ON THE FEFFERMAN-PHONG INEQUALITY AND A WIENER-TYPE ALGEBRA OF PSEUDODIFFERENTIAL OPERATORS

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Abstract. We provide an extension of the Fefferman-Phong inequality to nonnegative symbols whose fourth derivative belongs to a Wiener-type algebra of pseudodifferential operators introduced by J.Sjöstrand. As a byproduct, we obtain that the number of derivatives needed to get the classical Fefferman-Phong inequality in d dimensions is bounded above by $2d+4+\epsilon$. Our method relies on some refinements of the Wick calculus, which is closely linked to Gabor wavelets. Also we use a decomposition of $C^{3,1}$ nonnegative functions as a sum of squares of $C^{1,1}$ functions with sharp estimates. In particular, we prove that a $C^{3,1}$ nonnegative function a can be written as a finite sum $\sum b_j^2$, where each b_j is $C^{1,1}$, but also where each function b_j^2 is $C^{3,1}$. A key point in our proof is to give some bounds on $(b_j' b_j'')'$ and on $(b_j b_j'')''$.

Contents

1. Introduction and statement of the results

1.1. The Fefferman-Phong inequality and Bony's result. Let us consider a classical second-order symbol $a(x,\xi)$, i.e. a smooth function defined on $\mathbb{R}^n \times \mathbb{R}^n$ such that, for all multi-indices α, β

$$|(\partial_\xi^\alpha \partial_x^\beta a)(x,\xi)| \le C_{\alpha\beta}(1+|\xi|)^{2-|\alpha|}. \tag{1.1.1}$$

The Fefferman-Phong inequality states that, if a satisfies (1.1.1) and is a nonnegative function, there exists C such that, for all $u \in \mathcal{S}(\mathbb{R}^n)$,

$$\operatorname{Re}\langle a(x,D)u,u\rangle_{L^2(\mathbb{R}^n)} + C\,\|u\|^2_{L^2(\mathbb{R}^n)} \geq 0, \tag{1.1.2}$$

or equivalently (with an a priori different constant C)

$$a^w + C \geq 0, \tag{1.1.3}$$

where a^w stands for the Weyl quantization[1] of a,

$$(a^w u)(x) = \iint e^{2i\pi(x-y)\xi} a\big(\frac{x+y}{2},\xi\big)u(y)dyd\xi.$$

The constant C in (1.1.2-3) depends only a finite number of $C_{\alpha\beta}$ in (1.1.1). Let us ask our first question:

$$\textit{How many derivatives of } a \textit{ in } (1.1.1) \textit{ are needed to control } C \textit{ in } (1.1.2)? \tag{1.1.4}$$

Looking at the proof by C.Fefferman and D.H.Phong [FP](see also the *Theorem 18.6.8* in the third volume of [H2]), it seems clear that the number N of derivatives of a needed to control C should be

$$N = 4 + \nu(n), \quad \nu \text{ depending on the dimension } n.$$

Since the proof is using an induction on the dimension, it is not completely obvious to answer to our question with a reasonably simple ν. On the other hand, J.-M. Bony proved in [Bo1]*(Théorème 3.2)* the following result: if $a(x,\xi)$ is a nonnegative smooth function defined on $\mathbb{R}^n \times \mathbb{R}^n$ such that

$$|(\partial_\xi^\alpha \partial_x^\beta a)(x,\xi)| \leq C_{\alpha\beta}, \quad \text{for } |\alpha| + |\beta| \geq 4, \tag{1.1.5}$$

then the conclusions (1.1.2-3) hold. This result shows an interesting twofold phenomenon:

- Only derivatives with order larger than 4 are needed.
- The control of these derivatives is quite weak, of type $S^0_{0,0}$. In particular, the derivatives of large order do not get small (the class $S^0_{0,0}$ does not have an asymptotic calculus).

Our answer to the question (1.1.4) is $4 + 2n + \epsilon$ (for any positive ϵ). However, we shall in fact prove a much more precise result involving a Wiener-type algebra introduced by J.Sjöstrand in [S1]. To formulate our result, we need first to introduce that algebra.

1.2. Sjöstrand algebra of pseudodifferential operators. In [S1] and [S2], J.Sjöstrand introduced a Wiener-type algebra of pseudodifferential operators as follows. Let $\mathbb{Z}^{2n}$ be the standard lattice in $\mathbb{R}^{2n}_X$ and let $1 = \sum_{j\in\mathbb{Z}^{2n}} \chi_0(X-j), \chi_0 \in C^\infty_c(\mathbb{R}^{2n})$, be a partition of unity. We note $\chi_j(X) = \chi_0(X-j)$.

[1] The standard quantization $a(x,D)$ reads $(a(x,D)u)(x) = \int e^{2i\pi x\xi} a(x,\xi)\hat{u}(\xi)d\xi$.

Proposition 1.2.1. *Let a be a tempered distribution on $\mathbb{R}^{2n}$. We shall say that a belongs to the class $\mathcal{A}$ if $\omega_a \in L^1(\mathbb{R}^{2n})$, with $\omega_a(\Xi) = \sup_{j\in\mathbb{Z}^{2n}} |\mathcal{F}(\chi_j a)(\Xi)|$, where $\mathcal{F}$ is the Fourier transform*[2]. *Moreover, we have*

$$S^0_{0,0} \subset S^0_{0,0;2n+1} \subset \mathcal{A} \subset C^0(\mathbb{R}^{2n}) \cap L^\infty(\mathbb{R}^{2n}), \tag{1.2.1}$$

where $S_{0,0;2n+1}$ is the set of functions defined on $\mathbb{R}^{2n}$ such that $|(\partial_\xi^\alpha \partial_x^\beta a)(x,\xi)| \le C_{\alpha\beta}$ for $|\alpha| + |\beta| \le 2n+1$. $\mathcal{A}$ is a Banach algebra for the multiplication with the norm $\|a\|_{\mathcal{A}} = \|\omega_a\|_{L^1(\mathbb{R}^{2n})}$.

Proof. In fact, we have the implications $a \in \mathcal{A} \Longrightarrow \mathcal{F}(\chi_j a) \in L^1(\mathbb{R}^{2n}) \Longrightarrow \chi_j a \in C^0 \cap L^\infty$, and, since the sum is locally finite with a fixed overlap[3], we get $a \in C^0 \cap L^\infty$. Moreover, if $a \in S^0_{0,0;2n+1}$, i.e. is bounded as well as all its derivatives of order $\le 2n+1$, we have, with $P(\Xi) = (1 + \|\Xi\|^2)^n$ the formula $\mathcal{F}(\chi_j a)(\Xi) = P(\Xi)^{-1}\mathcal{F}\big(P(D_X)(\chi_j a)\big)$. We get the identity

$$\mathcal{F}(\chi_j a)(\Xi) = P(\Xi)^{-1}(\Xi_1 + i)^{-1}\mathcal{F}\big((D_{X_1} + i)P(D_X)(\chi_j a)\big).$$

This entails, in the cone $\{\Xi \in \mathbb{R}^{2n}, 2n|\Xi_1| \ge \|\Xi\|\}$ and thus everywhere[4]

$$|\mathcal{F}(\chi_j a)(\Xi)| \le \underbrace{P(\Xi)^{-1}(1 + \|\Xi\|)^{-1}}_{\in L^1(\mathbb{R}^{2n})} \operatorname{mes}(\operatorname{supp}\chi_0) \sup_{0\le k\le 2n+1} \|a^{(l)}\|_{L^\infty} C_n,$$

yielding the result. □

Remark 1.2.2. Since $1 \in \mathcal{A}$, $\mathcal{A}$ is not included in $\mathcal{F}(L^1(\mathbb{R}^{2n}))$. Moreover $\mathcal{A}$ contains $\mathcal{F}(L^1)$: let a be a function in $\mathcal{F}(L^1)$. With the above notations, we have

$$|\mathcal{F}(\chi_j a)(\Xi)| = \left|\int \hat\chi_0(\Xi - N)\hat a(N) e^{2i\pi j(N-\Xi)} dN\right| \le \int |\hat\chi_0(\Xi - N)||\hat a(N)| dN,$$

and thus $\int |\omega_a(\Xi)| d\Xi \le \|\hat a\|_{L^1} \|\widehat{\chi_0}\|_{L^1}$, which gives the inclusion. Moreover, $\mathcal{A}$ is a Banach commutative algebra for the multiplication.

Proposition 1.2.3. *The algebra $\mathcal{A}$ is stable by change of quantization, i.e. for all t real, $a \in \mathcal{A} \Longleftrightarrow J^t a = \exp(2i\pi t D_x \cdot D_\xi) a \in \mathcal{A}$. The bilinear map $a_1, a_2 \mapsto a_1 \sharp a_2$ is defined on $\mathcal{A} \times \mathcal{A}$ and continuous valued in $\mathcal{A}$, which is a (noncommutative) Banach algebra for $\sharp$. The maps $a \mapsto a^w, a(x,D)$ are continuous from $\mathcal{A}$ to $\mathcal{L}(L^2(\mathbb{R}^n))$.*

The proof is given in [S1]. *A.Boulkhemair established a lot more results on this algebra in his paper* [B1]. *In our appendix A.2, we give a few more properties of the algebra $\mathcal{A}$, which will be useful later on in this article.*

We recall that $(a_1 \sharp a_2)^w = a_1^w a_2^w$ with

$$(a_1 \sharp a_2)(X) = 2^{2n} \iint_{\mathbb{R}^{2n}\times\mathbb{R}^{2n}} a_1(Y_1) a_2(Y_2) e^{-4i\pi[X - Y_1, X - Y_2]} dY_1 dY_2. \tag{1.2.2}$$

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[2] $(\mathcal{F}a)(\Xi) = \int e^{-2i\pi X\Xi} a(X) dX$. We use also the notation $D_{X_j} = \frac{1}{2i\pi}\partial_{X_j}$, so that $\mathcal{F}(D^\alpha a) = \Xi^\alpha \mathcal{F}a$.

[3] If $\cap_{j\in J} \operatorname{supp}\chi_j \ne \emptyset$ then $\operatorname{card} J \le N_0$, where N_0 depends only on the compact set $\operatorname{supp}\chi_0$.

[4] $\mathbb{R}^{2n} = \cup_{1\le k\le 2n}\{\Xi \in \mathbb{R}^{2n}, 2n|\Xi_k| \ge \|\Xi\|\}$ since the complement of that union is empty: it is not possible to find Ξ so that $\max_{1\le k\le 2n} 2n|\Xi_k| < \|\Xi\| \le 2n \max_{1\le k\le 2n} |\Xi_k|$.

Comments on the Wiener Lemma. The standard Wiener's Lemma states that if $a \in \ell^1(\mathbb{Z}^d)$ is such that $u \mapsto a * u = C_a u$ is invertible as an operator on $\ell^2(\mathbb{Z}^d)$, then the inverse operator is of the form C_b for some $b \in \ell^1(\mathbb{Z}^d)$. In [S2] the author is proving several types of Wiener lemma for $\mathcal{A}$. First a commutative version, saying that if $a \in \mathcal{A}$ and $1/a$ is a bounded function, then $1/a$ belongs to $\mathcal{A}$. Next, the theorem 4.1 of [S2] provides a noncommutative version of the Wiener lemma for the algebra $\mathcal{A}$: if an operator a^w with $a \in \mathcal{A}$ is invertible as a continuous operator on L^2, then the inverse operator is b^w with $b \in \mathcal{A}$. In the paper [GL], K.Gröchenig and M.Leinert prove several versions of the noncommutative Wiener lemma, and their definition of the twisted convolution ((1.1) in [GL]) is indeed very close to (a discrete version of) the composition formula (1.2.2) above. It would be interesting to compare the methods used to prove these noncommutative versions of the Wiener lemma in the papers [GL] and [S2].

Back to the Gårding inequalities. Also J.Sjöstrand proved in the proposition 5.1 of [S2] the standard Gårding inequality with gain of one derivative for his class, in the semi-classical setting, where h is a small parameter in $(0, 1]$:

$$a \geq 0, a'' \in \mathcal{A} \Longrightarrow a(x, h\xi)^w + Ch \geq 0. \tag{1.2.3}$$

A consequence of the result (1.1.5) of [Bo1] is that[5]

$$a \geq 0, a^{(4)} \in S^0_{0,0} \Longrightarrow a(x, h\xi)^w + Ch^2 \geq 0. \tag{1.2.4}$$

Let us ask our second question. Is it possible to get an inequality with gain of 2 derivatives as in (1.2.4) and also to generalize Bony's result by replacing $S^0_{0,0}$ by $\mathcal{A}$? That would mean that

$$a \geq 0, a^{(4)} \in \mathcal{A} \Longrightarrow a(x, h\xi)^w + Ch^2 \geq 0. \tag{1.2.5}$$

From the first two inclusions in (1.2.1), we see that (1.2.5) implies (1.2.4). Moreover the constant C in (1.2.5) will depend only on the dimension and on the norm of $a^{(4)}$ in $\mathcal{A}$, which is much more precise than the dependence of C in (1.2.4), which depends on a finite number of semi-norms of a in $S^0_{0,0}$. Although (1.2.5) looks stronger than (1.2.3) since $h^2 \ll h$, it is not obvious to actually *deduce* (1.2.3) from (1.2.5). Anyhow we shall see that they are both true and that the proof of (1.2.3) is an immediate consequence of the most elementary properties of the so-called Wick quantization exposed in our section 2. Note also that a version of the Hörmander-Melin inequality with gain of 6/5 of derivatives was given, in the semi-classical setting, by F.Hérau in [Hé]: this author used the assumption (6.4) of the theorem 6.2 of [H1], but with a limited regularity on the symbol a, which is only such that $a^{(3)} \in \mathcal{A}$.

1.3. The main result.

We can state our main result.

Theorem 1.3.1. *Let n be a positive integer. There exists a constant C_n such that, for all nonnegative functions a defined on $\mathbb{R}^{2n}$ satisfying $a^{(4)} \in \mathcal{A}$, the operator a^w is semi-bounded from below and, more precisely, satisfies*

$$a^w + C_n \|a^{(4)}\|_{\mathcal{A}} \geq 0. \tag{1.3.1}$$

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[5] In fact the operator $h^{-2}a(x, h\xi)^w$ is unitarily equivalent to $h^{-2}a(h^{1/2}x, h^{1/2}\xi)^w$ and the function $b(x, \xi) = h^{-2}a(h^{1/2}x, h^{1/2}\xi)$ is nonnegative and satisfies $b^{(4)}(x, \xi) = a^{(4)}(h^{1/2}x, h^{1/2}\xi)$ which is uniformly in $S^0_{0,0}$ whenever h is bounded and $a^{(4)} \in S^0_{0,0}$.

The Banach algebra $\mathcal{A}$ is defined in the proposition 1.2.1. Note that the constant C_n depends only on the dimension n.

The proof is given in section 3.2.

Corollary 1.3.2. *Let n be a positive integer.*

(i) *Let $a(x,\xi)$ be a nonnegative function defined on $\mathbb{R}^n\times\mathbb{R}^n$ such that* (1.1.1) *is satisfied for $|\alpha|+|\beta|\le 2n+5$. Then* (1.1.2) *and* (1.1.3) *hold with a constant C depending only on n and on $\max_{|\alpha|+|\beta|\le 2n+5} C_{\alpha\beta}$.*

(ii) *Let $a(x,\xi,h)$ be a nonnegative function defined on $\mathbb{R}^n\times\mathbb{R}^n\times(0,1]$ such that*

$$|(\partial_\xi^\alpha\partial_x^\beta a)(x,\xi,h)|\le h^{|\alpha|}C_{\alpha\beta},\qquad \text{for } 4\le|\alpha|+|\beta|\le 2n+5.$$

Then $a^w+Ch^2\ge 0$ and $\operatorname{Re} a(x,D)+Ch^2\ge 0$ hold with a constant C depending only on n and on $\max_{4\le|\alpha|+|\beta|\le 2n+5} C_{\alpha\beta}$.

