continuous*** *critical subsolution u is strict at x if and only if $u(x) < T_c^- u(x) + \alpha[0]$ and $u(x) > T_c^+ u(x) - \alpha[0]$.*

def 1.2. Let u from M to $\mathbb{R}$ verify $u \prec c + \alpha[0]$. We will say that a chain $(x_i)_{0 \leqslant i \leqslant n}$ is $(u,c,\alpha[0])$-calibrated if

$$u(x_n) = u(x_0) + c(x_0,x_1) + \cdots + c(x_{n-1},x_n) + n\alpha[0].$$

Notice that a sub-chain of a calibrated chain formed by consecutive elements is again calibrated.

Following Bernard and Buffoni [BB07c] we will call Aubry set of u, $\widetilde{\mathcal{A}}_u$ the subset of $M^{\mathbb{Z}}$ consisting of the sequences whose finite sub-chains are $(u,c,\alpha[0])$-calibrated. We set

$$\widehat{\mathcal{A}}_u = \{(x,y) \in M \times M,\ \exists (x_n)_{n\in\mathbb{Z}} \in \widetilde{\mathcal{A}}_u \ \text{ with } \ x_0 = x \text{ and } x_1 = y\},$$

and we define the projected Aubry set of u by

$$\mathcal{A}_u = \{x \in M,\ \exists (x_n)_{n\in\mathbb{Z}},\ (u,c,\alpha[0])\text{-calibrated with } x_0 = x\}.$$

We then define the Aubry set:

$$\widetilde{\mathcal{A}} = \bigcap_{u \prec c+\alpha[0]} \widetilde{\mathcal{A}}_u,$$

the projected Aubry sets

$$\widehat{\mathcal{A}} = \{(x,y) \in M \times M, \exists (x_n)_{n\in\mathbb{Z}} \in \widetilde{\mathcal{A}}, x = x_0 \text{ and } y = x_1\},$$

and

$$\mathcal{A} = \bigcap_{u \prec c+\alpha[0]} \mathcal{A}_u$$

where in all cases, the intersection is taken over all critically dominated functions.

We recall some further facts obtained in [Zav08]:

lemma 18. *Let $u \prec c + \alpha[0]$ be a dominated function and $(x, y) \in M \times M$. If the following identity is verified:*

$$u(x) - u(y) = c(y, x) + \alpha[0]$$

then $u(x) = T_c^- u(x) + \alpha[0]$.
If the following identity is verified:

$$T_c^- u(x) - T_c^- u(y) = c(y, x) + \alpha[0]$$

then $u(y) = T_c^- u(y) + \alpha[0]$ and $T_c^- u(x) = u(y) + c(y, x)$.

proposition 1.3. *Let $u \prec c+\alpha[0]$ be a dominated function, then $\widetilde{\mathcal{A}}_u = \widetilde{\mathcal{A}}_{T_c^- u}$. In particular, the following equalities hold: $\widehat{\mathcal{A}}_u = \widehat{\mathcal{A}}_{T_c^- u}$ and $\mathcal{A}_u = \mathcal{A}_{T_c^- u}$.*

Theorem 13. *Let $u \prec c + \alpha[0]$ be a critically dominated function. There is a continuous subsolution u' which is strict at every $(x, y) \in M \times M - \widehat{\mathcal{A}}_u$ and which is equal to u on $\mathcal{A}_u$. In particular, we have*

$$\widehat{\mathcal{A}}_u = \widehat{\mathcal{A}}_{u'}.$$

There is a continuous subsolution u_0 which is strict at every $(x, y) \in M \times M - \widehat{\mathcal{A}}$. In particular

$$\widehat{\mathcal{A}} = \widehat{\mathcal{A}}_{u_0}.$$

proposition 1.4. *Let $u : M \to \mathbb{R}$ be a critical subsolution. If u is strict at every $(x, y) \in M \times M - \widehat{\mathcal{A}}_u$ then u is strict at every $x \in M - \mathcal{A}_u$. In particular, if u is continuous, the following inequalities hold:*

$$\forall x \in M - \mathcal{A}_u, \ u(x) < T_c^- u + \alpha[0],$$

$$\forall x \in M - \mathcal{A}_u, \ u(x) > T_c^+ u - \alpha[0].$$

Chapter 2

More regularity, the twist conditions and the partial dynamic

We will now suppose that the cost function is locally semi-concave, see [FF07] or [CS04] for a definition. In this text we will use the term locally semi-concave to refer to what is usually called locally semi-concave with linear modulus. Let us begin with some basic properties of locally semi-concave functions that we will need later.

proposition 2.1 (differentiability properties). *The following properties hold*

(i) Let f be a locally semi-concave function from M to $\mathbb{R}$ and let x_0 be a local minimum of f, then f is differentiable at x_0 and $\mathrm{d}_{x_0} f = 0$.

(ii) Let f and g be two locally semi-concave functions from M to $\mathbb{R}$ and x_0 be a point where $f + g$ is differentiable, then both f and g are differentiable at x_0.

Proof. (i) Since the result is local, we can suppose f is defined on an open subset $U \subset \mathbb{R}^n$, that it is semi-concave, and that $x_0 = 0$ is a global minimum. Moreover, since the problem is invariant by addition of a constant to f, we will assume $f(0) = 0$. Let $K \in \mathbb{R}$ such that $x \to f(x) - K\|x\|^2$ is concave on U. By the Hahn-Banach theorem, there is a linear form p such that

$$\forall x \in U, 0 \leqslant f(x) \leqslant p(x) + K\|x\|^2. \tag{2.1}$$

The positive function $p(x) + K\|x\|^2$ admits a local minimum at 0. Its differential at 0 must vanish so $p = 0$ and

$$\forall x \in U, 0 \leqslant f(x) \leqslant K\|x\|^2 \tag{2.2}$$

therefore f is differentiable at 0 with $\mathrm{d}_0 f = 0$.
(ii) Once more, let us assume that f and g are defined on an open subset $U \subset \mathbb{R}^n$, that they are semi-concave and that $x_0 = 0$. It is clear that if p and q are linear forms respectively in the super-differential at 0 of f and g then $p + q$ is in the super-differential at 0 of $f + g$. Since $f + g$ is differentiable at 0, its super-differential at 0 is a singleton. Moreover, f and g's super-differentials at 0 are non empty by the Hahn-Banach theorem and must also be singletons. This proves that f and g are differentiable at 0. □

def 2.1 (minimizing chains)**.** Let $(x, y) \in M \times M$ and $k \in \mathbb{N}^*$, we will say that $(x_1, \dots, x_k) \in M^k$ is a minimizing chain between x and y if, setting $x_0 = x$ and $x_{k+1} = y$,

$$\forall (y_1, \dots, y_k) \in M^k,\ \sum_{i=0}^{k} c(x_i, x_{i+1}) \leqslant c(x, y_1) + \sum_{i=1}^{k-1} c(y_i, y_{i+1}) + c(y_k, y).$$

Notice that any sub-chain of a minimizing chain formed by consecutive elements is again minimizing.
We will say that a sequence $(x_n)_{n\in\mathbb{Z}}$ is a minimizing sequence if all sub-chains formed by consecutive elements are minimizing.

A straightforward consequence of the previous results is the following theorem.

Theorem 14. *If $(x, x_1, y) \in M \times M \times M$ is a minimizing chain then $\partial c/\partial y(x, x_1)$ and $\partial c/\partial x(x_1, y)$ exist and verify*

$$\frac{\partial c}{\partial y}(x, x_1) + \frac{\partial c}{\partial x}(x_1, y) = 0. \tag{EL}$$

The equation above may be considered as a discrete analog of the Euler-Lagrange equation. It was already introduced in works on twist maps such as [Mat86]. By analogy, we therefore can define extremal chains and extremal sequences.

def 2.2 (extremal chains)**.** We will say that $(x, x_1, \dots, x_{k-1}, y)$ is an extremal chain if for every $i \in [1, k-1]$, (x_{i-1}, x_i, x_{i+1}) verify (EL) that is

$$\frac{\partial c}{\partial y}(x_{i-1}, x_i) + \frac{\partial c}{\partial x}(x_i, x_{i+1}) = 0,$$

where $x_0 = x$ et $x_k = y$.
We will say that a sequence $(x_n)_{n\in\mathbb{Z}}$ is extremal if for every $i \in \mathbb{Z}$, (x_{i-1}, x_i, x_{i+1}) verify (EL), that is

$$\frac{\partial c}{\partial y}(x_{i-1}, x_i) + \frac{\partial c}{\partial x}(x_i, x_{i+1}) = 0.$$

remark 4. Notice that minimizing chains and sequences are extremal.

It seems now natural to try and define a dynamic on M as follows: given two points x_1 and x_2, we would like to find an x_3 such that the triplet (x_1, x_2, x_3) verifies the discrete Euler-Lagrange equation (EL). However such an x_3 if it exists is not necessarily unique. To solve this problem, we introduce an additional constraint. It has already been introduced in the optimal transportation setting (see[BB07b, Lemma 29], [FF07] or even in earlier works in less explicit form [Car03]) and it is reminiscent of twist maps of the circle (see [MF94] or [Ban88]):

def 2.3. We will say that c verifies the *right twist condition* if for every $x \in M$, the function $y \mapsto \partial c/\partial x(x, y)$ is injective where it is defined.

Similarly, we will say c verifies the *left twist condition* if for every $y \in M$, the function $x \mapsto \partial c/\partial y(x, y)$ is injective where it is defined.
Finally we say c verifies the *twist condition* if c verifies the left and right twist conditions.

For more explanations about this definition see [FF07]. Let us just state that costs coming from time-periodic Tonelli Lagrangians satisfy the twist condition as is explained in the next section.

It is possible under the right twist condition to define a partial dynamic on $M \times M$ in the future and to define one in the past using the left twist condition. Let us be more precise on those points. Following [FF07], let us define the *skew Legendre transforms*:

def 2.4. We define the left skew Legendre transform as the partial map

$$\Lambda_c^l : M \times M \to T^*M,$$
$$(x, y) \mapsto \left(x, -\frac{\partial c}{\partial x}(x, y)\right),$$

whose domain of definition is

$$\mathcal{D}(\Lambda_c^l) = \left\{(x, y) \in M \times M,\ \frac{\partial c}{\partial x}(x, y) \text{ exists}\right\}.$$

Similarly, let us define the right skew Legendre transform as the partial map

$$\Lambda_c^r : M \times M \to T^*M,$$
$$(x, y) \mapsto \left(y, \frac{\partial c}{\partial y}(x, y)\right),$$

whose domain of definition is

$$\mathcal{D}(\Lambda_c^r) = \left\{(x, y) \in M \times M,\ \frac{\partial c}{\partial y}(x, y) \text{ exists}\right\}.$$

Note that saying that c verifies the left (resp. right) twist condition amounts to saying that the left (resp. right) skew Legendre transform is injective. Now we define the partial dynamics on $M \times M$.

def 2.5 (partial dynamics)**.** Let $c : M \times M \to \mathbb{R}$ be a locally semi-concave cost function which verifies the left twist condition. Set $\varphi_{-1} : M \times M \to M \times M$ the partial map defined by

$$\varphi_{-1}(x, y) = (\Lambda_c^l)^{-1} \circ \Lambda_c^r(x, y).$$

Similarly, if $c : M \times M \to \mathbb{R}$ is a locally semi-concave cost function which verifies the right twist condition, set $\varphi_{+1} : M \times M \to M \times M$ the partial map defined by

$$\varphi_{+1}(x, y) = (\Lambda_c^r)^{-1} \circ \Lambda_c^l(x, y).$$

remark 5. If both left an right twist conditions are verified, it is clear that φ_{-1} and φ_{+1} are inverses of each other on the intersection of their domains of definition.

remark 6. Let us assume here $M = \mathbb{R}^n$. If the cost function c is C^2 and verifies the stronger conditions that both Legendre transforms are global diffeomorphisms we may define a diffeomorphism $\widetilde{F}$ of $T^*\mathbb{R}^n \simeq \mathbb{R}^n \times \mathbb{R}^n$ by

$$\forall (x, p) \in T^*\mathbb{R}^n, \quad \widetilde{F}(x, p) = \Lambda_c^r \circ (\Lambda_l^r)^{-1}(x, p).$$

Assume now that $\widetilde{F}$ is the lift of a diffeomorphism F of $\mathbb{A}^n = T^*\mathbb{T}^n \simeq \mathbb{T}^n \times \mathbb{R}^n$. This diffeomorphism preserves the canonical symplectic form $\mathrm{d}\, x \wedge \mathrm{d}\, p$ and is then called an exact symplectic twist map. This particular case was studied in the founding paper [Her89]. Our cost function in this case is then called the generating function of the twist map. If moreover c verifies that the second derivative $\partial^2 c/\partial x \partial y$ is everywhere symmetric and negative non-degenerate, Bialy and Polterovitch proved that the twist map is in fact the time one map of a Tonelli Hamiltonian, periodic in time. A proof of this theorem as well as a study of symplectic twist maps can be found in [Gol01]. The particular example of costs coming from Tonelli Hamiltonians (or equivalently Tonelli Lagrangians) will be the subject of this next section. Let us finally mention that in this case of costs coming from autonomous Tonelli Hamiltonians, existence of regular subsolutions is proved in [Ber07] and [FFR09] in the non compact case. These results were extended recently by Patrick Bernard and Laurent Nocquet to the non-autonomous case ([Ber09b]).

Chapter 3

Example: costs coming from Tonelli Lagrangian

This section is devoted to explaining how these notions apply to costs coming from Tonelli Lagrangians. A convenient reference for the proofs of these results is the appendix of [FF07]. Many similar results also appear in [Ber08]
Let $L : TM \times \mathbb{R} \to \mathbb{R}$ be a time periodic Tonelli Lagrangian, that is a C^2 function verifying

1. **uniform super-linearity**: for every $K > 0$, there exists $C^*(K) \in \mathbb{R}$ such that
$$\forall (x, v, t) \in TM \times \mathbb{R},\ L(x, v, t) \geqslant K\|v\| - C^*(K),$$
2. **uniform boundedness**: for every $R \geqslant 0$, we have
$$A^*(R) = \sup\{L(x, v, t), \|v\| \leqslant R\} < +\infty,$$
3. **C^2-strict convexity in the fibers**: for every $(x, v, t) \in TM \times \mathbb{R}$, the second derivative along the fibers $\partial^2 L/\partial v^2(x, v, t)$ is positive strictly definite,
4. **time periodicity**: for every $(x, v, t) \in TM \times \mathbb{R}$, we have the relation $L(x, v, t) = L(x, v, t + 1)$,
5. **completeness**: the Euler-Lagrange flow associated to L is complete.

Then we can define a cost function c_L by
$$\forall (x, y) \in M \times M,\ c_L(x, y) = \inf_{\substack{\gamma(0)=x \\ \gamma(1)=y}} \int_{s=0}^{1} L(\gamma(s), \dot\gamma(s), s)\, \mathrm{d}\, s,$$
where the infimum is taken over all absolutely continuous curves.

proposition 3.1. *The cost c_L verifies conditions 1 and 2 and is locally semi-concave.*

Let $(x, y) \in M \times M$ and let $\gamma_{x,y}$ verify that

$$c_L(x, y) = \int_{s=0}^{1} L(\gamma_{x,y}(s), \dot{\gamma}_{x,y}(s), s) \, \mathrm{d}\, s,$$

with $\gamma_{x,y}(0) = x$ and $\gamma_{x,y}(1) = y$ then the following holds:

proposition 3.2. *The linear form on $TM \times TM$ defined by*

$$(v, w) \mapsto \frac{\partial L}{\partial v}(y, \dot{\gamma}_{x,y}(1), 0)w - \frac{\partial L}{\partial v}(x, \dot{\gamma}_{x,y}(0), 0)v$$

is a super-differential of c_L at (x, y). In particular, if $\partial c_L/\partial x(x, y)$ exists then it must be equal to $-\partial L/\partial v(x, \dot{\gamma}_{x,y}(0), 0)$ and similarly, if $\partial c_L/\partial y(x, y)$ exists then it must be equal to $\partial L/\partial v(y, \dot{\gamma}_{x,y}(1), 0)$.

Therefore, if either of the partial derivatives exists, the curve $\gamma_{x,y}$ realizing the minimum is unique (since L is strictly convex, the mapping $\partial L/\partial v$ is injective in each fiber and since $\gamma_{x,y}$ is an action minimizing curve for L and the flow is complete, it is a trajectory of the Euler-Lagrange flow). As a corollary, we have:

Theorem 15. *The cost c_L verifies both left and right twist conditions.*

We may now compute the skew Legendre transforms (when they exist). From the previous results we have the following:

$$\begin{aligned}\forall (x, y) \in \mathcal{D}(\Lambda_c^l),\ \Lambda_c^l(x, y) &= \left(x, -\frac{\partial c}{\partial x}(x, y)\right) \\ &= \left(x, \frac{\partial L}{\partial v}(x, \dot{\gamma}_{x,y}(0), 0)\right) = \mathcal{L}_L(x, \dot{\gamma}_{x,y}(0), 0),\end{aligned}$$

$$\begin{aligned}\forall (x, y) \in \mathcal{D}(\Lambda_c^r),\ \Lambda_c^r(x, y) &= \left(y, \frac{\partial c}{\partial y}(x, y)\right) \\ &= \left(y, \frac{\partial L}{\partial v}(y, \dot{\gamma}_{x,y}(1), 0)\right) = \mathcal{L}_L(y, \dot{\gamma}_{x,y}(1), 0),\end{aligned}$$

where we recall that the mapping $\mathcal{L}_L$ is the classical Legendre transform from TM to T^*M defined by

$$\forall (x, v, t) \in TM \times \mathbb{R},\ \mathcal{L}_L(x, v, t) = \left(x, \frac{\partial L}{\partial v}(x, v, t)\right).$$

Finally, let us study the partial dynamics for the cost c_L. Let $(x,y) \in M \times M$ be such that $\partial c_L/\partial y(x,y)$ exists, let us compute (if it exists) $\varphi_{+1}(x,y)$. We are looking for a z such that

$$\frac{\partial c_L}{\partial y}(x,y) = -\frac{\partial c_L}{\partial x}(y,z),$$

where all the partial derivatives exist, that is, using the previous notations,

$$\frac{\partial L}{\partial v}(\gamma_{x,y}(1), \dot\gamma_{x,y}(1), 0) = \frac{\partial L}{\partial v}(\gamma_{y,z}(0), \dot\gamma_{y,z}(0), 0),$$

which proves, since $\partial L/\partial v$ is injective, that $\dot\gamma_{x,y}(1) = \dot\gamma_{y,z}(0)$. Moreover, since all the above curves are minimizers, they are trajectories of the Euler-Lagrange flow of L which we denote by φ_L. To put it all in a nutshell, if z exists, then

$$(z, \dot\gamma_{y,z}(1), 1) = \varphi_L^1(y, \dot\gamma_{x,y}(1), 1) = \varphi_L^2(x, \dot\gamma_{x,y}(0), 0).$$

From this discussion, we obtain the following result:

proposition 3.3. *The point $(y,z) = \varphi_{+1}(x,y)$ exists if and only if the trajectory γ defined by*

$$\forall s \in [0,1],\ (\gamma(s), \dot\gamma(s), s) = \varphi_L(s)(y, \dot\gamma_{x,y}(1), 1)$$

is the only action minimizing curve between y and $\gamma(1) = z$ (defined on a time interval of length 1).

