UNIQUENESS OF L^∞ SOLUTIONS FOR A CLASS OF CONORMAL BV VECTOR FIELDS

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Abstract. Let X be a bounded vector field with bounded divergence defined in an open set Ω of $\mathbb{R}^d$, transverse to a hypersurface S. Let Ω_0 be an open subset of Ω such that the Hausdorff measure $\mathcal{H}^{d-1}(\Omega\backslash\Omega_0) = 0$. We suppose that the vector field X belongs to $BV_{\text{loc}}(\Omega_0)$ "conormally", an assumption made precise in the text, which is satisfied whenever the gradients of the coefficients of X have locally only a single component which is actually a Radon measure. This class can be invariantly defined and contains the so-called piecewise $W^{1,1}$ functions studied in [Li]. We prove the uniqueness of L^∞ solutions for the Cauchy problem related to X across the hypersurface S. We use for the proof some simple arguments of geometric measure theory to get rid of closed sets with codimension > 1. Next, we need an anisotropic regularization argument analogous to the one used in [Bo].

Contents

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1. Introduction

Let us give first the general framework of our study: we shall consider a real vector field X defined in an open set Ω of $\mathbb{R}^d$ and a Lipschitz oriented hypersurface S so that

$$X \in L^\infty_{\rm loc},\ \operatorname{div} X \in L^\infty_{\rm loc},\ X \text{ is positively transverse to } S. \tag{1.1}$$

Note that the first condition is simply the requirement of finite speed of propagation so that a local problem makes sense for the equation $Xu = f$. The second condition is essentially necessary to get a uniqueness result: in our appendix A1, we give an example of a two-dimensional $W^{1,1} \cap L^\infty$ vector field whose divergence is a positive unbounded L^1 function so that no uniqueness property is satisfied (see also section $IV.1$ of [DL]). Let us clarify the third condition. Let ν be a unit vector field conormal to the oriented Lipschitz hypersurface S. The vector field X is said to be positively tranverse to S if for all $x_0 \in S$, there exists a neighborhood V_0 of x_0 such that

$$\operatorname{essinf}_{V_0} X(x) \cdot \nu(x) > 0. \tag{1.2}$$

When the dimension is ≤ 2, the conditions (1.1) essentially ensure uniqueness of L^∞ solutions. In one dimension, the autonomous ODE

$$\dot{x} = f(x), \quad x(0) = x_0, \tag{1.3}$$

has a unique solution, provided f is merely continuous and $f(x_0) \neq 0$. The existence is given by Peano's theorem whereas the uniqueness follows from the direct integration of $\frac{dx}{f(x)} = dt$. In fact, setting $G(x) = \int_{x_0}^x \frac{dy}{f(y)}$, we find a neighborhood of x_0 in which $G \in C^1$, $G' \neq 0$ so that G has an inverse function $g \in C^1$. Then for a C^1 solution $x(t)$ defined near 0 of (1.3), we get

$$\frac{d}{dt}(G(x(t)) = \frac{\dot{x}(t)}{f(x(t))} = 1$$

which implies $G(x(t)) = t$ and thus $x(t) = g(t)$ and the uniqueness. In two dimensions, let us examine a divergence-free L^∞ vector field X. It is then a Hamiltonian vector field H_σ

$$X = \frac{\partial \sigma}{\partial y}\frac{\partial}{\partial x} - \frac{\partial \sigma}{\partial x}\frac{\partial}{\partial y},$$

where σ is a Lipschitz function. Denoting by ρ the Lipschitz equation of the hypersurface, one can assume from (1.2) that $\{\sigma, \rho\} \geq 1$ near 0. It means that the Jacobian determinant of the mapping κ given by $(\rho, \sigma) = \kappa(x, y)$ is

$$\det \kappa' = \begin{vmatrix} \partial_x \rho & \partial_y \rho \\ \partial_x \sigma & \partial_y \sigma \end{vmatrix} = X(\rho) \geq 1,$$

and κ is thus a bi-Lipschitz local homeomorphism. Setting $\nu = \kappa^{-1}$, and for a function $F \in C^1_c$, we get for $u \in L^\infty$ such that $Xu = 0$,

$$\begin{aligned} 0 = \langle Xu, F(\rho,\sigma)\rangle &= -\iint u(x,y)\frac{\partial F}{\partial\rho}(\rho(x,y),\sigma(x,y))X(\rho)(x,y)dxdy \\ &= -\iint (u\circ\nu)(\rho,\sigma)\frac{\partial F}{\partial\rho}(\rho,\sigma)\det\kappa'(\nu(\rho,\sigma))\det\nu'(\rho,\sigma)d\rho d\sigma \\ &= -\iint (u\circ\nu)(\rho,\sigma)\frac{\partial F}{\partial\rho}(\rho,\sigma)d\rho d\sigma. \end{aligned}$$

It means

$$\frac{\partial(u\circ\nu)}{\partial\rho} = 0$$

and since $u\circ\nu_{|\rho<0} = 0$, we obtain $u = 0$.

In dimension ≥ 3, the Cauchy uniqueness under the sole conditions (1.1) does not appear to be true. In fact a three-dimensional counterexample, due to M.Aizenman [Ai], shows that the existence of a flow is not guaranteed for a divergence-free L^∞ vector field. It is then natural to require some additional regularity for the coefficients of the vector field. A standard result in this direction is the Eulerian version of the classical Cauchy-Lipschitz theorem, ensuring the uniqueness of L^1_{loc} solutions for Lipschitz vector fields satisfying (1.1). In 1989, an important step forward was accomplished by R.DiPerna and P.-L.Lions, who proved in [DL] a uniqueness result for $W^{1,1}$ vector fields. Let us give a local version of their theorem.

Theorem 1.1. *Let X be a vector field and S be an hypersurface satisfying* (1.1) *on an open set of $\mathbb{R}^d$. Assume moreover that $X \in W^{1,1}_{\text{loc}}$ and let c be a L^1_{loc} function. Let u be a L^∞_{loc} function such that*

$$Xu = cu, \qquad \operatorname{supp} u \subset S_+,$$

where S_+ is the half-space above the oriented S. Then if c_+ belongs to L^∞_{loc}, the function u vanishes in a neighborhood of S. The same conclusion holds if we replace in (1.1) *the condition* $\operatorname{div} X \in L^\infty_{\text{loc}}$ *by* $(\operatorname{div} X)_+ \in L^\infty_{\text{loc}}$.

A natural question raised in [PP], [PR], [Li], with important implications in fluid mechanics, is to know if the same result holds, replacing $W^{1,1}$ by BV. In [CoL], the authors proved the following theorem.

Theorem 1.2. *Let X be a vector field and S be an hypersurface satisfying* (1.1) *on an open set Ω of $\mathbb{R}^d$. Assume moreover that $X \in BV_{\text{loc}}$ and let c be a Radon measure on Ω. Let u be a continuous function on Ω such that*

$$Xu = cu, \qquad \operatorname{supp} u \subset S_+,$$

where S_+ *is the half-space above the oriented* S*. Then, if* c_+ *belongs to* L^∞_{loc}*, the function* u *vanishes in a neighborhood of* S*. The same conclusion holds if we replace in* (1.1) *the condition* $\operatorname{div} X \in L^\infty_{\text{loc}}$ *by* $(\operatorname{div} X)_+ \in L^\infty_{\text{loc}}$.

Remarks. If $X \in L^1_{\text{loc}}$, $\operatorname{div} X \in L^1_{\text{loc}}$ and $u \in L^\infty_{\text{loc}}$, we *define*

$$Xu = \sum_{1 \le j \le n} \frac{\partial}{\partial x_j}(a_j u) - u \operatorname{div} X.$$

The same formula can be used if $X \in L^1_{\text{loc}}(\Omega)$, $\operatorname{div} X \in \mathcal{D}'^{(0)}(\Omega)$ (the Radon measures $\mathcal{M}(\Omega)$) and $u \in C^0(\Omega)$. These definitions are of course consistent with the usual definition of Xu whenever u is smooth and with the weak definition

$$\langle Xu, \phi \rangle = -\int u\big(X(\phi) + \phi \operatorname{div} X\big) dm, \quad \forall \phi \in C^1_c(\Omega),$$

where dm stands for the Lebesgue measure. In fact, if (M, ω) is a smooth oriented manifold, and X a locally bounded measurable vector field on M, the divergence of X can be defined by the equality

$$\operatorname{div} X = -{}^tX - X. \tag{1.4}$$

For φ, ψ, C^1_c test functions, we define

$$\langle {}^tX\varphi, \psi \rangle = \langle \varphi, X\psi \rangle = \int \varphi(X\psi)\omega.$$

In both theorems above, the one-sided condition can be replaced by the more elegant $(c + \operatorname{div} X)_+ \in L^\infty_{\text{loc}}$. On may remark that an unbounded divergence makes a real vector field an irreversible equation, since the divergence acts as a diffusion term. Theorem 1.2 gives uniqueness of continuous solutions, which are indeed weak solutions, but whose existence is not guaranteed. We are seeking uniqueness of L^∞ solutions, whose existence is known for vector fields satisfying (1.1) (see prop.II.1. in [DL]).

2. A new result

We shall now describe a new result, giving uniqueness of L^∞ solutions for a class of BV vector fields going beyond the so-called piecewise $W^{1,1}$ vector fields introduced in [Li]. We refer the reader to the appendix A2 for the trivial verification that the assumptions (1.1) and $X \in BV_{\text{loc}}$ are indeed invariant by $C^{1,1}$ diffeomorphism.

Definition 2.1. Let Ω_0 be an open set of $\mathbb{R}^d$. The space $conormalBV_{\text{loc}}(\Omega_0)$ is defined as a subspace of $BV_{\text{loc}}(\Omega_0)$.

(i) A function $a \in BV_{\text{loc}}(\Omega_0)$ belongs to $conormalBV_{\text{loc}}(\Omega_0)$ if each $x \in \Omega_0$ has a neighborhood $V \subset \Omega_0$ such that, on V, there exist $C^{1,1}$ coordinates $x_1, \dots, x_d$ so that

$$\frac{\partial a}{\partial x_1} \in \mathcal{M}(V), \qquad \frac{\partial a}{\partial x_k} \in L^1_{\text{loc}}(V), \quad \text{for } k \geq 2.$$

(ii) A vector field $X \in BV_{\text{loc}}(\Omega_0)$ belongs to $conormalBV_{\text{loc}}(\Omega_0)$ if each $x \in \Omega_0$ has a neighborhood $V \subset \Omega_0$ such that, on V, there exist $C^{1,1}$ coordinates $x_1, \dots, x_d$ so that, whenever $X = \sum_{1 \leq j \leq d} a_j \partial_{x_j}$,

$$\forall j \in \{1, \dots, d\}, \quad \frac{\partial a_j}{\partial x_1} \in \mathcal{M}(V), \qquad \frac{\partial a_j}{\partial x_k} \in L^1_{\text{loc}}(V), \quad \text{for } k \geq 2.$$

We shall denote the space $conormalBV_{\text{loc}}(\Omega_0)$ by $CBV_{\text{loc}}(\Omega_0)$.