(iii) *Let $a(x,\xi)$ be a nonnegative function defined on $\mathbb{R}^n\times\mathbb{R}^n$ such that $a^{(4)}$ belong to $\mathcal{A}$. Then $a(x,h\xi)^w+C\|a^{(4)}\|_{\mathcal{A}}h^2\ge 0$ and $\operatorname{Re} a(x,hD)+C\|a^{(4)}\|_{\mathcal{A}}h^2\ge 0$ hold with a constant C depending only on n.*

(iv) *Let $a(x,\xi,h)$ be a nonnegative function defined on $\mathbb{R}^n\times\mathbb{R}^n\times(0,1]$ such that, for $|\alpha|+|\beta|=4$, the functions $(x,\xi)\mapsto(\partial_1^\beta\partial_2^\alpha a)(xh^{1/2},\xi h^{-1/2},h)h^{-|\alpha|}$ belong to $\mathcal{A}$ with a norm bounded above by ν_0 for all $h\in(0,1]$. Then $a^w+C\nu_0h^2\ge 0$ and $\operatorname{Re} a(x,D)+C\nu_0h^2\ge 0$ hold with a constant C depending only on n.*

That corollary is proven in section 3.3.

Remark. It is possible to lower the requirement on the number of derivatives down to $2n+4+\epsilon$ (any positive ϵ) in the statements above, by using conditions on some fractional derivatives as in the theorem 1.1 of [B2].

2. The Wick calculus of pseudodifferential operators

2.1. Definitions. We recall here some facts on the so-called Wick quantization (see e.g. [L1]). That tool was introduced by F.A.Berezin in [Be], and used by many authors. In particular its role and effectiveness in proving the Gårding inequality with gain of one derivative (once called *sharp Gårding inequality*) was highlighted by the papers of A.Córdoba & C.Fefferman [CF] and A.Unterberger [Un].

Definition 2.1.1. Let $Y=(y,\eta)$ be a point in $\mathbb{R}^n\times\mathbb{R}^n$.

(i) The operator Σ_Y is defined as $\left[2^n e^{-2\pi|\cdot-Y|^2}\right]^w$. This is a rank-one orthogonal projection: $\Sigma_Y u=(Wu)(Y)\tau_Y\varphi$ with $(Wu)(Y)=\langle u,\tau_Y\varphi\rangle_{L^2(\mathbb{R}^n)}$, where $\varphi(x)=2^{n/4}e^{-\pi|x|^2}$ and $(\tau_{y,\eta}\varphi)(x)=\varphi(x-y)e^{2i\pi\langle x-\frac{y}{2},\eta\rangle}$.

(ii) Let a be in $L^\infty(\mathbb{R}^{2n})$. The Wick quantization of a is defined as

$$a^{\text{Wick}}=\int_{\mathbb{R}^{2n}} a(Y)\Sigma_Y dY. \tag{2.1.1}$$

(iii) Let m be a real number. We define S^m as the set of smooth functions $p(X,\Lambda)$ defined on $\mathbb{R}^{2n}\times[1,+\infty)$ such that, for all $k\in\mathbb{N}$,

$$\sup_{\Lambda\ge1,X\in\mathbb{R}^{2n}}|(\partial_X^k p)(X,\Lambda)\Lambda^{-m+\frac{k}{2}}|=\gamma_k(p)<\infty.$$

The following proposition is classical and easy (see e.g. section 5 in [L1]).

Proposition 2.1.2.

(i) *Let a be in $L^\infty(\mathbb{R}^{2n})$. Then $a^{\mathrm{Wick}} = W^* a^\mu W$ and $1^{\mathrm{Wick}} = Id_{L^2(\mathbb{R}^n)}$ where W is the isometric mapping from $L^2(\mathbb{R}^n)$ to $L^2(\mathbb{R}^{2n})$ given above, and a^μ the operator of multiplication by a in $L^2(\mathbb{R}^{2n})$. The operator $\pi_H = WW^*$ is the orthogonal projection on a closed proper subspace H of $L^2(\mathbb{R}^{2n})$. Moreover, we have*

$$\left\|a^{\mathrm{Wick}}\right\|_{\mathcal{L}(L^2(\mathbb{R}^n))} \leq \|a\|_{L^\infty(\mathbb{R}^{2n})}, \tag{2.1.2}$$

$$a(X) \geq 0 \text{ for all } X \text{ implies } a^{\mathrm{Wick}} \geq 0. \tag{2.1.3}$$

(ii) *Let m be a real number and $p \in S^m$. Then $p^{\mathrm{Wick}} = p^w + r(p)^w$, with $r(p) \in S^{m-1}$ so that the mapping $p \mapsto r(p)$ is continuous. More precisely, one has*

$$r(p)(X) = \int_0^1 \int_{\mathbb{R}^{2n}} (1-\theta) p''(X+\theta Y) Y^2 e^{-2\pi|Y|^2} 2^n dY d\theta. \tag{2.1.4}$$

Note that $r(p) = 0$ if p is affine.

(iii) *For $a \in L^\infty(\mathbb{R}^{2n})$, the Weyl symbol of a^{Wick} is*

$$a * 2^n \exp -2\pi|\cdot|^2 \quad \text{which belongs to } S^0 \text{ with } k^{th}\text{-seminorm } c(k)\|a\|_{L^\infty}. \tag{2.1.5}$$

(iv) *With the operator Σ_Y given in the definition 2.1.1, we have the estimate*

$$\|\Sigma_Y \Sigma_Z\|_{\mathcal{L}(L^2(\mathbb{R}^n))} \leq 2^n e^{-\frac{\pi}{2}|Y-Z|^2}. \tag{2.1.6}$$

(v) *More precisely, the Weyl symbol of $\Sigma_Y \Sigma_Z$ is, as a function of the variable $X \in \mathbb{R}^{2n}$,*

$$e^{-\frac{\pi}{2}|Y-Z|^2} e^{-2i\pi[X-Y, X-Z]} 2^n e^{-2\pi|X-\frac{Y+Z}{2}|^2}. \tag{2.1.7}$$

The proposition 2.1.2 is sufficient to prove the standard Gårding inequality with gain of one derivative, and in fact the following improvement was given by J.Sjöstrand in [S2].

Theorem 2.1.3. *Let a be a nonnegative function defined on $\mathbb{R}^{2n}$ such that the second derivatives a'' belongs to $\mathcal{A}$. Then we have*

$$a^w + C_n \|a''\|_{\mathcal{A}} \geq 0. \tag{2.1.8}$$

Proof. Although a proof of this result is given in [S2]*(proposition 5.1)*, it is a nice and simple introduction to our more complicated argument of section 3. From the proposition 2.1.2, we have

$$a^w = a^{\mathrm{Wick}} - r(a)^w \geq -r(a)^w,$$

with $r(a)(X) = \int_0^1 \int_{\mathbb{R}^{2n}} (1-\theta) a''(X+\theta Y) Y^2 e^{-2\pi|Y|^2} 2^n dY d\theta$. Since $\mathcal{A}$ is stable by translation (see the lemma A.2.1), we see that $r(a) \in \mathcal{A}$ and thus $r(a)^w$ is bounded on $L^2(\mathbb{R}^n)$ from the proposition 1.2.3. □

Remark 2.1.4. This theorem implies as well the following semi-classical version; let a be function satisfying the assumption of Theorem 2.1.3. For $h \in (0,1]$, we define $A_h(x,\xi) = h^{-1} a(xh^{1/2}, \xi h^{1/2})$. The function A_h is nonnegative with a second derivative bounded in $\mathcal{A}$ by cst$\times \|a''\|_{\mathcal{A}}$ (see the lemma A.2.1), so that the previous theorem implies, with C depending only on the dimension, that $A_h^w + C\|a''\|_{\mathcal{A}} \geq 0$. Since A_h^w is unitarily equivalent to $h^{-1} a(x, h\xi)^w$, this gives

$$a(x, h\xi)^w + hC\|a''\|_{\mathcal{A}} \geq 0. \tag{2.1.9}$$

2.2. Sharp estimates for the remainders. Proposition 2.1.2 falls short of providing a proof for the Fefferman-Phong inequality, which gains two derivatives.

Lemma 2.2.1. *Let a be a function defined on $\mathbb{R}^{2n}$ such that the fourth derivatives $a^{(4)}$ belong to $\mathcal{A}$. Then we have*

$$a^w = \Big(a - \frac{1}{8\pi}\operatorname{trace} a''\Big)^{\mathrm{Wick}} + \rho_0(a^{(4)})^w,$$

with $\rho_0(a^{(4)}) \in \mathcal{A}$ and more precisely $\|\rho_0(a^{(4)})\|_{\mathcal{A}} \le C_n \|a^{(4)}\|_{\mathcal{A}}$.

Proof. The Weyl symbol σ_a of a^{Wick} is

$$\begin{aligned}\sigma_a(X) &= \int a(X+Y)2^n e^{-2\pi|Y|^2}dY\\ &= a(X) + \int \frac{1}{2}a''(X)Y^2 2^n e^{-2\pi|Y|^2}dY\\ &\qquad + \frac{1}{3!}\iint_0^1 (1-\theta)^3 a^{(4)}(X+\theta Y)Y^4 2^n e^{-2\pi|Y|^2}dY d\theta\\ &= a(X) + \frac{1}{8\pi}\operatorname{trace} a''(X) + \frac{1}{3!}\iint_0^1 (1-\theta)^3 a^{(4)}(X+\theta Y)Y^4 2^n e^{-2\pi|Y|^2}dY d\theta.\end{aligned}$$

Moreover the Weyl symbol θ_a of $(\operatorname{trace} a'')^{\mathrm{Wick}}$ is, from proposition 2.1.2,

$$\theta_a(X) = \operatorname{trace} a''(X) + \int_0^1\int_{\mathbb{R}^{2n}} (1-\theta)(\operatorname{trace} a'')''(X+\theta Y)Y^2 e^{-2\pi|Y|^2}2^n dY d\theta.$$

As a result, the Weyl symbol of the operator $\Big(a - \frac{1}{8\pi}\operatorname{trace} a''\Big)^{\mathrm{Wick}}$ is

$$\begin{aligned}a &+ \frac{1}{3!}\iint_0^1 (1-\theta)^3 a^{(4)}(X+\theta Y)Y^4 2^n e^{-2\pi|Y|^2}dY d\theta\\ &\qquad - \frac{1}{8\pi}\int_0^1\int_{\mathbb{R}^{2n}} (1-\theta)(\operatorname{trace} a'')''(X+\theta Y)Y^2 e^{-2\pi|Y|^2}2^n dY d\theta.\end{aligned}$$

We get the equality in the lemma with

$$\begin{aligned}\rho_0(a^{(4)})(X) &= \frac{1}{8\pi}\int_0^1\int_{\mathbb{R}^{2n}} (1-\theta)(\operatorname{trace} a'')''(X+\theta Y)Y^2 e^{-2\pi|Y|^2}2^n dY d\theta\\ &\qquad - \frac{1}{3!}\iint_0^1 (1-\theta)^3 a^{(4)}(X+\theta Y)Y^4 2^n e^{-2\pi|Y|^2}dY d\theta. \quad (2.2.1)\end{aligned}$$

We note now that ρ_0 depends linearly on $a^{(4)}$ and that

$$\rho_0(a^{(4)})(X) = \iint_0^1 a^{(4)}(X+\theta Y)\underbrace{M(\theta,Y)}_{\substack{\text{polynomial}\\ \text{in } Y,\theta.}} e^{-2\pi|Y|^2}dY d\theta. \quad (2.2.2)$$

Looking now at the formula (2.2.2) and applying the lemma A.2.1, we get

$$\|\rho_0(a^{(4)})\|_{\mathcal{A}} \le \iint_0^1 M(\theta,Y)e^{-2\pi|Y|^2}dY d\theta C_0\|a^{(4)}\|_{\mathcal{A}} = C_1\|a^{(4)}\|_{\mathcal{A}}.$$

The proof of the lemma 2.2.1 is complete. □

Remark 2.2.2. We note that, from the lemma 2.2.1 and the L^2 boundedness of operators with symbols in $\mathcal{A}$, the theorem 1.3.1 is reduced to proving

$$a \geq 0, a^{(4)} \in \mathcal{A} \Longrightarrow \Big(a - \frac{1}{8\pi}\operatorname{trace} a''\Big)^{\text{Wick}} \text{is semi-bounded from below.} \tag{2.2.3}$$

Naturally, one should not expect the quantity $a - \frac{1}{8\pi}\operatorname{trace} a''$ to be nonnegative: this quantity will take negative values even in the simplest case $a(x,\xi) = x^2 + \xi^2$, so that the positivity of the quantization expressed by (2.1.3) is far from enough to get our result. We shall prove in section 3 a stronger version of (2.2.3), but before this, we need to investigate more closely the composition formula for the Wick quantization.

2.3. On the composition formula for the Wick quantization. In this section, we prove some formulas of composition for operators with very irregular Wick symbols. The first lemma below was already proven in [L1], but we give here a complete proof for the convenience of the reader, since these (easy) computations are not completely standard.

Lemma 2.3.1. *For $p, q \in L^\infty(\mathbb{R}^{2n})$ real-valued with $p'' \in L^\infty(\mathbb{R}^{2n})$, we have*

$$\operatorname{Re}\big(p^{\text{Wick}} q^{\text{Wick}}\big) = \Big(pq - \frac{1}{4\pi}\nabla p \cdot \nabla q\Big)^{\text{Wick}} + R,$$
$$\|R\|_{\mathcal{L}(L^2(\mathbb{R}^n))} \leq C(n)\, \|p''\|_{L^\infty} \|q\|_{L^\infty}.$$

The product $\nabla p \cdot \nabla q$ above makes sense (see our appendix A.3) as a tempered distribution since ∇p is a Lipschitz continuous function and ∇q is the derivative of an L^∞ function: in fact, we shall use as a definition (see our appendix A.3) $\nabla p \cdot \nabla q = \nabla \cdot (\underbrace{q}_{L^\infty} \underbrace{\nabla p}_{\text{Lip.}}) - \underbrace{q}_{L^\infty} \underbrace{\Delta p}_{L^\infty}$.

Proof. Using the definition 2.1.1, we see that

$$\begin{aligned} p^{\text{Wick}} q^{\text{Wick}} &= \iint_{\mathbb{R}^{2n}\times\mathbb{R}^{2n}} p(Y) q(Z) \Sigma_Y \Sigma_Z dY dZ \\ &= \iint \Big(p(Z) + p'(Z)(Y-Z) + p_2(Z,Y)(Y-Z)^2\Big) q(Z)\Sigma_Y\Sigma_Z dYdZ \\ &= \int (pq)(Z)\Sigma_Z dZ + \iint p'(Z)(Y-Z)\Sigma_Y dY q(Z)\Sigma_Z dZ + R_0, \\ \text{with } R_0 &= \iiint_0^1 (1-\theta) p''(Z+\theta(Y-Z))(Y-Z)^2 q(Z)\Sigma_Y\Sigma_Z dYdZd\theta. \end{aligned}$$

Claim 2.3.2. Let ω be a measurable function defined on $\mathbb{R}^{2n}\times\mathbb{R}^{2n}$ such that

$$|\omega(Y,Z)| \leq \gamma_0 \big(1 + |Y-Z|\big)^{N_0}.$$

Then the operator $\iint \omega(Y,Z)\Sigma_Y\Sigma_Z dYdZ$ is bounded on $L^2(\mathbb{R}^n)$ with $\mathcal{L}(L^2(\mathbb{R}^n))$ norm bounded above by a constant depending on γ_0, N_0. This is an immediate consequence of Cotlar's lemma (see e.g. Lemme 4.2.3 in [BL] *or Lemma 18.6.5 in* [H2]*) and of the formula* (2.1.6).

Using that claim, we obtain that

$$\|R_0\|_{\mathcal{L}(L^2(\mathbb{R}^n))} \le C_1(n)\,\|p''\|_{L^\infty(\mathbb{R}^{2n})}\,\|q\|_{L^\infty(\mathbb{R}^{2n})}\,. \tag{2.3.1}$$

We check now $\int(Y-Z)\Sigma_Y dY$ whose Weyl symbol is, as a function of X,

$$\int (Y-Z)2^n e^{-2\pi|X-Y|^2}dY = \int (X-Z)2^n e^{-2\pi|X-Y|^2}dY = X-Z.$$

So with $L_Z(X)=X-Z$, we have $\int(Y-Z)\Sigma_Y dY\Sigma_Z=(X-Z)^w\Sigma_Z=L_Z^w\Sigma_Z$ and thus

$$\operatorname{Re}\int (Y-Z)\Sigma_Y dY\Sigma_Z = \operatorname{Re}(L_Z^w\Sigma_Z)=\big((X-Z)2^n e^{-2\pi|X-Z|^2}\big)^w=\frac{1}{4\pi}\partial_Z(2^n e^{-2\pi|X-Z|^2}),$$

so that

$$\operatorname{Re}\int (Y-Z)\Sigma_Y dY\Sigma_Z = \frac{1}{4\pi}\partial_Z(\Sigma_Z). \tag{2.3.2}$$

Using that p and q are real-valued, the formula for $\operatorname{Re}(p^{\rm Wick}q^{\rm Wick})$ becomes

$$\begin{aligned}\operatorname{Re}\big(p^{\rm Wick}q^{\rm Wick}\big) &= \int (pq)(Z)\Sigma_Z dZ + \int p'(Z)q(Z)\frac{1}{4\pi}\partial_Z\Sigma_Z dZ + \operatorname{Re}R_0\\ &= \int\Big((pq)(Z)-\frac{1}{4\pi}p'(Z)\cdot q'(Z)\Big)\Sigma_Z dZ - \int\frac{1}{4\pi}\operatorname{trace}p''(Z)q(Z)\Sigma_Z dZ + \operatorname{Re}R_0\end{aligned}$$

that is the result of the lemma, using (2.3.1) and (2.1.2) for the penultimate term on the line above. □

The next lemma is more involved.