Proof. It only remains to prove the "if" part, therefore, let us assume that γ is the only action minimizing curve between $y = \gamma(0)$ and $z = \gamma(1)$.
We first prove that if $(y_n, z_n)_{n\in\mathbb{N}}$ is a sequence converging to (y,z) and if $(\gamma_n)_{n\in\mathbb{N}}$ verifies, $\gamma_n(0) = y_n$, $\gamma_n(1) = z_n$ and

$$\forall n \in \mathbb{N},\ c_L(y_n, z_n) = \int_0^1 L(\gamma_n(s), \dot{\gamma_n}(s), s)\,\mathrm{d}\,s,$$

then the $(\gamma_n, \dot\gamma_n)$ converge uniformly to $(\gamma, \dot\gamma)$ when $n \to +\infty$.
As a matter of fact, since M is compact and the γ_n are action minimizing curves defined for length time of 1, by the a priori compactness lemma (see [Fat08]), the sequence $(\gamma_n(0), \dot\gamma_n(0))_{n\in\mathbb{N}}$ is bounded. We obtain, by continuity of the Euler-Lagrange flow that the sequence of functions, $(\gamma_n, \dot\gamma_n)_{n\in\mathbb{N}}$ is relatively compact for the compact open topology. Therefore we only have

to prove that any converging subsequence converges to $(\gamma, \dot\gamma)$. Up to an extraction, let us assume that $(\gamma_n, \dot\gamma_n)$ converges to some $(\delta, \dot\delta) \in TM^{[0,1]}$. By continuity of the Euler-Lagrange flow, we necessarily have

$$\forall s \in [0,1],\ (\delta(s), \dot\delta(s), s) = \varphi_L(s)(\delta(0), \dot\delta(0), 0).$$

By continuity of the function c_L, we therefore obtain that

$$c_L(y,z) = \int_0^1 L(\delta(s), \dot\delta(s), s)\,\mathrm{d}\,s$$

which proves that $\delta = \gamma$ by uniqueness of γ and therefore that the $(\gamma_n, \dot\gamma_n)$ converge uniformly to $(\gamma, \dot\gamma)$.
As a direct corollary of the previous result, we have that if $(y_n, z_n)_{n\in\mathbb{N}}$ is a sequence converging to (y,z) and such that c_L is differentiable at each (y_n, z_n), then

$$\begin{aligned}\lim_{n\to+\infty} \mathrm{d}_{(y_n,z_n)}\, c_L \\ &= \lim_{n\to+\infty} \frac{\partial L}{\partial v}(z_n, \dot\gamma_{n y_n,z_n}(1), 1)\,\mathrm{d}\,y - \frac{\partial L}{\partial v}(y_n, \dot\gamma_{n y_n,z_n}(0), 0)\,\mathrm{d}\,x \\ &= \frac{\partial L}{\partial v}(z, \dot\gamma_{y,z}(1), 1)\,\mathrm{d}\,y - \frac{\partial L}{\partial v}(y, \dot\gamma_{y,z}(0), 0)\,\mathrm{d}\,x.\end{aligned}$$

Since c_L is a locally semi-concave function, it follows from basic properties of the Clarke super-differential ([CLSW98]) that c_L is differentiable at (x,y). □

As an immediate corrolary we obtain the following result that has been widely known for some time ([Fat09]). A similar statement appears in [Ber08]:

corollary 3.4. *For a cost coming from a Lagrangian, let $(x,y) \in M \times M$, if either $\partial c_L/\partial x(x,y)$ or $\partial c_L/\partial y(x,y)$ exists then c_L is in fact differentiable at (x,y).*

In the Lagrangian case, the partial dynamic φ_{+1} may be recovered from the restriction of the Euler-Lagrange flow, φ_L^1 to the right subset. Of course, the same holds for the negative time dynamic φ_{-1} which is closely related to the restriction to some set of φ_L^{-1}.

Chapter 4

Existence of $C^{1,1}$ critical subsolutions

We will now suppose c verifies the left and right twist conditions. Our goal from now on will be to construct more regular strict subsolutions. Let us state the main result of this section along with a sketch of its proof. The rest of the section will mostly be devoted to taking care of the technical details of the proof.

Theorem 16. *If u is a critical subsolution, then there exists a $C^{1,1}$ critical subsolution u' such that u and u' coincide on $\mathcal{A}_u$ and u' is strict outside of $\widehat{\mathcal{A}}_u$.*
There exists a $C^{1,1}$ critical subsolution which is strict outside of $\widehat{\mathcal{A}}$.

sketch of proof. First, instead of working with u, we construct a continuous function u_1 such that $\widehat{\mathcal{A}}_u = \widehat{\mathcal{A}}_{u_1}$, u and u_1 coincide on $\mathcal{A}_u$ and u_1 is strict outside of $\widehat{\mathcal{A}}_u$.

The idea is then to consider the two functions $T_c^- u_1 + \alpha[0] \geqslant T_c^+ T_c^- u_1$. The first one, $T_c^- u_1 + \alpha[0]$ is locally semi-concave while the second one $T_c^+ T_c^- u_1$ is locally semi-convex.

A classical lemma of Ilmanen stated below (18) asserts that under these hypothesis, there exists a $C^{1,1}$ function u' such that $T_c^- u_1 + \alpha[0] \geqslant u' \geqslant T_c^+ T_c^- u_1$. Adding one extra minor constraint on the way we chose u', we will prove it solves the requirements of 16.

□

proposition 4.1. *Let $u \prec c + \alpha[0]$ be a dominated function and (x_1, x_2, x_3) be a calibrated chain, then u is differentiable at x_2. Moreover,*

$$\mathrm{d}_{x_2} u = \frac{\partial c}{\partial y}(x_1, x_2) = -\frac{\partial c}{\partial x}(x_2, x_3).$$

Proof. By definition of domination, the following inequalities hold:

$$\forall x \in M,\ u(x_1) + c(x_1, x) + \alpha[0] \geqslant u(x) \geqslant u(x_3) - c(x, x_3) - \alpha[0]$$

where both inequalities are equalities at x_2. Define the functions

$$\varphi(x) = u(x_1) + c(x_1, x) + \alpha[0] \text{ and } \psi(x) = u(x_3) - c(x, x_3) - \alpha[0].$$

Clearly, φ is locally semi-concave and ψ is locally semi-convex, $\varphi \geqslant \psi$ with equality at x_2. The function $\varphi - \psi$ is always non-negative and vanishes at x_2 (which is a global minimum). Moreover, it is locally semi-concave therefore it is differentiable at x_2 and $\mathrm{d}_{x_2}(\varphi - \psi) = 0$. Finally, since both φ and $-\psi$ are locally semi-concave, both of them are differentiable at x_2 and from the inequalities $\varphi \geqslant u \geqslant \psi$ we deduce that u is differentiable at x_2 with $\mathrm{d}_{x_2} u = \mathrm{d}_{x_2} \varphi = \mathrm{d}_{x_2} \psi$. □

As a corollary we have the following:

corollary 4.2. *Suppose c satisfies the right and left twist conditions. If $u : M \to \mathbb{R}$ is a critically dominated function and $x \in \mathcal{A}_u$, then $\mathrm{d}_x u$ exists. Moreover there is a unique point x_1 such that $\partial c/\partial x(x, x_1)$ exists and verifies*

$$\mathrm{d}_x u = -\frac{\partial c}{\partial x}(x, x_1).$$

This point x_1 is also the unique point such that $(x, x_1) \in \widehat{\mathcal{A}}_u$. In particular it is necessarily in $\mathcal{A}_u$.
In the same way, there is a unique point x_{-1} such that $\partial c/\partial y(x_{-1}, x)$ exists and verifies

$$\mathrm{d}_x u = \frac{\partial c}{\partial y}(x_{-1}, x).$$

This point x_{-1} is also the unique point such that $(x_{-1}, x) \in \widehat{\mathcal{A}}_u$. In particular it is necessarily in $\mathcal{A}_u$.

Theorem 17 (Mather's graph theorem)**.** *Let $u \prec c + \alpha[0]$ be a dominated function then u is differentiable on $\mathcal{A}$. Moreover, the differential of u is independent of the dominated function u. In particular, the canonical projections from $\widetilde{\mathcal{A}}$ to $\mathcal{A}$ and from $\widehat{\mathcal{A}}$ to $\mathcal{A}$ are bijective.*

Proof. The first part is a straightforward consequence of the previous corollary (4.2). To prove the second part, notice that if $x \in \mathcal{A}$ then there is a sequence $(x_n)_{n\in\mathbb{Z}} \in \widetilde{\mathcal{A}}$ with $x_0 = x$. Therefore, $\mathrm{d}_x u = \partial c/\partial y(x_{-1}, x)$ which is independent from u. The last part is now a straightforward consequence of the twist condition. □

remark 7. Originally, in [Mat91], Mather obtains in his graph theorem that the projection, from the Aubry set to the projected Aubry set, is a bi-Lipschitz homomorphism. In the previous theorem, this is not necessarily the case, due to the fact that in the general framework we propose, the Skew Legendre transforms need not be bi-Lipschitz on their domain of definition. We will however give a bi-Lipschitz version of the graph theorem at the end of this section (see 19).

We now would like to obtain some regularity results about the differential of u on $\mathcal{A}_u$. One way to obtain that is to look for a u which is locally semi-concave. Here is a lemma that will help us to do so.

proposition 4.3. *If $u \prec c + \alpha[0]$ then $T_c^- u$ is locally semi-concave.*

Proof. The proof actually goes along the same lines as the proof that the image of a dominated function is continuous. The function $T_c^- u$ is locally a finite infimum of equi-locally semi-concave functions and is therefore itself locally semi-concave. For more details, see [FF07] or [Zav08]. □

The next proposition shows that in order to achieve our goal, we can consider $T_c^- u$ instead of u. Let us recall that by 1.3 we have $\mathcal{A}_u = \mathcal{A}_{T_c^- u}$ as soon as u is dominated. Here is a complement when c is locally semi-concave.

lemma 19. *Let $u \prec c + \alpha[0]$ be a dominated function, then if $x \notin \mathcal{A}_u$ and $\tilde{x} \in M$ verifies $T_c^- u(x) = u(\tilde{x}) + c(\tilde{x}, x)$ then $\tilde{x} \notin \mathcal{A}_u = \mathcal{A}_{T_c^- u}$.*
If $\tilde{x} \in M$ verifies $T_c^+ u(x) = u(\tilde{x}) - c(x, \tilde{x})$ then $\tilde{x} \notin \mathcal{A}_u = \mathcal{A}_{T_c^+ u}$.

Proof. Assume by contradiction $\tilde{x} \in \mathcal{A}_u$. By definition of the Lax-Oleinik semi-group, from

$$\forall z \in M, \; T_c^- u(x) \leqslant u(z) + c(z, x),$$

we obtain that

$$\forall z \in M, \; T_c^- u(x) - u(z) \leqslant c(z, x).$$

At $z = \tilde{x} \in \mathcal{A}_u$ the differential $\mathrm{d}_{\tilde{x}}\, u$ exists, therefore the sub-differential of the locally semi-concave function $z \mapsto c(z, x)$ is not empty at $\tilde{x}$. This implies that the partial derivative $\partial c / \partial x(\tilde{x}, x)$ exists and verifies

$$\mathrm{d}_{\tilde{x}}\, u = -\frac{\partial c}{\partial x}(\tilde{x}, x).$$

By corollary 4.2, we have necessarily $x \in \mathcal{A}_u$, a contradiction.
The proof of the second part is similar. □

proposition 4.4. *If $u \prec c + \alpha[0]$ is a continuous subsolution which is strict outside of $\widehat{\mathcal{A}}_u$ then $T_c^- u$ and $T_c^+ u$ are also subsolutions strict outside of $\widehat{\mathcal{A}}_{T_c^- u} = \widehat{\mathcal{A}}_u = \widehat{\mathcal{A}}_{T_c^+ u}$.*

Proof. We already know that $T_c^- u$ is a subsolution. Let $(x, x') \in M \times M$ verify $T_c^- u(x) - T_c^- u(x') = c(x', x) + \alpha[0]$. We therefore must have

$$T_c^- u(x') + \alpha[0] = u(x')$$

as seen in 18.

Since u is continuous and strict outside of $\widehat{\mathcal{A}}_u$, by proposition 1.4 we necessarily have $x' \in \mathcal{A}_u$. Using now that $u(x') = T_c^- u(x') + \alpha[0]$, we obtain the fact that

$$T_c^- u(x) = u(x') + c(x', x).$$

By 19 we must have $x \in \mathcal{A}_u$ and therefore $T_c^- u(x) = u(x) + \alpha[0]$. To put it all in a nutshell, we obtained that

$$u(x) - u(x') = c(x', x) + \alpha[0].$$

Since u is strict outside of $\widehat{\mathcal{A}}_u = \widehat{\mathcal{A}}_{T_c^- u}$ we finally get that $(x, x') \in \widehat{\mathcal{A}}_{T_c^- u}$. The proof for $T_c^+ u$ is the same. □

Using the previous result with 13 we obtain the following:

lemma 20. *Given a continuous critical subsolution u, there is a locally semi-concave critical subsolution u' which is strict outside of $\widehat{\mathcal{A}}_u$ and equal to u on $\mathcal{A}_u$. Moreover, there is a locally semi-concave subsolution u_0 which is strict outside of $\widehat{\mathcal{A}}$. The same holds replacing locally semi-concave with locally semi-convex.*

We now show how to construct $C^{1,1}$ critical subsolutions. Following the ideas of [Ber07], we will apply successively the negative and positive Lax-Oleinik semi group, as in a Lasry-Lions regularization. Nevertheless, some difficulty arise for this procedure is not regularizing in our case. Let us begin with a lemma:

lemma 21. *Let u be a continuous subsolution which is strict outside of $\widehat{\mathcal{A}}_u$, and v verify that*

$$u \leqslant v \leqslant T_c^- u + \alpha[0].$$

Assume moreover that $u(x) = v(x)$ if and only if $x \in \mathcal{A}_u$ then v itself is a critical subsolution, v and u coincide on $\mathcal{A}_u = \mathcal{A}_v$ and v is strict outside of the set $\widehat{\mathcal{A}}_u = \widehat{\mathcal{A}}_v$.

Proof. That v is a subsolution is a direct consequence of the following inequality which comes from the monotony of the Lax-Oleinik semi-group

$$u \leqslant v \leqslant T_c^- u + \alpha[0] \leqslant T_c^- v + \alpha[0].$$

Now let us prove that v is strict. Assume that for some $(x, y) \in M \times M$, the following holds: $v(x) - v(y) = c(y, x) + \alpha[0]$. Since v is critically dominated we have $v(x) = T_c^- v(x) + \alpha[0]$ and therefore, by the above inequality,

$$v(x) = T_c^- u(x) + \alpha[0] = T_c^- v(x) + \alpha[0]$$

The following inequalities are also true:

$$\begin{aligned} c(y, x) + \alpha[0] &= v(x) - v(y) \\ &= T_c^- u(x) + \alpha[0] - v(y) \\ &\leqslant T_c^- u(x) + \alpha[0] - u(y) \\ &\leqslant u(y) + c(y, x) + \alpha[0] - u(y). \end{aligned}$$

Therefore all inequalities are equalities and $v(y) = u(y)$. By the assumption we made, this proves that $y \in \mathcal{A}_u$ and from $T_c^- u(x) = u(y) + c(y, x)$ that $x \in \mathcal{A}_u$ too (by 19). Hence we have that $u(x) = T_c^- u(x) + \alpha[0]$ which yields that $u(x) - u(y) = c(y, x) + \alpha[0]$ and finally that $(y, x) \in \widehat{\mathcal{A}}_u$ since u is strict outside of $\widehat{\mathcal{A}}_u$. Consequently, we have that $\mathcal{A}_v \subset \mathcal{A}_u$ and v is strict outside of $\widehat{\mathcal{A}}_u$. Now, since $u = v$ on $\mathcal{A}_u$ (because $u = T_c^- u + \alpha[0]$ on $\mathcal{A}_u$) we have in fact $\widehat{\mathcal{A}}_u = \widehat{\mathcal{A}}_v$ which finishes the proof. □

remark 8. In the previous lemma, a similar argument shows that the hypothesis $u(x) = v(x)$ if and only if $x \in \mathcal{A}_u$ may be replaced by the following one: $T_c^- u(x) + \alpha[0] = v(x)$ if and only if $x \in \mathcal{A}_u$.

Therefore, given a critical subsolution u, by 20, we can construct a locally semi-concave critical subsolution u_1 which coincide with u on $\mathcal{A}_u$ and which is strict outside of $\widehat{\mathcal{A}}_u$ and a locally semi-convex function u_2 having the same properties such that $u_2 \leqslant u_1$ by setting $u_2 = T_c^+ u_1 - \alpha[0]$. Moreover, starting with u strict outside of $\widehat{\mathcal{A}}$ we are able to construct a locally semi-convex function $T_c^+ u - \alpha[0]$ and a locally semi-concave function $T_c^- T_c^+ u$ which are both strict outside of $\widehat{\mathcal{A}}$ and such that $T_c^+ u - \alpha[0] \leqslant T_c^- T_c^+ u$. Now the idea will be to consider a $C^{1,1}$ function in between which is the one we are looking for.

