Cyclotomic Root Systems

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En hommage à Serge Bouc – mais sans catégories ni foncteurs

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Variations on an old work of Gabi Nebe

A joint work with Ruth Corran and Jean Michel

A few words about motivation

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On can classify the root systems $\mathcal{R}$ and then classify the Weyl groups $G(\mathcal{R})$. Or proceed the opposite way: it's what we did for cyclotomic root systems and their (cyclotomic) Weyl groups.

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Let k be a subfield of $\mathbb{C}$ which is stable under complex conjugation.

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and s is the automorphism of V defined by

$$s(v) = v - \langle v, \alpha^\vee \rangle \alpha$$

whenever $\alpha \in L_s$ and $\alpha^\vee \in M_s$ are such that $\langle \alpha, \alpha^\vee \rangle = 1 - \zeta_s$.

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- 34 exceptional irreducible groups in dimension 2 to 8.

  Their field of definition (all subfields of "small" cyclotomic fields) have the remarkable property that all their rings of integers are principal ideal domains.

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$$Q_{\mathcal{R}} := \sum_{\mathfrak{r} \in \mathcal{R}} I_{\mathfrak{r}} \quad \text{and} \quad Q_{\mathcal{R}^{\vee}} := \sum_{\mathfrak{r} \in \mathcal{R}} J_{\mathfrak{r}}$$

$$P_{\mathcal{R}} := \{x \in V \mid \forall y \in Q_{\mathcal{R}^{\vee}}\,,\ \langle x, y \rangle \in \mathbb{Z}_k\} \quad \text{and} \quad P_{\mathcal{R}^{\vee}} := \ldots$$

There is a $\mathrm{Aut}(\mathcal{R})/G(\mathcal{R})$-invariant natural pairing

$$(P_{\mathcal{R}}/Q_{\mathcal{R}}) \times (P_{\mathcal{R}^{\vee}}/Q_{\mathcal{R}^{\vee}}) \to k/\mathbb{Z}_k\,.$$

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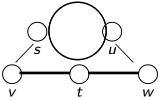
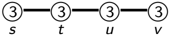
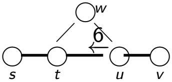
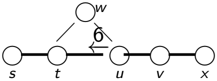
Then the Cartan matrix of $\mathcal{S}$ determines $\mathcal{R}$ up to genera.

Name	Diagram	Cartan matrix	Orbits	$\mathbb{Z}_k$	connection index
G_{31}		$\begin{pmatrix} 2 & i+1 & 1-i & -i & 0 \\ 1-i & 2 & 1-i & -1 & -1 \\ i+1 & i+1 & 2 & 0 & -1 \\ i & -1 & 0 & 2 & 0 \\ 0 & -1 & -1 & 0 & 2 \end{pmatrix}$	s	$\mathbb{Z}[i]$	1
G_{32}		$\begin{pmatrix} 1-\zeta_3 & \zeta_3^2 & 0 & 0 \\ -\zeta_3^2 & 1-\zeta_3 & \zeta_3^2 & 0 \\ 0 & -\zeta_3^2 & 1-\zeta_3 & \zeta_3^2 \\ 0 & 0 & -\zeta_3^2 & 1-\zeta_3 \end{pmatrix}$	s	$\mathbb{Z}[\zeta_3]$	1
G_{33}		$\begin{pmatrix} 2 & -1 & 0 & 0 & 0 \\ -1 & 2 & -1 & -\zeta_3^2 & 0 \\ 0 & -1 & 2 & -1 & 0 \\ 0 & -\zeta_3 & -1 & 2 & -1 \\ 0 & 0 & 0 & -1 & 2 \end{pmatrix}$	s	$\mathbb{Z}[\zeta_3]$	2
G_{34}		$\begin{pmatrix} 2 & -1 & 0 & 0 & 0 & 0 \\ -1 & 2 & -1 & -\zeta_3^2 & 0 & 0 \\ 0 & -1 & 2 & -1 & 0 & 0 \\ 0 & -\zeta_3 & -1 & 2 & -1 & 0 \\ 0 & 0 & 0 & -1 & 2 & -1 \\ 0 & 0 & 0 & 0 & -1 & 2 \end{pmatrix}$	s	$\mathbb{Z}[\zeta_3]$	1

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and the corresponding reflecting lines are

$$\begin{aligned} L_s &= kv_s \text{ with } v_s = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \qquad \text{and} \qquad & L_t &= kv_t \text{ with } v_t = \begin{pmatrix} -1 \\ 1 \end{pmatrix} \\ M_s &= kv_s^\vee \text{ with } v_s^\vee = (2,0) \qquad \text{and} \qquad & M_t &= kv_t^\vee \text{ with } v_t^\vee = (-1,1). \end{aligned}$$

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