Definition 2.1 is equivalent to the more intrinsic

Definition 2.1'. Let Ω_0 be $C^{1,1}$ oriented manifold, equipped with a $C^{1,1}$ local foliation of codimension 1. A vector field $X \in BV_{\text{loc}}(\Omega_0)$ belongs to $CBV_{\text{loc}}(\Omega_0)$ if, for all Lipschitz continuous vector fields Y tangent to the foliation, the bracket $[X, Y]$ is in $L^1_{\text{loc}}(\Omega_0)$. A function $a \in BV_{\text{loc}}(\Omega_0)$ belongs to $CBV_{\text{loc}}(\Omega_0)$ if, for all Lipschitz continuous vector fields Y tangent to the foliation, the function $Y(a)$ is in $L^1_{\text{loc}}(\Omega_0)$.

Note that in definition 2.1, the foliation is simply given by the hypersurfaces $\{x_1 = c\}$, which can always be assumed locally. As said in the abstract, we shall need the BV assumption only on a "small" open set Ω_0, included in our reference open set Ω. This is the reason for introducing the following

Definition 2.2. Let Ω be an open set of $\mathbb{R}^d$. We define the class $\mathcal{B}(\Omega)$ as

$$(2.1)\quad \mathcal{B}(\Omega) = \{a \in L^\infty_{\text{loc}}(\Omega) |\ \exists \Omega_0 \text{ open} \subset \Omega \text{ with } \mathcal{H}^{d-1}(\Omega \backslash \Omega_0) = 0 \text{ and } a \in CBV_{\text{loc}}(\Omega_0)\},$$

where $\mathcal{H}^{d-1}$ stands for the $d-1$ dimensional Hausdorff measure. A vector field $X \in L^\infty_{\text{loc}}(\Omega)$ belongs to $\mathcal{B}(\Omega)$ if there exists an open set $\Omega_0 \subset \Omega$ such that $\mathcal{H}^{d-1}(\Omega \backslash \Omega_0) = 0$ and $X \in CBV_{\text{loc}}(\Omega_0)$.

In the appendix A2, it is shown that the assumptions (1.1) and $X \in \mathcal{B}(\Omega)$ are invariant by a $C^{1,1}$ diffeomorphism. It might be helpful for the reader to get a couple of examples of CBV functions and vector fields.

Examples 2.3.

(a) The class $\mathcal{B}(\Omega)$ contains the so-called piecewise $W^{1,1}$ class accepting jumps across $C^{1,1}$ hypersurfaces (see proposition 6.4 of the appendix A3 for a proof of this statement). In fact, an important new feature of our result is that we can get rid of subsets whose $(d-1)$ dimensional Hausdorff measure is zero, so that our BV assumption is made only on a "small" open set $\Omega_0 \subset \Omega$, such that

$$\mathcal{H}^{d-1}(\Omega \backslash \Omega_0) = 0, \tag{2.2}$$

disregarding the geometric complexity coming from singular subsets of codimension > 1.

(b) The first simple example of a *CBV* function in $\mathbb{R}^d = \mathbb{R}_{x_1} \times \mathbb{R}^{d-1}_{x_2}$ is the tensor product

$$a(x_1, x_2) = b_1(x_1) b_2(x_2), \quad \text{where} \quad b_1 \in BV(\mathbb{R}), b_2 \in W^{1,1}(\mathbb{R}^{d-1}).$$

Note that this example is not in general piecewise $W^{1,1}$ since there is no restriction on the singularity with respect to the variable x_1, beyond the BV assumption.

(c) The class $\mathcal{B}(\Omega)$ is not included in $BV_{\text{loc}}(\Omega)$, because our regularity assumption is made only on an open subset Ω_0 of Ω such that (2.2) holds. In particular, we are able to handle vector fields which are not locally BV, but only L^∞ and conormal BV on a "small" open set Ω_0 such that (2.2) holds. For instance, we provide an example of a function in our class $\mathcal{B}(\mathbb{R}^2)$, which is not in $BV_{\text{loc}}(\mathbb{R}^2)$. Let us consider for $x = (x_1, x_2) \in \mathbb{R}^2$, $r = \sqrt{x_1^2 + x_2^2}$, the function

$$a(x_1, x_2) = \cos(r^{-2}).$$

The function a belongs to $L^\infty(\mathbb{R}^2)$ but is not in $BV_{\text{loc}}(\mathbb{R}^2)$ since, on $\{x \neq 0\}$,

$$\frac{\partial a}{\partial x_1} = 2r^{-4} \sin(r^{-2}) x_1$$

and testing[1] this distribution against the $C^0_c(\mathbb{R}^2)$ function $\mathbf{1}_{[0,\pi^{-1/2}]}(r) H(x_1) x_1 \sin(r^{-2})$, where H is the characteristic function of $\mathbb{R}_+$, we get

$$\int_0^{\pi^{-1/2}} 2r^{-1} \sin^2(r^{-2}) dr \int_{-\pi/2}^{\pi/2} \cos^2 \theta d\theta = \frac{\pi}{2} \int_\pi^{+\infty} \frac{\sin^2 s}{s} ds = +\infty.$$

Nevertheless, since the function a belongs to $C^\infty(\mathbb{R}^2 \backslash \{0\})$, it is indeed in $CBV_{\text{loc}}(\mathbb{R}^2 \backslash \{0\})$ and since $\mathcal{H}^1(\{0\}) = 0$, the function a belongs to $\mathcal{B}(\mathbb{R}^2)$. Although we have the inclusion $L^\infty_{\text{loc}}(\mathbb{R}^2) \cap C^\infty(\mathbb{R}^2 \backslash \{0\}) \subset \mathcal{B}(\mathbb{R}^2)$ we should also notice that, for a line Δ,

$$L^\infty_{\text{loc}}(\mathbb{R}^2) \cap C^\infty(\mathbb{R}^2 \backslash \Delta) \not\subset \mathcal{B}(\mathbb{R}^2), \quad \text{(e.g. } x_1 \sin(1/x_1) \notin \mathcal{B}(\mathbb{R}^2), \text{see the ftnt of lemma 6.5).}$$

[1] We should use a sequence of smooth test functions $\chi_k(x) x_1 \sin(r^{-2})$ where $\chi_k \in C^\infty_c(\mathbb{R}^2; [0,1])$ is supported in $\{x \in \mathbb{R}^2, 0 < |x| < \pi^{-1/2}$ and $x_1 \geq 0\}$, and such that, for almost all x, $\lim_k \chi_k(x) = \mathbf{1}_{[0,\pi^{-1/2}]}(|x|) H(x_1)$.

(d) One can go much beyond the second example (which is already not piecewise $W^{1,1}$) as shown by the following example. If $a(x_1,x_2)=b_1(x_1)b_2(x_2)$ with $b_1\in BV(\mathbb{R}), b_2\in BV(\mathbb{R})$ such that

$$b_1'=\sum_{k\in\mathbb{N}}\alpha_k\delta_{y_k}+L^1_{\text{loc}},\quad b_2'=\sum_{l\in\mathbb{N}}\beta_l\delta_{z_l}+L^1_{\text{loc}},\quad (\alpha_k)_{k\in\mathbb{N}},(\beta_l)_{l\in\mathbb{N}}\in\ell^1,$$

so that the sequences of real numbers $(y_k)_{k\in\mathbb{N}}$ and $(z_l)_{l\in\mathbb{N}}$ have both limit 0 then the set

$$\Omega_0^c=\{(0,0)\}\cup_{(k,l)\in\mathbb{N}^2}\{(y_k,z_l)\}\cup_{k\in\mathbb{N}}\{(y_k,0)\}\cup_{l\in\mathbb{N}}\{(0,z_l)\}$$

is closed in $\mathbb{R}^2$, and $\mathcal{H}^1(\mathbb{R}^2\backslash\Omega_0)=0$. Consequently, to check that the function a belongs to $\mathcal{B}(\mathbb{R}^2)$, it is enough to verify that $a\in CBV(\Omega_0)$. In fact, for $x\in\Omega_0$, there exists a neighborhood V of x such that $V\subset\Omega_0$ and $a_{|V}$ is as in example (b) above (or $a(x_2,x_1)$ is as in example (b)). This example is not piecewise $W^{1,1}$ because of the accumulation of singular lines at $\{0\}\times\mathbb{R}$ and $\mathbb{R}\times\{0\}$. Note however that the worst point $(0,0)$ belongs to Ω_0^c as well as the simple intersections (y_k,z_l). On the other hand, there is no difficulty to handle the points $(0,x_2)$ provided that x_2 does not belong to the closure of the set $\{z_l\}_{l\in\mathbb{N}}$ since there is then a neighborhood of $(0,x_2)$ in which a is of the type of example (b). In the appendix A4, we provide a picture of the singular support of the function a which illustrate the importance and flexibility of our assumption of *local* foliation on Ω_0 in the definition 2.1'.

(e) Let a be a function in $L^\infty_{\text{loc}}(\mathbb{R}^d)$ such that for all $j\in\{1,\dots,d\}$

$$\frac{\partial a}{\partial x_j}=\sum_{k\in\mathbb{N}}\alpha_{k,j}\delta_{S_k^j}+L^1_{\text{loc}},\quad (\alpha_{k,j})_{k\in\mathbb{N}}\in\ell^1,$$

where for each j, $(S_k^j)_{k\in\mathbb{N}}$ is a countable family of parallel hyperplanes, $\delta_{S_k^j}$ is the simple layer at S_k^j. We assume that for $j\neq j'$, the hyperplanes S_k^j and $S_l^{j'}$ are transverse and such that, for each j, the sequence $(S_k^j)_{k\in\mathbb{N}}$ converges. Argumenting as in example (d), one finds with

$$\Omega_0^c=\overline{\cup_{1\le j\neq j'\le d}\cup_{k,l\in\mathbb{N}}(S_k^j\cap S_l^{j'})}$$

that $\mathcal{H}^{d-1}(\mathbb{R}^d\backslash\Omega_0)=0$ and $a\in\mathcal{B}(\mathbb{R}^d)$.