From the discussion above, the proof is a direct consequence of the following lemma which appears in [Ilm93].

Theorem 18. *Given a locally semi-concave function $f : M \mapsto \mathbb{R}$ and a locally semi-convex function $g : M \mapsto \mathbb{R}$ such that $f \geqslant g$, there exists a $C^{1,1}$ function $h : M \mapsto \mathbb{R}$ such that $f \geqslant h \geqslant g$. Moreover, h can be constructed in such a way that $h(x) = g(x)$ implies $f(x) = g(x)$.*

Let us mention that the previous theorem (18) is equivalent to Ilmanen's insertion lemma proved in [Car01]. Following Cardaliaguet's observation, two independent proofs of the claim were obtained in [Ber09a, FZ09].

We conclude this section by giving another analogue of Mather's graph theorem in this discrete setting. Let us define yet another Aubry set:

def 4.1. *Given a critical subsolution, let us set $\mathcal{A}^*_u \subset T^*M$ by*

$$\mathcal{A}^*_u = \Lambda^l_c(\widehat{\mathcal{A}}_u).$$

Finally, let us set

$$\mathcal{A}^* = \Lambda^l_c(\widehat{\mathcal{A}}).$$

Theorem 19 (Mather's graph theorem bis)**.** *Given a critical subsolution u, the canonical projection π from T^*M to $\mathbb{R}$ induces a bi-Lipschitz homeomorphism from $\mathcal{A}^*_u$ to $\mathcal{A}_u$.*
The canonical projection π from T^*M to $\mathbb{R}$ induces a bi-Lipschitz homeomorphism from $\mathcal{A}^*$ to $\mathcal{A}$.

Proof. By 16, we can without loss of generality assume that u is $C^{1,1}$. By 4.2 and by definition of the skew Legendre transform Λ^l_c, the application π^{-1} from $\mathcal{A}_u$ to $\mathcal{A}^*_u$ is nothing but the following:

$$\forall x \in \mathcal{A}_u,\ \pi^{-1}(x) = (x, \mathrm{d}_x u)$$

which is therefore Lipschitz since u is $C^{1,1}$.
The second part is proved similarly starting with a $C^{1,1}$ strict subsolution (given by 16) whose Aubry set is $\mathcal{A}$. □

Chapter 5

Invariant and equivariant weak KAM solutions

In this section, following very closely the ideas of [FM07], we consider the case of invariant cost functions. This case arises naturally when studying covering spaces with the group of deck transformations as group of symmetries (we will study this case in the next and last section). Let us notice that most results of this section can be proved in the much more general setting exposed in [Zav08], when M is merely a length space at large scale.

Let G be a group of homeomorphisms that preserve c that is

$$\forall g \in G, \forall (x,y) \in M \times M,\ c(g(x), g(y)) = c(x,y).$$

We will denote by $\mathcal{I}$ the set of G-invariant functions that is

$$\mathcal{I} = \left\{ f \in \mathbb{R}^M, \forall g \in G,\ f \circ g = f \right\}.$$

For each $\mathfrak{C} \in \mathbb{R}$ let

$$\mathcal{H}_{inv}(\mathfrak{C}) = \mathcal{H}(\mathfrak{C}) \cap \mathcal{I}$$

be the set of the invariant functions which are $\mathfrak{C}$-dominated. It is clear that $\mathcal{H}_{inv}(\mathfrak{C}) \cap C^0(M,\mathbb{R})$ is a closed (for the topology of uniform convergence on compact subsets) and convex subset of $\mathcal{H}(\mathfrak{C}) \cap C^0(M,\mathbb{R})$. It is also clear that, if q denotes the canonical projection from $C^0(M,\mathbb{R})$ to $C^0(M,\mathbb{R})/\mathbb{R}\mathbb{1}_M$ ($\mathbb{1}_M$ denotes the constant function equal to 1 on M), and if we let $\widehat{\mathcal{H}}(\mathfrak{C}) = q(\mathcal{H}(\mathfrak{C}) \cap C^0(X,\mathbb{R}))$, then we may define

$$\widehat{\mathcal{H}}_{inv}(\mathfrak{C}) = q(\mathcal{H}_{inv}(\mathfrak{C}) \cap C^0(X,\mathbb{R})) = \widehat{\mathcal{H}}(\mathfrak{C}) \cap q(\mathcal{I}),$$

where the last equality follows from the fact that $\mathcal{I}$ contains the constant functions. Finally, since the Lax-Oleinik semi-group T_c^- commutes with the addition of constants, it induces canonically a semi-group $\widehat{T}_c^-$ on the quotient $C^0(M,\mathbb{R})/\mathbb{R}\mathbb{1}_M$.

proposition 5.1. *If $u \in \mathcal{I}$, then $T_c^- u \in \mathcal{I}$. Moreover, $\mathcal{H}_{inv}(\mathfrak{C}) \neq \varnothing$ for all $\mathfrak{C} \geqslant C(0)$.*

Recall that $C(0)$ is a constant introduced in 1 at the beginning of this paper.

Proof. The last part of this proposition is immediate since constant functions are dominated by $c + C(0) \geqslant 0$.
To prove the first part, let $u \in \mathcal{I}$ and $g \in G$. Then

$$T_c^- u(g(x)) = \inf_{y \in M} u(y) + c(y, g(x)) = \inf_{y \in M} u(g(y)) + c(g(y), g(x)) = \inf_{y \in M} u(y) + c(y, x)$$

where we have first used the fact that g is a bijection and then the invariance of u and c by g. □

We now define the invariant critical value for the action of the group G as the constant

$$\mathfrak{C}_{inv} = \inf\{\mathfrak{C} \in \mathbb{R},\ \mathcal{H}_{inv}(\mathfrak{C}) \neq \varnothing\}.$$

Clearly, we have that $-A(0) \leqslant \alpha[0] \leqslant \mathfrak{C}_{inv} \leqslant C(0)$, where $A(0)$ is introduced in 2 at the beginning of this paper. We are now able to prove the invariant weak KAM theorem:

Theorem 20 (invariant weak KAM)**.** *There exists a G-invariant function u such that $u = T_c^- u + \mathfrak{C}_{inv}$.*

Proof. We only sketch the proof since it is very similar to the proof of the weak KAM theorem ([Zav08]). We know that $\mathcal{I}$ is stable by T_c^-. This implies that $\widehat{\mathcal{I}}$ is stable by $\widehat{T}_c^-$. Therefore $\widehat{\mathcal{H}}_{inv}(\mathfrak{C})$ is stable by $\widehat{T}_c^-$ and so is $H_{inv}(\mathfrak{C}) = \mathrm{conv}(\widehat{T}_c^-(\widehat{\mathcal{H}}_{inv}(\mathfrak{C})))$, for each $\mathfrak{C} \in \mathbb{R}$. It is obvious that $H_{inv}(\mathfrak{C}) \neq \varnothing$ if and only if $\widehat{\mathcal{H}}_{inv}(\mathfrak{C}) \neq \varnothing$. It can be checked, using the Ascoli theorem, that $H_{inv}(\mathfrak{C})$ is convex and compact for the quotient of the topology of uniform convergence on compact subsets. As a consequence,

$$\bigcap_{\mathfrak{C} > \mathfrak{C}_{inv}} H_{inv}(\mathfrak{C}) \neq \varnothing$$

as the intersection of a decreasing family of compact nonempty sets. Therefore, $\widehat{\mathcal{H}}_{inv}(\mathfrak{C}_{inv})$ is nonempty. Moreover, $\widehat{T}_c^-$ induces a continuous mapping from $H_{inv}(\mathfrak{C}_{inv})$ into itself, so applying the Schauder-Tykhonoff theorem, we obtain a fixed point, that is a function $u_{inv} \in \mathcal{H}_{inv}(\mathfrak{C}_{inv})$ and a constant C' such that $T_c^- u_{inv} = u_{inv} + C'$. Finally, using the minimality of $\mathfrak{C}_{inv}$, it is easy to prove that in fact $-C' = \mathfrak{C}_{inv}$ which ends the proof of the theorem. □

Instead of looking at functions invariant by the group of symmetries G we can consider functions whose projections to $C^0(X,\mathbb{R})\backslash\mathbb{R}\mathbb{1}_M$ are invariant that is functions u such that for each $g \in G$ there is a $\rho(g)$ such that $u \circ g = u + \rho(g)$. Obviously, $\rho : G \to \mathbb{R}$ is a group homomorphism. We will denote by $\mathrm{Hom}(G,\mathbb{R})$ the set of group homomorphisms from G to $\mathbb{R}$. Given a $\rho \in \mathrm{Hom}(G,\mathbb{R})$ we will say that a function u is ρ-equivariant if it satisfies $u \circ g = u + \rho(g)$ for all g in G, we will denote by $\mathcal{I}_\rho$ the set of continuous ρ-equivariant functions. It is obvious that $\mathcal{I}_\rho$ is an affine subset of $C^0(X,\mathbb{R})$, in fact, it is either empty or equal to $u + \mathcal{I}$ where $u \in \mathcal{I}_\rho$. In particular $\mathcal{I}_0 = \mathcal{I}$. For $\mathfrak{C} \in \mathbb{R}$, $\rho \in \mathrm{Hom}(G,\mathbb{R})$, we set $\mathcal{H}_\rho(\mathfrak{C}) = \mathcal{H}(\mathfrak{C}) \cap C^0(M,\mathbb{R}) \cap \mathcal{I}_\rho$ and we define the ρ-equivariant critical value

$$\mathfrak{C}_\rho = \inf\{\mathfrak{C} \in \mathbb{R},\ \mathcal{H}_\rho(\mathfrak{C}) \neq \varnothing\} \in \mathbb{R} \cup \{+\infty\}.$$

Notice that the value $+\infty$ is reached if and only if there is no $\mathfrak{C}$ such that $\mathcal{H}_\rho(\mathfrak{C}) \neq \varnothing$. For example, the 0-equivariant critical value or invariant critical value is nothing but $\mathfrak{C}_0 = \mathfrak{C}_{inv}$. First, we notice that since the Lax-Oleinik semi-group commutes with addition of constants, we have, as in 5.1, the following:

proposition 5.2. *Let us consider a morphism $\rho \in \mathrm{Hom}(G,\mathbb{R})$. If $u \in \mathcal{I}_\rho$, then $T_c^- u \in \mathcal{I}_\rho$.*

def 5.1. We will say that a homomorphism $\rho : G \to \mathbb{R}$ is tame if the inequality $\mathfrak{C}_\rho < +\infty$ is verified and we will denote by $\mathrm{Hom}_{\mathrm{tame}}(G,\mathbb{R})$ the set of tame homomorphisms.

Since $\mathcal{I}_\rho$ is closed for the compact open topology and invariant by the Lax-Oleinik semi-group, we can easily adapt the proof of 20 to obtain the following equivariant weak KAM theorem:

Theorem 21 (equivariant weak KAM)**.** *For each $\rho \in \mathrm{Hom}_{\mathrm{tame}}(G,\mathbb{R})$, we have $\mathcal{H}_\rho(\mathfrak{C}_\rho) \neq \varnothing$. Moreover, we can find a ρ-equivariant weak KAM solution in $\mathcal{H}_\rho(\mathfrak{C}_\rho)$ that is a continuous function u such that $u = T_c^- u + \mathfrak{C}_\rho$ and for all $g \in G$, $u \circ g = u + \rho(g)$.*

Here are some properties of tame homomorphisms and of the function $\rho \mapsto \mathfrak{C}_\rho$.

proposition 5.3. *The set $\mathrm{Hom}_{\mathrm{tame}}(G,\mathbb{R})$ is a vector subspace of $\mathrm{Hom}(G,\mathbb{R})$. The restriction of the function $\mathfrak{C}$ to $\mathrm{Hom}_{\mathrm{tame}}(G,\mathbb{R})$ is convex. Moreover, if $\mathrm{Hom}_{\mathrm{tame}}(G,\mathbb{R})$ is finite dimensional, then the function $\mathfrak{C}$ is super-linear.*

Proof. Let ρ_1 and ρ_2 be two tame homomorphisms, λ_1 and λ_2 be real numbers. Let $u_1 \in \mathcal{H}_{\rho_1}(\mathfrak{C}_1)$ and $u_2 \in \mathcal{H}_{\rho_2}(\mathfrak{C}_2)$ where $\mathfrak{C}_1$ and $\mathfrak{C}_2$ have been chosen such that $\mathcal{H}_{\rho_1}(\mathfrak{C}_1) \neq \varnothing$ and $\mathcal{H}_{\rho_2}(\mathfrak{C}_2) \neq \varnothing$. Then $\lambda_1 u_1 + \lambda_2 u_2 \in \mathcal{I}_{\lambda_1\rho_1+\lambda_2\rho_2}$ (as a matter of fact, $\lambda_1\mathcal{I}_{\rho_1} + \lambda_2\mathcal{I}_{\rho_2} \subset \mathcal{I}_{\lambda_1\rho_1+\lambda_2\rho_2}$). Moreover, we clearly have that $\lambda_1 u_1 + \lambda_2 u_2 \in \mathcal{H}(|\lambda_1|\mathfrak{C}_1 + |\lambda_2|\mathfrak{C}_2)$ which proves that $\mathrm{Hom}_{\mathrm{tame}}(G,\mathbb{R})$ is a vector subspace of $\mathrm{Hom}(G,\mathbb{R})$.

If now $\lambda_1, \lambda_2 \geqslant 0$ and $\lambda_1 + \lambda_2 = 1$ then the inclusion

$$\lambda_1\mathcal{H}(\mathfrak{C}_1) + \lambda_2\mathcal{H}(\mathfrak{C}_2) \subset \mathcal{H}(\lambda_1\mathfrak{C}_1 + \lambda_2\mathfrak{C}_2)$$

holds. Altogether with the inclusion

$$\lambda_1\mathcal{I}_{\rho_1} + \lambda_2\mathcal{I}_{\rho_2} \subset \mathcal{I}_{\lambda_1\rho_1+\lambda_2\rho_2},$$

this proves the convexity of the function $\mathfrak{C}$.

We now prove the super-linearity when $\mathrm{Hom}_{\mathrm{tame}}(G,\mathbb{R})$ is finite dimensional. For each $g \in G$, consider the linear form

$$\hat{g} : \mathrm{Hom}_{\mathrm{tame}}(G,\mathbb{R}) \to \mathbb{R},$$

$$\rho \mapsto \rho(g).$$

These linear forms span a sub-vector space of the dual of $\mathrm{Hom}_{\mathrm{tame}}(G,\mathbb{R})$ which is therefore finite dimensional. Let $g_1, \dots, g_k$ be such that any $\hat{g}$ is a linear combination of the $\hat{g}_i$. In particular, it follows that if $\rho \in \mathrm{Hom}_{\mathrm{tame}}(G,\mathbb{R})$ then $\rho = 0$ if only if $\rho(g_1) = \cdots = \rho(g_k) = 0$. Thus we can use as a norm on $\mathrm{Hom}_{\mathrm{tame}}(G,\mathbb{R})$, $\|\rho\| = \max_{i=1}^k |\rho(g_i)|$. If ρ is given, let u be a ρ-equivariant weak KAM solution such that $u = T_c^- u + \mathfrak{C}_\rho$. We have $n\rho(g_i) = \rho(g_i^n) = u(g_i^n(x_0)) - u(x_0)$ for $n \in \mathbb{N}$, $i = 1, \dots, k$ and some x_0 fixed. We now have using the domination $u \prec c + \mathfrak{C}_\rho$

$$n\rho(g_i) = u(g_i^n(x_0)) - u(x_0) \leqslant c(x_0, g_i^n(x_0)) + \mathfrak{C}_\rho.$$

The constant $A_{i,n} = c(x_0, g_i^n(x_0))$ is independant of ρ. Arguing in the same way as above with g_i^{-1} instead of g_i, we obtain a constant $A'_{i,n}$ independant of ρ such that

$$-n\rho(g_i) = u(g_i^{-n}(x_0)) - u(x_0) \leqslant A'_{i,n} + \mathfrak{C}_\rho.$$

If we set $A_n = \max(A_{1,n}, \dots, A_{k,n}, A'_{1,n}, \dots, A'_{k,n})$ we have obtained a constant independant of ρ such that

$$n\|\rho\| = n\max(\rho(g_1), \dots, \rho(g_k), -\rho(g_1), \dots, -\rho(g_k)) \leqslant A_n + \mathfrak{C}_\rho.$$

Since n is an arbitrary integer, this proves the super-linearity of $\rho \mapsto \mathfrak{C}_\rho$.

□

We set

$$\mathfrak{C}_{G,min} = \inf\{\mathfrak{C}_\rho,\ \rho \in \mathrm{Hom}(G,\mathbb{R})\} = \inf\{\mathfrak{C}_\rho,\ \rho \in \mathrm{Hom}_{\mathrm{tame}}(G,\mathbb{R})\}.$$

lemma 22. *There exists $\rho \in \mathrm{Hom}_{\mathrm{tame}}(G,\mathbb{R})$ such that $\mathfrak{C}_{G,min} = \mathfrak{C}_\rho$.*

Proof. Of course, when $\mathrm{Hom}_{\mathrm{tame}}(G,\mathbb{R})$ is finite dimensional, this follows from the super-linearity of the function $\mathfrak{C}$.

