# Rouquier Blocks for Cyclotomic Hecke Algebras

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July 2007

After work of

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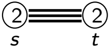
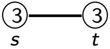
$G_2$ : 

, $G_4$ : 


and a field of realisation $\mathbb{Q}_G$ (ring of integers $\mathbb{Z}_G$) :

$$\mathbb{Q}_{G_2} = \mathbb{Q} \quad \text{and} \quad \mathbb{Q}_{G_4} = \mathbb{Q}(\zeta_3)\,.$$

# HECKE ALGEBRAS OF COMPLEX REFLECTION GROUPS

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$$G_2 \;:\; \underset{s}{\textcircled{2}} \equiv\!\equiv \underset{t}{\textcircled{2}} \qquad , \qquad G_4 \;:\; \underset{s}{\textcircled{3}} \text{——} \underset{t}{\textcircled{3}}$$

  and a field of realisation $\mathbb{Q}_G$ (ring of integers $\mathbb{Z}_G$) :

$$\mathbb{Q}_{G_2} = \mathbb{Q} \quad \text{and} \quad \mathbb{Q}_{G_4} = \mathbb{Q}(\zeta_3)\,.$$

- The associated generic Hecke algebra is defined from such a presentation :

$$\mathcal{H}(G_2) := <\; S, T \;;\; \begin{cases} STSTST = TSTSTS \\ (S-u_0)(S-u_1) = 0 \\ (T-v_0)(T-v_1) = 0 \end{cases} >$$

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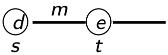
if $G =$  $\cdots$ ,


then for

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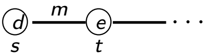
   if $G =$  ,

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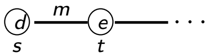
if $G =$ 

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- the algebra $\mathbb{Q}_G((x_i), (y_j), \dots)\mathcal{H}(G)$ is split semisimple,
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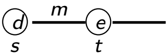
   if $G =$ 

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The above specialisation defines a bijection

$$\mathrm{Irr}(G) \xrightarrow{\sim} \mathrm{Irr}(\mathcal{H}(G)) \quad , \quad \chi \mapsto \chi_{\mathcal{H}} \,.$$

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5. The Schur elements of the irreducible characters of $G$ are the elements $s_\chi \in \mathbb{Z}_G[(x_i^{\pm 1}), (y_j^{\pm 1}), \dots]$ defined by

$$t = \sum_{\chi \in \mathrm{Irr}(G)} \frac{1}{s_\chi} \chi_{\mathcal{H}} \,.$$

## Theorem (M. Chlouveraki)

We have

$$s_\chi(\mathbf{x}) = \xi_\chi N_\chi \prod_{i \in I_\chi} \Psi_{\chi,i}(M_{\chi,i})^{n_{\chi,i}}$$

where

- $\xi_\chi \in \mathbb{Z}_G$, $N_\chi$ is a degree zero monomial in $\mathbb{Z}_G[\mathbf{x}, \mathbf{x}^{-1}]$, $(n_{\chi,i})_{i \in I_\chi}$ are integers,
- $(\Psi_{\chi,i})_{i \in I_\chi}$ is a family of $K$-cyclotomic polynomials,
- $(M_{\chi,i})_{i \in I_\chi}$ is a family of degree zero primitive monomials in $\mathbb{Z}_G[\mathbf{x}, \mathbf{x}^{-1}]$.

That factorisation is unique in $K[\mathbf{x}, \mathbf{x}^{-1}]$ and the monomials $(M_{\chi,i})_{i \in I_\chi}$ are unique (up to inversion).

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Elements of the orbit under $\mathfrak{S}(x_0, x_1) \times \mathfrak{S}(y_0, y_1)$ of

$$\begin{cases} s_1 := \Phi_2(x_0x_1^{-1})\Phi_2(y_0y_1^{-1})\Phi_3(x_0y_0x_1^{-1}y_1^{-1})\Phi_6(x_0y_0x_1^{-1}y_1^{-1})) \\ \quad \text{(corresponding to the four degree 1 irreducible characters)} \\ \\ s_2 := 2\Phi_6(x_0y_0x_1^{-1}y_1^{-1})\Phi_3(x_0y_1x_1^{-1}y_0^{-1}) \\ \quad \text{(corresponding to the two degree 2 irreducible characters)} \end{cases}$$

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Obtained by permutations of the parameters from

$$
\begin{cases}
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\qquad \text{(corresponding to the three characters of degree 1)} \\
\\
s_2 = -\,\zeta_3^2 x_2^6 x_1^{-6} \\
\qquad \Phi_9'(x_1x_0^{-1})\,\Phi_{18}''(x_1x_0^{-1})\,\Phi_9''(x_2x_0^{-1})\,\Phi_{18}'(x_2x_0^{-1}) \\
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s_3 = \Phi_4(x_0^2x_1^{-1}x_2^{-1})\,\Phi_{12}'(x_0^2x_1^{-1}x_2^{-1})\,\Phi_{12}''(x_0^2x_1^{-1}x_2^{-1}) \\
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\qquad \text{(corresponding to the unique character of degree 3)}
\end{cases}
$$