(f) Let us give a vector field example in $\mathbb{R}^d$ of the previous type. Assume that, for all $j\in\{1,\dots,d\}$, there exists a countable family $(S_k^j)_{k\in\mathbb{N}}$ of parallel hyperplanes so that, for $j\neq j'$, the hyperplanes S_k^j and $S_l^{j'}$ are transverse and such that, for each j, the sequence $(S_k^j)_{k\in\mathbb{N}}$ converges. Let $X=\sum_{1\le i\le d}a_i\partial_{x_i}$ be a bounded measurable vector field such that, for all $i,j\in\{1,\dots,d\}$

$$\frac{\partial a_i}{\partial x_j}=\sum_{k\in\mathbb{N}}\alpha^i_{k,j}\delta_{S_k^j}+L^1_{\text{loc}},\quad (\alpha^i_{k,j})_{k\in\mathbb{N}}\in\ell^1.$$

Then X belongs to $\mathcal{B}(\mathbb{R}^d)$.

Since the notion of conormal BV makes sense on a $C^{1,1}$ manifold, all the previous examples can be pushed-forward by a $C^{1,1}$ diffeomorphism.

The main theorem of this paper is the following

Theorem 2.4. *Let Ω be an open subset of $\mathbb{R}^d$, X be a vector field and S be an hypersurface satisfying* (1.1) *on Ω. Assume moreover that X belongs to $\mathcal{B}(\Omega)$ (cf.def.2.2) and let c be a function in $L^1_{\text{loc}}(\Omega)$. Let u be a function in $L^\infty_{\text{loc}}(\Omega)$ such that*

$$Xu = cu, \qquad \operatorname{supp} u \subset S_+,$$

where S_+ is the half-space above the oriented S. Then if c_+ belongs to $L^\infty_{\text{loc}}(\Omega)$, the function u vanishes in a neighborhood of S. The same conclusion holds if we replace in (1.1) *the condition* $\operatorname{div} X \in L^\infty_{\text{loc}}(\Omega)$ *by* $(\operatorname{div} X)_+ \in L^\infty_{\text{loc}}(\Omega)$.

3. First part of the proof

We collect in this section the standard facts related to the proof, postponing the introduction of the new ingredients to the next section.

Step 1: non-negative solutions are unique. The following lemma is proved in [CoL]. For the convenience of the reader, we recall its statement here and the proof in the appendix A5. Let Ω be an open set of $\mathbb{R}^d$ and X be a L^∞_{loc} vector field on Ω with divergence in L^1_{loc}, $c \in L^1_{\text{loc}}$ and $w \in L^\infty_{\text{loc}}$. We shall say that

$$Xw \le cw$$

if for all non-negative test functions $\theta \in C^1_c(\Omega)$,

$$-\int w(X\theta + \theta \operatorname{div} X)dm = \langle Xw, \theta\rangle \le \int cw\theta dm.$$

Lemma 3.1. *Let Ω be an open set of $\mathbb{R}^d$, X be a vector field on Ω, and S a Lipschitz oriented hypersurface of Ω such that $X \in L^\infty_{\text{loc}}$, $\operatorname{div} X \in L^1_{\text{loc}}$ and X is positively transverse to S. Let w be a L^∞_{loc} function such that, for some function $c \in L^1_{\text{loc}}$,*

$$Xw \le cw, \qquad \operatorname{supp} w \subset S_+ \qquad \text{and} \qquad w \ge 0.$$

Then if $(c + \operatorname{div} X)_+ \in L^\infty_{\text{loc}}$, the function w vanishes in a neighborhood of S. Note that the assumptions of this lemma are satisfied whenever $c_+ \in L^\infty_{\text{loc}}$ and (1.1) *is fulfilled.*

Remark. This result does not require any regularity for X, besides finite speed, bounded divergence and transversality to S.

Step 2: prove that $\mathbf{Xu = 0}$ implies $\mathbf{X(u^2) = 0}$. Then use Step 1 to get $u = 0$. More pedantically, one could say that Leibnizian vector fields satisfying (1.1) have unique L^∞ solutions across transverse hypersurface, where the following property would stand as a definition of a Leibnizian vector field:

$$u, v \in L^\infty, Xu, Xv \in L^1 \Longrightarrow X(uv) = X(u)v + uX(v). \tag{3.1}$$

Let us now assume that $u \in L^\infty_{\text{loc}}$ satisfies $Xu = cu$ where c is an L^1_{loc} function such that $c_+ \in L^\infty_{\text{loc}}$, X is a vector field and S a hypersurface satisfying (1.1). We compute Xu^2 using the property (3.1) and we get, since $Xu = cu \in L^1_{\text{loc}}$

$$Xu^2 = 2uXu = 2cu^2.$$

We can now use step 1 to get the answer since $0 \le u^2$, $c_+ \in L^\infty_{\text{loc}}$ and X, S satisfy (1.1) with $\operatorname{supp} u^2 = \operatorname{supp} u \subset S_+$. In fact, the following lemma asserts that an even weaker statement than (3.1) will be sufficient to get our uniqueness result.

Lemma 3.2. *Let Ω be an open set of $\mathbb{R}^d$, X be a vector field on Ω, and S a Lipschitz oriented hypersurface of Ω such that $X \in L^\infty_{\text{loc}}$, $\operatorname{div} X \in L^1_{\text{loc}}$, $(\operatorname{div} X)_+ \in L^\infty_{\text{loc}}$ and X is positively transverse to S. Let u be a L^∞_{loc} function such that, for some function $c \in L^1_{\text{loc}}$ with $c_+ \in L^\infty_{\text{loc}}$,*

$$Xu = cu, \qquad \operatorname{supp} u \subset S_+. \tag{3.2}$$

Assume also that there exist a L^∞_{loc} function C such that for all non-negative test functions $\theta \in C^1_c(\Omega)$ there exists a sequence (u_ϵ) of Lipschitz continuous functions, bounded in $L^\infty(\operatorname{supp}\theta)$ by $\|u\|_{L^\infty(\operatorname{supp}\theta)}$ and converging a.e. in $\operatorname{supp}\theta$ to u such that

$$\limsup_\epsilon \left\{ \int X(u_\epsilon) u \theta dm \right\} \le \int C u^2 \theta dm. \tag{3.3}$$

Then the function u vanishes in a neighborhood of S.

Proof. We shall prove that there exists a non-negative L^∞_{loc} function $\widetilde{c}$ such that

$$Xu^2 \le \widetilde{c} u^2. \tag{3.4}$$

To get the conclusion, we shall use lemma 3.1. It means that we need to check that, for all non-negative test functions $\theta \in C^1_c(\Omega)$, we have, with some non-negative L^∞_{loc} function $\widetilde{c}$,

$$-\int u^2 (X\theta + \theta \operatorname{div} X) dm = \langle Xu^2, \theta \rangle \le \int \widetilde{c} u^2 \theta dm,$$

i.e.

$$0 \leq \int u^2 \Big(X\theta + \theta(\operatorname{div} X + \widetilde{c}) \Big) dm. \tag{3.5}$$

Let u_ϵ be a sequence of Lipschitz continuous functions, bounded in $L^\infty(\operatorname{supp}\theta)$ by $\|u\|_{L^\infty(\operatorname{supp}\theta)}$, converging a.e. to u. Then we have, since u_ϵ is Lipschitz continuous, for all bounded measurable $\widetilde{c}$,

$$\begin{aligned}
\int u^2 \Big(X\theta + \theta(\operatorname{div} X + \widetilde{c}) \Big) dm &= \lim_\epsilon \int u u_\epsilon \Big(X\theta + \theta(\operatorname{div} X + \widetilde{c}) \Big) dm \\
&= \lim_\epsilon \left\{ -\langle X(u_\epsilon u), \theta \rangle + \int u u_\epsilon \theta \widetilde{c} dm \right\} \\
&= \lim_\epsilon \left\{ -\langle X(u_\epsilon) u, \theta \rangle - \langle u_\epsilon X(u), \theta \rangle + \int u u_\epsilon \theta \widetilde{c} dm \right\} \\
&= \lim_\epsilon \left\{ - \int X(u_\epsilon) u \theta dm \right\} + \int u^2 \theta (\widetilde{c} - c) dm \\
&\geq \lim_\epsilon \left\{ - \int X(u_\epsilon) u \theta dm \right\} + \int u^2 \theta (\widetilde{c} - c) dm \\
&\geq \int u^2 \theta (\widetilde{c} - c - C) dm. \qquad (3.6)
\end{aligned}$$

We can take $\widetilde{c} = (c + C)_+$ to infer (3.5) from (3.6). $\square$

The proof of theorem 2.3 is thus reduced to proving the estimate (3.3), which will be done in section 5.

4. Getting rid of subsets whose $\mathcal{H}^{d-1}$ measure is zero

We focus our attention in this section on the first new feature of our proof. We shall show first that we need only to prove this estimate for non-negative test functions θ in $C^1_c(\Omega_0)$ where Ω_0 is an open subset of Ω such that $\mathcal{H}^{d-1}(\Omega \backslash \Omega_0) = 0$. It is an important aspect of our argument to reduce checking (3.3) in an open set Ω_0 such that $\mathcal{H}^{d-1}(\Omega \backslash \Omega_0) = 0$, somehow getting rid a priori of subsets whose $(d-1)$ Hausdorff measure is zero.

Lemma 4.1. *Let $\Omega_0 \subset \Omega$ be open subsets of $\mathbb{R}^d$ such that $\mathcal{H}^{d-1}(\Omega \backslash \Omega_0) = 0$. Let X be a vector field in $L^\infty_{\text{loc}}(\Omega)$ such that $\operatorname{div} X \in L^1_{\text{loc}}(\Omega)$, and let $v \in L^\infty_{\text{loc}}(\Omega)$ be a (weak) solution on Ω_0 of the equation $Xv = f$ where f belongs to $L^1_{\text{loc}}(\Omega)$. It means that for all $\varphi \in C^1_c(\Omega_0)$,*

$$\int f \varphi dm = - \int v \big(X(\varphi) + \varphi \operatorname{div} X \big) dm. \tag{4.1}$$

Then the equation $Xv = f$ is satisfied weakly on Ω, i.e. (4.1) *is true for all* $\varphi \in C^1_c(\Omega)$.