For the general case, pick a decreasing sequence $\mathfrak{C}_{\rho_n}$ which converges to $\mathfrak{C}_{G,min}$. For each $n \in \mathbb{N}$, pick a function $u_n \in T_c^-(\mathcal{H}_{\rho_n}(\mathfrak{C}_{\rho_n}))$. The functions are locally equicontinous because they all belong to $T_c^-(\mathcal{H}(\mathfrak{C}_{\rho_0}))$. Substracting a constant from each u_n and extracting a subsequence if necessary, we can assume that u_n converges uniformly on each compact subset of M to a function u. Since for $n \geqslant n_0$, u_n is in the closed set $\mathcal{H}(\mathfrak{C}_{\rho_{n_0}})$, we must have $u \in \mathcal{H}(\mathfrak{C}_{\rho_{n_0}})$ for each n_0. Hence, $u \in \mathcal{H}(\mathfrak{C}_{G,min})$. Since for $x \in M$ we have $\rho_n(g) = u_n(g(x)) - u_n(x)$ we conclude that ρ_n converges (pointwise) to a $\rho \in \mathrm{Hom}(G,\mathbb{R})$ and $u \in \mathcal{I}_\rho$. It follows that $\mathfrak{C}_\rho \leqslant \mathfrak{C}_{G,min}$ but the reverse inequality follows from the definition of $\mathfrak{C}_{G,min}$. □

Chapter 6

Application: Mather's α function on the cohomology

In this final section, following Mather's ideas ([Mat91]), we apply the preceding results to the case when the group of symmetries arises from a covering of M. Let us consider M a smooth, finite dimensional, connected riemmanian manifold, g_M its metric. Let $\widetilde{\mathcal{M}}$ be its covering space verifying

$$\pi_1\left(\widetilde{\mathcal{M}}\right) = \ker(\mathfrak{H})$$

where $\mathfrak{H} : \pi_1(M) \to H_1(M, \mathbb{R})$ is the Hurewicz homomorphism. We consider then a cost function $\tilde{c} : \widetilde{\mathcal{M}} \times \widetilde{\mathcal{M}} \to \mathbb{R}$ which verifies 1 and 2. Let us assume moreover that $\tilde{c}$ is invariant by the diagonal action of the group of deck transformations $\mathfrak{T}$. This means that if T is a deck transformation, the following holds:

$$\forall (\tilde{x}, \tilde{y}) \in M \times M,\ \tilde{c}(\tilde{x}, \tilde{y}) = \tilde{c}(T(\tilde{x}), T(\tilde{y})).$$

Let $p : \widetilde{\mathcal{M}} \to M$ be the cover, we may define a cost function $c : M \times M \to \mathbb{R}$ by

$$\forall (x, y) \in M \times M,\ c(x, y) = \inf_{\substack{p(\tilde{x})=x \\ p(\tilde{y})=y}} \tilde{c}(\tilde{x}, \tilde{y}).$$

proposition 6.1. *The cost function c is continuous, uniformly super-linear and uniformly bounded in the sense of 1 and 2. Moreover, if $(x, y) \in M \times M$ then for each $\tilde{x} \in \widetilde{\mathcal{M}}$ verifying $p(\tilde{x}) = x$ there is a $\tilde{y} \in \widetilde{\mathcal{M}}$ such that $p(\tilde{y}) = y$ and $c(x, y) = \tilde{c}(\tilde{x}, \tilde{y})$.*

Proof. The proof of the continuity of c is much similar to the proofs of regularity of the Lax-Oleinik semi-groups (see [Zav08]) therefore we will sketch it briefly. Let us consider $K \subset M$ a compact subset of M and $\widetilde{K} \subset \widetilde{\mathcal{M}}$ compact verifying $p(\widetilde{K}) = K$. Since $\tilde{c}$ is invariant by the diagonal action of the group of deck transformations $\mathfrak{T}$ we have the following:

$$\forall (x,y) \in K \times M,\ c(x,y) = \inf_{\substack{\tilde{x} \in \widetilde{K}, p(\tilde{x})=x \\ p(\tilde{y})=y}} \tilde{c}(\tilde{x},\tilde{y}).$$

Let us now consider another compact set $K_1 \subset M$. It may be proved, using the super-linearity of $\tilde{c}$, that there exists a compact set $\widetilde{K}_1$ such that $K_1 \subset p(\widetilde{K}_1)$ and

$$\forall (x,y) \in K \times K_1,\ c(x,y) = \inf_{\substack{\tilde{x} \in \widetilde{K}, p(\tilde{x})=x \\ \tilde{y} \in \widetilde{K}_1, p(\tilde{y})=y}} \tilde{c}(\tilde{x},\tilde{y}).$$

Since $\widetilde{K} \times \widetilde{K}_1$ is compact, the function $\tilde{c}$ restricted to $\widetilde{K} \times \widetilde{K}_1$ is uniformly continuous and the function c restricted to $K \times K_1$ is a finite infimum (in fact this infimum is achieved) of uniformly continuous functions, therefore it is continuous. Note that since we managed to restrict ourselves to compact sets, we may apply the previous result to $K = \{x\}$ and $\widetilde{K} = \{\tilde{x}\}$ to obtain the last point of the proposition.

Let $\mathrm{d}(.,.)$ be the riemannian distance on M and $\tilde{\mathrm{d}}(.,.)$ the induced distance on $\widetilde{\mathcal{M}}$. The following is verified:

$$\forall (\tilde{x},\tilde{y}) \in \widetilde{\mathcal{M}} \times \widetilde{\mathcal{M}},\ \mathrm{d}(p(\tilde{x}),p(\tilde{y})) \leqslant \tilde{\mathrm{d}}(\tilde{x},\tilde{y}).$$

Since $\tilde{c}$ is uniformly super-linear we have that for every $k \geqslant 0$, there exists $C(k) \in \mathbb{R}$ such that

$$\forall (\tilde{x},\tilde{y}) \in \widetilde{\mathcal{M}} \times \widetilde{\mathcal{M}},\ \tilde{c}(\tilde{x},\tilde{y}) \geqslant k\tilde{\mathrm{d}}(\tilde{x},\tilde{y}) - C(k).$$

Let us pick $(x_0,y_0) \in M \times M$ and $(\tilde{x}_0,\tilde{y}_0)$ such that $p(\tilde{x}_0) = x_0$, $p(\tilde{y}_0) = y_0$ and $c(x_0,y_0) = \tilde{c}(\tilde{x}_0,\tilde{y}_0)$. The following holds:

$$c(x_0,y_0) = \tilde{c}(\tilde{x}_0,\tilde{y}_0) \geqslant k\tilde{\mathrm{d}}(\tilde{x}_0,\tilde{y}_0) - C(k) \geqslant k\,\mathrm{d}(x_0,y_0) - C(k),$$

which proves the super-linearity of c.

Similarly, for every $R \in \mathbb{R}$, there exists $A(R) \in \mathbb{R}$ such that

$$\tilde{\mathrm{d}}(\tilde{x},\tilde{y}) \leqslant R \Rightarrow \tilde{c}(\tilde{x},\tilde{y}) \leqslant A(R).$$

If $\mathrm{d}(x_0,y_0) \leqslant R$, we can find $(\tilde{x}_0,\tilde{y}_0)$ such that $p(\tilde{x}_0) = x_0$, $p(\tilde{y}_0) = y_0$ and $\mathrm{d}(x_0,y_0) = \tilde{\mathrm{d}}(\tilde{x}_0,\tilde{y}_0) \leqslant R$. Therefore, using the definition of c we obtain

$$c(x_0,y_0) \leqslant \tilde{c}(\tilde{x}_0,\tilde{y}_0) \leqslant A(R)$$

which proves that c is uniformly bounded in the sense of 2. □

Let us now consider a bounded (with respect to the metric g_M) closed 1-form ω on M. This form lifts to an exact form $\tilde\omega = \mathrm{d}\tilde f$ on $\widetilde{\mathcal{M}}$. Moreover, the function $\tilde f$ is globally Lipschitz hence has linear growth. We may therefore define a cost function $\tilde c_{\tilde\omega}$ by

$$\forall(\tilde x,\tilde y)\in\widetilde{\mathcal{M}}\times\widetilde{\mathcal{M}},\ \tilde c_{\tilde\omega}(\tilde x,\tilde y)=\tilde c(\tilde x,\tilde y)-\tilde f(\tilde y)+\tilde f(\tilde x).$$

Note that this cost function is still super-linear and uniformly bounded and that it does not depend on the choice of the primitive $\tilde f$. Let us fix a point $\tilde x\in\widetilde{\mathcal{M}}$ and define now the morphism $\rho_{\tilde\omega}:\mathfrak{T}\to\mathbb{R}$ by

$$\forall T\in\mathfrak{T},\ \rho_{\tilde\omega}(T)=\tilde f(T(x))-\tilde f(x).$$

It is straightforward to check that $\rho_{\tilde\omega}$ is indeed a morphism and that it is independent from x by Stoke's formula. Finally, the map $\omega\to\rho_{\tilde\omega}$ is linear in ω and vanishes if and only if ω is exact. Therefore it induces an injective morphism from the g_M-bounded cohomology of order 1, $H^1_{g_M,b}(M,\mathbb{R})$, to $\mathrm{Hom}(\mathfrak{T},\mathbb{R})$. We still denote by ρ this morphism. We now have the following lemma:

lemma 23. *The following inclusion holds:*

$$\mathrm{Im}(\rho)\subset\mathrm{Hom}_{\mathrm{tame}}(\mathfrak{T},\mathbb{R}).$$

Proof. It follows from the discussion above that, if $[\omega]\in H^1_{g_M,b}(M,\mathbb{R})$ and ω is a bounded 1-form whose cohomology class is $[\omega]$ then $\tilde c_{\tilde\omega}$ verifies 1 and 2. Therefore, by the invariant weak KAM theorem (20) applied to the cost $\tilde c_{\tilde\omega}$ there exist a function $\tilde u$ and a constant C such that $\tilde u = T^-_{\tilde c_{\tilde\omega}}\tilde u + C$ and $\tilde u\in\mathcal{I}$. This means exactly that $\tilde u+\tilde f = T^-_{\tilde c}(\tilde u+\tilde f)+C$ and $\tilde u+\tilde f\in\mathcal{I}_{\rho_{\tilde\omega}}$. □

We now introduce Mather's alpha function:

def 6.1. Let $[\omega]\in H^1_{g_M,b}(M,\mathbb{R})$ be the cohomology class of a bounded 1-form ω, we define the constant $\alpha[\omega]\in\mathbb{R}$ by the relation $\alpha[\omega]=\mathfrak{C}_{\rho_{\tilde\omega}}$. In other words, the value $\alpha[\omega]$ is the invariant critical value of the cost $\tilde c_{\tilde\omega}$.

In an analogous way to what we already did, if ω is a closed bounded 1-form on M, we may define a cost function c_ω by

$$\forall(x,y)\in M\times M,\ c_\omega(x,y)=\inf_{\substack{p(\tilde x)=x\\ p(\tilde y)=y}}\tilde c_{\tilde\omega}(\tilde x,\tilde y).$$

The constant $\alpha[\omega]$ is also the critical value of the cost c_ω. Moreover, this constant depends only on the cohomology class $[\omega]$ of the form ω. As a matter of fact, as in the proof of 23, if $\omega = \mathrm{d} f$ is exact, then $u : M \to \mathbb{R}$ is a critical subsolution for c_ω if and only if $u + f$ is a critical subsolution for c. This also justifies a posteriori the notation $\alpha[\omega]$.

From now on, we will assume, without loss of generality, that all the forms considered are smooth. The end of this paper will be devoted to checking that it is possible to adapt the machinery of sections 1, 2 and 4 to this cohomological setting.

proposition 6.2. *Assume the cost $\tilde{c} : \widetilde{\mathcal{M}} \times \widetilde{\mathcal{M}} \to \mathbb{R}$ is locally semi-concave then the cost $c : M \times M \to \mathbb{R}$ is also locally semi-concave. Assume moreover that $\tilde{c}$ verifies the left and right twist conditions, then so does c. Finally, in the latter case, if ω is a smooth closed 1-form on M, the costs $\tilde{c}_{\tilde{\omega}}$ and c_ω are locally semi-concave and verify the left and right twist conditions.*

Proof. As in the proof of 6.1, the function c is locally semi-concave because it is locally a finite infimum of equi-semi-concave functions (everything can locally be reduced to taking infimums over relatively compact sets).

For the second part of the proposition, let us prove only the left twist condition. Consider a point $x_0 \in M$ and a lift $\tilde{x}_0 \in \widetilde{\mathcal{M}}$ such that $p(\tilde{x}_0) = x_0$. By the last part of 6.1, the following holds:

$$\forall y \in M, \ c(x_0, y) = \inf_{\tilde{y} \in p^{-1}\{y\}} \tilde{c}(\tilde{x}_0, \tilde{y}).$$

Assume now that for some $y \in M$ the partial derivative $\partial c/\partial x(x_0, y)$ exists and consider $\tilde{y} \in \widetilde{\mathcal{M}}$ such that $c(x_0, y) = \tilde{c}(\tilde{x}_0, \tilde{y})$. Since $\tilde{c}$ is locally semi-concave, it follows that the partial derivative $\partial\tilde{c}/\partial\tilde{x}(\tilde{x}_0, \tilde{y})$ also exists and verifies (identifying the cotangent fibers $T_{(\tilde{x}_0,\tilde{y})}\widetilde{\mathcal{M}} \times \widetilde{\mathcal{M}}$ and $T_{(x_0,y)}M \times M$ via the cover p which is a local diffeomorphism)

$$\frac{\partial \tilde{c}}{\partial \tilde{x}}(\tilde{x}_0, \tilde{y}) = \frac{\partial c}{\partial x}(x_0, y). \tag{6.1}$$

Now, since $\tilde{c}$ verifies the left twist condition, it follows that the map

$$\tilde{y} \mapsto \Lambda^l_{\tilde{c}}(\tilde{x}_0, \tilde{y}) = \left(\tilde{x}_0, -\frac{\partial \tilde{c}}{\partial \tilde{x}}(\tilde{x}_0, \tilde{y})\right)$$

is injective on its domain of definition, and it follows immediately from 6.1 that the left Legendre transform

$$y \mapsto \Lambda^l_c(x_0, y) = \left(x_0, -\frac{\partial c}{\partial x}(x_0, y)\right)$$

is also injective on its domain of definition, which means that c verifies the left twist condition.

The last part of the proposition is now straightforward. Indeed, if ω is smooth, then so will be the function $\tilde{f}$, and the function

$$\tilde{c}_{\tilde{\omega}} : (\tilde{x}, \tilde{y}) \mapsto \tilde{c}(\tilde{x}, \tilde{y}) - \tilde{f}(\tilde{y}) + \tilde{f}(\tilde{x})$$

remains locally-semi-concave. Moreover the left Legendre transform of $\tilde{c}_{\tilde{\omega}}$ is defined if and only if the left Legendre transform of $\tilde{c}$ is defined and it is given by the formula

$$\Lambda^l_{\tilde{c}_{\tilde{\omega}}}(\tilde{x}, \tilde{y}) = \left(\tilde{x}, -\frac{\partial \tilde{c}_{\tilde{\omega}}}{\partial x}(\tilde{x}, \tilde{y})\right) = \left(\tilde{x}, -\frac{\partial \tilde{c}}{\partial x}(\tilde{x}, \tilde{y}) - \mathrm{d}_{\tilde{x}} \tilde{f}\right)$$

which clearly gives that $\tilde{c}$ verifies the left twist condition if and only if $\tilde{c}_{\tilde{\omega}}$ does. □

Thanks to 6.2, it is possible to associate to each cohomology class $[\omega] \in H^1_{g_M,b}(M, \mathbb{R})$ Aubry sets $\mathcal{A}_{[\omega]}$, $\widehat{\mathcal{A}}_{[\omega]}$ and $\widetilde{\mathcal{A}}_{[\omega]}$ by using the already introduced notions to the cost c_ω. Notice that these sets depend only on the cohomology class for, as in the time-continuous case (see [Mat91]), minimizers with fixed endpoints are unchanged by the addition of an exact form to the cost c. Theorem 16 then applies, proving the existence of $C^{1,1}$ strict subsolutions associated to each cohomology class.

Part III

Weak KAM for commuting Hamiltonians

IOP Publishing Nonlinearity
Nonlinearity **23** (2010) 793–808 doi:10.1088/0951-7715/23/4/002

Weak KAM for commuting Hamiltonians

M Zavidovique

UMPA, ENS Lyon, 46 allée d'Italie, 69007, Lyon, France

E-mail: maxime.zavidovique@umpa.ens-lyon.fr

Received 4 November 2009, in final form 26 December 2009
Published 25 February 2010
Online at stacks.iop.org/Non/23/793

Recommended by A Chenciner

Abstract
For two commuting Tonelli Hamiltonians, we recover the commutation of the Lax–Oleinik semi-groups, a result of Barles and Tourin (2001 *Indiana Univ. Math. J.* **50** 1523–44), using a direct geometrical method (Stoke's theorem). We also obtain a 'generalization' of a theorem of Maderna (2002 *Bull. Soc. Math. France* **130** 493–506). More precisely, we prove that if the phase space is the cotangent of a compact manifold then the weak KAM solutions (or viscosity solutions of the critical stationary Hamilton–Jacobi equation) for G and for H are the same. As a corollary we obtain the equality of the Aubry sets and of the Peierls barrier. This is also related to works of Sorrentino (2009 *On the Integrability of Tonelli Hamiltonians* Preprint) and Bernard (2007 *Duke Math. J.* **136** 401–20).

Mathematics Subject Classification: 35F21, 37J50, 37L50, 49L20, 49L25, 70H20

Introduction

It has been known for quite some time that the existence of first integrals affects the dynamics of Hamiltonian flows on the cotangent of a manifold. Indeed, the famous Arnol'd–Liouville theorem [Arn89] states the remarkable fact that under very mild compactness and connectedness conditions, if a Tonelli Hamiltonian H defined on the cotangent of an n-dimensional manifold M has n everywhere independent first integrals in involution, then the manifold is necessarily a torus. Moreover, the Hamiltonian flow is conjugated to a geodesic flow and T^*M is foliated by invariant tori on which the flow is linear.

