SOME FACTS ABOUT THE WICK CALCULUS

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Abstract. This is a slightly expanded version of a five-hour lecture series given at Cetraro during the CIME session of June 2006 dedicated to the topics of Pseudodifferential operators, Quantization and Signal.

Contents

1. Elementary Fourier analysis via wave packets

1.1. The Fourier transform of Gaussian functions. Let u be a function in the Schwartz class of rapidly decreasing functions $\mathcal{S}(\mathbb{R}^n)$: it means that u is a C^∞ function on $\mathbb{R}^n$ such that for all multi-indices[1] α, β

$$\sup_{x\in\mathbb{R}^n} |x^\alpha \partial_x^\beta u(x)| = C_{\alpha\beta} < \infty.$$

A simple example of such a function is $e^{-|x|^2}$, ($|x|$ is the Euclidean norm of x) and more generally if A is a symmetric positive definite $n \times n$ matrix the function

$$v_A(x) = e^{-\pi\langle Ax,x\rangle} \tag{1.1.1}$$

belongs to the Schwartz class. The Fourier transform of u is defined as

$$\hat{u}(\xi) = \int_{\mathbb{R}^n} e^{-2i\pi x\cdot\xi} u(x)dx. \tag{1.1.2}$$

It is an easy matter to check that the Fourier transform sends $\mathcal{S}(\mathbb{R}^n)$ into itself[2]. Moreover, for A as above, we have

$$\widehat{v_A}(\xi) = (\det A)^{-1/2} e^{-\pi\langle A^{-1}\xi,\xi\rangle}. \tag{1.1.3}$$

In fact, diagonalizing the symmetric matrix A, it is enough to prove the one-dimensional version of (1.1.3), i.e. to check

$$\int e^{-2i\pi x\xi} e^{-\pi x^2} dx = \int e^{-\pi(x+i\xi)^2} dx e^{-\pi\xi^2} = e^{-\pi\xi^2},$$

where the second equality can be obtained by taking the ξ-derivative of $\int e^{-\pi(x+i\xi)^2}dx$. Using (1.1.3) we calculate for $u \in \mathcal{S}(\mathbb{R}^n)$ and $\epsilon > 0$, dealing with absolutely converging integrals,

$$\begin{aligned} u_\epsilon(x) &= \int e^{2i\pi x\xi} \hat{u}(\xi) e^{-\pi\epsilon^2|\xi|^2} d\xi \\ &= \iint e^{2i\pi x\xi} e^{-\pi\epsilon^2|\xi|^2} u(y) e^{-2i\pi y\xi} dy d\xi \\ &= \int u(y) e^{-\pi\epsilon^{-2}|x-y|^2} \epsilon^{-n} dy \\ &= \int \underbrace{\big(u(x+\epsilon y) - u(x)\big)}_{\text{with absolute value}\le\epsilon|y|\|u'\|_{L^\infty}} e^{-\pi|y|^2} dy + u(x). \end{aligned}$$

Taking the limit when ϵ goes to zero, we get the Fourier inversion formula

$$u(x) = \int e^{2i\pi x\xi} \hat{u}(\xi) d\xi. \tag{1.1.4}$$

So far we have just proved that the Fourier transform is an isomorphism of the Schwartz class and provided an explicit inversion formula. This was devised to refresh our memory on this topic and we want now to move forward with the definition of our wave packets.

[1] $\alpha = (\alpha_1, \dots, \alpha_n) \in \mathbb{N}^n, x^\alpha = x_1^{\alpha_1} \dots x_n^{\alpha_n},\ \beta \in \mathbb{N}^n, \partial_x^\beta = \partial_{x_1}^{\beta_1} \dots \partial_{x_n}^{\beta_n}$.

[2] Just notice that

$$\xi^\alpha \partial_\xi^\beta \hat{u}(\xi) = \int e^{-2i\pi x\xi} \partial_x^\alpha (x^\beta u)(x) dx (2i\pi)^{|\beta|-|\alpha|} (-1)^{|\beta|}.$$

1.2. Wave packets and the Poisson summation formula. We define for $x \in \mathbb{R}^n$, $(y,\eta) \in \mathbb{R}^n \times \mathbb{R}^n$

$$\varphi_{y,\eta}(x) = 2^{n/4} e^{-\pi(x-y)^2} e^{2i\pi(x-y)\cdot\eta} = 2^{n/4} e^{-\pi(x-y-i\eta)^2} e^{-\pi\eta^2} \tag{1.2.1}$$

where for $\zeta = (\zeta_1, \dots, \zeta_n) \in \mathbb{C}^n$, we set

$$\zeta^2 = \sum_{1\le j\le n} \zeta_j^2. \tag{1.2.2}$$

We note that the function $\varphi_{y,\eta}$ is in $\mathcal{S}(\mathbb{R}^n)$ and with L^2 norm 1. In fact, $\varphi_{y,\eta}$ appears as a *phase translation* of a normalized Gaussian. The following lemma introduces the wave packets transform as a Gabor wavelet.

Lemma 1.2.1. *Let u be a function in the Schwartz class $\mathcal{S}(\mathbb{R}^n)$. We define*

$$\begin{aligned} Wu(y,\eta) = \langle u, \varphi_{y,\eta}\rangle_{L^2(\mathbb{R}^n)} &= 2^{n/4} \int u(x) e^{-\pi(x-y)^2} e^{-2i\pi(x-y)\cdot\eta} dx \\ &= 2^{n/4} \int u(x) e^{-\pi(y-i\eta-x)^2} dx e^{-\pi\eta^2}. \end{aligned} \tag{1.2.3}$$