Lemma 2.3.3. *For p measurable real-valued function such that $p'', (p'p'')', (pp'')'' \in L^\infty$, we have*

$$p^{\rm Wick}p^{\rm Wick} = \int\Big[p(Z)^2-\frac{1}{4\pi}|\nabla p(Z)|^2\Big]\Sigma_Z dZ + S, \tag{2.3.3}$$

$$\|S\|_{\mathcal{L}(L^2(\mathbb{R}^n))} \le C(n)\Big(\|p''\|^2_{L^\infty}+\|(p''p')'\|_{L^\infty}+\|(pp'')''\|_{L^\infty}\Big). \tag{2.3.4}$$

Here p'' stands for the vector (tensor) with components $(\partial_X^\alpha p)_{|\alpha|=2}$, whereas the components of $(p''p')'$ are $\partial_X^\alpha\big(\partial_X^\beta\partial_X^\gamma p\big)_{|\alpha|=1,|\beta|=2,\ |\gamma|=1}$ and those of $(pp'')''$ are $\partial_X^\alpha\big(p\partial_X^\beta p\big)_{|\alpha|=|\beta|=2}$.

Proof. We have

$$\begin{aligned}p^{\rm Wick}p^{\rm Wick} &= \iint p(Y)p(Z)\Sigma_Y\Sigma_Z dYdZ\\ &= \iint p(Y)\big(p(Y)+p'(Y)(Z-Y)\big)\Sigma_Y\Sigma_Z dYdZ\\ &\quad + \iiint_0^1 p(Y)(1-\theta)p''(Y+\theta(Z-Y))d\theta(Z-Y)^2\Sigma_Y\Sigma_Z dYdZd\theta\end{aligned}$$

so that using (2.3.2), we get, noting $\operatorname{trace}(p'')=\Delta p$,

$$p^{\rm Wick}p^{\rm Wick} = \big(p^2-\frac{1}{4\pi}|\nabla p|^2-\frac{1}{4\pi}p\Delta p\big)^{\rm Wick}+\operatorname{Re}(\Omega_0+\Omega_1+\Omega_2), \tag{2.3.5}$$

with

$$\Omega_0 = \iiint_0^1 p\big(Y+\theta(Z-Y)\big)p''\big(Y+\theta(Z-Y)\big)(Z-Y)^2\Sigma_Y\Sigma_Z dY dZ(1-\theta)d\theta, \quad (2.3.6)$$

$$\Omega_1 = \iiint_0^1 p'\big(Y+\theta(Z-Y)\big)\theta(Y-Z) \times p''\big(Y+\theta(Z-Y)\big)(Z-Y)^2\Sigma_Y\Sigma_Z dY dZ(1-\theta)d\theta \quad (2.3.7)$$

and from the claim (2.3.2),

$$\|\Omega_2\|_{\mathcal{L}(L^2(\mathbb{R}^n))} \le C_1(n)\,\|p''\|^2_{L^\infty}. \quad (2.3.8)$$

We write now $\Omega_0 = \Omega_{00} + \Omega_{01},\ \Omega_1 = \Omega_{10} + \Omega_{11}$ with

$$\Omega_{00} = \frac{1}{2}\iint p(Y)p''(Y)(Z-Y)^2\Sigma_Y\Sigma_Z dY dZ,$$

$$\Omega_{01} = \iiint_0^1 \Big((pp'')\big(Y+\theta(Z-Y)\big) - (pp'')(Y)\Big)(Z-Y)^2\Sigma_Y\Sigma_Z dY dZ(1-\theta)d\theta$$

$$\Omega_{10} = -\frac{1}{6}\iiint_0^1 p'(Y)(Z-Y)p''(Y)(Z-Y)^2\Sigma_Y\Sigma_Z dY dZ$$

$$\|\Omega_{11}\|_{\mathcal{L}(L^2(\mathbb{R}^n))} \le C_2(n)\,\|(p'p'')'\|_{L^\infty}. \quad (2.3.9)$$

We have also $\Omega_{01} = \Omega_{010} + \Omega_{011}$ with

$$\Omega_{010} = \frac{1}{6}\iint (pp'')'(Y)(Z-Y)(Z-Y)^2\Sigma_Y\Sigma_Z dY dZ,$$

$$\|\Omega_{011}\|_{\mathcal{L}(L^2(\mathbb{R}^n))} \le C_3(n)\,\|(pp'')''\|_{L^\infty}. \quad (2.3.10)$$

From (2.3.5-6-7-8-9-10), it suffices to check that the following term is a remainder satisfying the estimate (2.3.4) to get the result of the lemma 2.3.2:

$$\widetilde{\Omega} = -\frac{1}{4\pi}\int p(Y)\operatorname{trace} p''(Y)\Sigma_Y dY + \frac{1}{2}\operatorname{Re}\iint (pp'')(Y)(Z-Y)^2\Sigma_Y\Sigma_Z dY dZ + \frac{1}{6}\operatorname{Re}\iint (pp'')'(Y)(Z-Y)(Z-Y)^2\Sigma_Y\Sigma_Z dY dZ - \frac{1}{6}\operatorname{Re}\iiint_0^1 p'(Y)(Z-Y)p''(Y)(Z-Y)^2\Sigma_Y\Sigma_Z dY dZ. \quad (2.3.11)$$

The real part of the Weyl symbol of $\int (Z_j-Y_j)(Z_k-Y_k)(Z_l-Y_l)\Sigma_Y\Sigma_Z dZ$ is (see (2.1.7))

$$\int (Z_j-Y_j)(Z_k-Y_k)(Z_l-Y_l)e^{-\frac{\pi}{2}|Y-Z|^2}\cos(2\pi[X-Y,X-Z])2^n e^{-2\pi|X-\frac{Y+Z}{2}|^2}dZ$$
$$= \int T_jT_kT_l e^{-2\pi|T/2|^2}\cos(2\pi[X-Y,T])2^n e^{-2\pi|X-Y-\frac{T}{2}|^2}dT$$
$$= \int T_jT_kT_l\cos(2\pi[X-Y,T])e^{-\pi|X-Y-T|^2}dT\,2^n e^{-\pi|X-Y|^2} = \nu_{jkl}(X-Y)$$

with

$$\begin{aligned}\nu_{jkl}(S) &= \int T_j T_k T_l \cos(2\pi[S,T]) e^{-\pi|S-T|^2} dT 2^n e^{-\pi|S|^2} \\ &= 2^n e^{-\pi|S|^2} \int (T_j + S_j)(T_k + S_k)(T_l + S_l) \cos(2\pi[S,T]) e^{-\pi|T|^2} dT \\ &= 2^n e^{-\pi|S|^2} \int (T_j T_k S_l + T_k T_l S_j + T_l T_j S_k + S_j S_k S_l) \cos(2\pi[S,T]) e^{-\pi|T|^2} dT.\end{aligned} \tag{2.3.12}$$

We notice that the function $S \mapsto \int_{\mathbb{R}^{2n}} T_j T_k \exp(2i\pi[S,T]) e^{-\pi|T|^2} dT$ is a second-order derivative of $S \mapsto \int_{\mathbb{R}^{2n}} \exp(2i\pi[S,T]) e^{-\pi|T|^2} dT = e^{-\pi|S|^2}$ so that

$$2^n e^{-\pi|S|^2} S_l \int_{\mathbb{R}^{2n}} T_j T_k \cos(2\pi[S,T]) e^{-\pi|T|^2} dT = e^{-2\pi|S|^2} S_l P_{jk}(S),$$

with P_{jk} even, second-order and real polynomial. The function $S_{l_1} S_{l_2} S_{l_3} e^{-2\pi|S|^2}$ is always a linear combination of derivatives of Schwartz functions on $\mathbb{R}^{2n}$, since

if $l_1 < l_2 \le l_3$ it is the derivative with respect to S_{l_1} of $S_{l_2} S_{l_3} e^{-2\pi|S|^2}(-4\pi)^{-1}$,
if $l_1 = l_2 < l_3$ it is the derivative with respect to S_{l_3} of $S_{l_1} S_{l_2} e^{-2\pi|S|^2}(-4\pi)^{-1}$,
if $l_1 = l_2 = l_3 = l$ it is a linear combination of the third and first derivative with respect to S_l of $e^{-2\pi|S|^2}$, since

$$(e^{t^2})''' = (12t + 8t^3)e^{t^2}, \quad t^3 e^{t^2} = \frac{1}{8}(e^{t^2})''' - \frac{3}{4}(e^{t^2})'.$$

As a result the function ν_{jkl} defined by (2.3.12) is a linear combination of derivatives with respect to S_j, S_k or S_l of Schwartz functions on $\mathbb{R}^{2n}$. Integrating by parts in the last two terms of (2.3.11), we see that their $\mathcal{L}(L^2)$ norm is bounded from above by $C_4(n)(\|(pp'')''\|_{L^\infty} + \|(p'p'')'\|_{L^\infty})$. Looking at (2.3.11), we see that we are left with

$$\widetilde{\Omega}_0 = -\frac{1}{4\pi} \int p(Y) \operatorname{trace} p''(Y) \Sigma_Y dY + \frac{1}{2} \operatorname{Re} \iint (pp'')(Y)(Z-Y)^2 \Sigma_Y \Sigma_Z dY dZ. \tag{2.3.13}$$

The real part of the operator $\int (Z_j - Y_j)(Z_k - Y_k) \Sigma_Y \Sigma_Z dY dZ$ has the Weyl symbol (function of X)

$$\begin{aligned}&\int T_j T_k e^{-\pi|X-Y-T|^2} \cos(2\pi[X-Y,T]) dT 2^n e^{-\pi|X-Y|^2} \\ &= \int \Big((X_j - Y_j)(X_k - Y_k) + T_j T_k\Big) e^{-\pi|T|^2} \cos(2\pi[X-Y,T]) dT 2^n e^{-\pi|X-Y|^2} \\ &= \int \big(S_j S_k + T_j T_k\big) e^{-\pi|T|^2} \cos(2\pi[S,T]) dT 2^n e^{-\pi|S|^2}, \quad S = X - Y.\end{aligned} \tag{2.3.14}$$

• If $j \ne k$, both terms in (2.3.14) are second order derivatives with respect to Y of a Schwartz function in $\mathbb{R}^{2n}$. In fact the first term is

$$S_j S_k 2^n e^{-2\pi|S|^2} = \partial_{S_j} \partial_{S_k} \big(2^n e^{-2\pi|S|^2}/16\pi^2\big) = \partial_{Y_j} \partial_{Y_k} \big(2^n e^{-2\pi|S|^2}/16\pi^2\big)$$

and the second term is equal to $-S_{j'}S_{k'}2^n e^{-2\pi|S|^2}$, with $j' \neq k'$, also a second-order derivative. The contribution of these terms in (2.3.13) is then, after integration by parts, an L^2 bounded operator with norm $\le C_5(n)\,\|(pp'')''\|_{L^\infty}$.

• If $j = k$, with $j' = j \pm n$ (in fact $j' = j+n$ if $1 \le j \le n$ and $j' = j-n$ if $1+n \le j \le 2n$), we note that (2.3.14) is equal to

$$S_j^2 2^n e^{-2\pi|S|^2} - \frac{1}{4\pi^2} e^{-\pi|S|^2}\partial^2_{S_{j'}}\left(2^n e^{-\pi|S|^2}\right) = 2^n e^{-2\pi|S|^2}\Big(S_j^2 - \frac{1}{4\pi^2}(4\pi^2 S_{j'}^2 - 2\pi)\Big).$$

Taking into account the contribution of these terms in (2.3.13), we see that we are left with

$$-\frac{1}{4\pi}\int p(Y)\,\mathrm{trace}\,p''(Y)\Sigma_Y dY + \frac{1}{2}\iint \frac{1}{2\pi}\,\mathrm{trace}(pp'')(Y)\Sigma_Y dY = 0.$$

The proof of the lemma 2.3.3 is complete. □

3. The proof

3.1. Nonnegative functions as sum of squares.

Theorem 3.1.1. *Let m be a nonnegative integer. There exists an integer N and a positive constant C such that the following property holds. Let a be a nonnegative $C^{3,1}$ function*[6] *defined on $\mathbb{R}^m$ such that $a^{(4)} \in L^\infty$; then we can write*

$$a = \sum_{1\le j\le N} b_j^2 \tag{3.1.1}$$

where the b_j are $C^{1,1}$ functions such that $b_j'', (b_j' b_j'')', (b_j b_j'')'' \in L^\infty$. More precisely, we have

$$\|b_j''\|^2_{L^\infty} + \|(b_j' b_j'')'\|_{L^\infty} + \|(b_j b_j'')''\|_{L^\infty} \le C\|a^{(4)}\|_{L^\infty}. \tag{3.1.2}$$

Note that this implies that each function b_j is such that b_j^2 is $C^{3,1}$ and that N and C depend only on the dimension m.

Remark 3.1.2. We shall use the following notation: let A be a symmetric k-linear form on real normed vector space V. We define the norm of A by

$$\|A\| = \sup_{\|T\|=1} |AT^k|.$$

Since the symmetrized products of $T_1 \otimes \cdots \otimes T_k$ can be written as a linear combination of k-th powers, that norm is equivalent to the natural norm

$$|||A||| = \sup_{\substack{\|T_j\|=1,\\ 1\le j\le k}} |AT_1 \dots T_k|$$

and in fact, when V is Euclidean, we have the equality $\|A\| = |||A|||$ (see the paper by O.D.Kellogg [Ke]). In our appendix A.4, we prove that for an arbitrary normed space, the best estimate is $|||A||| \le \frac{k^k}{k!}\,\|A\|$.

[6] A $C^{3,1}$ function is a C^3 function whose third-order derivatives are Lipschitz continuous.

Comment. Part of this theorem is a consequence of the classical proof of the Fefferman-Phong inequality and of the more refined analysis of Bony in [Bo1] (see also [Gu] and [Ta]). However the control of the L^∞ norm of the quantities $(b_j' b_j'')', (b_j b_j'')''$ is more difficult to achieve and seems to be new. Naturally the inequality (3.1.2) is a key element of our proof, since it is connected with the estimates (2.3.4). We shall thus focus our attention on the new elements of the proof, referring the reader to our appendix or to the literature for the more standard points.

Proof of the theorem 3.1.1. We define

$$\rho(x) = \bigl(|a(x)| + |a''(x)|^2\bigr)^{1/4}, \quad \Omega = \{x, \rho(x) > 0\}, \tag{3.1.3}$$

assuming as we may $\|a^{(4)}\|_{L^\infty} \le 1$. Note that, since ρ is continuous, the set Ω is open. The metric $|dx|^2/\rho(x)^2$ is slowly varying in Ω (see the lemma A.1.2): $\exists r_0 > 0, C_0 \ge 1$ such that

$$x \in \Omega, |y - x| \le r_0 \rho(x) \Longrightarrow y \in \Omega, C_0^{-1} \le \frac{\rho(x)}{\rho(y)} \le C_0. \tag{3.1.4}$$

The constants r_0, C_0 can be chosen as "universal" constants, thanks to the normalization on $a^{(4)}$ above. Moreover, using the lemma A.1.1, the nonnegativity of a implies with $\gamma_j = 1$ for $j = 0, 2, 4$, $\gamma_1 = 3, \gamma_3 = 4$,

$$|a^{(j)}(x)| \le \gamma_j \rho(x)^{4-j}, \quad 1 \le j \le 4. \tag{3.1.5}$$

We refer the reader to the section 1.4 in [H2] for the basic properties of slowly varying metrics as well as for the following lemma.