Proof. We first make the more restrictive assumption

$$\mathcal{M}^{d-1}(\Omega \backslash \Omega_0) = 0$$

where $\mathcal{M}^{d-1}$ stands for the $(d-1)$ Minkowski content i.e., setting $F = \Omega \backslash \Omega_0$,

$$\mathcal{L}^d(F + \epsilon B_1) = \epsilon\alpha(\epsilon), \quad \text{with} \lim_{\epsilon \to 0_+} \alpha(\epsilon) = 0,$$

where $\mathcal{L}^d$ stands for the d-dimensional Lebesgue measure and B_1 is the closed unit ball of $\mathbb{R}^d$. Let φ be a test function $\in C^1_c(\Omega)$. Using the notation A^c for the complement of A in $\mathbb{R}^d$, we note that the set

$$K = F \cap \operatorname{supp} \varphi = \Omega_0^c \cap \operatorname{supp} \varphi$$

is a compact subset of Ω satisfying $\mathcal{M}^{d-1}(K) = 0$. We define the Lipschitz continuous function

$$\sigma(t) = \min\bigl((2t-1)_+, 1\bigr), \quad \text{so that} \begin{cases} \sigma(t) = 0 & \text{if } t \le 1/2, \\ \sigma(t) = 1 & \text{if } t \ge 1, \\ 0 \le \sigma' \le 2. \end{cases} \tag{4.2}$$

We note that $X\varphi$ belongs to $L^\infty_{\text{comp}}(\Omega)$, $\operatorname{supp} X\varphi \subset \operatorname{supp} \varphi$ and $\sigma(\epsilon^{-1}|x-K|)$ tends to 1 on the complement of K when ϵ goes to 0_+. Since the Lebesgue measure of K is zero, we have[2]

$$\begin{aligned} -\int v\bigl(X(\varphi) + \varphi \operatorname{div} X\bigr) dm \\ = -\lim_{\epsilon \to 0_+} \int v(x)\bigl((X\varphi)(x) + \varphi(x)(\operatorname{div} X)(x)\bigr)\sigma(\epsilon^{-1}|x-K|)dx. \end{aligned} \tag{4.3}$$

Setting for $\epsilon > 0$, $\omega_\epsilon(x) = \sigma(\epsilon^{-1}|x-K|)$, we note that the function ω_ϵ is Lipschitz continuous and

$$\operatorname{supp} \omega_\epsilon \subset \{|x-K| \ge \epsilon/2\} \subset K^c = F^c \cup (\operatorname{supp} \varphi)^c = \Omega^c \cup \Omega_0 \cup (\operatorname{supp} \varphi)^c$$

and thus

$$\operatorname{supp} \omega_\epsilon \cap \operatorname{supp} \varphi \subset \Omega_0. \tag{4.4}$$

[2] In the sequel, we shall use the notation dx for the Lebesgue measure when the variable x appears in the integrand.

We obtain from (4.3), with $T = \nabla_x(|x-K|)$ (note that $\|T\|_{L^\infty} \le 1$), integrating by parts as we may on the open set Ω_0 (that is using the assumption of the lemma),

$$(4.5)\quad -\int v\big(X(\varphi)+\varphi \operatorname{div} X\big)dm \\ = \lim_{\epsilon\to 0_+}\left\{\int \sigma(\epsilon^{-1}|x-K|)f\varphi dx + \int \varphi v \sigma'(\epsilon^{-1}|x-K|)\langle T(x)\cdot X(x)\rangle \epsilon^{-1}dx\right\}.$$

We have

$$\limsup_{\epsilon\to 0_+}\int \big|\varphi v\sigma'(\epsilon^{-1}|x-K|)\langle T(x)\cdot X(x)\rangle\big|\,\epsilon^{-1}dm \\ \le 2\,\|v\varphi X\|_{L^\infty}\limsup_{\epsilon\to 0_+}\epsilon^{-1}\mathcal{L}^d(K+\epsilon B_1)=0,$$

since our assumption is precisely $\epsilon^{-1}\mathcal{L}^d(K+\epsilon B_1)=\alpha(\epsilon)\to 0$ with ϵ. Thus (4.5) implies

$$-\int v\big(X(\varphi)+\varphi \operatorname{div} X\big)dm = \int f\varphi dm, \quad \text{q.e.d.}$$

Let us now make the less stringent[3] assumption $\mathcal{H}^{d-1}(F)=0$. Let φ be a test function $\in C^1_c(\Omega)$ and consider as before the compact set $K = F\cap \operatorname{supp}\varphi \subset \Omega$. The fact that $\mathcal{H}^{d-1}(K)=0$ means that for any $\epsilon>0$, there exist $\delta_\epsilon>0$ such that, for any $\delta\in]0,\delta_\epsilon]$, there exist a sequence of

$$(4.6)\qquad \text{open sets } S_j \text{ with diameter} \le \delta \text{ such that } S_j\cap K\neq\emptyset$$

and

$$(4.7)\qquad K\subset \cup_{j\in\mathbb{N}}S_j,\quad \sum_{j\in\mathbb{N}}(\operatorname{diam} S_j)^{d-1}\le\epsilon.$$

In particular, one can assume that $\delta\in]0,\epsilon]$. Let us choose from now on some $\delta\in]0,\epsilon]$. Since K is compact, we can assume that $K\subset\cup_{j\in J}S_j$ with a finite set of indices J. Setting $\delta_j=\operatorname{diam}S_j$ (note that $0<\delta_j\le\delta$ since S_j is a non-empty open set), we consider

$$\phi_j\in C^\infty_c(S_j+2\delta_jB_1;[0,1])$$

a smooth function equal to 1 on $\overline{S_j}+\delta_jB_1$ with $\|\nabla\phi_j\|_{L^\infty}\le\delta_j^{-1}C(d)$, where $C(d)$ depends only on the dimension. We consider now the Lipschitz continuous function

$$\chi=\sigma\big(\sum_{j\in J}\phi_j\big),\quad \text{where } \sigma \text{ is defined in (4.2).}$$

[3] We are indebted to Giovanni Alberti for this improvement.

For $x \in K + (\min_{j\in J} \delta_j)B_1$, there exists at least an index $j \in J$ such that $x \in S_j + \delta_j B_1$ and thus $\phi_j(x) = 1$. Consequently, the function χ is equal to 1 on $K + (\min_j \delta_j)B_1$ and is supported in the set

$$\{\sum_{j\in J} \phi_j \geq 1/2\} \subset \cup_{j\in J} \operatorname{supp} \phi_j \subset \cup_{j\in J}(S_j + 2\delta_j B_1) \subset K + 3\delta B_1,$$

where the last inclusion is due to the assumption (4.6). Moreover, the gradient of χ satisfies

$$|\nabla\chi| = |\sigma'(\sum_{j\in J} \phi_j) \sum_{j\in J} \nabla\phi_j| \leq 2C(d) \sum_{j\in J} \delta_j^{-1} \mathbf{1}_{S_j+2\delta_j B_1}$$

which implies the following estimate for its integral

$$\begin{aligned}\int_{\mathbb{R}^d} |\nabla\chi(x)|dx \leq 2^{1-d} C(d)\mathcal{L}^d(B_1) \sum_{j\in J} \delta_j^{-1}(\text{diam} S_j + 4\delta_j)^d \\ = C_1(d) \sum_{j\in J} \delta_j^{d-1} \leq C_1(d)\epsilon,\end{aligned}$$

where $C_1(d)$ depends only on the dimension. Eventually, for any $\epsilon > 0$, we were able to construct a Lipschitz continuous function χ_ϵ, valued in [0,1], supported in $K + 3\epsilon B_1$, equal to 1 in $K + \rho_\epsilon B_1$ with some $\rho_\epsilon \in]0, \epsilon]$ such that

$$\int |\nabla\chi_\epsilon(x)|dx \leq C_1(d)\epsilon. \tag{4.8}$$

In particular, since $\mathcal{L}^d(K) = 0$, we obtain, with

$$\omega_\epsilon = 1 - \chi_\epsilon, \tag{4.9}$$

$$\lim_{\epsilon\to 0_+} \omega_\epsilon(x) = 1,\ \mathcal{L}^d - \text{a.e. and } \|\omega_\epsilon\|_{L^\infty} \leq 1. \tag{4.10}$$

It is then easy to start over the computations in (4.2–5). We check, using (4.10) and $vX(\varphi), \varphi v \in L^\infty_{\text{comp}}(\Omega), v\varphi \operatorname{div} X, f\varphi \in L^1$,

$$\begin{aligned}-\int v\big(X(\varphi) + \varphi \operatorname{div} X\big)dm &= -\lim_{\epsilon\to 0_+} \int \omega_\epsilon v\big(X(\varphi) + \varphi \operatorname{div} X\big)dm \\ &= \lim_{\epsilon\to 0_+} \left\{\int \omega_\epsilon f\varphi dm + \int \varphi v\langle\nabla\omega_\epsilon(x) \cdot X(x)\rangle dx\right\} \\ &= \int f\varphi dm\end{aligned}$$

from (4.10) and the following consequence of (4.8)

$$\limsup_{\epsilon\to 0_+}\int|\varphi v\langle\nabla\omega_\epsilon(x)\cdot X(x)\rangle|\,dx\le\|\varphi vX\|_{L^\infty}\limsup_{\epsilon\to 0_+}\int|\nabla\chi_\epsilon(x)|dx=0. \tag{4.11}$$

The proof of lemma 4.1 is complete.

Remark. This proof shows also that it would be possible to weaken the assumption $\mathcal{H}^{d-1}(\Omega\backslash\Omega_0)=0$. As a matter of fact, a sufficient requirement on F concerns its so-called $W^{1,1}$ capacity. We need only to assume that for all $\epsilon>0$, and all K compact subset of F, there exists a $W^{1,1}$ function χ_ϵ, supported in Ω, valued in [0,1], such that

$$K\subset\operatorname{int}\big(\{\chi_\epsilon=1\}\big),\quad\lim_{\epsilon\to 0_+}\chi_\epsilon(x)=0\ \mathcal{L}^d-\text{a.e.},\quad\lim_{\epsilon\to 0_+}\int|\nabla\chi_\epsilon|dm=0. \tag{4.12}$$

This assumption amounts essentially to require that the $W^{1,1}$ capacity of $\Omega\backslash\Omega_0$ is 0. However, we shall stick on our hypothesis involving the $(d-1)$ Hausdorff measure since we believe that this condition is easier to understand and more explicit than (4.12). We can view the previous arguments as proofs of the implications[4]

$$\mathcal{M}^{d-1}(F)=0\Longrightarrow\mathcal{H}^{d-1}(F)=0\Longrightarrow\operatorname{cap}_{W^{1,1}}(F)=0.$$

In fact, we shall use the following lemma, dealing with an inequality, whose proof is identical to lemma 4.1's.