For $u \in L^2(\mathbb{R}^n)$, the function Tu defined by

$$(Tu)(y+i\eta) = e^{\pi\eta^2} Wu(y,-\eta) = 2^{n/4} \int u(x) e^{-\pi(y+i\eta-x)^2} dx$$

is an entire function. The mapping $u \mapsto Wu$ is continuous from $\mathcal{S}(\mathbb{R}^n)$ to $\mathcal{S}(\mathbb{R}^{2n})$ and isometric from $L^2(\mathbb{R}^n)$ to $L^2(\mathbb{R}^{2n})$. Moreover, we have the reconstruction formula

$$u(x) = \iint_{\mathbb{R}^n\times\mathbb{R}^n} Wu(y,\eta)\varphi_{y,\eta}(x) dy d\eta. \tag{1.2.4}$$

Proof. For u in $\mathcal{S}(\mathbb{R}^n)$, we have

$$Wu(y,\eta) = e^{2i\pi y\eta} \widehat{\Omega}^1(\eta, y)$$

where $\widehat{\Omega}^1$ is the Fourier transform with respect to the first variable of the $\mathcal{S}(\mathbb{R}^{2n})$ function $\Omega(x,y) = u(x) e^{-\pi(x-y)^2} 2^{n/4}$. Thus the function Wu belongs to $\mathcal{S}(\mathbb{R}^{2n})$. It makes sense to compute

$$2^{-n/2}\langle Wu, Wu\rangle_{L^2(\mathbb{R}^{2n})} = \lim_{\epsilon\to 0_+} \int u(x_1)\overline{u}(x_2) e^{-\pi[(x_1-y)^2+(x_2-y)^2+2i(x_1-x_2)\eta+\epsilon^2\eta^2]} dy d\eta dx_1 dx_2.$$

Now the last integral on $\mathbb{R}^{4n}$ converges absolutely and we can use the Fubini theorem. Integrating with respect to η involves the Fourier transform of a Gaussian function and we get $\epsilon^{-n}e^{-\pi\epsilon^{-2}(x_1-x_2)^2}$. Since

$$2(x_1-y)^2+2(x_2-y)^2=(x_1+x_2-2y)^2+(x_1-x_2)^2,$$

integrating with respect to y yields a factor $2^{-n/2}$. We are left with

$$\langle Wu, Wu\rangle_{L^2(\mathbb{R}^{2n})} = \lim_{\epsilon\to 0_+}\int u(x_1)\quad \overline{u}(x_2)e^{-\pi(x_1-x_2)^2/2}\epsilon^{-n}e^{-\pi\epsilon^{-2}(x_1-x_2)^2}dx_1dx_2.$$

Changing the variables, the integral is

$$\lim_{\epsilon\to 0_+}\int u(s+\epsilon t/2)\ \overline{u}(s-\epsilon t/2)e^{-\pi\epsilon^2t^2/2}e^{-\pi t^2}dtds=\|u\|^2_{L^2(\mathbb{R}^n)}$$

by Lebesgue's dominated convergence theorem: the triangle inequality and the estimate $|u(x)|\le C(1+|x|)^{-n-1}$ imply, with $v=u/C$,

$$\begin{aligned}|v(s+\epsilon t/2)\ \overline{v}(s-\epsilon t/2)| &\le (1+|s+\epsilon t/2|)^{-n-1}(1+|s+\epsilon t/2|)^{-n-1}\\ &\le (1+|s+\epsilon t/2|+|s-\epsilon t/2|)^{-n-1}\\ &\le (1+2|s|)^{-n-1}.\end{aligned}$$

Eventually, this proves that

$$\|Wu\|^2_{L^2(\mathbb{R}^{2n})}=\|u\|^2_{L^2(\mathbb{R}^n)} \tag{1.2.5}$$

i.e.

$$W: L^2(\mathbb{R}^n)\to L^2(\mathbb{R}^{2n})\qquad\text{with}\qquad W^*W=\mathrm{id}_{L^2(\mathbb{R}^n)}.$$

Noticing first that $\iint Wu(y,\eta)\varphi_{y,\eta}dyd\eta$ belongs to $L^2(\mathbb{R}^n)$ (with a norm smaller than $\|Wu\|_{L^1(\mathbb{R}^{2n})}$) and applying Fubini's theorem, we get from the polarization of (1.2.5) for $u,v\in\mathcal{S}(\mathbb{R}^n)$,

$$\begin{aligned}\langle u,v\rangle_{L^2(\mathbb{R}^n)} &= \langle Wu, Wv\rangle_{L^2(\mathbb{R}^{2n})}\\ &=\iint Wu(y,\eta)\langle\varphi_{y,\eta},v\rangle_{L^2(\mathbb{R}^n)}dyd\eta\\ &=\left\langle\iint Wu(y,\eta)\varphi_{y,\eta}dyd\eta, v\right\rangle_{L^2(\mathbb{R}^n)}\end{aligned}$$

yielding the result of the lemma $u=\iint Wu(y,\eta)\varphi_{y,\eta}dyd\eta$.

The following lemma is in fact the Poisson summation formula for Gaussian functions in one dimension.

Lemma 1.2.2. *For all complex numbers z, the following series are absolutely converging and*

$$\sum_{m\in\mathbb{Z}} e^{-\pi(z+m)^2} = \sum_{m\in\mathbb{Z}} e^{-\pi m^2} e^{2i\pi mz}. \tag{1.2.6}$$

Proof. We set $\omega(z) = \sum_{m\in\mathbb{Z}} e^{-\pi(z+m)^2}$. The function ω is entire and 1-periodic since for all $m \in \mathbb{Z}$, $z \mapsto e^{-\pi(z+m)^2}$ is entire and for $R > 0$

$$\sup_{|z|\le R} |e^{-\pi(z+m)^2}| \le \sup_{|z|\le R} |e^{-\pi z^2}| e^{-\pi m^2} e^{2\pi|m|R} \in l^1(\mathbb{Z}).$$

Consequently, for $z \in \mathbb{R}$, we obtain, expanding ω in Fourier series[3],

$$\omega(z) = \sum_{k\in\mathbb{Z}} e^{2i\pi kz} \int_0^1 \omega(x) e^{-2i\pi kx} dx.$$