Lemma 3.1.3. *Let a, ρ, Ω, r_0 be as above. There exists a positive number $r_0' \le r_0$, such that for all $r \in]0, r_0']$, there exists a sequence $(x_\nu)_{\nu \in \mathbb{N}}$ of points in Ω and a positive number M_r, such that the following properties are satisfied. We define $U_\nu, U_\nu^*, U_\nu^{**}$ as the closed Euclidean balls with center x_ν and radius $r\rho_\nu, 2r\rho_\nu, 4r\rho_\nu$ with $\rho_\nu = \rho(x_\nu)$. There exist two families of nonnegative smooth functions on $\mathbb{R}^m$, $(\varphi_\nu)_{\nu \in \mathbb{N}}$, $(\psi_\nu)_{\nu \in \mathbb{N}}$ such that*

$$\sum_\nu \varphi_\nu^2(x) = \mathbf{1}_\Omega(x), \ \operatorname{supp} \varphi_\nu \subset U_\nu, \qquad \psi_\nu \equiv 1 \ \text{ on } U_\nu^*, \ \operatorname{supp} \psi_\nu \subset U_\nu^{**} \subset \Omega.$$

Moreover, for all integers l, we have $\sup_{x \in \Omega, \nu \in \mathbb{N}} \|\varphi_\nu^{(l)}(x)\| \rho_\nu^l + \sup_{x \in \Omega, \nu \in \mathbb{N}} \|\psi_\nu^{(l)}(x)\| \rho_\nu^l < \infty$. The overlap of the balls U_ν^{**} is bounded, i.e.

$$\bigcap_{\nu \in \mathcal{N}} U_\nu^{**} \ne \emptyset \qquad \Longrightarrow \qquad \#\mathcal{N} \le M_r.$$

Moreover, $\rho(x) \sim \rho_\nu$ all over U_ν^{**} (i.e. the ratios $\rho(x)/\rho_\nu$ are bounded above and below by a fixed constant, provided that $x \in U_\nu^{**}$).

Since a is vanishing on Ω^c, we obtain

$$a(x) = \sum_{\nu \in \mathbb{N}} a(x) \varphi_\nu^2(x). \tag{3.1.6}$$

Definition 3.1.4. Let a, ρ, Ω be as above. Let θ be a positive number $\le \theta_0$, where θ_0 is a fixed constant satisfying the requirements of the lemma A.1.5. A point $x \in \Omega$ is said to be

(i) θ-elliptic whenever $a(x) \ge \theta\rho(x)^4$,

(ii) θ-nondegenerate whenever $a(x) < \theta\rho(x)^4$: we have then $\|a''(x)\|^2 \ge \rho(x)^4/2$.

We go on now with the proof of the theorem 3.1.1. We choose a positive number θ satisfying the condition in the definition 3.1.4. We choose a positive number $r \le r_0'$ as defined in the lemma 3.1.3 and we consider a sequence (x_ν) as in that lemma. We assume also that $4r \le \theta/8$, so that the lemma 3.1.3 can be applied on the ball U_ν^{**}.

Let us first consider the "elliptic" indices ν such that x_ν is θ-elliptic. According to the lemma A.1.3, for $x \in U_\nu^{**}$, we have $a(x) \sim \rho_\nu^4$, so that with

$$b_\nu(x) = a(x)^{1/2}\psi_\nu(x), \quad b_\nu^2 = a\psi_\nu^2, \quad \varphi_\nu^2 b_\nu^2 = a\varphi_\nu^2 \tag{3.1.7}$$

and on $\operatorname{supp}\varphi_\nu$ (where $\psi_\nu \equiv 1$),

$$\begin{cases} b_\nu' &= 2^{-1}a^{-1/2}a', \\ b_\nu'' &= -2^{-2}a^{-3/2}a'^2 + 2^{-1}a^{-1/2}a'', \\ b_\nu''' &= 3\times 2^{-3}a^{-5/2}a'^3 - \frac34 a^{-3/2}a'a'' + 2^{-1}a^{-1/2}a''', \\ b_\nu^{(4)} &= -\frac{15}{16}a^{-7/2}a'^4 + \frac94 a^{-5/2}a'^2a'' - \frac34 a^{-3/2}a''^2 - a^{-3/2}a'a''' + \frac12 a^{-1/2}a^{(4)}, \end{cases}$$

yielding

$$\begin{cases} |b_\nu'| &\le 2^{-1}a^{-1/2}|a'| \lesssim a^{-1/2}\rho^3 \lesssim \rho, \\ |b_\nu''| &\lesssim a^{-3/2}\rho^6 + a^{-1/2}\rho^2 \lesssim 1, \\ |b_\nu'''| &\lesssim a^{-5/2}\rho^9 + a^{-3/2}\rho^3\rho^2 + a^{-1/2}\rho \lesssim \rho^{-1}, \\ |b_\nu^{(4)}| &\lesssim a^{-7/2}\rho^{12} + a^{-5/2}\rho^6\rho^2 + a^{-3/2}\rho^4 + a^{-3/2}\rho^3\rho + a^{-1/2} \lesssim \rho^{-2}. \end{cases}$$

Note in particular that

$$|b_\nu b_\nu^{(4)}| + |b_\nu^{(1)}b_\nu^{(3)}| + |b_\nu^{(2)}b_\nu^{(2)}| \le C(\theta). \tag{3.1.8}$$

The whole difficulty is concentrated on the next case.

The nondegenerate indices ν are those for which x_ν is θ-nondegenerate. Since $4r \le \theta/8 \le \theta^{1/2}$, we can apply the remark A.1.6 on the product[7]

$$Q_\nu = [-\theta^{1/4}\rho_\nu + x_{\nu 1}, \theta^{1/4}\rho_\nu + x_{\nu 1}] \times B_{\mathbb{R}^{m-1}}(x_\nu', \theta^{1/2}\rho_\nu) \text{ (here } x_\nu = (x_{\nu 1}, x_\nu') \in \mathbb{R}\times\mathbb{R}^{m-1}).$$

There exists $\alpha : B_{\mathbb{R}^{m-1}}(x_\nu', \theta^{1/2}\rho_\nu) \to [x_{\nu 1} - \theta^{1/4}\rho_\nu, x_{\nu 1} + \theta^{1/4}\rho_\nu]$ such that

$$\partial_1 a(\alpha(x'), x') = 0 \tag{3.1.9}$$

and $\partial_1^2 a(x) \ge \rho_\nu^2/2$ for $|x - x_\nu| \le R_0\rho_\nu$ where $R_0 = 10^{-2}$ according to the lemma A.1.4. We have on Q_ν

$$\begin{aligned} a(x) &= a(x_1, x') \\ &= \int_0^1 (1-t)\partial_1^2 a\big(\alpha(x') + t(x_1 - \alpha(x')), x'\big)dt\big(x_1 - \alpha(x')\big)^2 + a(\alpha(x'), x'). \end{aligned} \tag{3.1.10}$$

[7] Naturally the choice of the linear coordinates depends on the index ν, according to the remark A.1.6. Note also that $U_\nu^{**} \subset Q_\nu \subset B(x_\nu, R_0\rho_\nu)$ since $4r \le \theta^{1/2} \le \theta^{1/4} \le R_0$, according to the previous requirements on r and θ and also to the condition on θ in the lemma A.1.5.

According to the remark A.1.6, we recall that we have for $|x'-x'_\nu|\le\theta^{1/2}\rho_\nu$,

$$\left.\begin{aligned}&|\alpha(x')-x_{\nu 1}|\le\theta^{1/4}\rho_\nu,\\&|\alpha'(x')|\le 2\rho_\nu^{-2}\rho(\alpha(x'),x')^2\le 2C_0^2=C_1,\\&|\alpha''(x')|\le 2\rho_\nu^{-2}\big(4^2C_0^4+4^2C_0^2+12\big)\rho(\alpha(x'),x')\le C_2\rho_\nu^{-1},\\&|\alpha'''(x')|\le C_3\rho_\nu^{-2},\end{aligned}\right\}\tag{3.1.11}$$

with universal constants C_j. Let us now compute the derivatives of the function

$$B'=B_{\mathbb{R}^{m-1}}(x'_\nu,\theta^{1/2}\rho_\nu)\ni x'\mapsto a(\alpha(x'),x')=c(x').\tag{3.1.12}$$

We have, denoting by ∂_2 the partial derivative with respect to x',

$$\begin{aligned}c'&=\alpha'\partial_1 a+\partial_2 a=\partial_2 a\qquad \textbf{(here we use the identity } \partial_{\mathbf 1}\mathbf{a}(\alpha(\mathbf{x}'),\mathbf{x}')\equiv\mathbf{0}\textbf{)},\\c''&=\alpha'\partial_1\partial_2 a+\partial_2^2 a,\\c'''&=\alpha''\partial_1\partial_2 a+{\alpha'}^2\partial_1^2\partial_2 a+2\alpha'\partial_1\partial_2^2 a+\partial_2^3 a,\\c''''&=\alpha'''\partial_1\partial_2 a+3\alpha''\alpha'\partial_1^2\partial_2 a+3\alpha''\partial_1\partial_2^2 a+{\alpha'}^3\partial_1^3\partial_2 a+3{\alpha'}^2\partial_1^2\partial_2^2 a+3\alpha'\partial_1\partial_2^3 a+\partial_2^4 a\end{aligned}$$

and we obtain

$$|c'|\lesssim\rho^3,\quad |c''|\lesssim\rho^2,\quad |c'''|\lesssim\rho^{-1}\rho^2+\rho\sim\rho,\quad |c''''|\lesssim\rho^{-2}\rho^2+\rho^{-1}\rho+1\sim 1$$

so that

$$c\in C^{3,1}(B'),\quad |c^{(j)}|\lesssim\rho_\nu^{4-j},0\le j\le 4.\tag{3.1.13}$$

Since $\partial_1^2 a\gtrsim\rho^2$ on Q_ν, we can define

$$R(x)=\omega(x)^{1/2},\quad \omega(x)=\int_0^1(1-t)\partial_1^2 a\big(\alpha(x')+t(x_1-\alpha(x')),x'\big)dt.\tag{3.1.14}$$

Note also that the identity (on Q_ν), $a=R(x)^2(x_1-\alpha)^2+a(\alpha(x'),x')$ forces the function

$$B(x)=R(x)^2(x_1-\alpha)^2$$

to be $C^{3,1}(Q_\nu)$ with a j-th derivative bounded above in absolute value by ρ_ν^{4-j} $(0\le j\le 4)$ since it is the case for a and c (this fact is not obvious since the function R is a priori only $C^{1,1}$). Defining on Q_ν

$$b(x)=R(x)\big(x_1-\alpha(x')\big)\tag{3.1.15}$$

we see that

$$a=b^2+c,\quad |(b^2)^{(j)}|=|B^{(j)}|\lesssim\rho_\nu^{4-j},\ 0\le j\le 4.\tag{3.1.16}$$

As a consequence with $\beta=x_1-\alpha(x')$, $b^2=R^2\beta^2=B\in C^{3,1}$,

$$R^2\beta^2=\overbrace{B(\alpha(x'),x')}^{=0}+\overbrace{\int_0^1\partial_1 B(\alpha(x')+\theta(x_1-\alpha(x')),x')d\theta\,\beta}^{\in C^{2,1}},\quad |\beta^{(j)}|\lesssim\rho^{1-j},0\le j\le 3,$$

and since β vanishes on an hypersurface

$$\left.\begin{array}{r}R^2\beta = \int_0^1 \partial_1 B(\alpha(x') + \theta(x_1 - \alpha(x')), x')d\theta \in C^{2,1},\\ |(R^2\beta)^{(j)}| \lesssim \rho_\nu^{3-j},\ 0 \le j \le 3, \quad \text{(from (3.1.16))}.\end{array}\right\} \tag{3.1.17}$$

Also we have $0 < R^2 = \omega \in C^{1,1}, \omega \sim \rho_\nu^2$ and from (3.1.14-11),

$$|\omega^{(j)}| \lesssim \rho_\nu^{2-j}, 0 \le j \le 2, \tag{3.1.18}$$

entailing that with $R = \omega^{1/2}$,

$$|R' = \frac{1}{2}\omega^{-1/2}\omega'| \lesssim 1, \quad |R'' = -\frac{1}{4}\omega^{-3/2}\omega'^2 + \frac{1}{2}\omega^{-1/2}\omega''| \lesssim \rho_\nu^{-3}\rho_\nu^2 + \rho_\nu^{-1} \sim \rho_\nu^{-1}. \tag{3.1.19}$$

Using Leibniz' formula, we get $(R^2\beta)''' = (\omega\beta)''' = \omega'''\beta + 3\omega''\beta' + 3\omega'\beta'' + \omega\beta'''$, which makes sense since ω''' is a distribution of order 1 and β is $C^{2,1}$ (see (3.1.11)). From (3.1.17), we know that $(\omega\beta)'''$ is L^∞, and since it is also the case of $\omega''\beta', \omega'\beta'', \omega\beta'''$ from (3.1.18) and (3.1.11), we get that $\omega'''\beta$ belongs to L^∞ and

$$|\omega'''\beta| \lesssim 1. \tag{3.1.20}$$

On the other hand we have

$$\omega''' = 2(RR')'' = 2(R'^2 + RR'')' = 4R'R'' + 2(RR'')' = 6R'R'' + 2\underbrace{R}_{C^{1,1}}\ \underbrace{R'''}_{\substack{\text{distribution}\\ \text{of order 1}}}$$

entailing from (3.1.20), that $\beta(6R'R'' + 2RR''')$ is L^∞ and since it is the case of $\beta R'R''$ (from (3.1.11) and (3.1.19)), we get that $\beta RR'''$ is L^∞ and, using the remark 3.1.2, we obtain

$$|\beta RR'''| \lesssim 1, \quad \text{i.e. for all multi-indices } \gamma \text{ with length 3, } |\beta R\partial_x^\gamma R| \lesssim 1. \tag{3.1.21}$$

With $b = R\beta$, we get $b'b'' = (R'\beta + R\beta')(R''\beta + 2R'\beta' + R\beta'')$ and to check that $(b'b'')'$ is in L^∞ with

$$|(b'b'')'| \lesssim 1, \tag{3.1.22}$$

it is enough (see (3.1.11), (3.1.19)) to check the derivatives of $R''\beta R'\beta,\ R''\beta R\beta'$ which are, up to bounded terms (see our appendix A.3 for the meaning of the products)

$$R'''\beta R'\beta = R'''\beta RR'\frac{\beta}{R}, \quad R'''\beta R\beta'$$

which are bounded according to 3.1.21-19-11. Note that b'' is bounded from (3.1.19) and (3.1.11). We want also to verify that $(bb'')''$ is bounded. We use that $(b^2)^{(4)}$ is bounded from (3.1.16) and since we have

$$\underbrace{(b^2)''''}_{\substack{\text{bounded}\\ (3.1.16)}} = 2(b' \otimes b' + bb'')'' = 2\underbrace{(b' \otimes b'' + b'' \otimes b')'}_{\substack{\text{bounded}\\ (3.1.22)}} + 2(bb'')'', \tag{3.1.23}$$

we obtain[8] the boundedness of $(bb'')''$.

Remark 3.1.5. Before going on, we should note that our functions b, c above are only defined on Q_ν where holds the identity $a(x) = b(x)^2 + c(x')$. We can replace the function c above by

$$\tilde c(x') = c(x')\chi\big((x' - x'_\nu)\theta^{-1/2}\rho_\nu^{-1}\big)$$

where $\chi \in C_c^\infty(\mathbb{R}^{m-1})$ supported in the unit ball and equal to 1 in the ball of radius 1/2, so that $\tilde c$ is defined on $\mathbb{R}^{m-1}$ and the identity $a = b^2 + \tilde c$ holds on

$$x_\nu + \frac{1}{2}(Q_\nu - x_\nu) \supset U_\nu^* \supset \operatorname{supp} \varphi_\nu.$$

The bounds on the derivatives are unchanged as long as θ is fixed, which is the case.