## Cyclotomic algebras

Let $\zeta$ be a root of unity. A $\zeta$–cyclotomic specialisation of the generic Hecke algebra is a morphism

$$\varphi \,:\, x_i \mapsto (\zeta^{-1}q)^{m_i} \,,\; y_j \mapsto (\zeta^{-1}q)^{n_j} \,, \dots \quad (m_i, n_j \in \mathbb{Z})\,,$$

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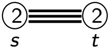
A 1–cyclotomic Hecke algebra for $G_2 =$ 

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A $\zeta_3$–cyclotomic Hecke algebra for $B_2(3) = \underset{s}{\textcircled{d}} = \underset{t}{\textcircled{2}}$ :

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### Relevance to character theory of finite reductive groups

The unipotent characters in a given $d$–Harish–Chandra series $\mathrm{UnCh}(\mathbf{G}^F\,;\,(\mathbf{L},\lambda))$ are described by a suitable $\zeta_d$–cyclotomic Hecke algebra $\mathcal{H}_{\mathbf{G}^F}(\mathbf{L},\lambda))$ for the corresponding $d$–cyclotomic Weyl group $W_{\mathbf{G}^F}(\mathbf{L},\lambda))$ :

$$\mathrm{UnCh}(\mathbf{G}^F\,;\,(\mathbf{L},\lambda)) \longleftrightarrow \mathrm{Irr}(\mathcal{H}_{\mathbf{G}^F}(\mathbf{L},\lambda)))\,.$$

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*Let $(e_k), (e'_k)_{k=1,2,\dots,|G|}$ be a pair of dual basis of $\mathcal{H}_\varphi(G)$ relative to the canonical trace form $t$.*

*Then a Rouquier block $b$ has the form*

$$b = \sum_k \left( \sum_{\chi\in b} \frac{\chi_{\mathcal{H}_\varphi}(e'_k)}{s_{\chi_\varphi}} \right) e_k \quad \textit{where} \quad \sum_{\chi\in b} \frac{\chi_{\mathcal{H}_\varphi}(e'_k)}{s_{\chi_\varphi}} \in R_G(q)$$

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For each integer $d$ and each $d$–cuspidal pair $(\mathbf{L}, \lambda)$, let us identify

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on the $d$–Harish–Chandra series $\mathrm{UnCh}(\mathbf{G}^F, (\mathbf{L}, \lambda))$ is the partition into Rouquier blocks of $\mathrm{Irr}(\mathcal{H}_{\mathbf{G}^F}(\mathbf{L}, \lambda))$ :

$$\mathrm{Irr}(\mathcal{H}(\mathbf{G}^F, (\mathbf{L}, \lambda))) = \dot{\bigcup}_{\mathcal{F} \in \mathrm{Fam}(\mathbf{G}^F)} \underbrace{\mathcal{F} \cap \mathrm{Irr}(\mathcal{H}_{\mathbf{G}^F}(\mathbf{L}, \lambda))}_{\text{Rouquier Block or } \emptyset}\,.$$

## Computation of Rouquier Blocks

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- For $\mathfrak{p}$ a prime ideal of $\mathbb{Z}_G$, the blocks of the localised algebra $\mathcal{H}(G)_\mathfrak{p}$ (resp. $\mathcal{H}_\varphi(G)_\mathfrak{p}$) are called $\mathfrak{p}$–blocks of $\mathcal{H}(G)$ (resp. $\mathcal{H}_\varphi(G)$).

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**Proposition**

*The Rouquier blocks of $\mathcal{H}_{\varphi}(G)$ are the union of the $\mathfrak{p}$–blocks of $\mathcal{H}_{\varphi}(G)$ for $\mathfrak{p}$ running over the set $\mathrm{bad}(\mathcal{H}_{\varphi}(G))$ of bad primes.*

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## Cyclotomic specialisation and $\mathfrak{p}$–singularity

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Let $\varphi$ be a cyclotomic specialisation of $\mathcal{H}(G)$, defined by the family of integers $(m_i), (n_j), \dots$. Let $M$ be a $\mathfrak{p}$–essential monomial.

We say that $\varphi$ is $\mathfrak{p}$–singular at $M$ (or at $H_M$) if

$$\varphi(M) = 1 \,,$$

*i.e.*, if the parameters of $\varphi$ belong to the $\mathfrak{p}$–essential hyperplane $H_M$ :

$$\sum_i a_i m_i + \sum_j b_j n_j + \cdots = 0 \,.$$

## $G_2$ : 2–essentiel in purple, 3–essentials in green

$$\begin{cases} s_1 := \Phi_2(x_0x_1^{-1})\Phi_2(y_0y_1^{-1})\Phi_3(x_0y_0x_1^{-1}y_1^{-1})\Phi_6(x_0y_0x_1^{-1}y_1^{-1})) \\ s_2 := 2\Phi_6(x_0y_0x_1^{-1}y_1^{-1})\Phi_3(x_0y_1x_1^{-1}y_0^{-1}) \end{cases}$$