We also check, using Fubini's theorem on $L^1(0,1) \times l^1(\mathbb{Z})$

$$\begin{aligned}\int_0^1 \omega(x) e^{-2i\pi kx} dx &= \sum_{m\in\mathbb{Z}} \int_0^1 e^{-\pi(x+m)^2} e^{-2i\pi kx} dx \\ &= \sum_{m\in\mathbb{Z}} \int_m^{m+1} e^{-\pi t^2} e^{-2i\pi kt} dt \\ &= \int_{\mathbb{R}} e^{-\pi t^2} e^{-2i\pi kt} = e^{-\pi k^2}.\end{aligned}$$

So (1.2.6) is proved for real z and since both sides are entire functions, we conclude by analytic continuation. □

It is now straightforward to get the n-th dimensional version of lemma 1.2.2: for all $z \in \mathbb{C}^n$, using the notation (1.2.2), we have

$$\sum_{m\in\mathbb{Z}^n} e^{-\pi(z+m)^2} = \sum_{m\in\mathbb{Z}^n} e^{-\pi m^2} e^{2i\pi m\cdot z}. \tag{1.2.7}$$

[3]Note that we use this expansion only for a C^∞ 1-periodic function. The proof is simple and requires only to compute $1 + 2\,\mathrm{Re} \sum_{1\le k\le N} e^{2i\pi kx} = \frac{\sin \pi(2N+1)x}{\sin \pi x}$. Then one has to show that for a smooth 1-periodic function ω such that $\omega(0) = 0$,

$$\lim_{\lambda\to+\infty} \int_0^1 \frac{\sin \lambda x}{\sin \pi x} \omega(x) dx = 0,$$

which is obvious since for a smooth ν (here we take $\nu(x) = \omega(x)/\sin \pi x$), $|\int_0^1 \nu(x) \sin \lambda x dx| = O(\lambda^{-1})$ by integration by parts.

Theorem 1.2.3. The Poisson summation formula. *Let n be a positive integer and u be a function in $\mathcal{S}(\mathbb{R}^n)$. Then*

$$\sum_{k\in\mathbb{Z}^n} u(k) = \sum_{k\in\mathbb{Z}^n} \hat{u}(k), \tag{1.2.8}$$

where $\hat{u}$ stands for the Fourier transform (1.1.2).

Proof. We write, according to (1.2.4) and to Fubini's theorem

$$\begin{aligned} \sum_{k\in\mathbb{Z}^n} u(k) &= \sum_{k\in\mathbb{Z}^n} \iint Wu(y,\eta)\varphi_{y,\eta}(k)dyd\eta \\ &= \iint Wu(y,\eta) \sum_{k\in\mathbb{Z}^n} \varphi_{y,\eta}(k)dyd\eta. \end{aligned} \tag{1.2.9}$$

Now, (1.2.7), (1.2.1) and (1.1.3) give

$$\sum_{k\in\mathbb{Z}^n} \varphi_{y,\eta}(k) = \sum_{k\in\mathbb{Z}^n} \widehat{\varphi}_{y,\eta}(k),$$

so that (1.2.9), (1.2.4) and Fubini's theorem imply (1.2.8). □

It is a simple matter to introduce at this point the dual space of the Fréchet $\mathcal{S}(\mathbb{R}^n)$, that is the space $\mathcal{S}'(\mathbb{R}^n)$ of tempered distributions (the continuous linear forms on $\mathcal{S}(\mathbb{R}^n)$). We can define the Fourier transform on $\mathcal{S}'(\mathbb{R}^n)$ by duality[4]:

$$\langle \widehat{T}, \varphi\rangle_{\mathcal{S}'(\mathbb{R}^n),\mathcal{S}(\mathbb{R}^n)} = \langle T, \widehat{\varphi}\rangle_{\mathcal{S}'(\mathbb{R}^n),\mathcal{S}(\mathbb{R}^n)}, \tag{1.2.10}$$

so that the inversion formula (1.1.4) still holds for $T \in \mathcal{S}'(\mathbb{R}^n)$ and reads

$$T = \check{\hat{\hat{T}}}, \qquad \text{with} \qquad \langle \check{T}, \varphi\rangle = \langle T, \check{\varphi}\rangle, \qquad \check{\varphi}(x) = \varphi(-x).$$

Using duality, it is a matter of routine left to the reader to give a version of lemma 1.2.1 for tempered distributions. Now theorem 1.2.3 can be given a more compact version saying that the tempered distribution $D_0 = \sum_{k\in\mathbb{Z}^n} \delta_k$ is such that $\widehat{D_0} = D_0$.

We shall need as well a parametric version of wave packets, and we state here a lemma analogous to lemma 1.2.1, whose proof is left to the reader.

1.3. Toeplitz operators. We define for $x \in \mathbb{R}^n$, $(\lambda, y, \eta) \in \mathbb{R}_+^* \times \mathbb{R}^n \times \mathbb{R}^n$,

$$\varphi_{y,\eta}^{\lambda}(x) = (2\lambda)^{n/4} e^{-\pi\lambda(x-y)^2} e^{2i\pi(x-y)\cdot\eta} = (2\lambda)^{n/4} e^{-\pi\lambda(x-y-i\lambda^{-1}\eta)^2} e^{-\pi\lambda^{-1}\eta^2}. \tag{1.3.1}$$

We note that the function $\varphi_{y,\eta}^{\lambda}$ is in $\mathcal{S}(\mathbb{R}^n)$ and with L^2 norm 1.

[4] In the formula below, we deal with real duality, so that, if T, φ are in $L^2(\mathbb{R}^n)$, $\langle T, \overline{\varphi}\rangle_{\mathcal{S}'(\mathbb{R}^n),\mathcal{S}(\mathbb{R}^n)} = \langle T, \varphi\rangle_{L^2(\mathbb{R}^n)}$.

