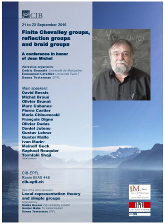
Action of $\mathrm{GL}_2(\mathbb{Z})$ on the Drinfeld double of a finite group

Michel Broué (Université Paris-Diderot Paris 7)

“Finite Chevalley groups, reflection groups and braid groups”

Centre Bernoulli, September 2016

in honour of **Jean Michel** (and François Digne)



The algebra $D_k G$

Let G be a finite group and let k be a field (at the beginning a commutative ring, at the end a large enough characteristic zero field).

- Let $F(G,k)$ be the k-algebra of functions on G with values in k. Thus

$$F(G,k) = \bigoplus_{s\in G} k\delta_s \quad \text{where } (\delta_s)_{s\in G} \text{ is a family of orthogonal idempotents}\,.$$

- G acts on $F(G,k)$ ($g\delta_s g^{-1} = \delta_{gsg^{-1}}$), and we set

$$D_k G := F(G,k) \rtimes G\,.$$

- Thus

$$D_k G = \bigoplus_{s,g\in G} k\delta_s g \quad \text{with} \quad \delta_s g \delta_t h = \begin{cases} \delta_s gh & \text{if } s = gtg^{-1} \\ 0 & \text{if not.} \end{cases}$$

and the unity element of $D_k G$ is $1_{D_k G} = \sum_{s\in G} \delta_s$.

The center ZD_kG

- Let $\mathrm{Com}(G \times G)$ denote the set of *commuting pairs* in G, and let $\mathcal{C}(G \times G)$ denote the set of *orbits of* $\mathrm{Com}(G \times G)$ under conjugation by G.

- For $\Gamma \in \mathcal{C}(G \times G)$, we set $\mathcal{S}_\Gamma := \sum_{(s,g)\in\Gamma} \delta_s g$.

Then $(\mathcal{S}_\Gamma)_{\Gamma\in\mathcal{C}(G\times G)}$ is a basis of ZD_kG.

- The first projection $G \times G \to G$, $(s, g) \mapsto s$ induces a bijection

$$\mathcal{C}(G \times G) \xrightarrow{\sim} \{(s, C) \mid {\scriptstyle (s\in G)(C\in \mathrm{Cl}(C_G(s))}\}/G\text{-conjugation}\,,$$

and the k-linear map

$$\bigoplus_{s\in[Cl(G)]} ZkC_G(s) \to ZD_kG \quad , \quad z_s \mapsto \mathrm{Tr}_{C_G(s)}^G(\delta_s z_s)$$

is an isomorphism.

The abelian category ${}_{D_k G}\mathbf{mod}$

Objects: the G-graded kG-modules $X = \bigoplus_{s\in G} {}_sX$.
Morphisms: the kG-morphisms $X \to Y$ such that ${}_sX \to_s Y$.

The following functors define "inverse" equivalences of abelian categories:

$$\begin{cases} {}_{D_k G}\mathbf{mod} \to \bigoplus_{s\in[\mathrm{Cl}(G)]} {}_{kC_G(s)}\mathbf{mod}\,, & \bigoplus_{s\in G} {}_sX \mapsto \bigoplus_{s\in[\mathrm{Cl}(G)]} {}_sX\,, \\ \bigoplus_{s\in[\mathrm{Cl}(G)]} {}_{kC_G(s)}\mathbf{mod} \to {}_{D_k G}\mathbf{mod}\,, & \bigoplus_{s\in[\mathrm{Cl}(G)]} S_s \mapsto \bigoplus_{s\in[\mathrm{Cl}(G)]} \mathrm{Ind}_{C_G(s)}^G S_s\,. \end{cases}$$

$\Rightarrow$ $D_k G$ is a *symmetric algebra*.

Actually, $\tau(\delta_s g) = \begin{cases} 0 & \text{if } g \neq 1\,, \\ 1 & \text{if } g = 1\,, \end{cases}$ is a symmetrizing form.

$\Rightarrow$ The map $z_s \mapsto \mathrm{Tr}_{C_G(s)}^G(\delta_s z_s)$ induces an *algebra isomorphism*

$$\bigoplus_{s\in[Cl(G)]} \mathrm{Tr}_{C_G(s)}^G(\delta_s \cdot) : \bigoplus_{s\in[Cl(G)]} ZkC_G(s) \to ZD_k G\,.$$

${}_{D_k G}\mathbf{mod}$ as a ribbon category

Tensor product:

$$X \otimes Y = \bigoplus_{s \in G} {}_s(X \otimes Y) \ \text{ where } \ {}_s(X \otimes Y) := \bigoplus_{t,u \mid tu=s} {}_tX \otimes {}_uY\,.$$

Dual:

$$X^* = \bigoplus_{s \in G} {}_s(X^*) \ \text{ where } \ {}_s(X^*) := ({}_{s^{-1}}X)^*\,,$$

with obvious evaluation $X^* \otimes X \to k$ and coevaluation $k \to X \otimes X^*$.