In the past decades, new techniques have been developed in order to study the dynamics of a single Tonelli Hamiltonian and existence of invariant sets. Aubry–Mather theory (see [MF94, Mat91, Mañ97, Ban88] for introductions) has had a huge development. More recently, thanks to Albert Fathi's weak KAM theory (see [Fat08, FM07] for introductions

0951-7715/10/040793+16$30.00 © 2010 IOP Publishing Ltd & London Mathematical Society Printed in the UK & the USA

and [FS04, Ber07a, Mad02] for further developments) the link between the geometrical point of view of the Aubry–Mather theory and the widely studied PDE approach of Hamilton–Jacobi equations has made it possible to simplify the proofs of already known results (in both fields) and obtain new ones (see for example [Fat98, Fat03, FFR09, Ber08]). Moreover, a discrete version of weak KAM has already appeared fruitful in the related subject of optimal transportation [BB07b, BB07a, BB06, FF07].

The connection between Aubry–Mather theory and first integrals has not, to our knowledge, yet been much studied. First results (although not formulated this way) appear in [Ber07b] where it is shown that given a Tonelli Hamiltonian H on the cotangent space of a closed compact Manifold, the Aubry, Mather and Mañé sets are symplectic invariants. This may be directly applied to the Hamiltonian flows of Tonelli first integrals of H which are exact symplectomorphisms which preserve H. Recently, in [Sor09], it is shown, thanks to Aubry–Mather theory, that in the Arnol'd–Liouville theorem, if the involution hypothesis between the first integrals is dropped, much information can still be recovered on the dynamics of H and on its first integrals.

From the PDE point of view, in [BT01], the authors study on $M = \mathbb{R}^n$ the so-called multi-time Hamilton–Jacobi equation, that is, given Tonelli Hamiltonians $H_1, \ldots, H_k$ and an initial value $u_0 : \mathbb{R}^n \to \mathbb{R}$, they look for solutions $u : \mathbb{R}^n \times \mathbb{R}^k \to \mathbb{R}$ of the equation

$$\begin{aligned} &\forall x \in \mathbb{R}^n,\ u(x, 0, \ldots, 0) = u_0(x),\\ &\frac{\partial u}{\partial t_1} + H_1(x, d_x u) = 0,\\ &\vdots\\ &\frac{\partial u}{\partial t_k} + H_k(x, d_x u) = 0. \end{aligned}$$

By proving the existence of such functions, they actually obtain a commutation property for the Lax–Oleinik semi-groups used in the weak KAM theory. The same problem is studied under a less stringent regularity hypothesis in [MR06]. Let us now explain the setting we use and the results we obtain. Let M be a finite dimensional C^2 complete connected Riemmanian manifold. We will say that a Hamiltonian $H : T^*M \to \mathbb{R}$ is Tonelli if it is C^2 and if it verifies the following conditions:

1. *uniform superlinearity*: for every $K > 0$, there exists $C^*(K) \in \mathbb{R}$ such that
$$\forall (x, p) \in T^*M, \qquad H(x, p) \geqslant K\|p\| - C^*(K),$$
2. *uniform boundedness*: for every $R \geqslant 0$, we have
$$A^*(R) = \sup\{H(x, p), \|p\| \leqslant R\} < +\infty,$$
3. C^2*-strict convexity in the fibres*: for every $(x, p) \in T^*M$, the second derivative along the fibres $\partial^2 H/\partial p^2(x, p)$ is positive strictly definite.

We recall that T^*M is equipped with a canonical symplectic structure by setting $\Omega = -\mathrm{d}\lambda$ where λ is the canonical Liouville form. We may then define the Hamiltonian vector field X_H by

$$\forall (x, p) \in T^*M, \qquad \Omega(X_H(x, p), .) = \mathrm{d}_{(x,p)}H.$$

We may then define the Lax–Oleinik semi-group: if $u : M \to \mathbb{R} \cup \{-\infty, +\infty\}$ is a function, we set

$$\forall x \in M, \quad \forall s > 0, \qquad T_H^{-s}u(x) = \inf_{\substack{\gamma\\ \gamma(s) = x}} u(\gamma(0)) + \int_0^s L_H(\gamma(\sigma), \dot\gamma(\sigma))\,\mathrm{d}\sigma,$$

where the infimum is taken over all absolutely continuous curves reaching x and where $L_H : TM \to \mathbb{R}$ is the Lagrangian associated with H defined by

$$\forall (x, v) \in TM, \qquad L_H(x, v) = \max_{p \in T^*M} p(v) - H(x, p).$$

Now, if G and H are two Tonelli Hamiltonians, we will say that G and H Poisson commute if the function $\Omega(X_G, X_H)$ vanishes everywhere. We will give a direct proof of the following theorem which also results from [BT01] for $M = \mathbb{R}^n$ and from [CV08].

Theorem 0.1. *If G and H are two Tonelli Hamiltonians which Poisson commute, then their Lax–Oleinik semi-groups commute.*

In order to state the second theorem, we need to introduce the notion of weak KAM solution:

Definition 0.2. We say that a function $u : M \to \mathbb{R}$ is a weak KAM solution for H (respectively G) if and only if there is a constant $\alpha \in \mathbb{R}$ such that for any $t > 0$, we have

$$u = T_H^{-t} u + t\alpha$$

(respectively $u = T_G^{-t} u + t\alpha$).

Theorem 0.3. *If M is compact, then any weak KAM solution for G is a weak KAM solution for H.*

We then introduce the notion of subsolution with the following definition: $u : M \to \mathbb{R}$ is an (α, H)-subsolution (respectively (α, G)-subsolution) if

$$\forall (x, y, t) \in M^2 \times \mathbb{R}_+, \qquad u(y) - u(x) \leqslant \inf \int_0^t L_H(\gamma(\sigma), \dot\gamma(\sigma))\, \mathrm{d}\sigma + t\alpha.$$

where the infimum is taken on all absolutely continuous curves $\gamma : [0, t] \to M$ such that $\gamma(0) = x$ and $\gamma(t) = y$ (respectively $u(y) - u(x) \leqslant \inf \int_0^t L_G(\gamma(\sigma), \dot\gamma(\sigma))\, \mathrm{d}\sigma + t\alpha$.).

Theorem 0.4. *If there exists a function $u : M \to \mathbb{R}$ which is both an (α, G)-subsolution and an (α', H)-subsolution for some constants (α, α') then there is a $C^{1,1}$ function u' which is also both an (α, G)-subsolution and an (α', H)-subsolution.*

In [Fat98], Fathi gives a canonical way to pair positive weak KAM solutions with negative weak KAM solutions in the compact case. We prove in the last section that this pairing is the same for commuting Hamiltonians (see theorem 4.4). As a corollary, we establish that the Aubry sets, the Mañé sets and the Peierls barrier (defined in section 4) coincide for both Hamiltonians.

Finally, in the last section, we study some links between the Mather α functions (or effective Hamiltonians) of commuting Hamiltonians. More precisely, we show that their flat parts are the same.

While this paper was being written, similar results were obtained independently by Cui and Li in a preprint. For the current version of their work see [CL09].

1. Commutation property for the Lax–Oleinik semi-groups

Let M be a C^2 complete connected manifold. In the following, we will denote by λ the canonical Liouville form defined on TT^*M and by $\Omega = -\mathrm{d}\lambda$ the canonical symplectic form. Let H and G be two Tonelli Hamiltonians which commute. More precisely, this means that $\{G, H\} = \Omega(X_G, X_H) = 0$ where $\{., .\}$ denotes the Poisson bracket and X_G,

X_H the Hamiltonian vector fields of G, H. By basic properties of the Poisson bracket, we have $[X_G, X_H] = X_{\{G,H\}} = 0$. Therefore the Hamiltonian flows commute. Finally, from $\Omega(X_G, X_H) = 0$, by definition of the Hamiltonian vector field, we deduce that $\mathrm{d}H(X_G) = 0$ and $\mathrm{d}G(X_H) = 0$, which means that G is constant on the trajectories of X_H (or in other terms, G is a first integral of H) and vice versa.

We will denote by L_G and L_H the Lagrangians associated with G and H and by $\mathcal{L}_G$ and $\mathcal{L}_H$ the respective Legendre transforms, φ_G, Φ_G and φ_H, Φ_H will be, respectively, the Lagrangian and Hamiltonian flows of, respectively, G and H. Finally, T_G^- and T_H^- are the Lax–Oleinik semi-groups associated with L_G and L_H, that is if $u : M \to \mathbb{R} \cup \{-\infty, +\infty\}$ is a function,

$$\forall x \in M, \quad \forall s > 0, \qquad T_G^{-s} u(x) = \inf u(\gamma(0)) + \int_0^s L_G(\gamma(\sigma), \dot\gamma(\sigma))\, \mathrm{d}\sigma,$$

where the infimum is taken over all absolutely continuous curves $\gamma : [0, s] \to M$ with $\gamma(s) = x$. Obviously, the definition of the Lax–Oleinik semi-group associated with H is similar. For an exposition of these definitions, see [Fat08].

The following has already appeared in [BT01] in a different setting, with a different formulation and in [MR06] for less regular Hamiltonians (see also [CV08] for related results). It is mainly 0.1, let us reformulate it:

Theorem 1.1 (Barles–Tourin). *The Lax–Oleinik semi-groups commute, that is, if $u : M \to \mathbb{R} \cup \{-\infty, +\infty\}$ is a function and s, t are two positive real numbers then*

$$T_G^{-s} T_H^{-t} u = T_H^{-t} T_G^{-s} u.$$

In order to prove this statement, let us introduce the action functionals (we define it here for G, the definition for H is the same):

Definition 1.2. Let $s > 0$ and $(x, y) \in M^2$, then we set

$$A_G^s(x, y) = \inf \int_0^s L_G(\gamma(\sigma), \dot\gamma(\sigma))\, \mathrm{d}\sigma,$$

where the infimum is taken on all absolutely continuous curves $\gamma : [0, s] \to M$ with $\gamma(0) = x$ and $\gamma(s) = y$.

The proof of theorem 1.1 will be a straight consequence of the following lemma.

Lemma 1.3. *Let $s, t > 0$ be two positive real numbers then the following holds:*

$$\forall (x, z) \in M^2, \qquad \inf_{y \in M} A_G^s(x, y) + A_H^t(y, z) = \inf_{y \in M} A_H^t(x, y) + A_G^s(y, z).$$

Proof. Let us begin by recalling that the action functionals are locally semi-concave functions (see [Fat08] or [Ber08]) and therefore, if $(x_0, z_0) \in M^2$ and if y_0 reaches the infimum (which is always the case for some y_0) in the following:

$$\inf_{y \in M} A_G^s(x_0, y) + A_H^t(y, z_0) = A_G^s(x_0, y_0) + A_H^t(y_0, z_0),$$

then the following is verified:

$$\frac{\partial A_G^s}{\partial y}(x_0, y_0) + \frac{\partial A_H^t}{\partial x}(y_0, z_0) = 0 \tag{1}$$

and the partial derivatives do exist. Actually, more can be said. Let γ_1 and γ_2 verify that $\gamma_1(0) = x_0$, $\gamma_1(s) = y_0$, $\gamma_2(0) = y_0$, $\gamma_2(t) = z_0$ and

$$A_G^s(x_0, y_0) + A_H^t(y_0, z_0) = \int_0^s L_G(\gamma_1(\sigma), \dot\gamma_1(\sigma))\, \mathrm{d}\sigma + \int_0^t L_H(\gamma_2(\sigma), \dot\gamma_2(\sigma))\, \mathrm{d}\sigma.$$

The following holds (see [Fat08, proposition 4.11.1], [FF07, corollary B.20] or [Ber08]):

$$\left(y_0, \frac{\partial A_G^s}{\partial y}(x_0, y_0)\right) = \mathcal{L}_G(y_0, \dot{\gamma}_1(s))$$

and

$$\left(y_0, -\frac{\partial A_H^t}{\partial x}(y_0, z_0)\right) = \mathcal{L}_H(y_0, \dot{\gamma}_2(0)).$$

Finally, using the fact that the γ_i are minimizers hence trajectories of the respective Euler–Lagrange flows and (1), setting

$$p_0 = \frac{\partial A_G^s}{\partial y}(x_0, y_0) = -\frac{\partial A_H^t}{\partial x}(y_0, z_0)$$

we obtain that

$$\forall \sigma \in [0, s], \qquad \mathcal{L}_G(\gamma_1(\sigma), \dot{\gamma}_1(\sigma)) = \mathcal{L}_G(\varphi_G^{\sigma-s}(y_0, \dot{\gamma}_1(s))) = \Phi_G^{\sigma-s}(y_0, p_0), \tag{2}$$

$$\forall \sigma \in [0, t], \qquad \mathcal{L}_H(\gamma_2(\sigma), \dot{\gamma}_2(\sigma)) = \mathcal{L}_H(\varphi_H^{\sigma}(y_0, \dot{\gamma}_2(0))) = \Phi_H^{\sigma}(y_0, p_0). \tag{3}$$

Using the definition of L_G we have

$$\begin{aligned} A_G^s(x_0, y_0) &= \int_0^s L_G(\gamma_1(\sigma), \dot{\gamma}_1(\sigma))\, \mathrm{d}\sigma \\ &= \int_0^s \frac{\partial L_G}{\partial v}(\gamma_1(\sigma), \dot{\gamma}_1(\sigma))\dot{\gamma}_1(\sigma)\, \mathrm{d}\sigma - \int_0^s G\left(\gamma_1(\sigma), \frac{\partial L_G}{\partial v}(\gamma_1(\sigma), \dot{\gamma}_1(\sigma))\right) \mathrm{d}\sigma. \end{aligned}$$

We now recognize in the first integral the image of the Hamiltonian vector field under the Liouville form. Therefore, also using (2), that Lagrangian and Hamiltonian flows are conjugated by the Legendre transform and that G is constant on its Hamiltonian trajectories, we get that

$$\begin{aligned} A_G^s(x_0, y_0) &= \int_0^s \frac{\partial L_G}{\partial v}(\gamma_1(\sigma), \dot{\gamma}_1(\sigma))\dot{\gamma}_1(\sigma)\, \mathrm{d}\sigma - \int_0^s G\left(\gamma_1(\sigma), \frac{\partial L_G}{\partial v}(\gamma_1(\sigma), \dot{\gamma}_1(\sigma))\right) \mathrm{d}\sigma \\ &= \int_0^s \lambda(X_G(\Phi_G^{\sigma-s}(y_0, p_0)))\, \mathrm{d}\sigma - \int_0^s G\left(\Phi_G^{\sigma-s}(y_0, p_0)\right) \mathrm{d}\sigma \\ &= \int_0^s \lambda(X_G(\Phi_G^{\sigma-s}(y_0, p_0)))\, \mathrm{d}\sigma - sG(y_0, p_0). \end{aligned}$$

Reasoning along the same lines yields similarly that

$$A_H^t(y_0, z_0) = \int_0^t \lambda(X_H(\Phi_H^{\sigma}(y_0, p_0)))\, \mathrm{d}\sigma - tH(y_0, p_0).$$

Summing up, we have proved that

$$\begin{aligned} \inf_{y\in M} A_G^s(x_0, y) + A_H^t(y, z_0) \\ &= \int_0^s \lambda(X_G(\Phi_G^{\sigma-s}(y_0, p_0)))\, \mathrm{d}\sigma + \int_0^t \lambda(X_H(\Phi_H^{\sigma}(y_0, p_0)))\, \mathrm{d}\sigma \\ &\quad -sG(y_0, p_0) - tH(y_0, p_0). \end{aligned} \tag{4}$$

Now, let us set $(y_1, p_1) = \Phi_H^t \circ \Phi_G^{-s}(y_0, p_0)$. We define

$$\forall \sigma \in [0, s], \qquad \gamma_3(\sigma) = \pi_1(\Phi_G^{\sigma}(y_1, p_1))$$

and

$$\forall \sigma \in [0, t], \qquad \gamma_4(\sigma) = \pi_1(\Phi_H^{\sigma - t}(y_1, p_1)),$$

where $\pi_1 : T^*M \to M$ denotes the canonical projection on the manifold. First of all, let us note that since G and H Poisson commute, their Hamiltonian vector fields also commute, which means that the Hamiltonian flows commute. As a direct consequence, we have that $\gamma_3(s) = z_0$. Moreover, it is obvious from the definitions that $\gamma_4(0) = x_0$. Let us now compute the quantity A defined below. The same arguments as those exposed previously give that

$$\begin{aligned} A = &\int_0^t L_H(\gamma_4(\sigma), \dot\gamma_4(\sigma))\, \mathrm{d}\sigma + \int_0^s L_G(\gamma_3(\sigma), \dot\gamma_3(\sigma))\, \mathrm{d}\sigma \\ &= \int_0^t \lambda(X_H(\Phi_H^{\sigma-t}(y_1, p_1)))\, \mathrm{d}\sigma + \int_0^s \lambda(X_G(\Phi_G^{\sigma}(y_1, p_1)))\, \mathrm{d}\sigma \\ &- tH(y_1, p_1) - sG(y_1, p_1). \end{aligned}$$

Since G and H commute, they are, respectively, first integral of the other which proves that $G(y_0, p_0) = G(y_1, p_1)$ and that $H(y_0, p_0) = H(y_1, p_1)$. Now let us consider the function ψ defined from $R = [0, s] \times [0, t] \subset \mathbb{R}^2$ to T^*M by

$$\psi(\sigma, \sigma') = \Phi_G^{\sigma - s} \circ \Phi_H^{\sigma'}(y_0, p_0).$$

Using Stokes' formula, the following holds:

$$\begin{aligned} &\int_0^s \lambda(X_G(\Phi_G^{\sigma-s}(y_0, p_0)))\, \mathrm{d}\sigma + \int_0^t \lambda(X_H(\Phi_H^{\sigma}(y_0, p_0)))\, \mathrm{d}\sigma \\ &\qquad - \int_0^t \lambda(X_H(\Phi_H^{\sigma-t}(y_1, p_1)))\, \mathrm{d}\sigma - \int_0^s \lambda(X_G(\Phi_G^{\sigma}(y_1, p_1)))\, \mathrm{d}\sigma \\ &\qquad = \int_{\partial R} \psi^*\lambda = \int_R \psi^* \mathrm{d}\lambda = -\int_R \psi^*\Omega = 0. \end{aligned} \tag{5}$$

As a matter of fact, Ω vanishes identically on the tangent space to $\psi(R)$ which is at each point spanned by X_G and X_H.