Lemma 1.3.1. *Let u be a function in the Schwartz class $\mathcal{S}(\mathbb{R}^n)$. We define, for $(\lambda, y, \eta) \in \mathbb{R}_+^* \times \mathbb{R}^n \times \mathbb{R}^n$,*

$$W_\lambda u(y,\eta) = \langle u, \varphi_{y,\eta}^\lambda \rangle_{L^2(\mathbb{R}^n)} = (2\lambda)^{n/4} \int u(x) e^{-\lambda\pi(x-y)^2} e^{-2i\pi(x-y)\cdot\eta} dx$$

$$= (2\lambda)^{n/4} \int u(x) e^{-\pi\lambda(y-i\lambda^{-1}\eta-x)^2} dx e^{-\pi\lambda^{-1}\eta^2}. \tag{1.3.2}$$

For $u \in L^2(\mathbb{R}^n)$, the function $T_\lambda u$ defined by

$$(T_\lambda u)(y+i\eta) = \lambda^{-n/4} e^{\pi\lambda\eta^2} W_\lambda u(y, -\lambda\eta) = 2^{n/4} \int u(x) e^{-\pi\lambda(y+i\eta-x)^2} dx \tag{1.3.3}$$

is an entire function. The mapping $u \mapsto W_\lambda u$ is continuous from $\mathcal{S}(\mathbb{R}^n)$ to $\mathcal{S}(\mathbb{R}^{2n})$ and isometric from $L^2(\mathbb{R}^n)$ to $L^2(\mathbb{R}^{2n})$. Moreover, we have the reconstruction formula for each positive λ,

$$u(x) = \iint_{\mathbb{R}^n \times \mathbb{R}^n} W_\lambda u(y,\eta) \varphi_{y,\eta}^\lambda(x) dy d\eta. \tag{1.3.4}$$

We shall see in the sequel that the actual rôle of the Gaussian functions is in fact quite limited, except for the very explicit inversion formulas, essentially due to (1.1.3).

2. On the Weyl calculus of pseudodifferential operators

2.1. A few classical facts. Let $a(x,\xi)$ be a classical Hamiltonian defined on $\mathbb{R}^n \times \mathbb{R}^n$. The Weyl quantization rule associates to this function the operator a^w defined on functions $u(x)$ as

$$(a^w u)(x) = \iint e^{2i\pi(x-y)\cdot\xi} \, a(\frac{x+y}{2}, \xi) \, u(y) dy d\xi. \tag{2.1.1}$$

For instance we have $(x \cdot \xi)^w = (x \cdot D_x + D_x \cdot x)/2$, with $D_x = \frac{1}{2i\pi} \frac{\partial}{\partial x}$ whereas the *ordinary* quantization rule would map the Hamiltonian $x \cdot \xi$ to the operator $x \cdot D_x$. A nice feature of the Weyl quantization rule, introduced in 1928 by Hermann Weyl in [Wy], is the fact that real Hamiltonians get quantized by (formally) self-adjoint operators. Let us recall that the classical quantization of the Hamiltonian $a(x,\xi)$ is given by the operator $\mathrm{Op}(a)$ acting on functions $u(x)$ by

$$(\mathrm{Op}(a)u)(x) = \int e^{2i\pi x\cdot\xi} \, a(x,\xi) \, \widehat{u}(\xi) d\xi. \tag{2.1.2}$$

In fact, introducing the following one-parameter group $J^t = \exp 2i\pi t D_x \cdot D_\xi$, given by the integral formula

$$(J^t a)(x,\xi) = |t|^{-n} \iint e^{-2i\pi t^{-1} y\cdot\eta} a(x+y, \xi+\eta) dy d\eta,$$

we see that

$$\big(\mathrm{Op}(J^t a)u\big)(x)=\iint e^{2i\pi(x-y)\cdot\xi}\ a\big((1-t)x+ty,\xi\big)u(y)dyd\xi.$$

In particular one gets $a^w=\mathrm{Op}(J^{1/2}a)$. Moreover since $\big(\mathrm{Op}(a)\big)^*=\mathrm{Op}(J\bar a)$ we obtain

$$(a^w)^*=\mathrm{Op}(J(\overline{J^{1/2}a}))=\mathrm{Op}(J^{1/2}\bar a)=(\bar a)^w,$$

yielding formal self-adjointness for real a.

Remark 2.1.1. Many other formulas of quantization yielding formal selfadjointness for real Hamiltonians have been used, e.g. the Feynman quantization $a\mapsto a^F$ defined by

$$(a^F u)(x)=\iint \frac{1}{2}\big(a(x,\xi)+a(y,\xi)\big)e^{2i\pi(x-y)\cdot\xi}u(y)dyd\xi.$$

Using the previous notations we see that $2a^F=\mathrm{Op}(a)+\mathrm{Op}(Ja)$, so that

$$\big(2a^F\big)^*=\mathrm{Op}(J\bar a)+\mathrm{Op}(J\ \overline{Ja})=\mathrm{Op}(J\bar a)+\mathrm{Op}(\bar a)=2\bar a^F$$

and for a real-valued, $\big(a^F\big)^*=a^F$. However, we shall see in section 2.2 that the important property of symplectic invariance is true for the Weyl quantization and fails for the Feynman and the ordinary quantizations. Since it turns out that this symplectic invariance is actually a very important property, we shall stick with the Weyl quantization as our quantization of reference.