Lemma 4.2. *Let $\Omega_0\subset\Omega$ be open subsets of $\mathbb{R}^d$ such that $\mathcal{H}^{d-1}(\Omega\backslash\Omega_0)=0$. Let X be a vector field in $L^\infty_{\mathrm{loc}}(\Omega)$ such that $\operatorname{div}X\in L^1_{\mathrm{loc}}(\Omega)$, and let $v\in L^\infty_{\mathrm{loc}}(\Omega)$ be a (weak) solution on Ω_0 of the inequality $Xv\le f$ where f belongs to $L^1_{\mathrm{loc}}(\Omega)$. It means that for all non-negative $\varphi\in C^1_c(\Omega_0)$,*

$$\int f\varphi dm+\int v\big(X(\varphi)+\varphi\operatorname{div}X\big)dm\ge 0. \tag{4.13}$$

Then the inequality $Xv\le f$ is satisfied weakly on Ω, i.e. (4.13) *is true for all $\varphi\in C^1_c(\Omega)$.*

Proof. Let φ be a non-negative function $\in C^1_c(\Omega)$, and let ω_ϵ satisfying $(4.9-10)$ with $K=\operatorname{supp}\varphi\cap\Omega_0^c$. Following the same lines as in the previous proofs, we get

$$\begin{aligned}\int f\varphi dm+\int v\big(X(\varphi)+\varphi\operatorname{div}X\big)dm&=\int f\varphi dm+\lim_{\epsilon\to 0_+}\int\omega_\epsilon v\big(X(\varphi)+\varphi\operatorname{div}X\big)dm\\&=\int f\varphi dm-\lim_{\epsilon\to 0_+}\int\left(X(\omega_\epsilon)v+\omega_\epsilon X(v)\right)\varphi dm\\&=\lim_{\epsilon\to 0_+}\left\{\int f\omega_\epsilon\varphi dm+\int v\big(X(\omega_\epsilon\varphi)+\omega_\epsilon\varphi\operatorname{div}X\big)dm\right\}\ge 0.\quad\square\end{aligned}$$

[4] We can note also that, for analytic sets (Suslin sets) we have the equivalence $\mathcal{H}^{d-1}(F)=0\Longleftrightarrow\operatorname{cap}_{W^{1,1}}(F)=0$ (see e.g. lemma 5.12.3 in [Zi]).

5. Commutation arguments

Preliminary remarks. When $X \in W^{1,1}_{\text{loc}}(\Omega)$, one can prove (see [DL]) a much stronger statement than (3.3). Let Ω be an open set of $\mathbb{R}^d$, X be a vector field on Ω, and S a C^1 oriented hypersurface of Ω such that $X \in L^\infty_{\text{loc}}$, $\operatorname{div} X \in L^1_{\text{loc}}$, $(\operatorname{div} X)_+ \in L^\infty_{\text{loc}}$ and X is positively transverse to S. Let u be a L^∞_{loc} function such that, for some function $c \in L^1_{\text{loc}}$ with $c_+ \in L^\infty_{\text{loc}}$,

$$Xu = cu, \qquad \operatorname{supp} u \subset S_+.$$

If $X \in W^{1,1}_{\text{loc}}(\Omega)$, then one can prove the strong convergence in L^1 of $\theta X(\chi u)_\epsilon$ to $\theta c u$, where χ, θ are smooth compactly supported functions, $\chi = 1$ on the support of θ, $(\chi u)_\epsilon$ is a regularization of χu by any standard mollifier. Here we took any smooth compactly supported function ρ with integral 1 and set

$$v_\epsilon(x) = \int \epsilon^{-n} \rho\Big(\frac{x-y}{\epsilon}\Big) v(y) dy.$$

The proof amounts to the computation of the commutator

$$[X, \widehat{\rho}(\epsilon D)].$$

It fails even in the piecewise $W^{1,1}$ case if a jump occurs on a curved hypersurface. In the latter case, one should use a pseudo-differential mollifier and not only a convolution operator, or equivalently, straighten first the jump hypersurface and after this use a convolution. However, one should be careful at choosing the various speeds: if the jump occurs on the hypersurface $\{x_1 = 0\}$ then it is natural to choose $\epsilon_1 \ll \epsilon_2, \dots, \epsilon_d$ if ϵ is a diagonal matrix. It is still not enough to handle the simplest BV example: we must pay more attention at choosing the $\epsilon' s$.

Beginning of the proof of the estimate (3.3). In order to obtain the result in theorem 2.4, we need only to prove (3.3) on Ω_0, since we shall then get that

$$X(u^2) \le (C + c_+) u^2$$

on Ω_0 and thus, from lemma 4.2, the same inequality in Ω. Then u^2 will satisfy the assumptions in lemma 3.1, which will give the result. Let us then consider a vector field on V open subset of Ω_0

$$X = \sum_{1 \le j \le d} a_j(x) \frac{\partial}{\partial x_j}$$

and assume for $1 \le j \le d$

$$\text{(5.1)} \qquad \frac{\partial a_j}{\partial x_1} \in \mathcal{M}(V) = \mathcal{D}^{'(0)}(V), \text{ and for } k \ge 2,\ \frac{\partial a_j}{\partial x_k} \in L^1_{\text{loc}}(V), \operatorname{div} X \in L^1_{\text{loc}}(V).$$

Let u be a $L^\infty_{\rm loc}(V)$, $c \in L^1_{\rm loc}(V)$ such that $Xu = cu$. We consider a non-negative function $\theta \in C^1_c(V)$ and $\chi \in C^1_c(V;[0,1])$ such that $\operatorname{supp}\theta \subset \{\chi = 1\}$. Let ρ be a smooth compactly supported function with integral 1, ϵ be a positive diagonal $d \times d$ matrix with $\epsilon_2 = \cdots = \epsilon_d$. We need to check (3.3), setting $v = \chi u$ and $v_\epsilon = v * \rho_\epsilon$

$$\begin{aligned}\langle X(u^2), \theta\rangle &= -\int u^2 (X(\theta) + \theta \operatorname{div} X) dm = -\int \chi u^2 (X(\theta) + \theta \operatorname{div} X) dm \\ &= -\lim_{\epsilon \to 0_+} \int v_\epsilon \; u\big(X(\theta) + \theta \operatorname{div} X\big) dm = \lim_{\epsilon \to 0_+} \int u \theta X v_\epsilon dm + \int c u^2 \theta dm \\ &\le \lim_{\epsilon \to 0_+} \int u\theta X v_\epsilon dm + \int u^2 \theta c_+ dm.\end{aligned}$$

We define, (for $x \in \operatorname{supp}\chi$ and ϵ small enough so that $\operatorname{supp}\chi + \epsilon \operatorname{supp}\rho \subset V$),

$$(R_\epsilon v)(x) = \sum_{1 \le j \le d} \int \Big(v(x - \epsilon z) - v(x)\Big) \epsilon_j^{-1} \Big(a_j(x) - a_j(x - \epsilon z)\Big) (\partial_j \rho)(z) dz \tag{5.2}$$

and we check easily that

$$\big(X(v * \rho_\epsilon) - (Xv) * \rho_\epsilon\big)(x) = (R_\epsilon v)(x) + \overbrace{\int (\operatorname{div} X)(x - \epsilon z)\rho(z)\Big(v(x - \epsilon z) - v(x)\Big) dz}^{(T_\epsilon v)(x)}. \tag{5.3}$$

Consequently, we have,

$$X(v_\epsilon) = (Xv) * \rho_\epsilon + T_\epsilon v + R_\epsilon v. \tag{5.4}$$

From the equation $Xv = cv + uX(\chi)$, we get that Xv belongs to L^1 and the strong convergence in L^1 of $\theta(Xv * \rho_\epsilon)$ to $c\theta u$. The term $T_\epsilon v$ is also easy[5] to handle since, using the notation $\tau_t w(x) = w(x - t)$, we have

$$\begin{aligned}\|\theta T_\epsilon v\|_{L^1} \le \int \|\theta(\tau_{\epsilon z} - \mathrm{Id})(\operatorname{div} X)\|_{L^1} \, 2 \, \|v\|_{L^\infty} |\rho(z)| dz \\ + \iint |\rho(z)| |v(x - \epsilon z) - v(x)| |(\operatorname{div} X)(x)| \theta(x) dx dz.\end{aligned} \tag{5.5}$$

Since $\operatorname{div} X$ belongs to $L^1_{\rm loc}$ and ρ is compactly supported, the first term in the right-hand side of the above inequality goes to 0 with ϵ. For the second term, we can use the assertion (5.6) of the following simple lemma.

[5] It is even trivial to see that $T_\epsilon v$ goes to zero strongly in L^1 if we assume that $\operatorname{div} X$ is bounded since v belongs to L^1. However, we want to show that our weaker assumption $(\operatorname{div} X)_+ \in L^\infty_{\rm loc}$ and $\operatorname{div} X \in L^1_{\rm loc}$ is enough.

Lemma 5.1. *Let $b \in L^1$, $a \in BV$ and $v \in L^\infty$. Then, we have*

$$\lim_{t\rightarrow 0}\int |b(x)||v(x+t)-v(x)|dx = 0 \quad \text{and} \tag{5.6}$$

$$\int |a(x+t)-a(x)||v(x)|dx \le \|\nabla a\|_{\mathcal{M}_b}\|v\|_{L^\infty}|t|. \tag{5.7}$$

Proof: see the appendix A6.

We get then

$$\langle X(u^2),\theta\rangle \le \lim_{\epsilon\rightarrow 0_+}\int u\theta R_\epsilon v dm + 2\int u^2\theta c_+ dm.$$

The key argument. We need to check the quantity

$$\int u\theta R_\epsilon v dm. \tag{5.8}$$

Looking at (5.2), we calculate, with a sequence u_ν of continuous functions converging a.e. on $\operatorname{supp}\chi$ to u with $\|u_\nu\|_{L^\infty(\operatorname{supp}\chi)} \le \|u\|_{L^\infty(\operatorname{supp}\chi)}$ and setting $v_\nu = \chi u_\nu$

$$\begin{aligned}&\int u(x)\theta(x)(R_\epsilon v)(x)dx \qquad (5.9)\\ &= \lim_\nu \sum_{1\le j\le d}\iint u_\nu(x)\theta(x)\Big(v_\nu(x-\epsilon z)-v_\nu(x)\Big)\epsilon_j^{-1}\Big(a_j(x)-a_j(x-\epsilon z)\Big)(\partial_j\rho)(z)dzdx.\end{aligned}$$