Braiding:

$$c_{X,Y} : X \otimes Y \xrightarrow{\sim} Y \otimes X\,, \ \begin{cases} {}_tX \otimes {}_uY \to {}_{tut^{-1}}Y \otimes {}_tX \\ x \otimes y \mapsto ty \otimes x \end{cases}$$

Twist:

$$\theta_X : X \xrightarrow{\sim} X\,, \ \begin{cases} {}_sX \to {}_sX \\ x \mapsto sx \end{cases}$$

We have $\theta_{X \otimes Y} = \theta_X \cdot \theta_Y \cdot c_{Y,X} \cdot c_{X,Y}$.

Graded characters and Grothendieck ring

From now on, K is a characteristic zero field, which contains the $|G|$-th roots of unity.

Graded character:

For X a $D_K G$-module, we set $\mathrm{grchar}_X := \sum_{s\in G} \mathrm{tr}(\cdot \mid {}_sX)s$, that is

$$\mathrm{grchar}_X(t) = \sum_{s\in C_G(t)} \mathrm{tr}(t \mid {}_sX)s \in ZKC_G(t)$$

and

$$\mathrm{grchar}_{X\otimes Y}(t) = \mathrm{grchar}_X(t)\mathrm{grchar}_Y(t)\,.$$

If $Y = X(t, T)$, define

$$\sigma_Y : \begin{cases} \mathrm{Gr}(D_K G) \to K\,, \\ X \mapsto \sigma(X) = \omega_T(\mathrm{grchar}_X(t)) \end{cases}$$

Then σ_Y is a ring morphism.

${}_{D_K G}\mathbf{mod}$ as a modular category

The S-matrix:

$$\mathbf{S}_{X,Y} := \frac{1}{|G|}\mathrm{tr}(c_{Y,X}\cdot c_{X,Y} \mid X\otimes Y)_{X,Y\in\mathrm{Irr}(D_K G)}\,.$$

${}_tX\otimes {}_uY$ contributes to the trace of $c_{Y,X}\cdot c_{X,Y}$ only if $tu=ut$, in which case

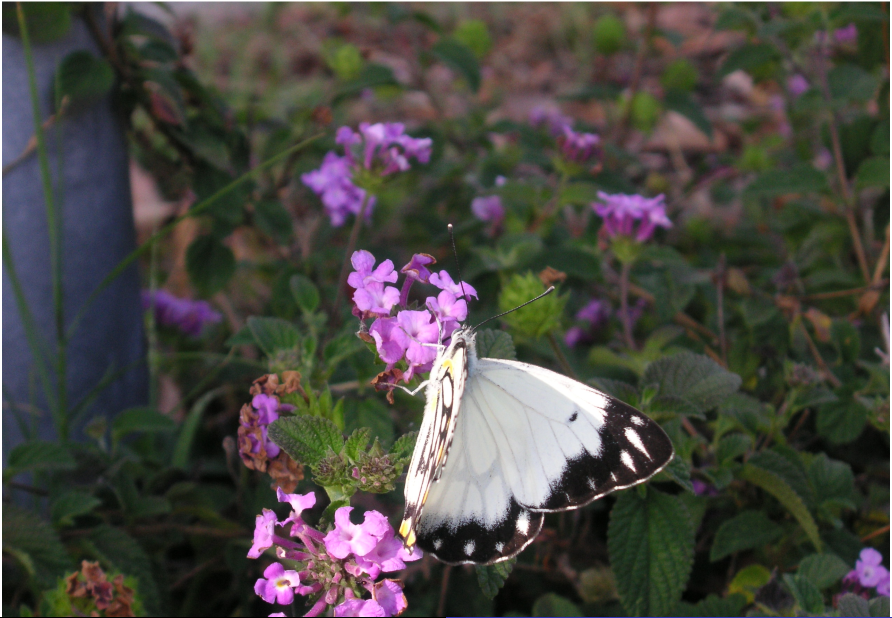
$$c_{Y,X}\cdot c_{X,Y} : \begin{cases} {}_tX\otimes {}_uY \to {}_tX\otimes {}_uY \\ x\otimes y \mapsto ux\otimes ty\,. \end{cases}$$

${}_{D_K G}\mathbf{mod}$ is a *modular category* since $\mathbf{S}$ is *nondegenerate*.

For $X, Y \in \mathrm{Irr}(D_K G)$,

$$\sigma_Y(X) = \frac{|G|}{\chi_Y(1)}\mathbf{S}_{X,Y}\,.$$

Now just for fun: name, where, when ?



Two bases of $\mathrm{CF}(D_K G)$

$\mathrm{CF}(D_K G)$ has two bases, both partitioned according to $s \in [\mathrm{Cl}(G)]$:

- $(\gamma_\Gamma)_{\Gamma \in \mathcal{C}(G \times G)}$,

where $\Gamma = \Gamma(s, C)$ for $s \in [\mathrm{Cl}(G)]$ and $C \in \mathrm{Cl}(C_G(s))$,
and $\gamma_{(s,C)} := \gamma_\Gamma$ denotes the characteristic function of Γ,

- $(\chi_X)_{X \in \mathrm{Irr}(D_K G)}$,

where $X = \mathrm{Ind}_{C_G(s)}^G S$ for $s \in [\mathrm{Cl}(G)]$ and $S \in \mathrm{Irr}_K(C_G(s))$, and we set $\chi_{(s,S)} := \chi_X$.