Formula (2.1.1) can be written as

$$(a^w u,v)=\iint a(x,\xi)\mathcal{H}(u,v)(x,\xi)dxd\xi, \tag{2.1.3}$$

where the *Wigner function* $\mathcal{H}$ is defined as

$$\mathcal{H}(u,v)(x,\xi)=\int u(x+\frac{y}{2})\bar v(x-\frac{y}{2})e^{-2i\pi y\cdot\xi}dy. \tag{2.1.4}$$

The mapping $(u,v)\mapsto\mathcal{H}(u,v)$ is sesquilinear continuous from $\mathcal{S}(\mathbb{R}^n)\times\mathcal{S}(\mathbb{R}^n)$ to $\mathcal{S}(\mathbb{R}^{2n})$ so that a^w makes sense for $a\in\mathcal{S}'(\mathbb{R}^{2n})$ (here $u,v\in\mathcal{S}(\mathbb{R}^n)$ and $\mathcal{S}^*$ stands for the antidual):

$$\langle a^w u,v\rangle_{\mathcal{S}^*(\mathbb{R}^n),\mathcal{S}(\mathbb{R}^n)}=\langle a,\mathcal{H}(u,v)\rangle_{\mathcal{S}'(\mathbb{R}^{2n}),\mathcal{S}(\mathbb{R}^{2n})}.$$

The Wigner function also satisfies, since $\mathcal{H}(u,v)$ is the partial Fourier transform of the function $(x,y)\mapsto u(x+y/2)\bar v(x-y/2)$,

$$\begin{gathered}\|\mathcal{H}(u,v)\|_{L^2(\mathbb{R}^{2n})}=\|u\|_{L^2(\mathbb{R}^n)}\|v\|_{L^2(\mathbb{R}^n)},\\ \mathcal{H}(u,v)(x,\xi)=2^n\langle\sigma_{x,\xi}u,v\rangle_{L^2(\mathbb{R}^n)},\\ \text{with}\quad(\sigma_{x,\xi}u)(y)=u(2x-y)\exp-4i\pi(x-y)\cdot\xi.\end{gathered} \tag{2.1.5}$$

and the phase symmetries σ_X are unitary and selfadjoint operators on $L^2(\mathbb{R}^n)$. We have also ([Un], [Wy]),

$$a^w = \int_{\mathbb{R}^{2n}} a(X) 2^n \sigma_X dX = \int_{\mathbb{R}^{2n}} \widehat{a}(\Xi) \exp(2i\pi \Xi \cdot M) d\Xi, \tag{2.1.6}$$

where $\Xi \cdot M = \hat{x} \cdot x + \hat{\xi} \cdot D_x$ (here $\Xi = (\hat{x}, \hat{\xi})$). These formulas give in particular

$$\|a^w\|_{\mathcal{L}(L^2)} \leq \min(2^n \|a\|_{L^1(\mathbb{R}^{2n})}, \|\widehat{a}\|_{L^1(\mathbb{R}^{2n})}), \tag{2.1.7}$$

where $\mathcal{L}(L^2)$ stands for the space of bounded linear maps from $L^2(\mathbb{R}^n)$ into itself.

2.2. Symplectic invariance. As shown below, the symplectic invariance of the Weyl quantization is actually its most important property. Let us consider a finite dimensional real vector space E (the configuration space $\mathbb{R}^n_x$) and its dual space E^* (the momentum space $\mathbb{R}^n_\xi$). The phase space is defined as $\Phi = E \oplus E^*$; its running point will be denoted in general by a capital letter $(X = (x, \xi), Y = (y, \eta))$. The symplectic form on Φ is given by

$$[(x, \xi), (y, \eta)] = \langle \xi, y \rangle_{E^*, E} - \langle \eta, x \rangle_{E^*, E}, \tag{2.2.1}$$

where $\langle \cdot, \cdot \rangle_{E^*, E}$ stands for the bracket of duality. The symplectic group is the subgroup of the linear group of Φ preserving (2.2.1). With

$$\sigma = \begin{pmatrix} 0 & \mathrm{Id}(E^*) \\ -\mathrm{Id}(E) & 0 \end{pmatrix},$$

we have for $X, Y \in \Phi$, $[X, Y] = \langle \sigma X, Y \rangle_{\Phi^*, \Phi}$, so that the equation of the symplectic group is $A^* \sigma A = \sigma$. One can describe a set of generators for the symplectic group $Sp(n)$, identifying Φ with $\mathbb{R}^n_x \times \mathbb{R}^n_\xi$: the mappings

(i) $(x, \xi) \mapsto (Tx, {}^t T^{-1} \xi)$, where T is an automorphism of E,

(ii) $(x_k, \xi_k) \mapsto (\xi_k, -x_k)$, and the other coordinates fixed,

(iii) $(x, \xi) \mapsto (x, \xi + Sx)$, where S is symmetric from E to E^*.

We then describe the metaplectic group, introduced by André Weil [Wi]. The metaplectic group $Mp(n)$ is the subgroup of the group of unitary transformations of $L^2(\mathbb{R}^n)$ generated by

(j) $(M_T u)(x) = |\det T|^{-1/2} u(T^{-1} x)$, where T is an automorphism of E,

(jj) Partial Fourier transformation, with respect to x_k for $k = 1, \ldots, n$,

(jjj) Multiplication by $\exp(i\pi \langle Sx, x \rangle)$, where S is symmetric from E to E^*.

There exists a two-fold covering (the π_1 of both $Mp(n)$ and $Sp(n)$ is $\mathbb{Z}$)

$$\pi : Mp(n) \to Sp(n)$$

such that, if $\chi = \pi(M)$ and u, v are in $L^2(\mathbb{R}^n)$, $\mathcal{H}(u, v)$ is their Wigner function,

$$\mathcal{H}(Mu, Mv) = \mathcal{H}(u, v) \circ \chi^{-1}.$$

This is Segal formula [Se] which could be rephrased as follows. Let $a \in \mathcal{S}'(\mathbb{R}^{2n})$ and $\chi \in Sp(n)$. There exists M in the fiber of χ such that

$$(a \circ \chi)^w = M^* a^w M. \tag{2.2.2}$$

In particular, the images by π of the transformations (j), (jj), (jjj) are respectively (i), (ii), (iii). Moreover, if χ is the phase translation, $\chi(x, \xi) = (x + x_0, \xi + \xi_0)$, (2.2.2) is fulfilled with $M = \tau_{x_0,\xi_0}$, the phase translation given by

$$(\tau_{x_0,\xi_0} u)(y) = u(y - x_0)\ e^{2i\pi\langle y - \frac{x_0}{2}, \xi_0\rangle}.$$

If χ is the symmetry with respect to (x_0, ξ_0), M in (2.2.2) is, up to a unit factor, the phase symmetry σ_{x_0,ξ_0} defined above.