To put it all in a nutshell, we have proved that

$$\begin{aligned} A &= \int_0^t L_H(\gamma_4(\sigma), \dot\gamma_4(\sigma))\, \mathrm{d}\sigma + \int_0^s L_G(\gamma_3(\sigma), \dot\gamma_3(\sigma))\, \mathrm{d}\sigma \\ &= \int_0^s \lambda(X_G(\Phi_G^{\sigma-s}(y_0, p_0)))\, \mathrm{d}\sigma + \int_0^t \lambda(X_H(\Phi_H^{\sigma}(y_0, p_0)))\, \mathrm{d}\sigma \\ &\quad - tH(y_1, p_1) - sG(y_1, p_1) \\ &= \int_0^s L_G(\gamma_1(\sigma), \dot\gamma_1(\sigma))\, \mathrm{d}\sigma + \int_0^t L_H(\gamma_2(\sigma), \dot\gamma_2(\sigma))\, \mathrm{d}\sigma \\ &= A_G^s(x_0, y_0) + A_H^t(y_0, z_0) = \inf_{y \in M} A_G^s(x_0, y) + A_H^t(y, z_0). \end{aligned}$$

Now, by the definition of the action functionals, the following inequality clearly holds:

$$\inf_{y \in M} A_G^s(x_0, y_0) + A_H^t(y_0, z_0) \geqslant \inf_{y \in M} A_H^t(x_0, y_0) + A_G^s(y_0, z_0).$$

By a symmetrical argument, the previous inequality is in fact an equality, which proves the lemma since (x_0, z_0) was taken arbitrarily. □

The proof of the theorem is now straightforward:

Proof of theorem 1.1. Let $u : M \to \mathbb{R} \cup \{-\infty, +\infty\}$ be any function. By definition of the Lax–Oleinik semi-groups if $x \in M$ is a point, the following equalities hold:

$$\begin{aligned} T_G^{-s} T_H^{-t} u(x) &= \inf_{z\in M} \inf_{y\in M} u(z) + A_H^t(z, y) + A_G^s(y, x) \\ &= \inf_{z\in M} \inf_{y\in M} u(z) + A_G^s(z, y) + A_H^t(y, x) = T_H^{-t} T_G^{-s} u(x). \end{aligned}$$ □

2. Subsolutions and weak KAM solutions

We explain here how most of the theory of subsolutions and viscosity solutions of the Hamilton–Jacobi equation can be adapted to the setting of two commuting Hamiltonians. Our presentation is mainly adapted from [FM07].

Until now, in order to prove the commutation of the Lax–Oleinik semi-groups in its full generality (theorem 1.1) , we did not assume any regularity or growth condition on the functions u on which the semi-groups act. As a counterpart, we had to consider functions taking values in $\mathbb{R} \cup \{-\infty, +\infty\}$. As a matter of fact, the image of any real valued function by the Lax–Oleinik semi-group of a Tonelli Hamiltonian may have infinite values. Let us stress the fact that from now on, we will only be dealing with globally bounded functions or sub-solutions which in particular are globally Lipschitz. It is known that starting with a globally bounded or a Lipschitz function, $u : M \to \mathbb{R}$, the families of functions $(T_G^{-s} u)_{s\geqslant 0}$ and $(T_H^{-t} u)_{t\geqslant 0}$ are real valued functions. Moreover, it can be proved that when u is Lipschitz, they are families of equi-Lipschitz functions (see [FM07, proposition 3.2]).

Let us recall that if α is a real number, we say that $u : M \to \mathbb{R}$ is an (α, G)-subsolution if

$$\forall (x, y, t) \in M^2 \times \mathbb{R}_+, \qquad u(y) - u(x) \leqslant A_G^t(x, y) + t\alpha.$$

We denote by $\mathcal{H}_G(\alpha)$ the set of (α, G)-subsolutions. Of course we can also define analogously (α, H)-subsolutions and we will denote by $\mathcal{H}_H(\alpha)$ the set of such functions. Finally, if $(\alpha, \alpha') \in \mathbb{R}^2$, we will denote by

$$\mathcal{H}(\alpha, \alpha') = \mathcal{H}_G(\alpha) \cap \mathcal{H}_H(\alpha').$$

Since G and H are Tonelli, for α and α' big enough, constant functions are both (α, G)-subsolutions and (α', H)-subsolutions, hence the set $\mathcal{H}(\alpha, \alpha')$ is not empty. As a matter of fact, it follows from the Tonelli hypothesis on the Hamiltonians that the associated Lagrangians are also uniformly superlinear (see [FM07, lemma 2.1]) hence bounded below. Therefore there is a constant C such that

$$\forall (x, v) \in TM, \qquad \min(L_G(x, v), L_H(x, v)) \geqslant C.$$

Hence, for any absolutely continuous curve $\gamma : [0, t] \to M$ the following inequality holds:

$$\int_0^t L_G(\gamma(\sigma), \dot\gamma(\sigma))\, \mathrm{d}\sigma \geqslant tC,$$

which may be rewritten as follows

$$0 \leqslant \int_0^t L_G(\gamma(\sigma), \dot\gamma(\sigma))\, \mathrm{d}\sigma - tC.$$

This implies directly that constant functions on M are $(-C, G)$-subsolutions and obviously the same holds for L_H.

If $\alpha \in \mathbb{R}$, following Fathi, we say a function $u : M \to \mathbb{R}$ is a (negative) (α, G)-weak KAM solution if

$$\forall t \geqslant 0, \qquad u = T_G^{-t} u + t\alpha.$$

We denote by $\mathcal{S}_G^-(\alpha)$ the set of (α, G)-weak KAM solutions. Obviously, we define analogously the notion of (α, H)-weak KAM solution and the set $\mathcal{S}_H^-(\alpha)$. Let us now state Fathi's weak KAM theorem (we state it for G) (see [Fat08] for a proof in the compact case and [FM07] for a proof in the non-compact case).

Theorem 2.1 (weak KAM). *There is a constant $\alpha_G[0]$ such that $\mathcal{H}_G(\alpha[0])$ is not empty and if $\alpha < \alpha_G[0]$ then $\mathcal{H}_G(\alpha)$ is empty. Moreover, the set $\mathcal{S}_G^-(\alpha_G[0])$ is not empty, that is,*

$$\exists u_- : M \to \mathbb{R}, \quad \forall s \geqslant 0, \qquad u_- = T_G^{-s} u_- + s\alpha_G[0].$$

Remark 2.2. Let us mention that $\alpha_G[0]$ is called Mañé's critical value, therefore we call an $(\alpha_G[0], G)$-subsolution a G-critical subsolution. Moreover, we will set $\mathcal{S}_G^-(\alpha_G[0]) = \mathcal{S}_G^-$. If M is compact, this notation is very natural, for if a function u is an (α, G)-weak KAM solution for some α then $\alpha = \alpha_G[0]$. However, let us stress that if M is not compact, then as soon as $\alpha \geqslant \alpha_G[0]$ then $\mathcal{S}_G^-(\alpha)$ is not empty. As a matter of fact, using the Mañé potential, it is possible to construct weak KAM solutions using a method inspired by the construction of Busemann functions in Riemannian geometry (see [Fat08, corollary 8.2.3]).

Let us begin with an easy lemma:

Lemma 2.3. *The following assertions are true:*

1. *Let $u : M \to \mathbb{R}$ be a real valued function and $\alpha \in \mathbb{R}$, then $u \in \mathcal{H}_G(\alpha)$ if and only if*
$$\forall s \geqslant 0, \qquad u \leqslant T_G^{-s} u + s\alpha.$$
2. *For any $t \geqslant 0$ and $\alpha \in \mathbb{R}$, the set $\mathcal{H}_G(\alpha)$ is stable by T_H^{-t}.*
3. *For any $t \geqslant 0$ and $\alpha \in \mathbb{R}$, the set $\mathcal{S}_G^-(\alpha)$ is stable by T_H^{-t}.*

Proof. The first part holds because by definition, u is an (α, G)-subsolution if and only if

$$\forall (x, y, s) \in M^2 \times \mathbb{R}_+, \qquad u(x) - u(y) \leqslant A_G^s(y, x) + s\alpha,$$

which by taking an infimum on y is equivalent to

$$\forall (x, s) \in M \times \mathbb{R}_+, \qquad u(x) \leqslant \inf_{y \in M} u(y) + A_G^s(y, x) + s\alpha = T_G^{-s} u(x) + s\alpha.$$

For the second part, by monotonicity of the Lax–Oleinik semi-group, using theorem 1.1, we obtain that if $u \leqslant T_G^{-s} u + s\alpha$ then

$$T_H^{-t} u \leqslant T_H^{-t}(T_G^{-s} u + s\alpha) = T_H^{-t}(T_G^{-s} u) + s\alpha = T_G^{-s}(T_H^{-t} u) + s\alpha.$$

The last point is a straightforward consequence of the commutation property of the semi-groups (theorem 1.1). If for any positive s, $u = T_G^{-s} u + s\alpha$ then

$$T_H^{-t} u = T_H^{-t}(T_G^{-s} u + s\alpha) = T_H^{-t}(T_G^{-s} u) + s\alpha = T_G^{-s}(T_H^{-t} u) + s\alpha. \qquad \square$$

We now prove a version of the weak KAM theorem for commuting Hamiltonians when the manifold M is compact. The proof is very similar to the proof of the classical weak KAM theorem (see [Fat08] or [FM07]). Let us recall that in the compact case, the Lax–Oleinik semi-groups are non-expansive for the infinity norm [Fat08, proposition 4.6.5]. This is in fact important since it will enable us to apply the following theorem of DeMarr [DeM63]:

Theorem 2.4 (DeMarr). *Let B be a Banach space and $(f_a)_{a\in A}$ a family of commuting non-expansive continuous functions on B which preserve a compact convex subset $C \subset B$; then these semi-groups have a common fixed point in C.*

Theorem 2.5 (double weak KAM). *Let us assume M is compact. There is a function $u_- : M \mapsto \mathbb{R}$ which is both a weak KAM solution for G and H.*

Proof. Take $(\alpha, \alpha') \in \mathbb{R}^2$ such that $\mathcal{H}(\alpha, \alpha')$ is not empty. It is known [FM07] that $\mathcal{H}(\alpha, \alpha')$ is made of equi-Lipschitz functions. Therefore, by the Arzela–Ascoli theorem, the set

$$\widehat{\mathcal{H}}(\alpha, \alpha') = \mathcal{H}(\alpha, \alpha')/\mathbb{R}1$$

is compact for the compact open topology (where 1 denotes the function constantly equal to 1 on M). Moreover, since $\mathcal{H}(\alpha, \alpha')$ is convex, the same holds for $\widehat{\mathcal{H}}(\alpha, \alpha')$. Finally, since by lemma 2.3, $\mathcal{H}(\alpha, \alpha')$ is stable by the semi-groups, which commute with the addition of constants, they induce two semi-groups which still commute and leave $\widehat{\mathcal{H}}(\alpha, \alpha')$ stable. Since the Lax–Oleinik semi-groups are non-expansive, we can therefore apply DeMarr's theorem for commutative families of non-expansive maps [DeM63]:

$$\exists u_- \in \mathcal{H}(\alpha, \alpha'), \quad \exists (\beta, \beta') \in \mathbb{R}^2, \quad \forall s \geqslant 0, \qquad u_- = T_G^{-s} u_- + s\beta$$
$$\forall t \geqslant 0, \qquad u_- = T_H^{-t} u_- + t\beta'.$$

The function u_- is the double weak KAM solution we are looking for. □

Remark 2.6. Since M is compact, using remark 2.2, we obtain that in the previous proof, $\beta = \alpha_G[0]$ and $\beta' = \alpha_H[0]$.

Actually, in the compact case the link between weak KAM solutions for G and H is much simpler to understand due to the following theorem which is a reformulation of 0.3.

Theorem 2.7. *If M is compact and $u_- : M \to \mathbb{R}$ is a weak KAM solution for G then it is also a weak KAM solution for H: $u_- \in \mathcal{S}^-$. In short, the following equalities hold:*

$$\mathcal{S}_G^- = \mathcal{S}_H^- = \mathcal{S}^-.$$

In order to prove this theorem we need to recall a few facts about Aubry–Mather theory. We call Mather set for G

$$\widehat{\mathcal{M}}_G = \overline{\bigcup_{\mu} \operatorname{supp}\mu}$$

the closed union of all supports of minimizing probability measures on T^*M invariant by Φ_G and $\mathcal{M}_G$ its projection on M. Clearly, $\widehat{\mathcal{M}}_G$ is invariant by Φ_G but it actually is a symplectic invariant (see [Ber07b, Sor09]) and therefore it is also invariant by Φ_H. Mather proved [Mat91] that $\widehat{\mathcal{M}}_G$ is a compact Lipschitz graph over $\mathcal{M}_G$. Finally, Fathi [Fat08] proved that if $u : M \to \mathbb{R}$ is a critical subsolution for G and if $x \in \mathcal{M}_G$ is in the projected Mather set then the function u is differentiable at x, and $(x, d_x u) \in \widehat{\mathcal{M}}_G$, therefore the differential is independent of the critical subsolution.

Finally, let us state that $\mathcal{M}_G$ is a uniqueness set for the stationary critical Hamilton–Jacobi equation associated with G, which means that if two G-weak KAM solutions coincide on $\mathcal{M}_G$, they are in fact equal.

With these facts in mind, we are now able to prove the theorem.

Proof of theorem 2.7. Let u_0 be a double weak KAM solution given by theorem 2.5. By what was mentioned above, for any $s \geqslant 0$ the function

$$v_s = T_H^{-s} u_- - T_H^{-s} u_0 = T_H^{-s} u_- + s\alpha_H[0] - u_0$$

is differentiable on $\mathcal{M}_G$ with a vanishing differential. Let $(x, p) \in \widehat{\mathcal{M}}_G$ and set

$$\forall s \in \mathbb{R}, \qquad (x(s), p(s)) = \Phi_H^s(x, p) \in \widehat{\mathcal{M}}_G,$$

then it is known (see [Fat08, 4.11.1], [FF07, corollary B.20], [Ber08]) that

$$\begin{aligned}\forall s > 0, \qquad v_s(x) &= T_H^{-s}u_-(x) - T_H^{-s}u_0(x)\\ &= u_-(x(-s)) + \int_{-s}^0 L_H(\mathcal{L}_H^{-1}(\Phi_H^\sigma(x, p)))\,\mathrm{d}\sigma\\ &\quad - u_0(x(-s)) - \int_{-s}^0 L_H(\mathcal{L}_H^{-1}(\Phi_H^\sigma(x, p)))\,\mathrm{d}\sigma\\ &= u_-(x(-s)) - u_0(x(-s)) = v_0(x(-s)).\end{aligned}$$

Since the trajectory $s \mapsto x(s)$, $s \in \mathbb{R}$ is C^2, has its image included in $\mathcal{M}_G$ and the function v_0 has a vanishing differential on it, we can deduce that v_0 is constant on the image of $s \mapsto x(s)$. Therefore,

$$\begin{aligned}\forall s > 0, \qquad v_s(x) &= T_H^{-s}u_-(x) - T_H^{-s}u_0(x)\\ &= T_H^{-s}u_-(x) + s\alpha_H[0] - u_0(x)\\ &= u_-(x) - u_0(x).\end{aligned}$$

In short,

$$\forall x \in \mathcal{M}_G, \qquad u_-(x) = T_H^{-s}u_-(x) + s\alpha_H[0]. \tag{6}$$

We have proved the invariance on $\mathcal{M}_G$, it remains to prove the same on its complementary. But the equality of u_- and of $T_H^{-s}u_- + s\alpha_H[0]$ everywhere follows directly from the facts that they are both G-weak KAM solutions (lemma 2.3) and that two G-weak KAM solutions that coincide on $\mathcal{M}_G$ must coincide everywhere [Fat08, theorem 4.12.6]. □

Corollary 2.8. *If M is compact and if u_- is a G-weak KAM solution then the graph of the differential of u_-, $\Gamma(u_-)$ verifies the following:*

$$\forall t \leqslant 0, \qquad \Phi_H^{-t}(\Gamma(u_-)) \subset \Gamma(u_-). \tag{7}$$

Note that $(\Phi_H^{-t})_{t\geqslant 0}$ is a one-parameter semi-group of symplectomorphisms preserving G. Moreover, the same holds for $\overline{\Gamma(u_-)}$.

Proof. It is proved in [Fat08, theorem 4.13.2] that if $u_- \in \mathcal{S}_H^-$ is a weak KAM solution for H, then

$$\forall t \leqslant 0, \qquad \Phi_H^{-t}(\Gamma(u_-)) \subset \Gamma(u_-).$$

Let us recall the main steps of this proof. If $x \in M$ is a differentiability point of u_-, the following holds:

$$\forall s > 0, \qquad u_-(x) = T_H^{-s}u_-(x) = u_-(x(-s)) + \int_{-s}^0 L_H(\mathcal{L}_H^{-1}(\Phi_H^\sigma(x, \mathrm{d}_x u_-)))\,\mathrm{d}\sigma,$$

where we have used the following notation:

$$\forall s \in \mathbb{R}, \qquad (x(s), p(s)) = \Phi_H^s(x, \mathrm{d}_x u).$$

Moreover, it can be proved that for all $s \geqslant 0$, the point $x(-s)$ is then a differentiability point u_- which verifies $\mathrm{d}_{x(-s)}u_- = p(-s)$ (one could use the fact that u_- is locally semi-concave and that $p(-s)$ is in the sub-differential of u_- at $x(-s)$ [Fat08, proposition 4.11.1]). This proves the inclusion (7) since by theorem 2.7 any weak KAM solution for G is also a weak KAM solution for H. The end of the corollary is straightforward. □

Remark 2.9. The proof of theorem 2.7 is very similar to the one given in [Mad02] of a result (in the compact case) concerning the stability of weak KAM solutions by diffeomorphisms of the base space. More precisely, let $\Gamma^G(M)$ denote the set of C^1 diffeomorphisms which preserve G equipped with the topology of uniform convergence on compact subsets. Let Γ_0^G denote the identity component of Γ^G. Then if $g \in \Gamma_0^G$, any weak KAM solution for G is stable by g. In this case, we have corollary 2.8 as a similar statement. Indeed it asserts that for a certain class of symplectomorphisms preserving G, the graph of the differential of a weak KAM solution is also, in a weak sense, stable by these symplectomorphisms.