From one basis to the other:
$$\begin{cases} \chi_{s,S} = \displaystyle\sum_{g \in [\mathrm{Cl}(C_G(s)]} \chi_S(g)\gamma_{(s,C_g)} \,, \\ \gamma_{\Gamma(s,g)} = \dfrac{|\Gamma(s,g)|}{|G|} \displaystyle\sum_{S \in \mathrm{Irr}_K(C_G(s))} \chi_S(g^{-1})\chi_{(s,S)} \,. \end{cases}$$

Action of $\mathrm{GL}_2(\mathbb{Z})$ on $\mathrm{CF}(D_K G)$

- Let $\mathbf{S}$ be the endomorphism of $\mathrm{CF}(D_K G)$ such that

$$\mathbf{S} : \chi_X \mapsto \sum_{Y \in \mathrm{Irr}(D_K G)} \mathbf{S}_{X,Y} \cdot \chi_Y \ \text{ for } X \in \mathrm{Irr}(D_K G)\,.$$

- Let $\mathbf{\Omega}$ be the endomorphism of $\mathrm{CF}(D_K G)$ such that

$$\mathbf{\Omega} : \chi_X \mapsto \theta_X \cdot \chi_X \ \text{ for } X \in \mathrm{Irr}(D_K G)\,.$$

- Let $\mathbf{\Delta}_n$ ($n \in (\mathbb{Z}/|G|\mathbb{Z})^\times$) be the endomorphism of $\mathrm{CF}(D_K G)$ such that

$$\mathbf{\Delta}_n : \chi_X \mapsto \chi_{{}^n X} \ \text{ for } X \in \mathrm{Irr}(D_K G)\,.$$

Action of $\mathrm{GL}_2(\mathbb{Z})$ on $\mathrm{CF}(D_K G)$, continued

- $\mathrm{GL}_2(\mathbb{Z})$ acts on the set $\mathrm{Com}(G \times G)$ of commuting pairs of elements of G :

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \cdot (s, g) := (s^a g^b, s^c g^d),$$

$\Rightarrow$ hence on the set $\mathcal{C}(G \times G)$ of its orbits under G,

$\Rightarrow$ hence on the basis $(\gamma_\Gamma)_{\Gamma \in \mathcal{C}(G \times G)}$ of $\mathrm{CF}(D_K G)$.

Theorem

$$\begin{cases} \mathbf{S} : \quad \gamma_{(s,g)} \mapsto \gamma_{(g,s^{-1})} \quad \text{hence acts like } \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \\ \mathbf{\Omega} : \quad \gamma_{(s,g)} \mapsto \gamma_{(s,gs^{-1})} \quad \text{hence acts like } \begin{pmatrix} 1 & 0 \\ -1 & 1 \end{pmatrix} \\ \mathbf{\Delta}_n : \quad \gamma_{(s,g)} \mapsto \gamma_{(s,g^n)} \quad \text{hence acts like } \begin{pmatrix} 1 & 0 \\ 0 & n \end{pmatrix} \end{cases}$$

Action of $\mathrm{GL}_2(\mathbb{Z})$ on $\mathrm{CF}(D_K G)$, continued

Hence

$$\text{the group } \langle \mathbf{S}, \boldsymbol{\Omega} \rangle \text{ acts like } \mathrm{SL}_2(\mathbb{Z}/|G|\mathbb{Z}),$$

$$\langle \mathbf{S}, \boldsymbol{\Omega}, \boldsymbol{\Delta}_n \rangle \text{ acts like } \mathrm{GL}_2(\mathbb{Z}/|G|\mathbb{Z}).$$

- $\mathbf{S}^2$ corresponds to permutation matrices

$$\chi_{(s,S)} \mapsto \chi_{(s^{-1},S^*)} \quad \text{and} \quad \gamma_{(s,g)} \mapsto \chi_{(s^{-1},g^{-1})}\,,$$

- For $\mathbf{Sh} := \mathbf{S}\boldsymbol{\Omega}\mathbf{S}^{-1}$, we have

$$\boldsymbol{\Omega} \cdot \mathbf{Sh} \cdot \boldsymbol{\Omega} = \mathbf{Sh} \cdot \boldsymbol{\Omega} \cdot \mathbf{Sh}\,.$$

- **Verlinde formula**: If $X \otimes Y \simeq \bigoplus_Z z^{N_{X,Y}^Z}$, then

$$N_{Y,Z}^W = \sum_X \frac{\mathbf{S}_{X,Y}\mathbf{S}_{X,Z}\mathbf{S}_{X,W}^{-1}}{\mathbf{S}_{X,1}} \in \mathbb{N}\,.$$

Again !