Remark 2.2.1. Going back to the remark 2.1.1 on the Feynman quantization, let us prove that this quantization is *not* invariant by the symplectic group: we assume that $n = 1$ and consider the symplectic mapping $\chi(x, \xi) = (x, \xi + Sx)$ where S is a non-zero real number. We shall prove now that one can find some $a \in \mathcal{S}(\mathbb{R}^{2n})$ such that

$$(a \circ \chi)^F \neq M^* a^F M,$$

where M is the unitary transformation of $L^2(\mathbb{R})$ given by $(Mu)(x) = e^{i\pi S x^2} u(x)$. We compute

$$\begin{aligned}
2\langle (a \circ \chi)^F u, v\rangle &= \int e^{2i\pi(x-y)\xi}\big(a(x, \xi + Sx) + a(y, \xi + Sy)\big)u(y)\overline{v(x)}dydxd\xi \\
&= \int e^{2i\pi(x-y)(\xi - Sx)} a(x, \xi)u(y)\overline{v(x)}dydxd\xi + \int e^{2i\pi(x-y)(\xi - Sy)} a(y, \xi)u(y)\overline{v(x)}dydxd\xi \\
&= \int e^{2i\pi(x-y)\xi}\big(a(x, \xi)e^{-i\pi S(x-y)^2} + a(y, \xi)e^{i\pi S(x-y)^2}\big)(Mu)(y)\overline{(Mv)(x)}dydxd\xi
\end{aligned}$$

so that $(a \circ \chi)^F = M^* K M$ where the kernel k of the operator K is given by

$$2k(x, y) = \widehat{a}^2(x, y - x)e^{-i\pi S(x-y)^2} + \widehat{a}^2(y, y - x)e^{i\pi S(x-y)^2}.$$

On the other hand the kernel l of the operator a^F is given by

$$2l(x, y) = \widehat{a}^2(x, y - x) + \widehat{a}^2(y, y - x).$$

Checking the case $S = 1, a(x, \xi) = e^{-\pi(x^2+\xi^2)}$, we see that

$$2k(1, 0) = -e^{-2\pi} - e^{-\pi}, \quad 2l(1, 0) = e^{-2\pi} + e^{-\pi},$$

proving that $K \neq a^F$ and the sought result.

2.3. Composition formulas. We have the following composition formula $a^w b^w = (a\sharp b)^w$ with

$$(a\sharp b)(X) = 2^{2n} \iint e^{-4i\pi[X-Y,X-Z]} a(Y)b(Z)dYdZ, \tag{2.3.1}$$

with an integral on $\mathbb{R}^{2n} \times \mathbb{R}^{2n}$. We can compare this with the ordinary composition formula,

$$\mathrm{Op}(a)\mathrm{Op}(b) = \mathrm{Op}(a \circ b)$$

(cf.(2.1.2)) with

$$(a \circ b)(x, \xi) = \iint e^{-2i\pi y\cdot\eta} a(x, \xi + \eta)b(y + x, \xi)dyd\eta, \tag{2.3.2}$$

with an integral on $\mathbb{R}^n \times \mathbb{R}^n$. It is convenient to give an asymptotic version of these compositions formulae, e.g. in the semi-classical[5] case. Let m be a real number. A smooth function $a(x, \xi, \lambda)$ defined on $\mathbb{R}^n_x \times \mathbb{R}^n_\xi \times [1, +\infty)$ is in the symbol class S^m_{scl} if for any multi-indices (α, β), we have

$$\gamma_{\alpha\beta}(a) = \sup_{(x,\xi)\in\mathbb{R}^{2n},\lambda\ge1} |D^\alpha_x D^\beta_\xi a(x, \xi, \lambda)|\lambda^{-m+|\beta|} < \infty. \tag{2.3.3}$$

Then one has for $a \in S^{m_1}_{\mathrm{scl}}$ and $b \in S^{m_2}_{\mathrm{scl}}$, the expansion

$$(a\sharp b)(x, \xi) = \sum_{0\le k<N} 2^{-k} \sum_{|\alpha|+|\beta|=k} \frac{(-1)^{|\beta|}}{\alpha!\beta!} D^\alpha_\xi \partial^\beta_x a \ D^\beta_\xi \partial^\alpha_x b + r_N(a, b), \tag{2.3.4}$$

with $r_N(a, b) \in S^{m_1+m_2-N}_{\mathrm{scl}}$. The beginning of this expansion is thus $ab + \frac{1}{2\iota}\{a, b\}$, where

$$\{a, b\} = \sum_{1\le j\le n} \partial_{\xi_j} a \ \partial_{x_j} b - \partial_{x_j} a \ \partial_{\xi_j} b$$

is the Poisson bracket and $\iota = 2\pi i$. The sums inside (2.3.4) with k even are symmetric in a, b and skew-symmetric for k odd. This can be compared to the classical expansion formula

$$(a \circ b)(x, \xi) = \sum_{|\alpha|<N} \frac{1}{\alpha!} D^\alpha_\xi a \ \partial^\alpha_x b + t_N(a, b), \tag{2.3.5}$$

with $t_N(a, b) \in S^{m_1+m_2-N}_{\mathrm{scl}}$.