Taking that remark into account, as well as the above estimates on the derivatives, we have finally, with E_2 standing for the nondegenerate indices,

$$\begin{aligned}
a(x) = \sum_{\nu\in\mathbb{N}} b_\nu(x)^2\varphi_\nu^2(x) + &\sum_{\nu\in E_2} a_\nu(x')\varphi_\nu^2(x)\\
|b_\nu| \lesssim \rho_\nu^2,\ |b'_\nu| \lesssim \rho_\nu,\ &|b''_\nu| \lesssim 1,\ |(b_\nu b''_\nu)''| + |(b'_\nu b''_\nu)'| \lesssim 1\\
|a_\nu| \lesssim \rho_\nu^4,\ |a'_\nu| \lesssim \rho_\nu^3,\ &|a''_\nu| \lesssim \rho_\nu^2,\ |a'''_\nu| \lesssim \rho_\nu,\ |a''''_\nu| \lesssim 1,\\
a_\nu \text{ is defined on } &\mathbb{R}^{m-1}.
\end{aligned}$$

Now, we consider the function $\mathbb{R}^{m-1} \ni t \mapsto A(t) = \rho_\nu^{-4} a_\nu\big(\rho_\nu t\big)$ and we have

$$|A_\nu| \lesssim 1,\ |A'_\nu| \lesssim 1,\ |A''_\nu| \lesssim 1,\ |A'''_\nu| \lesssim 1,\ |A''''_\nu| \lesssim 1.$$

Following the main argument in the proof by C.Fefferman and D.H.Phong, we can use an induction on the dimension m to get

$$A(t) = \sum_{1\le j\le N_{m-1})} B_j^2(t), \quad B_j \in C^{1,1}, \quad \text{and } B''_j, (B'_jB''_j)', (B_jB''_j)'' \in L^\infty.$$

Incorporated in the induction hypothesis is that the bounds on B depend only on the bounds on $A^{(4)}$. We obtain

$$a(x) = \sum_{\nu\in\mathbb{N}} b_\nu(x)^2\varphi_\nu^2(x) + \sum_{\nu\in E_2}\sum_{1\le j\le N_{m-1}} \rho_\nu^4 B_{j,\nu}^2(\frac{x'}{\rho_\nu})\varphi_\nu^2(x)$$

i.e.

$$a(x) = \sum_{1\le j\le N_{m-1}+1}\sum_{\nu\in\mathbb{N}} b_{\nu,j}(x)^2\varphi_\nu^2(x).$$

One needs to pass to a finite sum, which is quite standard since the overlap of the support of the functions φ_ν is bounded; this last argument is given in the appendix A.5. The proof of the theorem 3.1.1 is complete.

[8]The equality (3.1.23) is an equality between tensors (0,4) and it might look somewhat pedantic to resort to such notations: the reader may check directly the implication

$$\left.\begin{aligned}\forall\gamma, |\gamma| = 4, \partial_x^\gamma(b^2) \in L^\infty,\\ \forall\gamma_j, 1\le j\le 3, |\gamma_1| = 1 = |\gamma_2|, |\gamma_3| = 2, \partial_x^{\gamma_1}(\partial_x^{\gamma_2} b\partial_x^{\gamma_3} b) \in L^\infty,\end{aligned}\right\}$$
$$\Longrightarrow \forall\gamma_3,\gamma_4, |\gamma_3| = 2 = |\gamma_4|,\ \partial_x^{\gamma_3}(b\partial_x^{\gamma_4} b) \in L^\infty.$$

3.2. Application of the Wick calculus: proof of the theorem 1.3.1. Let a be a nonnegative function defined on $\mathbb{R}^{2n}$ such that $a^{(4)}$ belongs to $\mathcal{A}$ (defined in the proposition 1.2.1). Applying the lemma 2.2.1 and the L^2-boundedness of the operators with Weyl symbol in $\mathcal{A}$, we see that it suffices to prove that the operator with Wick symbol $a - \frac{1}{8\pi}\operatorname{trace} a''$ is semi-bounded from below. Since $\mathcal{A} \subset L^\infty(\mathbb{R}^{2n})$, it is enough to prove the following lemma.

Lemma 3.2.1. *Let a be a nonnegative function defined on $\mathbb{R}^{2n}$ such that $a^{(4)}$ belongs to $L^\infty(\mathbb{R}^{2n})$. The theorem 3.1.1 is providing a decomposition $a = \sum_{1\le j\le N} b_j^2$ along with the estimates* (3.1.2). *Then we have*

$$\Big(a - \frac{1}{8\pi}\operatorname{trace} a''\Big)^{\mathrm{Wick}} = \sum_{1\le j\le N}\Big[\Big(b_j - \frac{1}{8\pi}\operatorname{trace} b_j''\Big)^{\mathrm{Wick}}\Big]^2 + R$$

where R is a L^2-bounded operator such that $\|R\|_{\mathcal{L}(L^2(\mathbb{R}^n))} \le C\|a^{(4)}\|_{L^\infty(\mathbb{R}^{2n})}$, C depending only on the dimension n.

Proof. We have

$$a - \frac{1}{8\pi}\operatorname{trace} a'' = a - \frac{\Delta a}{8\pi} = \sum_{1\le j\le N} b_j^2 - \frac{1}{4\pi}|\nabla b_j|^2 - \frac{1}{4\pi} b_j \Delta b_j. \tag{3.2.1}$$

Then using the lemma 2.3.2, we get

$$b_j^{\mathrm{Wick}} b_j^{\mathrm{Wick}} = \Big(b_j^2 - \frac{1}{4\pi}|\nabla b_j|^2\Big)^{\mathrm{Wick}} + S_j, \tag{3.2.2}$$

with

$$\|S_j\|_{\mathcal{L}(L^2(\mathbb{R}^n))} \le C_1\Big(\big\|b_j''\big\|_{L^\infty}^2 + \big\|(b_j''b_j')'\big\|_{L^\infty} + \big\|(b_j b_j'')''\big\|_{L^\infty}\Big) \le C_2\|a^{(4)}\|_{L^\infty(\mathbb{R}^{2n})},$$

where C_1, C_2 depend only on the dimension. Moreover, we have, from the lemma 2.3.1,

$$\operatorname{Re}\Big(b_j^{\mathrm{Wick}}(\Delta b_j)^{\mathrm{Wick}}\Big) = \Big(b_j\Delta b_j - \frac{1}{4\pi}\nabla\cdot(\nabla b_j \Delta b_j) + \frac{1}{4\pi}(\Delta b_j)^2\Big)^{\mathrm{Wick}} + R_j, \tag{3.2.3}$$

with

$$\|R_j\|_{\mathcal{L}(L^2(\mathbb{R}^n))} \le C_3\big\|b_j''\big\|_{L^\infty(\mathbb{R}^{2n})}^2 \le C_4\|a^{(4)}\|_{L^\infty(\mathbb{R}^{2n})}.$$

As a consequence, from (3.2.2-3), we get

$$\begin{aligned}&\Big(b_j - \frac{1}{8\pi}\operatorname{trace} b_j''\Big)^{\mathrm{Wick}}\Big(b_j - \frac{1}{8\pi}\operatorname{trace} b_j''\Big)^{\mathrm{Wick}}\\ &= \Big(b_j^2 - \frac{1}{4\pi}|\nabla b_j|^2 - \frac{1}{4\pi}b_j\Delta b_j\Big)^{\mathrm{Wick}} + \frac{1}{16\pi^2}\Big(\nabla\cdot(\nabla b_j\Delta b_j)\Big)^{\mathrm{Wick}} - \frac{1}{16\pi^2}\Big((\Delta b_j)^2\Big)^{\mathrm{Wick}}\\ &\qquad + S_j - \frac{1}{4\pi}R_j + \frac{1}{64\pi^2}(\Delta b_j)^{\mathrm{Wick}}(\Delta b_j)^{\mathrm{Wick}},\end{aligned} \tag{3.2.4}$$

so that from (2.1.2), (3.1.2) and the estimates above for R_j, S_j, we obtain from (3.2.1) that

$$\sum_{1\le j\le N}\Big(b_j - \frac{1}{8\pi}\operatorname{trace} b_j''\Big)^{\mathrm{Wick}}\Big(b_j - \frac{1}{8\pi}\operatorname{trace} b_j''\Big)^{\mathrm{Wick}} = \Big(a - \frac{1}{8\pi}\operatorname{trace} a''\Big)^{\mathrm{Wick}} + S$$

with $\|S\|_{\mathcal{L}(L^2(\mathbb{R}^n))} \le C_5 \|a^{(4)}\|_{L^\infty(\mathbb{R}^{2n})}$ C_5 depending only on the dimension. This is the result of the lemma, completing as well the proof of the theorem 1.3.1. $\square$

Remark. The proof above is giving a slightly better result, since we prove the lemma for each $a_j = b_j^2$, provided the lhs of (3.1.2) is controlled.

Comment 3.2.2. One may ask the following question: why did we not apply this induction argument on the Sjöstrand algebra $\mathcal{A}$ directly, and avoid that complicated detour with the Wick calculus? The answer to that interrogation is simple: as seen above the Fefferman-Phong induction procedure requires a cutting process (this is the metric $dX^2/\rho(X)^2$) and also a bending of the phase space (the function α is not linear). Although the cutting part may respect $\mathcal{A}$, it is not very likely that the rigid affine structure of $\mathcal{A}$ would survive the bending.

3.3. Proof of the Corollary 1.3.2.

Let us begin with the statement (iv) in this corollary. Let us define

$$A(x,\xi) = h^{-2} a(xh^{1/2}, \xi h^{-1/2}, h). \tag{3.3.1}$$

The function A satisfies

$$\begin{aligned}(\partial_\xi^\alpha \partial_x^\beta A)(x,\xi) &= h^{-2-\frac{|\alpha|}{2}+\frac{|\beta|}{2}} (\partial_1^\beta \partial_2^\alpha a)(xh^{1/2}, \xi h^{-1/2}, h) \\ &= h^{\frac{|\alpha|+|\beta|-4}{2}} (\partial_1^\beta \partial_2^\alpha a)(xh^{1/2}, \xi h^{-1/2}, h) h^{-|\alpha|}\end{aligned}$$

so that for $|\alpha|+|\beta| = 4$, we have $(\partial_\xi^\alpha \partial_x^\beta A)(x,\xi) = (\partial_1^\beta \partial_2^\alpha a)(xh^{1/2}, \xi h^{-1/2}, h) h^{-|\alpha|}$. We have supposed that for $|\alpha|+|\beta| = 4$, the functions $(x,\xi) \mapsto (\partial_1^\beta \partial_2^\alpha a)(xh^{1/2}, \xi h^{-1/2}, h) h^{-|\alpha|}$ belongs to $\mathcal{A}$ with a norm bounded above independently by ν_0. As a result the function $A^{(4)}(x,\xi)$ belongs to $\mathcal{A}$ with a norm bounded above by ν_0. Since $A(x,\xi) \ge 0$, the theorem 1.3.1 implies that $A^w + C_n \nu_0 \ge 0$, i.e. $\big(a(xh^{1/2}, \xi h^{-1/2}, h)\big)^w + C_n \nu_0 h^2 \ge 0$ and since there is a unitary mapping U_h such that $U_h^* a(x,\xi,h)^w U_h = \big(a(xh^{1/2}, \xi h^{-1/2}, h)\big)^w$, we obtain

$$a(x,\xi,h)^w + C_n \nu_0 h^2 \ge 0, \text{ qed.} \tag{3.3.2}$$

To get that $\operatorname{Re} a(x,D,h) + Ch^2 \ge 0$, one[9] should note that the symbols $A^{(4)}$ defined above belong to $\mathcal{A}$, which implies that it is also the case for $J^{-1/2} A^{(4)}$ and $J^{1/2} \overline{A^{(4)}}$. Now we have

$$2 \operatorname{Re} a(x,D,h) = 2 \operatorname{Re} (J^{-1/2} a)^w = (J^{-1/2} a + J^{1/2} \bar{a})^w,$$

so that rescaling[10] the symbol $J^{-1/2} a + J^{1/2} \bar{a}$, we find $J^{-1/2} A + J^{1/2} \bar{A}$. Since we have

$$J^{-1/2} A = e^{-i\pi D_x \cdot D_\xi} A = A - i\pi D_x \cdot D_\xi A - \int_0^1 (1-\theta) e^{-i\pi\theta D_x \cdot D_\xi} d\theta \pi^2 (D_x \cdot D_\xi)^2 A,$$

[9]With the group J^t defined in the proposition 1.2.3, the formula linking the Weyl quantization with the ordinary quantization is $a(x,D) = (J^{-1/2} a)^w$.

[10]We define

$$B(x,\xi) = h^{-2}(J^{-1/2}a)(xh^{1/2}, \xi h^{-1/2}) + h^{-2}(J^{1/2}a)(xh^{1/2}, \xi h^{-1/2}) = (J^{-1/2}A)(x,\xi) + (J^{1/2}A)(x,\xi).$$

and that A is real-valued, we get

$$\operatorname{Re}(J^{-1/2}A) = A - \int_0^1 (1-\theta)e^{-i\pi\theta D_x\cdot D_\xi} d\theta \pi^2 \underbrace{(D_x\cdot D_\xi)^2 A}_{\in\mathcal{A}}.$$

Now we have from the previous identity, since $\mathcal{A}$ is stable by the group J^t (theorem 1.1 in [S1]), with a uniform constant for t in a compact set,

$$2\operatorname{Re} A(x,D) = \big(2\operatorname{Re}(J^{-1/2}A)\big)^w \in 2A^w + \mathcal{A}^w.$$

We can then apply the result (3.3.2) and the L^2 boundedness of $\mathcal{A}^w$ to conclude. The proof of *(iv)* in the corollary 1.3.2 is complete.

Let us show that (iv) implies (iii). We define $b(x,\xi,h) = a(x,h\xi)$, which is nonnegative; it is enough to check the functions $(x,\xi)\mapsto(\partial_1^\beta\partial_2^\alpha b)(xh^{1/2},\xi h^{-1/2},h)h^{-|\alpha|}$, for $|\alpha|+|\beta|=4$. We have in fact

$$(\partial_1^\beta\partial_2^\alpha b)(xh^{1/2},\xi h^{-1/2},h)h^{-|\alpha|} = (\partial_1^\beta\partial_2^\alpha a)(xh^{1/2},\xi h^{1/2}).$$

Now, from the lemma A.2.1 in our appendix, for $h\in(0,1]$, the functions

$$(x,\xi)\mapsto(\partial_1^\beta\partial_2^\alpha a)(xh^{1/2},\xi h^{1/2})$$

belong to $\mathcal{A}$ with a bounded norm since we have supposed that $a^{(4)}\in\mathcal{A}$. We can then apply the already proven result *(iv)* in the corollary to get

$$a(x,\xi h)^w + Ch^2\|a^{(4)}\|_{\mathcal{A}}\ge 0,\quad \operatorname{Re} a(x,hD) + Ch^2\|a^{(4)}\|_{\mathcal{A}}\ge 0,\quad \text{qed}.$$

Let us show that (iv) implies (ii). We assume that $a(x,\xi,h)$ is a nonnegative function satisfying the assumptions of *(ii)*. According to the already proven *(iv)*, we need only to check, for $|\alpha'|+|\beta'|=4$, the norm in $\mathcal{A}$ of

$$(x,\xi)\mapsto(\partial_1^{\beta'}\partial_2^{\alpha'}a)(xh^{1/2},\xi h^{-1/2},h)h^{-|\alpha'|} = c_{\alpha'\beta'}(x,\xi).$$

Because of the second inclusion in (1.2.1), it is enough to find an L^∞ bound on the $2n+1$ first derivatives of that function; we have, for $|\alpha''|+|\beta''|\le 2n+1$

$$(\partial_\xi^{\alpha''}\partial_x^{\beta''}c_{\alpha'\beta'})(x,\xi) = (\partial_1^{\beta'+\beta''}\partial_2^{\alpha'+\alpha''}a)(xh^{1/2},\xi h^{-1/2},h)h^{-|\alpha'|}h^{\frac{-|\alpha''|+|\beta''|}{2}} \tag{3.3.3}$$

and from the assumption in *(ii)*, we get, since $4\le|\alpha'+\alpha''|+|\beta'+\beta''|\le 2n+5$,

$$|(\partial_1^{\beta'+\beta''}\partial_2^{\alpha'+\alpha''}a)(xh^{1/2},\xi h^{-1/2},h)|\le C_{\alpha'+\alpha'',\beta'+\beta''}h^{|\alpha'|+|\alpha''|}, \tag{3.3.4}$$

so that (3.3.3-4) imply

$$\begin{aligned}|(\partial_\xi^{\alpha''}\partial_x^{\beta''}c_{\alpha'\beta'})(x,\xi)| &\le C_{\alpha'+\alpha'',\beta'+\beta''}h^{|\alpha'|+|\alpha''|}h^{-|\alpha'|}h^{\frac{-|\alpha''|+|\beta''|}{2}}\\ &= C_{\alpha'+\alpha'',\beta'+\beta''}h^{\frac{|\alpha''|+|\beta''|}{2}}\le C_{\alpha'+\alpha'',\beta'+\beta''}.\end{aligned}$$

yielding the sought bound. The proof of *(ii)* is complete.