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## $G_4$ : 2–essentiel in purple, 3–essentials in green

$$\begin{cases} s_1 = \Phi_9''(x_0x_1^{-1})\,\Phi_{18}'(x_0x_1^{-1})\,\Phi_4(x_0x_1^{-1})\,\Phi_{12}'(x_0x_1^{-1})\,\Phi_{12}''(x_0x_1^{-1})\,\Phi_{36}'(x_0x_1^{-1}) \\ \qquad \Phi_9'(x_0x_2^{-1})\,\Phi_{18}''(x_0x_2^{-1})\,\Phi_4(x_0x_2^{-1})\,\Phi_{12}'(x_0x_2^{-1})\,\Phi_{12}''(x_0x_2^{-1})\,\Phi_{36}''(x_0x_2^{-1}) \\ \qquad \Phi_4(x_0^2x_1^{-1}x_2^{-1})\,\Phi_{12}'(x_0^2x_1^{-1}x_2^{-1})\,\Phi_{12}''(x_0^2x_1^{-1}x_2^{-1}) \\ s_2 = -\zeta_3^2x_2^6x_1^{-6} \\ \qquad \Phi_9'(x_1x_0^{-1})\,\Phi_{18}''(x_1x_0^{-1})\,\Phi_9''(x_2x_0^{-1})\,\Phi_{18}'(x_2x_0^{-1}) \\ \qquad \Phi_4(x_1x_2^{-1})\,\Phi_{12}'(x_1x_2^{-1})\,\Phi_{12}''(x_1x_2^{-1})\,\Phi_{36}'(x_1x_2^{-1}) \\ \qquad \Phi_4(x_0^{-2}x_1x_2)\,\Phi_{12}'(x_0^{-2}x_1x_2)\,\Phi_{12}''(x_0^{-2}x_1x_2) \\ s_3 = \Phi_4(x_0^2x_1^{-1}x_2^{-1})\,\Phi_{12}'(x_0^2x_1^{-1}x_2^{-1})\,\Phi_{12}''(x_0^2x_1^{-1}x_2^{-1}) \\ \qquad \Phi_4(x_1^2x_2^{-1}x_0^{-1})\,\Phi_{12}'(x_1^2x_2^{-1}x_0^{-1})\,\Phi_{12}''(x_1^2x_2^{-1}x_0^{-1}) \\ \qquad \Phi_4(x_2^2x_0^{-1}x_1^{-1})\,\Phi_{12}'(x_2^2x_0^{-1}x_1^{-1})\,\Phi_{12}''(x_2^2x_0^{-1}x_1^{-1}) \end{cases}$$

- Let $\mathcal{H}_\varphi(G_2)$ be a cyclotomic algebra for $G_2$ defined by

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### Theorem (Maria Chlouveraki)

Assume that $G$ is an irreducible complex reflection group.

Let $\mathcal{B}\ell_{\mathfrak{p}}(\mathcal{H}(G))$ (resp. $\mathcal{B}\ell_{\mathfrak{p}}(\mathcal{H}_{\varphi}(G))$ be the partition of $\mathrm{Irr}(G)$ (identified with $\mathrm{Irr}(\mathcal{H}(G))$) into $\mathfrak{p}$–blocks of $\mathcal{H}(G)$ (resp. $\mathcal{H}_{\varphi}(G)$).

Whenever $M$ is a $\mathfrak{p}$-essential monomial, there exists a partition $\mathcal{B}\ell_{\mathfrak{p}}^{M}(\mathcal{H}(G))$ of $\mathrm{Irr}(G)$ with the following properties.

It is believed that the condition in 1 is an equivalence. That has been checked for $G$ exceptional.

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2. The partition $\mathcal{B}\ell_{\mathfrak{p}}(\mathcal{H}_\varphi(G))$ is the partition generated by the family of partitions $\mathcal{B}\ell_{\mathfrak{p}}^M(\mathcal{H}(G))$ where $M$ runs over the set of all $\mathfrak{p}$–essential monomial at which $\varphi$ is singular.

It is believed that the condition in 1 is an equivalence. That has been checked for $G$ exceptional.

Characters of $G_4$ are denoted $\chi_{d,b}$, where

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The cycle $(x_0\ x_1\ x_2)$ acts as follows on $\mathrm{Irr}(G)$ :

$$\begin{cases} \chi_{1,0} \mapsto \chi_{1,4} \mapsto \chi_{1,8} \\ \chi_{2,1} \mapsto \chi_{2,5} \mapsto \chi_{2,3} \end{cases} \quad \text{and fixes } \chi_{3,2}$$

The partitions associated with the 2–essential monomials are (up to permutation)

$$\mathcal{B}\ell_2^{(m_0=m_1)}(\mathcal{H}(G_4)) = \{\chi_{1,0}, \chi_{1,4}, \chi_{2,1}\} \bigcup \text{(singletons)}$$

$$\mathcal{B}\ell_2^{(2m_0=m_1+m_2)}(\mathcal{H}(G_4)) = \{\chi_{1,0}, \chi_{2,5}, \chi_{3,2}\} \bigcup \text{(singletons)}$$