Theorem 2.3.1. *Let m be a real number and $a(x, \xi, \lambda)$ be in S^m_{scl}. Then the operator $a^w\lambda^{-m}$ is bounded on $L^2(\mathbb{R}^n)$ with a norm bounded above by a semi-norm* (2.3.3) *of a.*

This theorem is a consequence of the much more general

[5]We use a *large* parameter λ instead of a small Planck constant h. Writing $\lambda = 1/h$ will give back the more familiar picture.

Lemma 2.3.2. *Let $b(x,\xi)$ be a function defined on $\mathbb{R}^{2n}$, bounded as well as all its derivatives. Then the operators a^w, $Op(a)$, $Op(J^t a)$ are bounded on $L^2(\mathbb{R}^n)$.*

Proof. Let us check the classical quantization with $u,v\in\mathcal{S}(\mathbb{R}^n)$, assuming that $b\in C_c^\infty(\mathbb{R}^{2n})$

$$\langle b(x,D)u,v\rangle=\int e^{2i\pi(x-y)\xi}b(x,\xi)u(y)\overline{\hat v(\eta)}e^{-2i\pi x\eta}d\eta dydxd\xi.$$

Integrating by parts with respect to x gives with a polynomial P

$$P(D_x)\big(e^{2i\pi x(\xi-\eta)}\big)=e^{2i\pi x(\xi-\eta)}P(\xi-\eta),$$

so that with $P_k(\xi)=(1+|\xi|^2)^{k/2}$ for $k\in 2\mathbb{N}$, we get

$$\langle b(x,D)u,v\rangle=\int e^{2i\pi(x-y)\xi}P_k(\xi-\eta)^{-1}(P_k(D_x)b)(x,\xi)u(y)\overline{\hat v(\eta)}e^{-2i\pi x\eta}d\eta dydxd\xi.$$

Now we integrate by parts with respect to ξ so that $\langle b(x,D)u,v\rangle=$

$$\int e^{2i\pi(x-y)\xi}P_k(x-y)^{-1}P_k(D_\xi)\Big(P_k(\xi-\eta)^{-1}(P_k(D_x)b)\Big)(x,\xi)u(y)\overline{\hat v(\eta)}e^{-2i\pi x\eta}d\eta dydxd\xi.$$

As a result we obtain that $\langle b(x,D)u,v\rangle=$

$$\sum_l\int e^{2i\pi(x-y)\xi}P_k(x-y)^{-1}\varphi_l(\xi-\eta)b_{kl}(x,\xi)u(y)\overline{\hat v(\eta)}e^{-2i\pi x\eta}d\eta dydxd\xi,$$

where the sum is finite, $|\varphi_l(\zeta)|\le(1+|\zeta|)^{-k}$, $b_{kl}=\partial_x^\alpha\partial_\xi^\beta b$, with $|\alpha|\le k,|\beta|\le k$. We get that $\langle b(x,D)u,v\rangle$ is a finite sum of terms of type

(2.3.6)
$$\iint \underbrace{e^{2i\pi x\xi}b_{kl}(x,\xi)}_{\text{bounded}}\Big(\int e^{-2i\pi y\xi}P_k(x-y)^{-1}u(y)dy\Big)\Big(\int\varphi_l(\xi-\eta)\overline{\hat v(\eta)}e^{-2i\pi x\eta}d\eta\Big)dxd\xi.$$

Assuming that $k>n/2$, we get that P_k and φ_l are in $L^2(\mathbb{R}^n)$. It is then enough to check that

$$\Big\|\int e^{-2i\pi y\xi}P_k(x-y)^{-1}u(y)dy\Big\|_{L^2(\mathbb{R}^{2n}_{x,\xi})}\le C\,\|u\|_{L^2(\mathbb{R}^n)}\,. \tag{2.3.7}$$

In fact, using (2.3.7) we will get from the Cauchy-Schwarz inequality in $L^2(\mathbb{R}^{2n})$ that (2.3.6) is bounded above by $C'\|u\|_{L^2(\mathbb{R}^n)}\|v\|_{L^2(\mathbb{R}^n)}$, which gives the result of the lemma. Finally, we have to verify (2.3.7), which is indeed obvious since the integral inside the norm on the lhs of (2.3.7) is the partial Fourier transform of the $L^2(\mathbb{R}^{2n}_{x,y})$ function $P_k(x-y)u(y)$ whose L^2 norm is $\|P_k^{-1}\|_{L^2(\mathbb{R}^n)}\|u\|_{L^2(\mathbb{R}^n)}$. □

Remark 2.3.3. The reader may have noticed that this quite original method of proof, due to I.L.Hwang [Hw], is also giving the sharp number of derivatives, as proven in [CM] and in particular the result

$$\|\mathrm{Op}(a)\|_{\mathcal{L}(L^2(\mathbb{R}^n))}\le C(n)\sup_{\substack{(x,\xi)\in\mathbb{R}^{2n}\\|\alpha|\le[n/2]+1,|\beta|\le[n/2]+1}}|\partial_x^\alpha\partial_\xi^\beta a(x,\xi)|$$

where $[x]$ stands for the largest integer $\le x$ and $C(n)$ depends only on the dimension n.

3. DEFINITION AND FIRST PROPERTIES OF THE WICK QUANTIZATION

3.1. Definitions. Let us consider a symplectic vector space Φ, i.e. a finite dimensional real vector space equipped with a nondegenerate alternate bilinear form σ. The form σ can be identified to an isomorphism

$$\sigma : \Phi \to \Phi^*, \quad \sigma^* = -\sigma, \quad \text{the form is } \langle \sigma X, Y\rangle_{\Phi^*,\Phi}.$$

Then the dimension of Φ is even: take e_1 a nonzero vector in Φ and define $\epsilon_1 = \sigma^{-1}e_1^*$ where $e_1^* \in \Phi^*$ is such that $\langle e_1^*, e_1\rangle_{\Phi^*,\Phi} = 1$. We have thus

$$\langle \sigma\epsilon_1, e_1\rangle_{\Phi^*,\Phi} = \langle e_1^*, e_1\rangle_{\Phi^*,\Phi} = 1.$$