Setting (when $\frac{\partial a_j}{\partial x_k}$ is a Radon measure, the integral stands for a bracket of duality)

$$(R_\epsilon v_\nu)(x) = \sum_{1\le j,k\le d}\int_0^1\int\Big(v_\nu(x-\epsilon z)-v_\nu(x)\Big)\epsilon_j^{-1}\epsilon_k\frac{\partial a_j}{\partial x_k}(x-s\epsilon z)z_k(\partial_j\rho)(z)dsdz, \tag{5.10}$$

and using the fact that ϵ is a positive diagonal matrix with

$$0<\epsilon_1\le\epsilon_2=\cdots=\epsilon_d, \tag{5.11}$$

we get

$$(R_\epsilon v_\nu)(x) =$$

$$\int_0^1\int\Big(v_\nu(x-\epsilon z)-v_\nu(x)\Big)\frac{\partial a_1}{\partial x_1}(x-s\epsilon z)z_1(\partial_1\rho)(z)dsdz \tag{5.12}$$

$$+\sum_{2\le k\le d}\int_0^1\int\Big(v_\nu(x-\epsilon z)-v_\nu(x)\Big)\epsilon_1^{-1}\epsilon_2\frac{\partial a_1}{\partial x_k}(x-s\epsilon z)z_k(\partial_1\rho)(z)dsdz \tag{5.13}$$

$$+\sum_{2\le j\le d}\int_0^1\int\Big(v_\nu(x-\epsilon z)-v_\nu(x)\Big)\epsilon_2^{-1}\epsilon_1\frac{\partial a_j}{\partial x_1}(x-s\epsilon z)z_1(\partial_j\rho)(z)dsdz \tag{5.14}$$

$$+\sum_{2\le j,k\le d}\int_0^1\int\Big(v_\nu(x-\epsilon z)-v_\nu(x)\Big)\frac{\partial a_j}{\partial x_k}(x-s\epsilon z)z_k(\partial_j\rho)(z)dsdz. \tag{5.15}$$

Note that from $\operatorname{div} X \in L^1_{\mathrm{loc}}$, and (5.1) we get $\frac{\partial a_1}{\partial x_1} \in L^1_{\mathrm{loc}}$ so that

$$\frac{\partial a_1}{\partial x_1}, \frac{\partial a_1}{\partial x_k} \text{ for } k \geq 2 \text{ and } \frac{\partial a_j}{\partial x_k} \text{ for } k \geq 2 \text{ are in } L^1_{\mathrm{loc}}. \tag{5.16}$$

This implies that the limit with $\nu \to +\infty$ of the terms (5.12), (5.13) and (5.15) are respectively (a.e. in x)

$$\int_0^1 \int \Big(v(x-\epsilon z) - v(x)\Big) \frac{\partial a_1}{\partial x_1}(x - s\epsilon z) z_1 (\partial_1 \rho)(z) ds dz,$$

$$\sum_{2\leq k\leq d} \int_0^1 \int \Big(v(x-\epsilon z) - v(x)\Big) \epsilon_1^{-1}\epsilon_2 \frac{\partial a_1}{\partial x_k}(x - s\epsilon z) z_k (\partial_1 \rho)(z) ds dz,$$

$$\sum_{2\leq j,k\leq d} \int_0^1 \int \Big(v(x-\epsilon z) - v(x)\Big) \frac{\partial a_j}{\partial x_k}(x - s\epsilon z) z_k (\partial_j \rho)(z) ds dz.$$

with domination by L^1 functions of the variable x independent of the index ν. Consequently, we obtain from (5.9),

$$\begin{aligned}
&\int u(x)\theta(x)(R_\epsilon v)(x)dx = \\
&\int_0^1 \iint u(x)\theta(x)\Big(v(x-\epsilon z) - v(x)\Big) \frac{\partial a_1}{\partial x_1}(x - s\epsilon z) z_1 (\partial_1 \rho)(z) ds dz dx, \\
&+ \sum_{2\leq k\leq d} \int_0^1 \iint u(x)\theta(x)\Big(v(x-\epsilon z) - v(x)\Big) \epsilon_1^{-1}\epsilon_2 \frac{\partial a_1}{\partial x_k}(x - s\epsilon z) z_k (\partial_1 \rho)(z) ds dz dx, \\
&+ \lim_\nu \sum_{2\leq j\leq d} \int_0^1 \iint u_\nu(x)\theta(x)\Big(v_\nu(x-\epsilon z) - v_\nu(x)\Big) \epsilon_2^{-1}\epsilon_1 \frac{\partial a_j}{\partial x_1}(x - s\epsilon z) z_1 (\partial_j \rho)(z) ds dz dx, \\
&+ \sum_{2\leq j,k\leq d} \int_0^1 \iint u(x)\theta(x)\Big(v(x-\epsilon z) - v(x)\Big) \frac{\partial a_j}{\partial x_k}(x - s\epsilon z) z_k (\partial_j \rho)(z) ds dz dx.
\end{aligned}$$

Thus, using lemma 5.1, we get

$$\begin{aligned}
\left|\int u(x)\theta(x)(R_\epsilon v)(x)dx\right| \leq{}& \sigma_{11}(\epsilon_1,\epsilon_2) + \sum_{2\leq k\leq d} \sigma_{1k}(\epsilon_1,\epsilon_2)\frac{\epsilon_2}{\epsilon_1} + \sum_{2\leq j,k\leq d} \sigma_{jk}(\epsilon_1,\epsilon_2) \\
&+ \frac{\epsilon_1}{\epsilon_2} 2 \left\|u^2\theta\right\|_{L^\infty} \sum_{2\leq j\leq d} \left\|\frac{\partial a_j}{\partial x_1}\right\|_{\mathcal{M}_b(\operatorname{supp}\chi)} \int |z_1(\partial_j\rho)(z)| dz,
\end{aligned} \tag{5.17}$$

where the functions $\sigma_{11}, \sigma_{1k}, \sigma_{jk}$ tend to 0 with ϵ. We infer from (5.17) that there exists a constant C_1 and a function σ such that

$$\left|\int u(x)\theta(x)(R_\epsilon v)(x)dx\right| \leq \sigma(\epsilon_1,\epsilon_2) + \sigma(\epsilon_1,\epsilon_2)\frac{\epsilon_2}{\epsilon_1} + C_1 \frac{\epsilon_1}{\epsilon_2}, \tag{5.18}$$

$$\text{where } 0 = \lim_{\epsilon\to 0} \sigma(\epsilon_1,\epsilon_2). \tag{5.19}$$

We set

$$\beta(\epsilon_2) = \sup_{0<\epsilon_1\le\epsilon_2} \sigma(\epsilon_1,\epsilon_2), \tag{5.20}$$

we note that $\lim_{\epsilon_2\to 0}\beta(\epsilon_2)=0$ and we choose in (5.18)

$$\epsilon_1 = \beta(\epsilon_2)^{1/2}\epsilon_2, \tag{5.21}$$

which is $\le \epsilon_2$, for ϵ_2 small enough, obtaining

$$\left|\int u(x)\theta(x)(R_\epsilon v)(x)dx\right| \le \beta(\epsilon_2)+\beta(\epsilon_2)^{1/2}+C_1\beta(\epsilon_2)^{1/2} \xrightarrow[\epsilon_2\to 0]{} 0. \tag{5.22}$$

We obtain

$$\langle X(u^2),\theta\rangle \le 2\int u^2\theta c_+ dm,$$

that is the inequality

$$X(u^2)\le 2c_+u^2$$

is satisfied on Ω_0. Following the previous remarks, this completes the proof of theorem 2.2.

Concluding remarks. Note that the choice of ϵ_1,ϵ_2 depends on the geometry: the condition (5.21) implies $\epsilon_1 \ll \epsilon_2$, a condition forced by the term ϵ_1/ϵ_2 in the right-hand-side of (5.18). But this natural geometric condition is not enough to handle that matter: the choice of ϵ_1,ϵ_2 depends also on the function u under scope, since in (5.20-21), the functions σ and β depend on u.

6. Appendix

A1. Three simple examples with pictures. We give in this section three simple examples, demonstrating that the condition $(\operatorname{div} X)_+ \in L^\infty_{\text{loc}}$ is necessary for the forward uniqueness property.

Example 6.1. *Let* $T_1 = \operatorname{sign} x\frac{\partial}{\partial x}+\frac{\partial}{\partial y}$ *be a vector field on* $\mathbb{R}^2$. *The vector field* T_1 *belongs to* $L^\infty\cap BV$, *is tranverse to the hypersurface* $S=\{y=0\}$ *since* $T_1(y)=1$, *and fails to have the uniqueness property across* S *since*

$$T_1\big((y-|x|)_+\big)=0, \quad \textit{and} \quad (y-|x|)_{+_{|y<0}}=0.$$

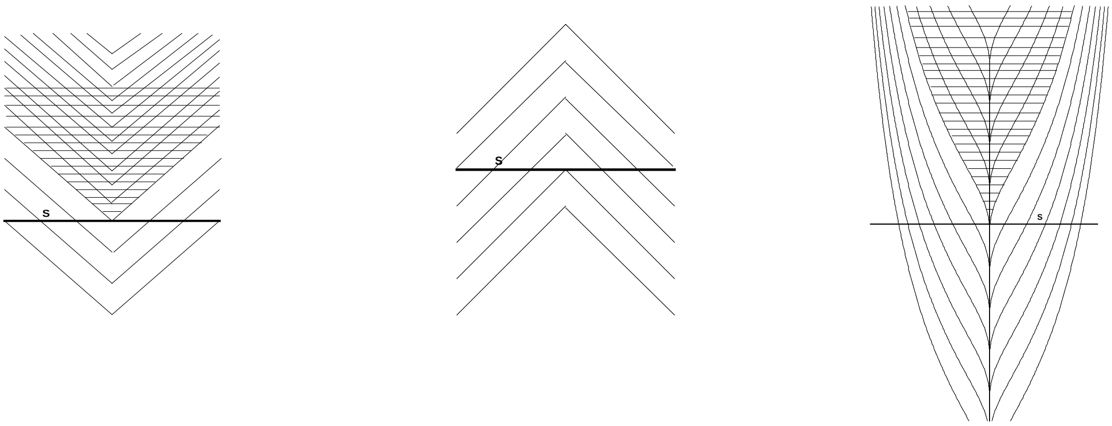
Note that $\operatorname{div} T_1 = 2\delta(x)$ which is non-negative and not in L^∞. The next example shows that the negative part of the divergence is unimportant for uniqueness.

Example 6.2. *Let* $T_2 = -\operatorname{sign} x \frac{\partial}{\partial x} + \frac{\partial}{\partial y}$ *be a vector field on* $\mathbb{R}^2$*. The vector field* T_2 *belongs to* $L^\infty \cap BV$*, is tranverse to the hypersurface* $S = \{y = 0\}$ *since* $T_2(y) = 1$*, and has the uniqueness property across* S *from theorem* 2.4 *since* $(\operatorname{div} T_2)_+ = 0$.