3. Positive time Lax–Oleinik semi-groups and $C^{1,1}$ subsolutions

As usual in weak KAM theory, there is a positive time analogue for every result proved. Let us see how this applies to commuting Hamiltonians. Here again, we follow the exposition from [Fat08].

Given a Tonelli Hamiltonian, $G : T^*M \to \mathbb{R}$ and its associated Lagrangian $L_G : TM \to \mathbb{R}$, we can define the symmetrical Hamiltonian and Lagrangian as follows:

$$\forall (x, p) \in T^*M, \qquad \widehat{G}(x, p) = H(x, -p),$$
$$\forall (x, v) \in TM, \qquad \widehat{L}_G(x, v) = L(x, -v).$$

Obviously, $\widehat{G}$ and $\widehat{L}_G$ are once again Legendre transform of one another and are still Tonelli. We may now define the positive time Lax–Oleinik semi-group of a function $u : M \to \mathbb{R}$:

$$\forall s > 0, \quad \forall x \in M, \qquad T_G^{+s}u(x) = -T_{\widehat{G}}^{-s}(-u)(x).$$

Now, if H is another Tonelli Hamiltonian which Poisson commutes with G, it is clear that $\widehat{G}$ and $\widehat{H}$ also Poisson commute. Therefore, we have the following results:

Theorem 3.1 (Barles–Tourin). *The positive time Lax–Oleinik semi-groups commute, that is, if $u : M \to \mathbb{R}$ is a function and s, t are two positive real numbers then*

$$T_G^{+s}T_H^{+t}u = T_H^{+t}T_G^{+s}u.$$

If $\alpha \in \mathbb{R}$, following Fathi, we say a function $u : M \to \mathbb{R}$ is a positive time (α, G)-weak KAM solution if

$$\forall t \geqslant 0, \qquad u = T_G^{+t}u - t\alpha.$$

We denote by $\mathcal{S}_G^+(\alpha)$ the set of positive time (α, G)-weak KAM solutions. Obviously, we define analogously the notion of positive time (α, H)-weak KAM solution and the set $\mathcal{S}_H^+(\alpha)$. Let us now state Fathi's weak KAM theorem (we state it for G).

Theorem 3.2 (positive time weak KAM). *The set $\mathcal{S}_G^+(\alpha_G[0])$ is not empty, that is,*

$$\exists u_+ : M \to \mathbb{R}, \qquad \forall s \geqslant 0, \qquad u_+ = T_G^{+s}u_+ - s\alpha_G[0].$$

Remark 3.3. Let us mention that $\alpha_G[0]$ is again Mañé's critical value. Moreover we will set $\mathcal{S}_G^+(\alpha_G[0]) = \mathcal{S}_G^+$. If M is compact, this notation is very natural, since as for negative time, if a function u is a positive time (α, G)-weak KAM solution for some α then $\alpha = \alpha_G[0]$.

Lemma 3.4. *The following assertions are true:*

1. *Let $u : M \to \mathbb{R}$ be a real valued function and $\alpha \in \mathbb{R}$, then $u \in \mathcal{H}_G(\alpha)$ if and only if*

$$\forall s \geqslant 0, \qquad u \geqslant T_G^{+s}u - s\alpha.$$

2. *For any $t \geqslant 0$ and $\alpha \in \mathbb{R}$, the set $\mathcal{H}_G(\alpha)$ is stable by T_H^{+t}.*
3. *For any $t \geqslant 0$ and $\alpha \in \mathbb{R}$, the set $\mathcal{S}_G^+(\alpha)$ is stable by T_H^{+t}.*

Theorem 3.5. *If M is compact and $u_+ : M \to \mathbb{R}$ is a positive time weak KAM solution for G then it is also a positive time weak KAM solution for H: $u \in \mathcal{S}^+$. In short, the following equalities hold:*

$$\mathcal{S}_G^+ = \mathcal{S}_H^+ = \mathcal{S}^+.$$

Equipped with the positive time Lax–Oleinik semi-groups, we are now able to prove existence theorems of $C^{1,1}$ common subsolutions for G and H (this is theorem 0.4).

Theorem 3.6 (existence of common $C^{1,1}$ subsolutions). *Assume that the pair $(\alpha, \alpha') \subset \mathbb{R}^2$ is such that $\mathcal{H}(\alpha, \alpha') \neq \varnothing$, then there is a locally $C^{1,1}$ function in $\mathcal{H}(\alpha, \alpha')$. Moreover, $\mathcal{H}(\alpha, \alpha') \cap C^{1,1}(M, \mathbb{R})$ is dense in $\mathcal{H}(\alpha, \alpha')$ for the compact open topology.*

Proof. The proof is just a simple adaptation of [FFR09] which itself is very much inspired from [Ber07a]. The idea is to use successively positive and negative Lax–Oleinik semi-groups in order to realize a kind of Lasry–Lions regularization. More precisely, if $u_0 \in \mathcal{H}(\alpha, \alpha')$, it is proved in [FFR09] that for a suitable choice of 'small' positive constants, $(\varepsilon_k^-)_{k\in\mathbb{N}^*}$ and $(\varepsilon_k^+)_{k\in\mathbb{N}^*}$, the sequence of functions

$$\forall n \in \mathbb{N}, \qquad u_n = T_G^{+\varepsilon_n} T_G^{-\varepsilon_n} \cdots T_G^{+\varepsilon_1} T_G^{-\varepsilon_1} u_0$$

converges (for the compact open topology) to a $C^{1,1}$ function u_∞ which is an (α, G)-subsolution.

Moreover, by lemmas 2.3 and 3.4, the functions u_n are also (α', H)-subsolutions, which proves that u_∞ is itself an (α', H)-subsolutions ($\mathcal{H}(\alpha, \alpha')$ is closed for the compact open topology). Therefore, u_∞ is a $C^{1,1}$ function which belongs to $\mathcal{H}(\alpha, \alpha')$.

To prove the density result, just note that by continuity of the Lax–Oleinik semi-groups as maps from $[0, +\infty[\times\mathcal{H}(\alpha, \alpha')$ to $\mathcal{H}(\alpha, \alpha')$ (for the compact open topology on $\mathcal{H}(\alpha, \alpha')$, see [FM07, proposition 3.3]), by taking smaller sequences $(\varepsilon_k^-)_{k\in\mathbb{N}^*}$ and $(\varepsilon_k^+)_{k\in\mathbb{N}^*}$, we can actually construct u_∞ arbitrarily close to u_0. □

4. More on the compact case

Throughout this section, we assume M is compact. Moreover, up to adding constants to G and H, we will assume that $\alpha_G[0] = \alpha_H[0] = 0$. In [Fat97] Fathi proved the following:

Theorem 4.1 (paired weak KAM solutions). *Given a critical subsolution u_G for G (respectively u_H for H), there exist a unique negative weak KAM solution u_G^- and a unique positive weak KAM solution u_G^+ (respectively u_H^- and u_H^+) such that*

$$u_{G|\mathcal{M}_G} = u^-_{G|\mathcal{M}_G} = u^+_{G|\mathcal{M}_G},$$

(respectively $u_{H|\mathcal{M}_H} = u^-_{H|\mathcal{M}_H} = u^+_{H|\mathcal{M}_H}$). We denote this relation by $u_G^- \sim_G u_G^+$ (respectively $u_H^- \sim_H u_H^+$). Moreover, we have that

$$\lim_{t\to+\infty} T_G^{-t} u_G = u_G^-,$$
$$\lim_{t\to+\infty} T_G^{+t} u_G = u_G^+$$

(respectively $\lim_{t\to+\infty} T_H^{-t} u_H = u_H^-$ and $\lim_{t\to+\infty} T_H^{+t} u_H = u_H^+$) where the limits hold for the infinity norm.

Finally, let us mention that paired weak KAM solutions are characterized by those limits, more precisely

$$u_G^- \sim_G u_G^+ \iff \lim_{t\to+\infty} T_G^{-t} u_G^+ = u_G^- \iff \lim_{t\to+\infty} T_G^{+t} u_G^- = u_G^+,$$

(respectively $u_H^- \sim_H u_H^+ \iff \lim_{t\to+\infty} T_H^{-t} u_H^+ = u_H^- \iff \lim_{t\to+\infty} T_H^{+t} u_H^- = u_H^+$).

Let us define (for G) the Aubry and Mañé sets and the Peierls barrier:

Definition 4.2. The Aubry set is defined by

$$\widehat{\mathcal{A}}_G = \bigcap_{u^- \sim_G u^+} \left\{(x, d_x u^-), u^-(x) = u^+(x)\right\},$$

the Mañé set is defined by

$$\widehat{\mathcal{N}}_G = \bigcup_{u^- \sim_G u^+} \left\{(x, d_x u^-), u^-(x) = u^+(x)\right\}$$

and finally the Peierls barrier is defined by

$$\forall (x, y) \in M^2,\ h(x, y) = \sup_{u^- \sim_G u^+} u^-(y) - u^+(x).$$

Remark 4.3. In the definitions of the Aubry and Mañé sets it must be justified why the differentials of u^- exist. This comes from the facts that if (u^-, u^+) are paired weak KAM solutions, then u^- (respectively u^+) is locally semi-concave (respectively locally semi-convex, see [Fat08, proposition 6.2.1]), with $u^- \geqslant u^+$. Hence u^- is differentiable on the set $\{x \in M, u^-(x) = u^+(x)\}$.

Usually, the Aubry, Mañé sets and the Peierls barrier are rather defined on the tangent bundle of M using the Lagrangian setting and the action functionals. However, for simplicity of the exposition, we only give here these equivalent definitions from the weak KAM point of view.

The result we are going to prove is that both relations $\sim_G$ and $\sim_H$ are the same:

Theorem 4.4. *Let u^- and u^+ be a negative and a positive weak KAM solution, then if $u^- \sim_G u^+$ then $u^- \sim_H u^+$.*

Before giving the proof of this theorem, let us recall another result of Fathi [Fat98]. Note that we are still assuming that $\alpha_G[0] = \alpha_H[0] = 0$.

Theorem 4.5. *Let $u : M \to \mathbb{R}$ be a continuous function, then the functions $T_G^{-t}u$ converge as t goes to $+\infty$ to a function u^- which moreover is a negative weak KAM solution. Obviously, the same holds for the time positive Lax–Oleinik semi-group and for the semi-groups associated with H.*

Now, using the last part of theorem 4.1, the proof of theorem 4.4 is a direct consequence of the following proposition (and of its analogue for the positive time Lax–Oleinik semi-groups):

Proposition 4.6. *Let $u : M \to \mathbb{R}$ be a continuous function, and let u_G^- be the limit of the $T_G^{-t}u$ (respectively u_H^- be the limit of the $T_H^{-t}u$). Then we have that $u_G^- = u_H^-$.*

Proof. We begin with the proof that for any positive s, the functions $T_G^{-t}(T_H^{-s}u)$ still converge to u_G^- as t goes to infinity.

It is a simple consequence of the commutation property of the semi-groups (0.1). As a matter of fact, for all t the following holds

$$T_G^{-t}(T_H^{-s}u) = T_H^{-s}(T_G^{-t}u)$$

and by continuity of the Lax–Oleinik semi-group, the functions $T_H^{-s}(T_G^{-t}u)$ converge to $T_H^{-s}(u_G^-) = u_G^-$ by theorem 2.7.

Now, let us recall that the Lax–Oleinik semi-groups are 1-Lipschitz for the infinity norm, therefore we have that

$$\forall (s, t) \in \mathbb{R}_+^2, \qquad \|T_G^{-t}(T_H^{-s}u) - u_H^-\|_\infty = \|T_G^{-t}(T_H^{-s}u) - T_G^{-t}u_H^-\|_\infty \leqslant \|(T_H^{-s}u) - u_H^-\|_\infty.$$

Letting t go to $+\infty$ we obtain that

$$\forall s \in \mathbb{R}_+, \qquad \|u_G^- - u_H^-\|_\infty \leqslant \|(T_H^{-s}u) - u_H^-\|_\infty \underset{s\to+\infty}{\longrightarrow} 0,$$

this proves the result. □

We end this section by the following theorem which is a straight consequence of the definitions and of theorem 4.4:

Theorem 4.7. *The following equalities hold:*

$$\widehat{\mathcal{A}}_G = \widehat{\mathcal{A}}_H, \qquad \widehat{\mathcal{N}}_G = \widehat{\mathcal{N}}_H, \qquad h_G = h_H.$$

5. Flats of Mather's α function

In this section, the underlying manifold M is still compact. We will need the following notation: if H is a Tonelli Hamiltonian, we will denote the set $\widetilde{\mathcal{A}}_H \subset TM$ defined as follows:

$$\widetilde{\mathcal{A}}_H = \mathcal{L}_H^{-1}\left(\widehat{\mathcal{A}}_H\right).$$

Given a Tonelli Hamiltonian H and a closed 1-form $\omega : T^*M \to \mathbb{R}$, Mather noticed in [Mat91] that the Hamiltonian H_ω defined by

$$\forall (x, p) \in T^*M, \qquad H_\omega(x, p) = H(x, p + \omega_x)$$

is still Tonelli, therefore it admits a critical value and this critical value depends only on the cohomology class of ω which we denote $[\omega] \in H^1(M, \mathbb{R})$. We call $\alpha_H[\omega]$ the critical value of H_ω. Mather also proves that the function $\alpha_H : H^1(M, \mathbb{R}) \to \mathbb{R}$ is convex and superlinear. The function α_H is sometimes called an effective Hamiltonian in homogenization theory. A flat of the function α_H is a convex set $C \subset H^1(M, \mathbb{R})$ on which α_H is linear. The following theorem is proved in [Mas03, Ber02]:

Theorem 5.1. *Assume that Mather's α_H function is affine between two cohomology classes $[\omega_0]$ and $[\omega_1]$, that is*

$$\forall t \in (0, 1), \qquad \alpha_H[t\omega_0 + (1-t)\omega_1] = t\alpha_H[\omega_0] + (1-t)\alpha_H[\omega_1]$$

then the following holds:

$$\forall t \in (0, 1), \qquad \widetilde{\mathcal{A}}_{H_{\omega_t}} \subset \widetilde{\mathcal{A}}_{H_{\omega_0}} \cap \widetilde{\mathcal{A}}_{H_{\omega_1}},$$

where we have used the notation $\omega_t = t\omega_0 + (1-t)\omega_1$.

In particular, the Aubry sets are constant in the relative interior of a flat of the α_H function and the α_H function must be constant on this flat.

We now want to study the relation between flats of α functions of two commuting Hamiltonians. The following proposition shows that the question is legitimate.

Proposition 5.2. *Let G and H be two commuting Tonelli Hamiltonians. Then for any closed 1-form, ω on M, we have that G_ω and H_ω Poisson commute.*

Proof. It is a direct consequence of the definition of the Poisson bracket and the fact that when ω is closed, the map $\psi_\omega : T^*M \to T^*M$ defined by

$$\forall (x, p) \in T^*M, \qquad \psi_\omega(x, p) = (x, p + \omega_x)$$

is symplectic. □

We may now state the main result of this section.

Theorem 5.3. *Let M be a compact C^2 closed manifold and G, H two commuting Tonelli Hamiltonians on T^*M. Let us denote by $C_G \subset H^1(M, \mathbb{R})$ a flat of α_G on which it is therefore constant. Then C_G is also a flat of α_H.*

Proof. Let us consider ω_1, ω_2 two closed forms whose cohomology classes belong to the relative interior of C_G. As seen in theorem 5.1, we then have the following equality:

$$\widetilde{\mathcal{A}}_{G_{\omega_1}} = \widetilde{\mathcal{A}}_{G_{\omega_2}}$$

which after taking the Legendre transform yields

$$\widehat{\mathcal{A}}_{G_{\omega_1}} + \omega_1 = \widehat{\mathcal{A}}_{G_{\omega_2}} + \omega_2 \tag{8}$$

(where we denote $\widehat{\mathcal{A}}_{G_{\omega_1}} + \omega_1 = \{(x, p + \omega_{1,x}), (x, p) \in \widehat{\mathcal{A}}_{G_{\omega_1}}\}$ and $\widehat{\mathcal{A}}_{G_{\omega_2}} + \omega_2 = \{(x, p + \omega_{2,x}), (x, p) \in \widehat{\mathcal{A}}_{G_{\omega_2}}\}$). Now using (8), proposition 5.2 and theorem 4.7 we obtain that

$$\widehat{\mathcal{A}}_{H_{\omega_1}} + \omega_1 = \widehat{\mathcal{A}}_{G_{\omega_1}} + \omega_1 = \widehat{\mathcal{A}}_{G_{\omega_2}} + \omega_2 = \widehat{\mathcal{A}}_{H_{\omega_2}} + \omega_2.$$

But we also know that if $(x, p) \in \widehat{\mathcal{A}}_{H_{\omega_1}}$ then $(x, p') = (x, p + (\omega_1 - \omega_2)(x)) \in \widehat{\mathcal{A}}_{H_{\omega_2}}$ and the following holds (where we use Carneiro's theorem [Car95] stating that the Aubry set lies in the critical energy level of the Hamiltonian):

$$\alpha_H[\omega_1] = H_{\omega_1}(x, p) = H(x, p + \omega_{1,x}) = H(x, p' + \omega_{2,x}) = H_{\omega_2}(x, p') = \alpha_H[\omega_2].$$

This proves that α_H is constant on C_G. □

Acknowledgments

I would first like to thank Nalini Anantharaman for pointing out to me that it would be interesting to study the weak KAM theory for commuting Hamiltonians. I would like to thank Albert Fathi for his careful reading of the manuscript and for his comments and remarks during my research on this subject. I also thank Bruno Sévennec for useful discussions. This paper was partially elaborated during a stay at the Sapienza University in Rome. I wish to thank Antonio Siconolfi, Andrea Davini for useful discussions and the Dipartimento di Matematica 'Guido Castelnuovo' for its hospitality while I was there. I also would like to thank Explora'doc which partially supported me during this stay. Finally, I would like to thank the ANR KAM faible (Project BLANC07-3_187245, Hamilton–Jacobi and Weak KAM theory) for its support during my research.