Proof of (i) in the corollary 1.3.2. Using a Littlewood-Paley decomposition, we have

$$1=\sum_{\nu\ge 0}\varphi_\nu^2(\xi),\quad \varphi_\nu\in C_c^\infty(\mathbb{R}^n),$$
$$\text{for } \nu\ge 1,\ \operatorname{supp}\varphi_\nu\subset\{2^{\nu-1}\le|\xi|\le 2^{\nu+1}\},\quad \sup_{\nu,\xi}|\partial_\xi^\alpha\varphi_\nu(\xi)|2^{\nu|\alpha|}<\infty.$$

We introduce also some smooth nonnegative compactly supported functions $\psi_\nu(\xi)$, satisfying the same uniform estimates than φ_ν and supported in $2^{\nu-3}\le|\xi|\le 2^{\nu+3}$ for $\nu\ge 1$, identically 1 on $2^{\nu-2}\le|\xi|\le 2^{\nu+2}$ (in particular on the support of φ_ν). We consider a nonnegative symbol a satisfying (1.1.1) for $4\le|\alpha|+|\beta|\le 2n+5$. We write

$$a=\sum_{\nu\ge 0}\varphi_\nu^2 a=\sum_{\nu\ge 0}(\psi_\nu\sharp\varphi_\nu^2 a\sharp\psi_\nu+r_\nu). \tag{3.3.5}$$

The proof relies on the following

Claim 3.3.1. *The operator with Weyl symbol $\sum_\nu r_\nu$ is bounded on $L^2(\mathbb{R}^n)$.*

As a matter of fact, if this claim is proven, we are left with the operator $\sum_\nu\psi_\nu^w(\varphi_\nu^2 a)^w\psi_\nu^w$ and we can apply the already proven result *(ii)* in this corollary to get that with a uniform C,

$$\sum_\nu\psi_\nu^w(\varphi_\nu^2 a)^w\psi_\nu^w=\sum_\nu\psi_\nu^w\underbrace{\big((\varphi_\nu^2 a)^w+C\big)}_{\ge 0}\psi_\nu^w-C\underbrace{(\sum_\nu\psi_\nu^2)^w}_{L^2\text{bounded}}$$

and so this operator is semi-bounded from below as well as a^w. Let us prove the claim. We leave as an exercise for the reader to check, using (1.2.2), the composition formula

$$(a_1\sharp a_2\sharp a_3)(X)$$
$$=2^{2n}\iint_{\mathbb{R}^{2n}\times\mathbb{R}^{2n}}a_1(Y_1)a_2(Y_2)a_3(X-Y_1+Y_2)e^{-4i\pi[X-Y_1,X-Y_2]}dY_1dY_2. \tag{3.3.6}$$

Applying this to $\psi_\nu\sharp a_\nu\sharp\psi_\nu$ with $a_\nu=\varphi_\nu^2 a$, we get

$$\begin{aligned}r_\nu(x,\xi)&=-2^n\iiint_0^1(1-\theta)e^{-4i\pi y\eta}\psi_\nu(\xi+\eta)\psi_\nu(\xi-\eta)(\partial_x^2 a_\nu)(x+\theta y,\xi)y^2dyd\eta d\theta\\&=\frac{2^n}{16\pi^2}\iiint_0^1(1-\theta)\partial_\eta^2(e^{-4i\pi y\eta})\psi_\nu(\xi+\eta)\psi_\nu(\xi-\eta)(\partial_x^2 a_\nu)(x+\theta y,\xi)dyd\eta d\theta\\&=\frac{2^n}{16\pi^2}\iiint_0^1(1-\theta)e^{-4i\pi y\eta}\partial_\eta^2\big(\psi_\nu(\xi+\eta)\psi_\nu(\xi-\eta)\big)(\partial_x^2 a_\nu)(x+\theta y,\xi)dyd\eta d\theta.\end{aligned}$$

From this formula we see that r_ν is supported where $2^{\nu-1}\le|\xi|\le 2^{\nu+1}$ since it is the case for a_ν ($\nu\ge 1$); since the overlap of the rings where $|\xi|\sim 2^\nu$ is bounded, it is enough to check some bounds on the derivatives of r_ν to get similar bounds on the $\sum_\nu r_\nu$. Moreover in the integrand, if the function $\psi_\nu(\xi+\eta)$ is differentiated, we get

$$2^{\nu+2}\le|\xi+\eta|\le 2^{\nu+3}\text{ or }2^{\nu-3}\le|\xi+\eta|\le 2^{\nu-2}.$$

As a result, in the first case, we have $|\eta|\ge|\xi+\eta|-|\xi|\ge 2^{\nu+2}-2^{\nu+1}=2^{\nu+1}$, whereas in the second case $|\eta|\ge|\xi|-|\xi+\eta|\ge 2^{\nu-1}-2^{\nu-2}=2^{\nu-2}$, which implies that we always have $|\eta| \ge 2^{\nu-2}$. Since we have also $|\eta| \le |\xi+\eta| + |\xi| \le 2^{\nu+3} + 2^{\nu+1}$, we obtain (note that the case when the other function $\psi(\xi-\eta)$ is differentiated is similar) on the integrand

$$2^{\nu-2} \le |\eta| \le 2^{\nu+4}. \tag{3.3.7}$$

We write now

$$\frac{1}{\alpha!}(\partial_\xi^\alpha \partial_x^\beta r_\nu)(x,\xi) = \sum_{\alpha'+\alpha''=\alpha} \frac{2^n}{\alpha'!\alpha''!16\pi^2} \times \iiint_0^1 (1-\theta)e^{-4i\pi y\eta}\partial_\xi^{\alpha'}\partial_\eta^2\big(\psi_\nu(\xi+\eta)\psi_\nu(\xi-\eta)\big)(\partial_\xi^{\alpha''}\partial_x^\beta\partial_x^2 a_\nu)(x+\theta y,\xi)dyd\eta d\theta$$

and since the integral above is, for N, k even integers, $N > n$,

$$\iiint_0^1 (1-\theta)e^{-4i\pi y\eta}(1+4|\eta|^2)^{-k/2} \times (1+4|y|^2)^{-N/2}(1+D_\eta^2)^{N/2}\partial_\xi^{\alpha'}\partial_\eta^2\big(\psi_\nu(\xi+\eta)\psi_\nu(\xi-\eta)\big) \times (1+D_y^2)^{k/2}\big((\partial_\xi^{\alpha''}\partial_x^\beta\partial_x^2 a_\nu)(x+\theta y,\xi)\big)dyd\eta d\theta$$

we get, for $|\alpha|+|\beta|+k \le 2n+3$,

$$|(\partial_\xi^\alpha \partial_x^\beta r_\nu)(x,\xi)| \le C_{\alpha\beta N}(2^\nu)^{-k-|\alpha'|-2+2-|\alpha''|+n} = C_{\alpha\beta N}(2^\nu)^{-|\alpha|+n-k}.$$

For α,β given such that $\max(|\alpha|,|\beta|) \le n+1$, we choose $k = n-|\alpha|$ or $k = n-|\alpha|+1$ so that k is even, and we get, uniformly in ν, $|(\partial_\xi^\alpha \partial_x^\beta r_\nu)(x,\xi)| \lesssim 1$; note that then we have indeed

$$|\alpha|+|\beta|+k \le |\beta|+n+1 \le 2n+2 \le 2n+3.$$

Eventually, from (a mild version of) the theorem 1.2 in [B2] we get the claim 3.3.1: we have proven that for $\max(|\alpha|,|\beta|) \le n+1$, $\partial_\xi^\alpha \partial_x^\beta r$ is bounded. The proof of (1.1.3) is complete, under the assumptions of the corollary.

Proof of (1.1.2). To obtain also the result for the ordinary quantization is not a direct consequence of the previous result, because of our limitation on the regularity of a. So we have to revisit our argument above, replacing at each step the Weyl quantization by the standard quantization. It is a bit tedious, but unavoidable. We write

$$a = \sum_{\nu\ge 0}\varphi_\nu^2 a = \sum_{\nu\ge 0}(\psi_\nu \circ \varphi_\nu^2 a \circ \psi_\nu + s_\nu). \tag{3.3.8}$$

The proof relies on the following

Claim 3.3.2. *The operator with standard symbol $\sum_\nu s_\nu$ is bounded on $L^2(\mathbb{R}^n)$.*

As a matter of fact, if this claim is proven, we are left with the operator

$$\operatorname{Re}\sum_\nu \operatorname{Op}(\psi_\nu)\operatorname{Op}(\varphi_\nu^2 a)\operatorname{Op}(\psi_\nu)$$

and we can apply the already proven result (ii) in this corollary to get that with a uniform C,

$$\sum_\nu \operatorname{Op}(\psi_\nu)\operatorname{Re}\operatorname{Op}(\varphi_\nu^2 a)\operatorname{Op}(\psi_\nu) = \sum_\nu \operatorname{Op}(\psi_\nu)\underbrace{\big(\operatorname{Re}\operatorname{Op}(\varphi_\nu^2 a)+C\big)}_{\ge 0}\operatorname{Op}(\psi_\nu) - C\underbrace{\operatorname{Op}(\sum_\nu \psi_\nu^2)}_{L^2\text{bounded}}$$

and so this operator is semi-bounded from below as well as $\operatorname{Re}\operatorname{Op}(a)$. Let us prove the claim. Reminding the ordinary composition formula, we have

$$(a\circ b)(x,\xi) = \iint_{\mathbb{R}^n\times\mathbb{R}^n} e^{-2i\pi y\eta} a(x,\xi+\eta) b(y+x,\xi) dy d\eta. \tag{3.3.9}$$

Applying this to $\psi_\nu\circ a_\nu\circ\psi_\nu$ with $a_\nu=\varphi_\nu^2 a$, we get $\psi_\nu\circ a_\nu\circ\psi_\nu=\psi_\nu\circ a_\nu\psi_\nu=\psi_\nu\circ a_\nu$ and

$$\begin{aligned}(\psi_\nu\circ a_\nu\circ\psi_\nu)(x,\xi) &= \iint e^{-2i\pi y\eta}\psi_\nu(\xi+\eta) a_\nu(y+x,\xi) dy d\eta\\ &= \iint e^{-2i\pi y\eta}\psi_\nu(\xi+\eta)\Big(a_\nu(x,\xi)+\int_0^1(1-\theta)\partial_x^2 a_\nu(x+\theta y,\xi)y^2 d\theta\Big)dy d\eta\\ &= (a_\nu\psi_\nu)(x,\xi)-s_\nu(x,\xi),\end{aligned}$$

with

$$s_\nu(x,\xi) = -\iiint_0^1 (1-\theta)e^{-2i\pi y\eta}\psi_\nu(\xi+\eta)(\partial_x^2 a_\nu)(x+\theta y,\xi)y^2 dy d\eta d\theta.$$

That formula is so similar to the defining formula of r_ν above that we can resume the discussion and use (a mild version of) the theorem 1.1 in [B2] we get the claim 3.3.3: The proof of (1.1.2) is complete, under the assumptions of the corollary.

Appendix

A.1. On nonnegative functions. Let a be a nonnegative $C^{3,1}$ function defined on $\mathbb{R}^m$ such that $\|a^{(4)}\|_{L^\infty}\le 1$; ρ and Ω are defined in (3.1.3).

Lemma A.1.1. *Let a,ρ,Ω be as above. For $0\le j\le 4$, we have $\|a^{(j)}(x)\|\le\gamma_j\rho(x)^{4-j}$, with $\gamma_0=\gamma_2=\gamma_4=1,\gamma_1=3,\gamma_3=4$.*

Proof. The inequalities for $j=0,2,4$ are obvious. Let us write Taylor's formula,

$$a(x+h)=a(x)+a'(x)h+\frac12 a''(x)h^2+\frac16 a^{(3)}(x)h^3+\int_0^1\frac{(1-\theta)^3}{3!}a^{(4)}(x+\theta h)d\theta h^4.$$

We get $a(x+h)-a(x)-\frac12 a''(x)h^2-\frac{|h|^4}{24}\le a'(x)h+\frac16 a^{(3)}(x)h^3$ and since $a(x+h)\ge 0$, we have $-a(x)-\frac12 a''(x)h^2-\frac{|h|^4}{24}\le a'(x)h+\frac16 a^{(3)}(x)h^3$. Since the rhs is odd in the variable h, we obtain

$$|a'(x)h+\frac16 a^{(3)}(x)h^3|\le a(x)+\frac12 a''(x)h^2+\frac{|h|^4}{24}. \tag{A.1.1}$$

Let us choose $h=\rho(x)sT$ where T is a unit vector and s is a real parameter. We have

$$|s\rho(x)a'(x)T+s^3\rho(x)^3\frac16 a^{(3)}(x)T^3|\le\rho(x)^4\Big(1+\frac12 s^2+\frac{s^4}{24}\Big). \tag{A.1.2}$$

Note. *Let $\alpha, \beta, \gamma \in \mathbb{R}$, and assume that $\forall s \in \mathbb{R}$, $|s\alpha + s^3\beta| \le \gamma(1 + \frac{1}{2}s^2 + \frac{s^4}{24})$. Applying that inequality for $s = 1, 3$ gives $|\alpha + \beta| \le \gamma\frac{37}{24}$, $|3\alpha + 27\beta| \le \gamma\frac{213}{24}$ and thus*

$$24|\beta| = |3\alpha + 27\beta - 3(\alpha+\beta)| \le \frac{324}{24}\gamma, \quad |\beta| \le \gamma\frac{324}{24^2},$$
$$|\alpha| = |\alpha + \beta - \beta| \le \gamma\frac{37 \times 24 + 324}{24^2} = \gamma\frac{1212}{576}.$$

As a result, from (A.1.2), we get for $\rho(x) > 0$, $\|a'(x)\| \le 3\rho(x)^3$, $\|a^{(3)}(x)\| \le 4\rho(x)$. If $\rho(x) = 0$, we use the inequality (A.1.1) with $h = \epsilon T$ where T is a unit vector and ϵ is a positive parameter, providing $|\epsilon a'(x)T + \epsilon^3\frac{1}{6}a^{(3)}(x)T^3| \le \frac{\epsilon^4}{24}$. Dividing by ϵ and letting it go to zero, we find $a'(x)T = 0$, for all T, i.e. $a'(x) = 0$. Next we find that for all vectors T, $a^{(3)}(x)T^3 = 0$, implying that the symmetric trilinear form $a^{(3)}(x)$ is zero (see the remark 3.1.2). The proof of the lemma is complete. □

Lemma A.1.2. *Let a, ρ, Ω be as above. The metric $\frac{|dx|^2}{\rho(x)^2}$ is slowly varying on the open set Ω, i.e. there exists $C_0 \ge 1 > r_0 > 0$ such that*

$$x \in \Omega \text{ and } |x - y| \le r_0\rho(x) \Longrightarrow y \in \Omega, \quad C_0^{-1} \le \frac{\rho(x)}{\rho(y)} \le C_0. \tag{A.1.3}$$

The constants r_0 and C_0 can be chosen as "universal" fixed constants (independently of the dimension and of the function a, which is normalized by the condition $\|a^{(4)}\|_{L^\infty} \le 1$).