Example 6.3. *Let* $T_3 = x \ln^2 |x| \frac{\partial}{\partial x} + \frac{\partial}{\partial y}$ *be a vector field on* $\mathbb{R}^2$*. The vector field* T_3 *belongs to* $\cap_{1 \le p < \infty} W^{1,p}_{\text{loc}}$*, is tranverse to the hypersurface* $S = \{y = 0\}$ *since* $T_3(y) = 1$*, and fails to have the uniqueness property across* S *since, on the open set* $\{|x| < 1\}$*,*

$$T_3\Big(\Big(y - \frac{1}{|\ln|x||}\Big)_+\Big) = 0 \quad \textit{and} \quad \Big(y - \frac{1}{|\ln|x||}\Big)_{+|y<0} = 0.$$

Note that $\operatorname{div} T_3 = \ln^2|x| + 2\ln|x|$ is non-negative near the origin and not bounded. The behaviour of the T_j is also apparent on the following pictures.



Integral curves of T_1, T_2, T_3

The integral curves of T_1 (resp.T_3) starting from $\{y < 0\}$ cannot penetrate the shaded region where $\{y > |x|\}$ (resp. $\{y > 1/|\ln|x||\}$). On the other hand, the integral curves of T_2 starting from $\{y < 0\}$ fill a neighborhood of the origin.

A2. Invariance properties. Let us first check that the assumptions (1.1) and $X \in BV_{\text{loc}}$ are invariant by a $C^{1,1}$ diffeomorphism. The boundedness and transversality properties in (1.1) are already invariantly stated. Moreover the BV regularity for a vector field makes sense on a $C^{1,1}$ manifold since with $x = \kappa(y)$, where κ is a local $C^{1,1}$ diffeomorphism,

$$X = \sum_j a_j \frac{\partial}{\partial x_j} = \sum_{k,j} a_j \frac{\partial y_k}{\partial x_j} \frac{\partial}{\partial y_k}$$

and

$$\frac{\partial}{\partial y_l}\left(a_j(\kappa(y))\frac{\partial y_k}{\partial x_j}(\kappa(y))\right) = \sum_i \overbrace{\frac{\partial a_j}{\partial x_i}}^{\in \mathcal{M}} \overbrace{\frac{\partial x_i}{\partial y_l}\frac{\partial y_k}{\partial x_j}}^{\in C^{0,1}} + \overbrace{a_j}^{\in L^1} \overbrace{\frac{\partial^2 y_k}{\partial x_j \partial x_i}\frac{\partial x_i}{\partial y_l}}^{\in L^\infty}$$

is indeed a Radon measure. The divergence is invariantly expressed in (1.4), but we can note directly that for ω a non vanishing Lipschitz continuous function, the divergence of the vector field X with respect to the n-form $\omega(x)dx_1 \wedge \cdots \wedge dx_n$ is

$$\operatorname{div} X = \omega^{-1}\sum_j \frac{\partial(a_j\omega)}{\partial x_j} = \sum_j \frac{\partial a_j}{\partial x_j} + X(\ln|\omega|).$$

Since X is also in L^∞_{loc}, the divergence of X is *a priori* a Radon measure and the condition $\operatorname{div} X \in L^\infty_{\text{loc}}$ (resp. $(\operatorname{div} X)_+ \in L^\infty_{\text{loc}}$) is simply $\sum_j \partial_{x_j}(a_j) \in L^\infty_{\text{loc}}$ (resp. $(\sum_j)_+ \in L^\infty_{\text{loc}}$), a condition that can be easily checked as above in another chart of $C^{1,1}$ coordinates. Note that, on a $C^{1,1}$ manifold, the regularity of the tangent bundle and of the bundle of n-forms is Lipschitz continuity.

The condition $X \in \mathcal{B}(\Omega)$ is also invariant by change of $C^{1,1}$ coordinates from the previous discussion and the fact that it is also the case for the condition on the Hausdorff dimension.

Remark. It is tempting to formulate analogous conditions on a Lipschitz manifold, where, for a vector field X, the conditions $X \in L^\infty_{\text{loc}}$, X positively transverse to a Lipschitz hypersurface S make sense. The vanishing of the Hausdorff measure is also invariant by bi-Lipschitz homeomorphism. However, the divergence condition does not have a simple expression (if any) in that framework, since the regularity of the function ω above could only be boundedness and measurability (for instance it could be the determinant of the Jacobian of a bi-Lipschitz homeomorphism). Moreover the notion of BV_{loc} regularity for a vector field on a Lipschitz manifold should be handled with caution since the change of coordinates formulas written above do not obviously make sense, since it is not possible to make the product of an L^∞ function with a Radon measure. Nevertheless, the Lipschitz framework would be certainly better, and in particular would allow a good approximation of the jump set of BV functions.

A3. The class $\mathcal{B}$ contains the piecewise $W^{1,1}$ functions. Let Ω be an open set of $\mathbb{R}^d$. Let us recall[6] the definition of the class $\mathcal{P}(\Omega) = L^\infty_{\text{loc}}(\Omega)\cap$ piecewise $W^{1,1}$ (see page

[6] In fact the points (i)—(iv) are consequences of the definition in [Li]. Anyhow our purpose is to prove that piecewise $W^{1,1}$ functions are indeed in our class $\mathcal{B}$.

836 of [Li]). A function $a \in L^\infty_{\rm loc}(\Omega)$ is said to belong to $\mathcal{P}(\Omega)$ if there exists a partition of Ω, $\Sigma_d, \Sigma_{d-1}, \dots, \Sigma_0, \Sigma_{-1}$ such that

(i) for all $0 \le k \le d$, $\cup_{d-k\le i\le d}\Sigma_i$ are open sets and $a \in W^{1,1}(\Sigma_d)$,

(ii) for all $0 \le k \le d-2$, for all $x \in \Sigma_k$, there exists a neighborhood U of x and a bi-Lipschitz continuous homeomorphism

$$\begin{array}{rcccc} \kappa & : & U & \to & B_1 \\ & & y & \to & \big(\kappa_1(y), \dots, \kappa_d(y)\big) \end{array}$$

such that $\quad \Sigma_k \cap U = \{y \in U |\ \forall j, 1 \le j \le d-k,\ \kappa_j(y) = 0\}$,

(iii) for all $x \in \Sigma_{d-1}$, there exists a neighborhood U of x and a $C^{1,1}$ diffeomorphism

$$\begin{array}{rcccc} \kappa & : & U & \to & B_1 \\ & & y & \mapsto & \big(\kappa_1(y), \dots, \kappa_d(y)\big) \end{array}$$

such that

$$\Sigma_{d-1} \cap U = \{y \in U |\ \ \kappa_1(y) = 0\}.$$

(iv) The points of Σ_{-1} are isolated.

Proposition 6.4. *Let Ω be an open set of $\mathbb{R}^d$ and $\mathcal{P}(\Omega)$ be the set of piecewise $W^{1,1}$ functions described above. Then $\mathcal{P}(\Omega) \subset \mathcal{B}(\Omega)$. In particular for all $a \in \mathcal{P}(\Omega)$, there exists an open subset Ω_0 of Ω such that*

$$\mathcal{H}^{d-1}(\Omega\backslash\Omega_0) = 0, \quad \textit{and} \quad a \in CBV_{\rm loc}(\Omega_0).$$

Note that the set Ω_0 may depend on the function a.

Proof. Note that if $\kappa : U \longrightarrow V$ is a bi-Lipschitz homeomorphism of open sets of $\mathbb{R}^d$, setting $\nu = \kappa^{-1}$, we obtain by regularization

$$\det \kappa'(x) \det \nu'(\kappa(x)) = 1,$$

so that since essup$\,|\det \kappa'(x)|$ is finite, we get also that essinf$\,|\det \nu'(\kappa(x))|$ is positive. We remark first that $\mathcal{M}^{d-1}(\Sigma_{d-2}) = 0$. In fact $\mathcal{L}^d(\Sigma_{d-2} + rB_1) = O(r^2)$ since for any point in Σ_{d-2} there exists a neighborhood V and local Lipschitz coordinates z such that

$$V \cap (\Sigma_{d-2} + rB_1) \subset \{z = (z_1, z_2, z'') \in \mathbb{R} \times \mathbb{R} \times \mathbb{R}^{d-2},\ |z_1| \le r, |z_2| \le r, |z''| \le M_0\}.$$

The same argument can be applied for $k \in \{2, \dots, d\}$ and with (iv), we get

$$\forall k \in \{2, \dots, d, d+1\}, \mathcal{M}^{d-1}(\Sigma_{d-k}) = 0.$$

Consequently, we have indeed

$$\Omega_0 = \overbrace{\Sigma_d \cup \Sigma_{d-1}}^{\text{open set}} \subset \Omega, \quad \text{with} \quad \mathcal{H}^{d-1}(\Omega\backslash\Omega_0) = 0,$$

and Σ_{d-1} is a $C^{1,1}$ hypersurface of Ω_0. Since $a \in W^{1,1}(\Sigma_d)$, we get that a belongs to *conormal*$BV_{\rm loc}(\Omega_0)$ from the following simple lemma.

Lemma 6.5. *Let $R_0 > r_0 > 0$ be given positive numbers and d be an integer ≥ 2. We define*

$$V_+ = \{x = (x_1, x') \in \mathbb{R} \times \mathbb{R}^{d-1}, 0 < x_1 < r_0, |x'| < R_0\},$$
$$V_- = \{x = (x_1, x') \in \mathbb{R} \times \mathbb{R}^{d-1}, -r_0 < x_1 < 0, |x'| < R_0\},$$
$$V = \{x = (x_1, x') \in \mathbb{R} \times \mathbb{R}^{d-1}, -r_0 < x_1 < r_0, |x'| < R_0\}.$$

Let a be an $L^\infty(V)$ function such that $a \in W^{1,1}(V_+) \cap W^{1,1}(V_-)$. Then a belongs to $BV(V)$ and more precisely[7]

$$\frac{\partial a}{\partial x_1} \in \mathcal{M}_b(V), \quad \nabla_{x'} a \in L^1(V).$$