References

[Arn89] Arnol'd V I 1989 *Mathematical Methods of Classical Mechanics* (*Graduate Texts in Mathematics* vol 60) 2nd edn (New York: Springer) (Translated from the Russian by Vogtmann K and Weinstein A)

[Ban88] Bangert V 1988 Mather sets for twist maps and geodesics on tori *Dynamics Reported* vol 1 (*Dynam. Report. Series Dynam. Systems Appl.* vol 1) (Chichester: Wiley) pp 1–56

[BB06] Bernard P and Buffoni B 2006 The Monge problem for supercritical Mañé potentials on compact manifolds *Adv. Math.* **207** 691–706

[BB07a] Bernard P and Buffoni B 2007 Optimal mass transportation and Mather theory *J. Eur. Math. Soc. (JEMS)* **9** 85–121

[BB07b] Bernard P and Buffoni B 2007 Weak KAM pairs and Monge-Kantorovich duality *Asymptotic Analysis and Singularities—Elliptic and Parabolic PDEs and Related Problems* (*Advanced Studies in Pure Mathematics* vol 47) (Japan: Mathematical Society of Japan) pp 397–420

[Ber02] Bernard P 2002 Connecting orbits of time dependent Lagrangian systems *Ann. Inst. Fourier (Grenoble)* **52** 1533–68

[Ber07a] Bernard P 2007 Existence of $C^{1,1}$ critical sub-solutions of the Hamilton-Jacobi equation on compact manifolds *Ann. Sci. École Norm. Sup. (4)* **40** 445–52

[Ber07b] Bernard P 2007 Symplectic aspects of Mather theory *Duke Math. J.* **136** 401–20

[Ber08] Bernard P 2008 The dynamics of pseudographs in convex Hamiltonian systems *J. Am. Math. Soc.* **21** 615–69

[BT01] Barles G and Tourin A 2001 Commutation properties of semigroups for first-order Hamilton-Jacobi equations and application to multi-time equations *Indiana Univ. Math. J.* **50** 1523–44

[Car95] Dias Carneiro M J 1995 On minimizing measures of the action of autonomous Lagrangians *Nonlinearity* **8** 1077–85

[CL09] Cui X and Li J 2009 *On commuting Tonelli Hamiltonians: Autonomous Case* arXiv:0911.3471v2 [math.DS]

[CV08] Cardin F and Viterbo C 2008 Commuting Hamiltonians and Hamilton–Jacobi multi-time equations *Duke Math. J.* **144** 235–84

[DeM63] DeMarr R 1963 Common fixed points for commuting contraction mappings *Pacific J. Math.* **13** 1139–41

[Fat97] Fathi A 1997 Solutions KAM faibles conjuguées et barrières de Peierls *C. R. Acad. Sci. Paris Sér. I Math.* **325** 649–52

[Fat98] Fathi A 1998 Sur la convergence du semi-groupe de Lax-Oleinik *C. R. Acad. Sci. Paris Sér. I Math.* **327** 267–70

[Fat03] Fathi A 2003 Regularity of C^1 solutions of the Hamilton-Jacobi equation *Ann. Fac. Sci. Toulouse Math.* **12** 479–516

[Fat08] Fathi A 2008 *Weak KAM Theorem in Lagrangian Dynamics* preliminary version 10 (Lyon), unpublished

[FF07] Fathi A and Figalli A 2010 Optimal transportation on non-compact manifolds *Israel J. Math.* to appear

[FFR09] Fathi A, Figalli A and Rifford L 2009 On the Hausdorff dimension of the Mather quotient *Commun. Pure Appl. Math.* **62** 445–500

[FM07] Fathi A and Maderna E 2007 Weak KAM theorem on non compact manifolds *NoDEA* **14** 1–27

[FS04] Fathi A and Siconolfi A 2004 Existence of C^1 critical subsolutions of the Hamilton–Jacobi equation *Invent. Math.* **155** 363–88

[Mad02] Maderna E 2002 Invariance of global solutions of the Hamilton–Jacobi equation *Bull. Soc. Math. France* **130** 493–506

[Mañ97] Mañé R 1997 Lagrangian flows: the dynamics of globally minimizing orbits *Bol. Soc. Brasil. Mat. (N.S.)* **28** 141–53

[Mas03] Massart D 2003 On Aubry sets and Mather's action functional *Israel J. Math.* **134** 157–71

[Mat91] Mather J N 1991 Action minimizing invariant measures for positive definite Lagrangian systems *Math. Z.* **207** 169–207

[MF94] Mather J N and Forni G 1994 Action minimizing orbits in Hamiltonian systems *Transition to Chaos in Classical and Quantum Mechanics (Montecatini Terme, 1991)* (*Lecture Notes in Mathematics* vol 1589) (Berlin: Springer) pp 92–186

[MR06] Motta M and Rampazzo F 2006 Nonsmooth multi-time Hamilton-Jacobi systems *Indiana Univ. Math. J.* **55** 1573–614

[Sor09] Sorrentino A 2009 *On the Integrability of Tonelli Hamiltonians* arXiv:0903.4300v1 [math.DS]

Bibliography

[Arn63a] V. I. Arnol′d. Proof of a theorem of A. N. Kolmogorov on the preservation of conditionally periodic motions under a small perturbation of the Hamiltonian. *Uspehi Mat. Nauk*, 18(5 (113)):13–40, 1963.

[Arn63b] V. I. Arnol′d. Small denominators and problems of stability of motion in classical and celestial mechanics. *Uspehi Mat. Nauk*, 18(6 (114)):91–192, 1963.

[Arn63c] V. I. Arnol′d. A theorem of Liouville concerning integrable problems of dynamics. *Sibirsk. Mat. Ž.*, 4:471–474, 1963.

[Arn89] V. I. Arnol′d. *Mathematical methods of classical mechanics*, volume 60 of *Graduate Texts in Mathematics*. Springer-Verlag, New York, second edition, 1989. Translated from the Russian by K. Vogtmann and A. Weinstein.

[Ban88] V. Bangert. Mather sets for twist maps and geodesics on tori. In *Dynamics reported, Vol. 1*, volume 1 of *Dynam. Report. Ser. Dynam. Systems Appl.*, pages 1–56. Wiley, Chichester, 1988.

[BB06] Patrick Bernard and Boris Buffoni. The Monge problem for supercritical Mañé potentials on compact manifolds. *Adv. Math.*, 207(2):691–706, 2006.

[BB07a] Patrick Bernard and Boris Buffoni. Optimal mass transportation and Mather theory. *J. Eur. Math. Soc. (JEMS)*, 9(1):85–121, 2007.

[BB07b] Patrick Bernard and Boris Buffoni. Weak KAM pairs and Monge-Kantorovich duality. In *Asymptotic analysis and singularities—elliptic and parabolic PDEs and related problems*, volume 47 of *Adv. Stud. Pure Math.*, pages 397–420. Math. Soc. Japan, Tokyo, 2007.

[BB07c] Patrick Bernard and Boris Buffoni. Weak KAM pairs and Monge-Kantorovich duality. In *Asymptotic analysis and singularities—elliptic and parabolic PDEs and related problems*, volume 47 of

Adv. Stud. Pure Math., pages 397–420. Math. Soc. Japan, Tokyo, 2007.

[Ber07] Patrick Bernard. Existence of $C^{1,1}$ critical sub-solutions of the Hamilton-Jacobi equation on compact manifolds. *Ann. Sci. École Norm. Sup. (4)*, 40(3):445–452, 2007.

[Ber08] Patrick Bernard. The dynamics of pseudographs in convex Hamiltonian systems. *J. Amer. Math. Soc.*, 21(3):615–669, 2008.

[Ber09a] Patrick Bernard. Lasry-Lions regularisation and a Lemma of Ilmanen. *preprint*, 2009.

[Ber09b] Patrick Bernard. Personal communication, 2009.

[BT01] Guy Barles and Agnès Tourin. Commutation properties of semigroups for first-order Hamilton-Jacobi equations and application to multi-time equations. *Indiana Univ. Math. J.*, 50(4):1523–1544, 2001.

[Car95] M. J. Dias Carneiro. On minimizing measures of the action of autonomous Lagrangians. *Nonlinearity*, 8(6):1077–1085, 1995.

[Car01] Pierre Cardaliaguet. Front propagation problems with nonlocal terms. II. *J. Math. Anal. Appl.*, 260(2):572–601, 2001.

[Car03] Guillaume Carlier. Duality and existence for a class of mass transportation problems and economic applications. In *Advances in mathematical economics. Vol. 5*, volume 5 of *Adv. Math. Econ.*, pages 1–21. Springer, Tokyo, 2003.

[CEL84] M. G. Crandall, L. C. Evans, and P.-L. Lions. Some properties of viscosity solutions of Hamilton-Jacobi equations. *Trans. Amer. Math. Soc.*, 282(2):487–502, 1984.

[Chi08] L. Chierchia. A. N. Kolmogorov's 1954 paper on nearly-integrable Hamiltonian systems. A comment on: "On conservation of conditionally periodic motions for a small change in Hamilton's function" [Dokl. Akad. Nauk SSSR (N.S.) **98** (1954), 527–530; mr0068687]. *Regul. Chaotic Dyn.*, 13(2):130–139, 2008.

[CISM00] Gonzalo Contreras, Renato Iturriaga, and Hector Sanchez-Morgado. Weak solutions of the Hamilton-Jacobi equation for time periodic Lagrangians. *preprint*, 2000.

[CL83] Michael G. Crandall and Pierre-Louis Lions. Viscosity solutions of Hamilton-Jacobi equations. *Trans. Amer. Math. Soc.*, 277(1):1–42, 1983.

[CL09] Xiaojun Cui and Ji Li. On commuting Tonelli Hamiltonians: Autonomous case. *preprint*, 2009.

[CLSW98] F. H. Clarke, Yu. S. Ledyaev, R. J. Stern, and P. R. Wolenski. *Nonsmooth analysis and control theory*, volume 178 of *Graduate Texts in Mathematics*. Springer-Verlag, New York, 1998.

[Con01] G. Contreras. Action potential and weak KAM solutions. *Calc. Var. Partial Differential Equations*, 13(4):427–458, 2001.

[CS04] Piermarco Cannarsa and Carlo Sinestrari. *Semiconcave functions, Hamilton-Jacobi equations, and optimal control.* Progress in Nonlinear Differential Equations and their Applications, 58. Birkhäuser Boston Inc., Boston, MA, 2004.

[CV08] Franco Cardin and Claude Viterbo. Commuting Hamiltonians and Hamilton-Jacobi multi-time equations. *Duke Math. J.*, 144(2):235–284, 2008.

[DP07] Andrea Davini and Marcello Ponsiglione. Homogenization of two-phase metrics and applications. *J. Anal. Math.*, 103:157–196, 2007.

[DS09] Andrea Davini and Antonio Siconolfi. Exact and approximate correctors for stochastic Hamiltonians: the 1-dimensional case. *Math. Ann.*, 345(4):749–782, 2009.

[Dug66] James Dugundji. *Topology.* Allyn and Bacon Inc., Boston, Mass., 1966.

[Fat08] Albert Fathi. Weak KAM Theorem in Lagrangian Dynamics, preliminary version 10, Lyon. unpublished, June 15 2008.

[Fat09] Albert Fathi. Personal communication, 2009.

[FF07] A. Fathi and A. Figalli. Optimal transportation on non-compact manifolds, 2007.

[FFR09] Albert Fathi, Alessio Figalli, and Ludovic Rifford. On the Hausdorff dimension of the Mather quotient. *Comm. Pure Appl. Math.*, 62(4):445–500, 2009.

[FGS08] Albert Fathi, Alessandro Giuliani, and Alfonso Sorrentino. Uniqueness of invariant lagrangian graphs in a homology or a cohomology class. 2008.

[FIM09a] N. Forcadel, C. Imbert, and R. Monneau. Homogenization of fully overdamped Frenkel-Kontorova models. *J. Differential Equations*, 246(3):1057–1097, 2009.

[FIM09b] Nicolas Forcadel, Cyril Imbert, and Régis Monneau. Homogenization of some particle systems with two-body interactions and of the dislocation dynamics. *Discrete Contin. Dyn. Syst.*, 23(3):785–826, 2009.

[FM07] A. Fathi and E. Maderna. Weak KAM theorem on non compact manifolds. *NoDEA*, 14(1):1–27, 2007.

[FS03] Albert Fathi and Antonio Siconolfi. A third proof of the existence of strict C^1 subsolutions adaptable to time-dependant Lagragians. *Unpublished manuscript*, 2003.

[FS04] Albert Fathi and Antonio Siconolfi. Existence of C^1 critical subsolutions of the Hamilton-Jacobi equation. *Invent. Math.*, 155(2):363–388, 2004.

[FZ09] Albert Fathi and Maxime Zavidovique. Insertion of $C^{1,1}$ functions and Ilmanen's lemma. *preprint*, 2009.

[Gol01] Christophe Golé. *Symplectic twist maps*, volume 18 of *Advanced Series in Nonlinear Dynamics*. World Scientific Publishing Co. Inc., River Edge, NJ, 2001. Global variational techniques.

[Gro99] Misha Gromov. *Metric structures for Riemannian and non-Riemannian spaces*, volume 152 of *Progress in Mathematics*. Birkhäuser Boston Inc., Boston, MA, 1999. Based on the 1981 French original With appendices by M. Katz, P. Pansu and S. Semmes, Translated from the French by Sean Michael Bates.

[Her83] Michael-R. Herman. *Sur les courbes invariantes par les difféomorphismes de l'anneau. Vol. 1*, volume 103 of *Astérisque*. Société Mathématique de France, Paris, 1983. With an appendix by Albert Fathi, With an English summary.

[Her86] Michael-R. Herman. Sur les courbes invariantes par les difféomorphismes de l'anneau. Vol. 2. *Astérisque*, (144):248, 1986. With a correction to: ıt On the curves invariant under diffeomorphisms of the annulus, Vol. 1 (French) [Astérisque No. 103-104, Soc. Math. France, Paris, 1983; MR0728564 (85m:58062)].

[Her89] Michael-R. Herman. Inégalités "a priori" pour des tores lagrangiens invariants par des difféomorphismes symplectiques. *Inst. Hautes Études Sci. Publ. Math.*, (70):47–101 (1990), 1989.

[Ilm93] Tom Ilmanen. The level-set flow on a manifold. In *Differential geometry: partial differential equations on manifolds (Los Angeles, CA, 1990)*, volume 54 of *Proc. Sympos. Pure Math.*, pages 193–204. Amer. Math. Soc., Providence, RI, 1993.

[IM08] Cyril Imbert and Régis Monneau. Homogenization of first-order equations with (u/ϵ)-periodic Hamiltonians. I. Local equations. *Arch. Ration. Mech. Anal.*, 187(1):49–89, 2008.

[IMR08] Cyril Imbert, Régis Monneau, and Elisabeth Rouy. Homogenization of first order equations with (u/ϵ)-periodic Hamiltonians. II. Application to dislocations dynamics. *Comm. Partial Differential Equations*, 33(1-3):479–516, 2008.

[Kan42] L. Kantorovitch. On the translocation of masses. *C. R. (Doklady) Acad. Sci. URSS (N.S.)*, 37:199–201, 1942.

[Kol54] A. N. Kolmogorov. On conservation of conditionally periodic motions for a small change in Hamilton's function. *Dokl. Akad. Nauk SSSR (N.S.)*, 98:527–530, 1954.

[LPV87] P.-L. Lions, G. Papanicolaou, and S.R.S. Varadhan. Homogenization of Hamilton-Jacobi equation. unpublished preprint, 1987.

[LR86] P.-L. Lions and J.-C. Rochet. Hopf formula and multitime Hamilton-Jacobi equations. *Proc. Amer. Math. Soc.*, 96(1):79–84, 1986.

[Mañ97] Ricardo Mañé. Lagrangian flows: the dynamics of globally minimizing orbits. *Bol. Soc. Brasil. Mat. (N.S.)*, 28(2):141–153, 1997.

[Mas03] Daniel Massart. On Aubry sets and Mather's action functional. *Israel J. Math.*, 134:157–171, 2003.

[Mas07] Daniel Massart. Subsolutions of time-periodic Hamilton-Jacobi equations. *Ergodic Theory Dynam. Systems*, 27(4):1253–1265, 2007.

[Mat86] John Mather. A criterion for the nonexistence of invariant circles. *Inst. Hautes Études Sci. Publ. Math.*, (63):153–204, 1986.

[Mat91] John N. Mather. Action minimizing invariant measures for positive definite Lagrangian systems. *Math. Z.*, 207(2):169–207, 1991.

[Mat93] John N. Mather. Variational construction of connecting orbits. *Ann. Inst. Fourier (Grenoble)*, 43(5):1349–1386, 1993.

[MF94] John N. Mather and Giovanni Forni. Action minimizing orbits in Hamiltonian systems. In *Transition to chaos in classical and quantum mechanics (Montecatini Terme, 1991)*, volume 1589 of *Lecture Notes in Math.*, pages 92–186. Springer, Berlin, 1994.

[Mon81] Gaspard Monge. Mémoire sur la théorie des déblais et des remblais, 1781.

[MR06] Monica Motta and Franco Rampazzo. Nonsmooth multi-time Hamilton-Jacobi systems. *Indiana Univ. Math. J.*, 55(5):1573–1614, 2006.

[Sor09] Alfonso Sorrentino. On the Integrability of Tonelli Hamiltonians. *preprint*, 2009.

[Vil03] Cédric Villani. *Topics in optimal transportation*, volume 58 of *Graduate Studies in Mathematics*. American Mathematical Society, Providence, RI, 2003.

[Vil09] Cédric Villani. *Optimal transport*, volume 338 of *Grundlehren der Mathematischen Wissenschaften [Fundamental Principles of Mathematical Sciences]*. Springer-Verlag, Berlin, 2009. Old and new.

[Zav08] Maxime Zavidovique. Strict subsolutions and Mañe potential in discrete weak KAM theory. *preprint*, 2008.