Proof. Using Taylor's formula, one gets, using (A.1.1), the lemma A.1.1, the remark 3.1.2,

$$\begin{aligned}
\rho(x+h)^4 &= a(x+h) + \|a''(x+h)\|^2 \\
&\le a(x) + a'(x)h + \frac{1}{2}a''(x)h^2 + \frac{1}{6}a'''(x)h^3 + \frac{1}{24}|h|^4 \\
&\qquad + 3\|a''(x)\|^2 + 3\|a'''(x)\|^2|h|^2 + 3\frac{1}{4}|h|^4 \\
&\le 2a(x) + a''(x)h^2 + \frac{1}{12}|h|^4 + 3\|a''(x)\|^2 + 3\|a'''(x)\|^2|h|^2 + 3\frac{1}{4}|h|^4 \\
&\le 2\rho(x)^4 + \rho(x)^2|h|^2 + \frac{1}{12}|h|^4 + 3\rho(x)^4 + 3\,2^4\rho(x)^2|h|^2 + \frac{3}{4}|h|^4 \\
&\le 5\rho(x)^4 + |h|^2\rho(x)^2(1 + 3\,2^4) + |h|^4(\frac{1}{12} + 3\,2^{-2}) \\
&\le 3^4\big(\rho(x) + |h|\big)^4.
\end{aligned}$$

This implies that

$$\rho(x+h) \le 3\big(\rho(x) + |h|\big). \tag{A.1.4}$$

As a consequence, we have for $\|T\| \le 1, r \ge 0$, $\rho\big(x + r\rho(x)T\big) \le 3(1+r)\rho(x)$, and thus

$$|y - x| \le r\rho(x) \Longrightarrow \rho(y) \le 3(1+r)\rho(x).$$

Moreover if $y = x + r\rho(x)T$ with $r \ge 0$ and $|T| \le 1$, (A.1.4) gives

$$\rho(x) = \rho\big(y - r\rho(x)T\big) \le 3\big(\rho(y) + r\rho(x)\big)$$

and if $r \le 1/6$ we find $\frac{1}{2}\rho(x) \le 3\rho(y) \le (9+\frac{3}{2})\rho(x)$ providing the result of the lemma with $C_0 = 1/r_0 = 6$. $\square$

Remark. When the normalisation condition $\|a^{(4)}\|_{L^\infty} \le 1$ is not satisfied, it is of course possible to divide a by a constant to get back to that normalization condition. When $\|a^{(4)}\|_{L^\infty} \neq 0$, the lemma A.1.1 is providing the inequalities

$$\|a'(x)\|^{4/3} \le 3^{4/3}\big(a(x)\|a^{(4)}\|_\infty^{1/3} + \|a''(x)\|^2\|a^{(4)}\|_\infty^{-2/3}\big), \tag{A.1.5}$$

$$\|a^{(3)}(x)\|^4 \le 4^4\big(a(x)\|a^{(4)}\|_\infty^3 + \|a''(x)\|^2\|a^{(4)}\|_\infty^2\big). \tag{A.1.6}$$

Note that if $\|a^{(4)}\|_{L^\infty} = 0$, i.e. $a^{(4)} \equiv 0$, a is a polynomial of degree ≤ 3, and the nonnegativity implies $a^{(3)} \equiv 0$ so that, if its minimum is realized at 0, a is the sum of a nonnegative quadratic form and of a nonnegative constant.

Lemma A.1.3. *Let a, ρ, Ω be as above. Let θ such that $0 < \theta \le 1/2$. If $y \in \Omega$ verifies $a(y) \ge \theta\rho(y)^4$, then*

$$|x - y| \le \theta\rho(y)2^{-3} \Longrightarrow a(x) \ge \theta\rho(y)^4/2.$$

Proof. We note that for $|x - y| \le r\rho(y)$, using the lemma A.1.1 and Taylor's formula, we have

$$a(x) \ge a(y) - r\rho(y)3\rho(y)^3 - \frac{1}{2}r^2\rho(y)^2\rho(y)^2 - \frac{1}{6}r^3\rho(y)^3 4\rho(y) - \frac{1}{24}r^4\rho(y)^4,$$

implying, for $a(y) \ge \theta\rho(y)^4$, that $a(x) \ge \rho(y)^4(\theta - 3r - \frac{r^2}{2} - \frac{2r^3}{3} - \frac{r^4}{24})$ and since for $r \le 1/2$, we have $3r + \frac{r^2}{2} + \frac{2r^3}{3} + \frac{r^4}{24} \le r(3 + 1/8 + 1/12 + 1/384) \le 4r$ we obtain indeed $a(x) \ge \frac{1}{2}\theta\rho(y)^4$ if $r \le \theta/8$. $\square$

Lemma A.1.4. *Let a, ρ, Ω be as above. There exists $R_0 > 0$ such that if $y \in \Omega$ verifies $a(y) < \rho(y)^4/2$, then there exists a unit vector T such that,*

$$|x - y| \le R_0\rho(y) \Longrightarrow a''(x)T^2 \ge 2^{-1}\rho(y)^2.$$

One can take $R_0 = 10^{-2}$.

Proof. We have $\|a''(y)\| \ge 2^{-1/2}\rho(y)^2$, so with the remark 3.1.2, we find a unit vector T such that $|a''(y)T^2| \ge 2^{-1/2}\rho(y)^2$. Then we have for all real s

$$0 \le a(y + s\rho(y)T) \le a(y) + s\rho(y)a'(y)T + \frac{s^2}{2}\rho(y)^2a''(y)T^2 + \frac{s^3}{6}\rho(y)^3a'''(y)T^3 + \frac{s^4}{24}\rho(y)^4.$$

The quantity $s\rho(y)a'(y)T + \frac{s^3}{6}\rho(y)^3a'''(y)T^3$ is odd in the variable s so that

$$a(y) + \frac{s^2}{2}\rho(y)^2a''(y)T^2 + \frac{s^4}{24}\rho(y)^4 \ge |s\rho(y)a'(y)T + \frac{s^3}{6}\rho(y)^3a'''(y)T^3| \ge 0,$$

and in particular,

$$\forall s \neq 0,\ a''(y)T^2 \ge -\frac{s^2}{12}\rho(y)^2 - s^{-2}2a(y)\rho(y)^{-2} \Longrightarrow a''(y)T^2 \ge -2 \times 6^{-1/2}a(y)^{1/2}.$$

Since $|a''(y)T^2| \geq 2^{-1/2}\rho(y)^2$, this implies

$$a''(y)T^2 \geq 2^{-1/2}\rho(y)^2, \tag{A.1.7}$$

otherwise we would have $-2 \times 6^{-1/2}a(y)^{1/2} \leq a''(y)T^2 \leq -2^{-1/2}\rho(y)^2$ and thus

$$a(y)^{1/2} \geq 6^{1/2}2^{-3/2}\rho(y)^2 \Longrightarrow a(y) \geq \frac{3}{4}\rho(y)^4$$

which is incompatible with $a(y) < \rho(y)^4/2$. Using the Taylor expansion for $x \mapsto a''(x)T^2$ yields the following; we write, for $|x-y| \leq \rho(y)s$,

$$a''(x)T^2 \geq a''(y)T^2 - |s|\rho(y)4\rho(y) - \frac{s^2}{2}\rho(y)^2 \geq \rho(y)^2\big(\frac{1}{\sqrt{2}} - 12|s| - 3s^2/2\big) \geq \rho(y)^2/2,$$

provided $|s| \leq 10^{-2}$. □

Lemma A.1.5. *Let $a, \rho, \Omega, C_0, r_0, R_0$ be as above. There exists a positive constant θ_0 such that if $0 < \theta \leq \theta_0$ and $y \in \Omega$ is such that $a(y) < \theta\rho(y)^4$, the following property is true. For all x such that $|x-y| \leq \theta^{1/2}\rho(y)$, the function $\tau \mapsto a'(x+\tau\rho(y)T)T$ has a unique zero on the interval $[-\theta^{1/4}, \theta^{1/4}]$. The constant θ_0 is a universal constant that will be chosen also $\leq \min(1/2, r_0^2, R_0^4)$.*

Proof. From the previous lemma, we know that for $y \in \Omega$ such that $a(y) < \rho(y)^4/2$ then there exists a unit vector T such that,

$$|x-y| \leq R_0\rho(y) \Longrightarrow a''(x)T^2 \geq 2^{-1}\rho(y)^2.$$

The second-order Taylor's formula gives, for $|t| \leq r_0$, using (A.1.3),

$$0 \leq a(y+t\rho(y)T) \leq a(y) + t\rho(y)a'(y)T + \frac{\rho(y)^2t^2}{2}C_0^2\rho(y)^2$$

and thus

$$|t|\rho(y)|a'(y)T| \leq a(y) + C_0^2\rho(y)^4t^2/2 \leq \theta\rho(y)^4 + C_0^2\rho(y)^4t^2/2.$$

As a result choosing $t = \theta^{1/2}$ (which is indeed smaller than r_0), we get

$$|a'(y)T| \leq \rho(y)^3(\theta^{1/2} + C_0^2\theta^{1/2}/2). \tag{A.1.8}$$

We have for s real

$$a'(y+s\rho(y)T)T = a'(y)T + s\rho(y)a''(y)T^2 + \frac{s^2}{2}\rho(y)^2a'''(y)T^3 + \int_0^1 \frac{1}{2}(1-t)^2a^{(4)}(y+ts\rho(y)T)T^4dts^3\rho(y)^3,$$

so that, using (A.1.8), we have

$$\begin{aligned} a'(y+s\rho(y)T)T &\leq \rho(y)^3\theta^{1/2}\overbrace{(1+C_0^2/2)}^{=C_1} + s\rho(y)a''(y)T^2 + \frac{s^2}{2}\rho(y)^3 4 + \frac{1}{6}|s|^3\rho(y)^3 \\ &\leq \rho(y)^3\Big(\theta^{1/2}C_1 + s\frac{a''(y)T^2}{\rho(y)^2} + 2s^2 + \frac{|s|^3}{6}\Big). \end{aligned}$$

The coefficient of s inside the bracket above belongs to the interval $[2^{-1/2}, 1]$. For $s = -\theta^{1/4}$, we get that

$$a'(y - \theta^{1/4}\rho(y)T)T \le \rho(y)^3\Big(\theta^{1/2}C_1 - \theta^{1/4}2^{-1/2} + 2\theta^{1/2} + \frac{|\theta|^{3/4}}{6}\Big) < 0$$

if θ is small enough with respect to a universal constant. Since we have also the inequality

$$\begin{aligned} a'(y + s\rho(y)T)T &\ge -\rho(y)^3\theta^{1/2}C_1 + s\rho(y)a''(y)T^2 - \frac{s^2}{2}\rho(y)^3 4 - \frac{1}{6}|s|^3\rho(y)^3 \\ &\ge \rho(y)^3\big(-\theta^{1/2}C_1 + s\frac{a''(y)T^2}{\rho(y)^2} - 2s^2 - \frac{|s|^3}{6}\big), \end{aligned}$$

the choice $s = \theta^{1/4}$ shows that $a'(y + \theta^{1/4}\rho(y)T)T > 0$. As a result the function ϕ defined by $\phi(\tau) = a'(y + \tau\rho(y)T)T$ vanishes for some τ with $|\tau| \le \theta^{1/4} \le R_0$. Moreover, from the lemma A.1.4, its derivative ϕ' satisfies

$$\phi'(\tau) = a''(y + \tau\rho(y)T)T^2\rho(y) \ge 2^{-1}\rho(y)^3 > 0,$$

so that ϕ is monotone increasing of τ on the interval $[-\theta^{1/4}, \theta^{1/4}]$, with a unique zero on that interval. Considering now for $|y - x| \le \theta^{1/2}\rho(y)$ the function

$$\psi(\tau, x) = a'(x + \tau\rho(y)T)T,$$

we get that

$$\phi(\tau) - \theta^{1/2}\rho(y)C_0^2\rho(y)^2 \le \psi(\tau, x) \le \phi(\tau) + \theta^{1/2}\rho(y)C_0^2\rho(y)^2$$

so that the same reasoning as before, we find that for all x such that $|x - y| \le \theta\rho(y)$, the function $\tau \mapsto a'(x + \tau\rho(y)T)T$ has a unique zero on the interval $[-\theta^{1/4}, \theta^{1/4}]$, provided that θ is smaller than a positive universal constant. $\square$

Remark A.1.6. Let $a, \rho, \Omega, r_0, C_0, R_0, \theta_0$ be as in the lemma A.1.5 and $0 < \theta \le \theta_0$. Let y be a point in Ω such that $a(y) < \theta\rho(y)^4$. We may choose the linear orthonormal coordinates such that the vector T given by the lemma A.1.5 is the first vector of the canonical basis of $\mathbb{R}^m$. Then a consequence of the lemma A.1.5 is that, for all $x' \in B_{\mathbb{R}^{m-1}}(y', \theta^{1/2}\rho(y))$ the map $\tau \mapsto \partial_1 a(\tau, x')$ has a unique zero $\alpha(x')$ on the interval $[-\theta^{1/4}\rho(y) + y_1, \theta^{1/4}\rho(y) + y_1]$. We have thus

$$|x' - y'| \le \theta^{1/2}\rho(y) \Longrightarrow \partial_1 a(\alpha(x'), x') \equiv 0, \quad |\alpha(x') - y_1| \le \theta^{1/4}\rho(y). \tag{A.1.9}$$

From the lemma A.1.4, we get also

$$\partial_1^2 a(\alpha(x'), x') \ge \rho(y)^2/2. \tag{A.1.10}$$

Since the function $\partial_1 a$ is $C^{2,1}$, the implicit function theorem entails that the function α is C^2; let us show in fact that α is $C^{2,1}$. Denoting by ∂_2 the x' derivative, with a and its derivatives always evaluated at $x_1 = \alpha(x')$, we obtain by differentiating the identity $\partial_1 a(\alpha(x'), x') \equiv 0$,

$$\alpha' \partial_1^2 a + \partial_1 \partial_2 a = 0, \tag{A.1.11}$$
$$\alpha'' \partial_1^2 a + {\alpha'}^2 \partial_1^3 a + 2\alpha' \partial_1^2 \partial_2 a + \partial_1 \partial_2^2 a = 0. \tag{A.1.12}$$

The identities (A.1.11-12) give for $|x' - y'| \le \theta^{1/2}\rho(y)$, using (A.1.10),

$$\left.\begin{aligned} &|\alpha(x') - y_1| \le \theta^{1/4}\rho(y),\\ &|\alpha'(x')| \le 2\rho(y)^{-2}\rho(\alpha(x'), x')^2 \le 2C_0^2 \lesssim 1,\\ &|\alpha''(x')| \le 2\rho(y)^{-2}\big(4^2C_0^4 + 4^2C_0^2 + 12\big)\rho(\alpha(x'), x') \lesssim \rho(y)^{-1}. \end{aligned}\right\} \tag{A.1.13}$$

We have also the identity, using (A.1.12),

$$\alpha''(x') = -(\partial_1^2 a(\alpha(x'), x'))^{-1}({\alpha'}^2 \partial_1^3 a(\alpha(x'), x') + 2\alpha' \partial_1^2 \partial_2 a(\alpha(x'), x') + \partial_1 \partial_2^2 a(\alpha(x'), x')), \tag{A.1.14}$$

so that the function α'' is Lipschitz continuous. Applying formally the chain rule from (A.1.12) would give the identity

$$\alpha''' \partial_1^2 a + 3\alpha''\alpha' \partial_1^3 a + 3\alpha'' \partial_1^2 \partial_2 a + {\alpha'}^3 \partial_1^4 a + 3{\alpha'}^2 \partial_1^3 \partial_2 a + 3\alpha' \partial_1^2 \partial_2^2 a + \partial_1 \partial_2^3 a = 0.$$

However the meaning of the last four terms above is not clear since the fourth derivative of a is only L^∞, so to restrict it to the hypersurface $x_1 = \alpha(x')$ does not make sense. In fact, we do not need that, but only the fact that the composition of Lipschitz continuous function gives a Lipschitz continuous functions with the obvious bound on the Lipschitz constant. We start over from (A.1.14) and we write the duality products with a smooth compactly supported test function χ, a_ϵ a regularized a,

$$\begin{aligned} \langle \alpha''', \chi\rangle = -\int \alpha'' \chi' dm &= \int \chi'({\alpha'}^2 \partial_1^3 a + 2\alpha' \partial_1^2 \partial_2 a + \partial_1 \partial_2^2 a)(\partial_1^2 a)^{-2} dm\\ &= \lim_{\epsilon \to 0} \int \chi'({\alpha'}^2 \partial_1^3 a_\epsilon + 2\alpha' \partial_1^2 \partial_2 a_\epsilon + \partial_1 \partial_2^2 a_\epsilon)(\partial_1^2 a)^{-2} dm\\ &\qquad = -\lim_{\epsilon \to 0} \int \chi\Big(({\alpha'}^2 \partial_1^3 a_\epsilon + 2\alpha' \partial_1^2 \partial_2 a_\epsilon + \partial_1 \partial_2^2 a_\epsilon)(\partial_1^2 a)^{-2}\Big)' dm. \end{aligned}$$

The computation of the derivative between the parenthesis above, with uniform bounds wih respect to ϵ gives indeed

$$|\alpha'''(x')| \lesssim \rho(y)^{-2}. \tag{A.1.15}$$

A.2. More properties of the algebra $\mathcal{A}$.