Proof. Let φ be a test function in $C^1_c(V)$. With brackets of duality, ω standing for a C^1 function of one real variable equal to 1 outside a neighborhood of the origin, setting

$$b_\pm = \frac{\partial a}{\partial x_1}|_{V_\pm} \in L^1(V_\pm),$$

we have,

$$\begin{aligned}
\langle \frac{\partial a}{\partial x_1}, \varphi\rangle &= -\iint \frac{\partial \varphi}{\partial x_1}(x_1, x') a(x_1, x') dx_1 dx' \\
&= -\lim_{\epsilon \to 0} \iint \frac{\partial \varphi}{\partial x_1}(x_1, x') a(x_1, x') \omega(x_1 \epsilon^{-1}) dx_1 dx' \\
&= \lim_{\epsilon \to 0} \Big\{ \iint \varphi(x_1, x') \Big[\frac{\partial a}{\partial x_1}(x_1, x') \omega(x_1 \epsilon^{-1}) + a(x_1, x') \epsilon^{-1} \omega'(x_1 \epsilon^{-1})\Big] dx_1 dx' \Big\} \\
&= \iint \varphi(x_1, x')[b_+(x_1, x') + b_-(x_1, x')] dx_1 dx' \\
&\qquad\qquad + \lim_{\epsilon \to 0} \iint \varphi(\epsilon x_1, x') a(\epsilon x_1, x') \omega'(x_1) dx_1 dx'. \\
&= \langle b_+ + b_-, \varphi\rangle + \lim_{\epsilon \to 0} \int \alpha(\epsilon x_1) \omega'(x_1) dx_1,
\end{aligned}$$

where α is the $L^\infty_{\text{comp}}(\mathbb{R})$ function $\alpha(t) = \int \varphi(t, x') a(t, x') dx'$. From the inequality

$$\|\alpha\|_{L^\infty(\mathbb{R})} \le \|\varphi\|_{L^\infty(V)} \sup_{|t| < r_0} \int_{|x'| < R_0} |a(t, x')| dx',$$

[7]Note that the assumptions are "up to the boundary" and that the hypothesis $a \in W^{1,1}_{\text{loc}}(V_+) \cap W^{1,1}_{\text{loc}}(V_-)$ is not sufficient to get the conclusion as shown by the following function $u \in C^0(\mathbb{R}) \cap C^\infty(\mathbb{R}^*)$ given by $u(x) = x \sin(1/x)$ whose distribution derivative is not a Radon measure.

we get that

$$\left|\lim_{\epsilon\to 0}\int \alpha(\epsilon x_1)\omega'(x_1)dx_1\right| \leq C\int |\omega'(x_1)|dx_1\,\|\varphi\|_{L^\infty(V)},$$

which implies that $\partial a/\partial x_1$ is a Radon measure on V. Finally, we need to check the x' derivatives ; setting

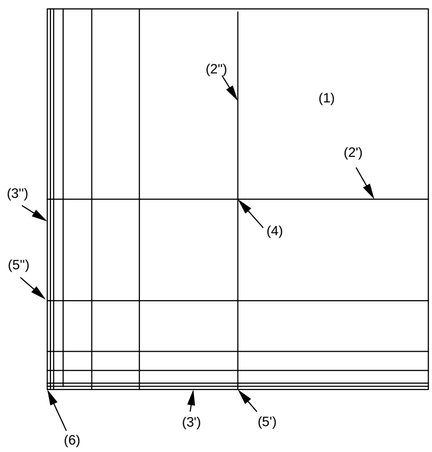
$$c_\pm = \nabla_{x'}a|_{V_\pm} \in L^1(V_\pm),$$

we write with the same notations as previously,

$$\begin{aligned}\langle\nabla_{x'}a,\varphi\rangle &= -\iint a\nabla_{x'}\varphi dx_1dx' \\ &= -\lim_{\epsilon\to 0}\iint a(x_1,,x')(\nabla_{x'}\varphi)(x_1,,x')\omega(x_1\epsilon^{-1})dx_1dx' \\ &= \lim_{\epsilon\to 0}\iint \varphi(x_1,x')\nabla_{x'}a(x_1,x')\omega(x_1\epsilon^{-1})dx_1dx' \\ &= \iint (c_+ + c_-)\varphi dx_1dx'\end{aligned}$$

concluding the proof of the lemma. $\square$

A4. A picture. We first describe the singular set of the two-dimensional example 2.3.(d)



Singular set of a in example 2.3.d

The points (1) are regular points (say $W^{1,1}$ points). The points (2) are the jump points: in the picture above, the foliation is horizontal at $(2')$, vertical at $(2'')$. The points (3') (resp. (3")) are accumulation points, but in Ω_0 with horizontal (resp. vertical) foliation. The open set Ω_0 is the reunion of points (1),(2),(3). The other points (4),(5),(6) are the compact set $\Omega\backslash\Omega_0$ whose $\mathcal{H}^1$ measure is zero.

A vector field X in $\mathbb{R}^2_{x,y}$ of the type defined in example (f) is transverse to the hypersurface $\{y=0\}$, and has jumps across the hypersurfaces

$$\Sigma_k = \{x = x_k\}, S_l = \{y = y_l\}$$

where (x_k) and (y_l) are sequences with limit 0. The divergence condition forces the jump to occur in the tangential part to the foliation. The foliation has not to be defined at the intersection of the singular hypersurfaces, is vertical near the vertical component and horizontal near the horizontal component. That type of example is not piecewise $W^{1,1}$.

A5. Proof of lemma 3.1. Let us consider a point $x_0 \in S$ and φ a defining function for S in a neighborhood of x_0 (i.e. $S \cap V_0 = \{x \in V_0, \varphi(x) = 0\}$). We know that, on an open neighborhood V_0 of x_0, with $w_0 = w_{|V_0} \geq 0$, $w_0 \in L^\infty(V_0)$, we have

$$Xw_0 \leq cw_0, \qquad \operatorname{supp} w_0 \subset \{\varphi \geq 0\}, \qquad X\varphi \geq \rho_0 > 0. \tag{6.1}$$

Let us consider the following Lipschitz continuous function defined on V_0

$$\psi(x) = \varphi(x) + |x - x_0|^2, \qquad \theta(\psi(x)) = \frac{1}{2}\Big(\big(\alpha^2 - \psi(x)\big)_+\Big)^2, \tag{6.2}$$

where α is a positive parameter such that the closed ball $B(x_0, \alpha)$ with center x_0 and radius α is included in V_0. We have

$$\operatorname{supp}(\theta(\psi)) \subset \{\psi \leq \alpha^2\}$$

and

$$\operatorname{supp}(w_0\theta(\psi)) \subset \{\varphi \geq 0\} \cap \{\psi \leq \alpha^2\} = K_\alpha \ni x_0$$

which is a compact subset of V_0 (as a closed subset of $B(x_0,\alpha)$). Let $\chi \in C^\infty_{\rm c}(V_0;[0,1])$, $\chi = 1$ on a neighborhood of K_α. Since ψ and $\theta(\psi)$ are Lipschitz continuous functions, and $X(\chi) = 0$ on a neighborhood of $\operatorname{supp} w_0\theta(\psi)$, we have

$$\sum_{1\leq j\leq n} a_j w_0 \partial_j\big(\theta(\psi)\chi\big) = w_0\chi\theta'(\psi)X(\psi) + \overbrace{w_0\theta(\psi)X(\chi)}^{=0}.$$

We calculate, dm standing for the Lebesgue measure,

$$\begin{aligned}\int cw_0\theta(\psi)\chi dm &\geq \langle Xw_0, \underbrace{\theta(\psi)\chi}_{\geq 0}\rangle_{\mathcal{D}'^{(1)}(V_0), C^1_c(V_0)}\\ &= -\sum_{1\leq j\leq n}\int a_j w_0 \partial_j(\theta(\psi)\chi)dm - \int w_0\theta(\psi)\chi \operatorname{div} X dm\\ &= \sum_{1\leq j\leq n}\int \chi w_0 a_j \partial_j(\psi)(\alpha^2-\psi)_+ dm - \int w_0\theta(\psi)\chi\operatorname{div} X dm.\end{aligned}$$

We obtain

$$0 \geq \int \chi w_0(\alpha^2-\psi)_+\Big[X(\psi)-\frac{1}{2}(\alpha^2-\psi)_+(\mathrm{div}X+c)\Big]dm. \tag{6.3}$$

Now on the set

$$\{x\in V_0,\ \varphi(x)+|x-x_0|^2=\psi(x)\leq \alpha^2\}\cap\{x,\varphi(x)\geq 0\}\subset B(x_0,\alpha)$$

we have, using now (6.1–2) and the assumption of the lemma, that

$$X(\psi)-\frac{1}{2}(\alpha^2-\psi)_+(\mathrm{div}\,X+c)\geq$$
$$\rho_0-2\left\|X\right\|_{L^\infty(B(x_0,\alpha))}\alpha-\frac{1}{2}\alpha^2\left\|(\mathrm{div}X+c)_+\right\|_{L^\infty(B(x_0,\alpha))}\geq \rho_0/2$$

if α is chosen small enough with respect to ρ_0 and $\|X\|_{L^\infty(V_0)}$. On the other hand, the term

$$\int \chi w_0\big((\alpha^2-\psi)_+\big)^2(\mathrm{div}\,X+c)_-dm$$

makes sense and is non-negative. This yields

$$0\geq \int \chi w_0(\alpha^2-\psi)_+dm,$$

and since the integrand is non-negative we get $\chi w_0(\alpha^2-\psi)_+=0$. Since on a neighborhood of x_0, we have $\chi=1$ and $\alpha^2-\psi>0$, we indeed obtain that w_0 vanishes near x_0. The proof of lemma 3.1 is complete.

A6. Proof of lemma 5.1. To prove (5.6), we note that for all $\kappa>0$, $R>0$ we have

$$\begin{aligned}
&\limsup_{t\to 0}\int |b(x)||v(x+t)-v(x)|dx\\
&\quad=\limsup_{t\to 0}\left\{\int_{|v(x+t)-v(x)|\leq\kappa}|b(x)||v(x+t)-v(x)|dx+\int_{|v(x+t)-v(x)|>\kappa}|b(x)||v(x+t)-v(x)|dx\right\}\\
&\quad\leq \kappa\left\|b\right\|_{L^1}+2\left\|v\right\|_{L^\infty}\limsup_{t\to 0}\int_{\substack{|v(x+t)-v(x)|>\kappa\\ |x|\leq R}}|b(x)|dx+2\left\|v\right\|_{L^\infty}\int_{|x|>R}|b(x)|dx\\
&\quad=\kappa\left\|b\right\|_{L^1}+2\left\|v\right\|_{L^\infty}\int_{|x|>R}|b(x)|dx
\end{aligned}$$

since $\mathcal{L}^d(A=\{x,|x|\leq R,\text{and }|v(x+t)-v(x)|>\kappa\})\to 0$ with t: in fact we have the estimates

$$\begin{aligned}
\mathcal{L}^d(A)&\leq \kappa^{-1}\int_{|x|\leq R,|x+t|\leq R}|v(x+t)-v(x)|dx+\mathcal{L}^d(\{x,|x|\leq R,|x+t|>R\})\\
&\leq \kappa^{-1}\left\|\tau_{-t}v_R-v_R\right\|_{L^1}+|t|R^{d-1}|S^{d-1}|,
\end{aligned}$$

with $v_R(x)=v(x)\mathbf{1}(|x|\leq R)$ which is an L^1 function. The assertion (5.7) is an immediate consequence of a.e. convergence of C^0_c functions to v with L^∞ bound $\|v\|_{L^\infty}$. $\square$

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