Lemma A.2.1. *Let b be a function in $\mathcal{A}$ and $T \in \mathbb{R}^{2n}, t \in \mathbb{R}$. Then the functions $\tau_T b$, b_t defined by $\tau_T b(X) = b(X - T), b_t(X) = b(tX)$ belong to $\mathcal{A}$ and*

$$\sup_{T \in \mathbb{R}^{2n}} \|\tau_T b\|_{\mathcal{A}} \le C \|b\|_{\mathcal{A}}, \qquad \|b_t\|_{\mathcal{A}} \le (1 + |t|)^{2n} C \|b\|_{\mathcal{A}}, \tag{A.2.1}$$

where C depends only on the dimension.

Proof. We check, using that $T = S + j_0$, $j_0 \in \mathbb{Z}^{2n}$, $S \in [0,1]^{2n}$,

$$\begin{aligned}\mathcal{F}(\chi_j \tau_T b)(\Xi) &= \int e^{-2i\pi X\Xi}\chi_j(X) b(X-T)dX \\ &= e^{-2i\pi T\Xi}\int e^{-2i\pi X\Xi}\chi_{j-j_0}(X+S)b(X)dX \\ &= e^{-2i\pi T\Xi}\int e^{-2i\pi X\Xi}\chi_{j-j_0}(X+S)\Big(\sum_{|k|\le R_0}\chi_{j-j_0+k}(X)b(X)\Big)dX \\ &= e^{-2i\pi T\Xi}\sum_{|k|\le R_0}\mathcal{F}\big((\tau_{-S}\chi_{j-j_0})(\chi_{j-j_0+k}b)\big)(\Xi) \\ &= e^{-2i\pi T\Xi}\sum_{|k|\le R_0}\big(\mathcal{F}(\tau_{-S}\chi_{j-j_0}) * \mathcal{F}(\chi_{j-j_0+k}b)\big)(\Xi)\end{aligned}$$

so that

$$\begin{aligned}|\mathcal{F}(\chi_j\tau_T b)(\Xi)| &\le C_0\int |\mathcal{F}(\tau_{-S}\chi_{j-j_0})(\Xi-\Xi')|\omega_b(\Xi')d\Xi' \\ &= C_0\int |\mathcal{F}(\chi_0)(\Xi-\Xi')|\omega_b(\Xi')d\Xi',\end{aligned}$$

entailing $\int \sup_{j\in\mathbb{Z}^{2n}} |\mathcal{F}(\chi_j\tau_T b)(\Xi)|d\Xi \le C_0 \|\widehat{\chi_0}\|_{L^1}\|b\|_{\mathcal{A}}$ and the first part of (A.2.1). The second part is obvious if $t = 0$ since $\mathcal{A}$ is continuously embedded in $C^0 \cap L^\infty$ (proposition 1.2.1). Assuming $t \ne 0$, we look now at

$$\begin{aligned}\mathcal{F}(\chi_j b_t)(\Xi) &= \int e^{-2i\pi X\Xi}\chi_0(X-j)b(tX)dX \\ &= \sum_{k\in\mathbb{Z}^{2n}}\int e^{-2i\pi X\Xi}\chi_0(X-j)\chi_k(tX)b(tX)dX,\end{aligned}$$

and since on the integrand, we have $|X-j| \le R_0$, $\quad |k-tX| \le R_0$ and thus $|k-tj| \le R_0 + |t|R_0$, we get

$$\begin{aligned}\mathcal{F}(\chi_j b_t)(\Xi) &= \int e^{-2i\pi X\Xi}\chi_0(X-j)b(tX)dX \\ &= \sum_{\substack{k \\ |k-tj|\le R_0(1+|t|)}}\int e^{-2i\pi X\Xi}\chi_0(X-j)b(tX)\chi_k(tX)dX \\ &= \sum_{|k-tj|\le R_0(1+|t|)}\iint e^{-2i\pi t^{-1}X\Xi}\widehat{\chi_0}(N)e^{2i\pi N(t^{-1}X-j)}(\chi_k b)(X)dXdN|t|^{-2n} \\ &= \sum_{|k-tj|\le R_0(1+|t|)}\iint e^{-2i\pi t^{-1}X\Xi}\widehat{\chi_0}(tN)e^{2i\pi N(X-tj)}(\chi_k b)(X)dXdN \\ &= \sum_{|k-tj|\le R_0(1+|t|)}\iint \widehat{\chi_0}(tN)e^{-2i\pi Ntj}e^{2i\pi X(N-t^{-1}\Xi)}(\chi_k b)(X)dXdN \\ &= \sum_{|k-tj|\le R_0(1+|t|)}\iint \widehat{\chi_0}(tN+\Xi)e^{-2i\pi Ntj}e^{-2i\pi\Xi j}e^{2i\pi XN}(\chi_k b)(X)dXdN \\ &= \sum_{|k-tj|\le R_0(1+|t|)}\int \widehat{\chi_0}(-tN+\Xi)e^{2i\pi Ntj}e^{-2i\pi\Xi j}\left(\int e^{-2i\pi XN}(\chi_k b)(X)dX\right)dN,\end{aligned}$$

so that

$$|\mathcal{F}(\chi_j b_t)(\Xi)| \le \sum_{|k-tj|\le R_0(1+|t|)} \int |\widehat{\chi_0}(-tN+\Xi)| \left|\int e^{-2i\pi XN}(\chi_k b)(X)dX\right| dN$$
$$\le C_n R_0^{2n} \int |\widehat{\chi_0}(-tN+\Xi)|\omega_b(N)dN(1+|t|)^{2n},$$

and finally the sought result $\int \sup_j |\mathcal{F}(\chi_j b_t)(\Xi)|d\Xi \le C_n R_0^{2n}(1+|t|)^{2n} \|\widehat{\chi_0}\|_{L^1} \|b\|_{\mathcal{A}}$. □

A.3. On Leibniz formulæ. Let a be a function in L^1_{loc} of some open set Ω of $\mathbb{R}^m$ and let u be a locally Lipschitz continuous function on Ω. Although a' may be a distribution of order 1 and u is not C^1, it is possible to define the product $T = a'u$ as follows (φ is a test function):

$$\langle T, \varphi\rangle = -\int a(u'\varphi + u\varphi')dx$$

so that T is a distribution of order 1 satisfying the identity $(au)' = T + au'$. As a matter of fact, we have $\langle (au)', \varphi\rangle = -\int au\varphi' dx = \langle T, \varphi\rangle + \int au'\varphi dx = \langle T + au', \varphi\rangle$. It means in particular that one can multiply the first-order distribution $\frac{d}{dx}(\ln|x|) = \text{pv}\frac{1}{x}$ by the Lipschitz continuous function $|x|$ and get $(\text{pv}\frac{1}{x})|x| = \frac{d}{dx}\Big((\ln|x|)|x|\Big) - (\ln|x|)\operatorname{sign} x = \operatorname{sign} x$ as it is easily verified. On the other hand it is not possible to multiply the first order distribution δ_0' by the Lipschitz continuous function $|x|$.

A.4. Symmetric k-tensors as sum of k-th powers. Since the symmetrized products of $T_1 \otimes \cdots \otimes T_k$ can be written as a linear combination of k-th powers, the norm of the k-linear symmetric form A given by $\|A\| = \sup_{\|T\|=1} |AT^k|$ is equivalent to the natural norm

$$\|\!|A|\!\| = \sup_{\substack{\|T_j\|=1,\\ 1\le j\le k}} |AT_1 \dots T_k|$$

and we have the inequalities $\|A\| \le \|\!|A|\!\| \le \kappa_k \|A\|$ with a constant κ_k depending only on k. The best constant constant in general is $\kappa_k = k^k/k!$. In fact, in a commutative algebra on a field with characteristic 0, using the polarization formula, the products $T_1 \dots T_k$ are linear combination of k-th powers

$$T_1 T_2 \dots T_k = \frac{1}{2^k k!} \sum_{\epsilon_j = \pm 1} \epsilon_1 \dots \epsilon_k (\epsilon_1 T_1 + \cdots + \epsilon_k T_k)^k.$$

Using the triangle inequality, we get $\|\!|A|\!\| \le \frac{1}{2^k k!} 2^k k^k \|A\|$, and thus $\kappa_k \le \frac{k^k}{k!}$. On the other hand, for $T_j \in \mathbb{R}^k$ and A defined by

$$A(T_1, \dots, T_k) = \frac{1}{k!} \sum_{\sigma \in \mathcal{S}_k} T_{\sigma(1),1} \dots T_{\sigma(k),k},$$

we have $A(e_1, \dots, e_k) = 1/k!$ so that $\|\!|A|\!\| \ge \frac{1}{k!}$ and for $\theta \in \mathbb{R}^k$ (with the norm $\sum |\theta_j|$),

$$|A\theta^k| = |\theta_1 \dots \theta_k| \le \left(\frac{\sum |\theta_j|}{k}\right)^k \Longrightarrow \|A\| \le k^{-k},$$

so that $\kappa_k \ge \frac{\|\!|A|\!\|}{\|A\|} \ge \frac{k^k}{k!}$.

A.5. From discrete sums to finite sums. At the end of the proof of the theorem 3.1.1, we have established that

$$a(x) = \sum_{1\le j\le 1+N_{m-1}} \sum_{\nu\in\mathbb{N}} b_{\nu,j}(x)^2 \varphi_\nu(x)^2 \tag{A.5.1}$$

with (φ_ν) satisfying the properties of the lemma 3.1.3 and the $b_{\nu,j}$ are $C^{1,1}$ functions such that

$$|b_{\nu,j}^{(l)}| \le c_0 \rho_\nu^{2-l}, \quad 0\le l\le 2, \quad |\big(b'_{\nu,j} b''_{\nu,j}\big)'| \le c_0 \quad |\big(b_{\nu,j} b''_{\nu,j}\big)''| \le c_0, \tag{A.5.2}$$

where c_0 is a universal constant (we keep the normalization assumption $\|a^{(4)}\|_{L^\infty(\mathbb{R}^m)} \le 1$). We want to write a as a finite sum with similar properties, using the slow variation of the metric $|dx|^2/\rho(x)^2$. We are given a positive number $r \le r'_0$, where r'_0 is defined in the lemma 3.1.3. We define a sequence (x_ν) and balls U_ν as in that lemma.

$\cdot\ \mathcal{N}_1$ = maximal subset of $\mathbb{N}$ containing 0 such that for $\nu' \ne \nu''$ both in $\mathcal{N}_1$, $U_{\nu'} \cap U_{\nu''} = \emptyset$. Let $\nu_2 = \min \mathcal{N}_1^c$.

$\cdot\ \mathcal{N}_2$ = maximal subset of $\mathcal{N}_1^c$ containing ν_2 such that for $\nu' \ne \nu''$ both in $\mathcal{N}_2$, $U_{\nu'} \cap U_{\nu''} = \emptyset$. Let $\nu_3 = \min(\mathcal{N}_1 \cup \mathcal{N}_2)^c$.

... Let $\nu_{k+1} = \min(\mathcal{N}_1 \cup \cdots \cup \mathcal{N}_k)^c$.

$\cdot\ \mathcal{N}_{k+1}$ = maximal subset of $(\mathcal{N}_1 \cup \cdots \cup \mathcal{N}_k)^c$ containing ν_{k+1} such that for $\nu' \ne \nu''$ both in $\mathcal{N}_{k+1}$, $U_{\nu'} \cap U_{\nu''} = \emptyset$. Let $\nu_{k+2} = \min(\mathcal{N}_1 \cup \cdots \cup \mathcal{N}_{k+1})^c$.

...

We observe the following.

• The sets $\mathcal{N}_j$ are two by two disjoint.

• For all j,k such that $1\le j\le k$, there exists $\nu\in\mathcal{N}_j$ so that $U_\nu \cap U_{\nu_{k+1}} \ne \emptyset$: otherwise, we could find $1\le j\le k$ so that for all $\nu\in\mathcal{N}_j$, $U_\nu\cap U_{\nu_{k+1}} = \emptyset$, so that the set $\mathcal{N}_j \cup \{\nu_{k+1}\}$ would satisfy the property that the maximal $\mathcal{N}_j$ should satisfy.

• For k large enough, we have $\mathcal{N}_1\cup\cdots\cup\mathcal{N}_k = \mathbb{N}$: otherwise ν_{k+1} is always well-defined and using the property above, we get that one can find $\mu_j \in \mathcal{N}_j, 1\le j\le k$, so that $U_{\mu_j} \cap U_{\nu_{k+1}} \ne \emptyset$. As a consequence, for $1\le j\le k$, we find $y_j \in U_{\mu_j}$ such that

$$|x_{\mu_j} - y_j| \le r\rho(x_{\mu_j}) \le C_0 r\rho(y_j), \quad |x_{\nu_{k+1}} - y_j| \le r\rho(x_{\nu_{k+1}}) \le C_0 r \rho(y_j) \le C_0^2 r\rho(x_{\nu_{k+1}})$$

and thus

$$|x_{\nu_{k+1}} - x_{\mu_j}| \le (C_0^2 r + r)\rho(x_{\nu_{k+1}}), \tag{A.5.3}$$

with distinct μ_j (they belong to two by two disjoint sets). On the other hand, we know by construction (see the lemma 1.4.9 in [H2]) that there exists a positive r_1 such that , for $\nu' \ne \nu''$,

$$|x_{\nu'} - x_{\nu''}| \ge r_1 \rho(x_{\nu'}),$$

so that, with a fixed $r_2 > 0$, the balls $\big(B(x_{\mu_j}, r_2\rho(x_{\mu_j}))\big)_{1\le j\le k}$ are two by two disjoint as well as $\big(B(x_{\mu_j}, r_3\rho(x_{\nu_{k+1}}))\big)_{1\le j\le k}$ with a fixed positive r_3. Thanks to (A.5.3), they are also all included in $B(x_{\nu_{k+1}}, r_4\rho(x_{\nu_{k+1}}))$ with a fixed positive r_4 so that $k \le r_4^m/r_3^m$ and thus k is bounded. We can thus write, with $M_m = \lambda_0^m$, since the balls $U_\nu(\supset \operatorname{supp}\varphi_\nu)$ are two by two disjoint for ν running in each $\mathcal{N}_k$,

$$a = \sum_{1\le j\le 1+N_{m-1}} \sum_{1\le k\le M_m} \Big(\sum_{\nu\in\mathcal{N}_k} b_{\nu,j}\varphi_\nu\Big)^2$$

and defining $B_{j,k} = \sum_{\nu\in\mathcal{N}_k} b_{\nu,j}\varphi_\nu$ we get

$$a = \sum_{1\le j\le 1+N_{m-1}} \sum_{1\le k\le M_m} B_{j,k}^2 \tag{A.5.4}$$

with $|B''_{j,k}| \le \sum_{\nu\in\mathcal{N}_k} c_0\psi_\nu \lesssim 1$. Moreover the identities

$$(B'_{j,k}B''_{j,k})' = \sum_{\nu\in\mathcal{N}_k} \big((b_{\nu,j}\varphi_\nu)'(b_{\nu,j}\varphi_\nu)''\big)'\psi_\nu,$$
$$(B_{j,k}B''_{j,k})'' = \sum_{\nu\in\mathcal{N}_k} \big((b_{\nu,j}\varphi_\nu)(b_{\nu,j}\varphi_\nu)''\big)''\psi_\nu$$

yield the sought estimates on the derivatives. As a final question, one may ask for some estimate of the Pythagorean number, i.e. the number of squares necessary for the decomposition. From the formula (A.5.4), we have the estimate

$$N_m \le \big(1 + N_{m-1}\big)\lambda_0^m, \quad \lambda_0 \text{ universal constant},$$

which gives $N_m \le \mu_0^{m^2}$, which is probably a very crude estimate, compared to the exponential bound known for the Artin theorem of decomposition as sum of squares of nonnegative rational fractions. As a matter of fact, a recent paper of Bony [Bo2] is providing the equality $N_1 = 2$, which is optimal in view of the Glaeser counterexample ([Gl]); however his proof is much more involved than our argument as exposed above with our set of indices $\mathcal{N}_k$.

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