Let us now consider the vector space $\mathcal{V}(\epsilon_1, e_1)$: since the form is alternate, this is a plane (ϵ_1 cannot be proportional to e_1 and satisfy $\langle \sigma\epsilon_1, e_1\rangle = 1$) and its symplectic orthogonal $\Psi = \mathcal{V}(\epsilon_1, e_1)^\sigma$ has dimension $\dim\Phi - 2$ (σ is nondegenerate); now we can restrict the form σ to Ψ. It is of course bilinear alternate; let us check that it is nondegenerate. Assuming $\sigma(X, Y) = 0$ for some $X \in \Psi$ and all $Y \in \Psi$, we get also since $X \in \mathcal{V}(\epsilon_1, e_1)^\sigma$

$$\sigma(X, e_1) = \sigma(X, \epsilon_1) = 0$$

so that X is σ-orthogonal to Φ and thus is zero. Using an induction on the dimension, we can indeed find a *symplectic basis* that is a basis $\epsilon_1, e_1, \dots, \epsilon_n, e_n$, such that

$$\langle \sigma\epsilon_j, e_k\rangle = \delta_{j,k}, \quad \langle \sigma\epsilon_j, \epsilon_k\rangle = \langle \sigma e_j, e_k\rangle = 0.$$

Writing $X = \sum_{1\le j\le n} \xi_j\vec{\epsilon_j} + x_j\vec{e_j},\ Y = \sum_{1\le j\le n} \eta_j\vec{\epsilon_j} + y_j\vec{e_j}$ we get back to the familiar

$$[X, Y] = \xi\cdot y - \eta\cdot x, \quad \text{or } \sigma = \sum_j d\xi_j \wedge dx_j.$$

Let us now assume that our symplectic vector space Φ is equipped with a positive definite quadratic form Q. The form σ can be identified to a skew-symmetric isomorphism A of the Q-Euclidean Φ, via the identity $Q(AX, Y) = \langle \sigma X, Y\rangle_{\Phi^*,\Phi}$, and this formula implies that

$$Q(X, AY) = Q(AY, X) = \langle \sigma Y, X\rangle_{\Phi^*,\Phi} = -\langle \sigma X, Y\rangle_{\Phi^*,\Phi} = -Q(AX, Y)$$

so that $A^* = -A$ (duality induced by Q). As a consequence the spectrum of A is purely imaginary and there exist $X_1, X_2 \in \Phi, \lambda \in \mathbb{R}^*$, such that $(X_1, X_2) \ne (0, 0)$ and

$$A(X_1 + iX_2) = i\lambda(X_1 + iX_2), \quad \text{i.e.} \quad AX_1 = -\lambda X_2, \quad AX_2 = \lambda X_1.$$

This implies that $\lambda Q(X_1, X_2) = Q(AX_2, X_2) = 0 = Q(X_1, X_2)$, so the vectors X_1, X_2 are Q-orthogonal and independent ($X_2 = \alpha X_1$ would imply $\alpha AX_1 = \lambda X_1$ and $\lambda Q(X_1, X_1) = 0$, i.e. $X_1 = 0 = X_2$). Moreover the plane $\mathcal{V}(X_1, X_2)$ is invariant by A and its Q-orthogonal (the X such that $Q(X, X_1) = Q(X, X_2) = 0$) coincides with its symplectic orthogonal (the X such that $Q(AX, X_1) = Q(AX, X_2) = 0$), so that $\mathcal{V}(X_1, X_2)^{\perp_Q}$ is also invariant by A. Using an induction on the dimension, we can find a symplectic basis $\epsilon_1, e_1, \dots, \epsilon_n, e_n$, which is also orthogonal for Q, whose expression will be

$$\sum_{1\le j\le n} \lambda_j(\xi_j^2 + x_j^2), \quad \lambda_j > 0. \tag{3.1.1}$$

A more concise argument (in fact the same) for this simultaneous diagonalization is that the Hermitian form $i\sigma$ can be diagonalized in the complex vector space equipped with the dot product Q. From this short discussion, we have to keep in mind that a positive definite quadratic form can be reduced to (3.1.1), via a suitable choice of symplectic coordinates. In the case of a semi-definite quadratic form or in the hyperbolic case, some normal forms are known but the discussion is much more involved; we refer the reader to the section 21.5 of [H2] and to the theorem 21.5.3 there.

Let Γ be an Euclidean norm on $\mathbb{R}^{2n}$, identified with a $2n \times 2n$ symmetric matrix ; we define $\Gamma^\sigma = \sigma^*\Gamma^{-1}\sigma$, where $\sigma = \begin{pmatrix} 0 & I_n \\ -I_n & 0 \end{pmatrix}$. We shall say that Γ is a symplectic norm whenever $\Gamma = \Gamma^\sigma$. The basic examples of symplectic norms that we are going to use are

$$\Gamma_\lambda = \lambda|dx|^2 + \frac{|d\xi|^2}{\lambda} = \begin{pmatrix} \lambda I_n & 0 \\ 0 & \lambda^{-1} I_n \end{pmatrix}, \tag{3.1.2}$$

where λ is a positive parameter. Our construction of the Wick quantization could be carried out for any symplectic norm, however, for simplicity, we shall limit ourselves to the norms (3.1.2). The following definition contains also some classical properties.

Definition 3.1.1. Let $Y = (y, \eta)$ be a point in $\mathbb{R}^{2n}$ and $\lambda > 0$. We define first the operator

$$\Sigma_Y^\lambda = \left[2^n e^{-2\pi\Gamma_\lambda(\cdot - Y)}\right]^w. \tag{3.1.3}$$

This is a rank-one orthogonal projection: using the notations (1.3.1-2), we have

$$\Sigma_Y^\lambda u = (W_\lambda u)(Y)\varphi_Y^\lambda = \langle u, \varphi_Y^\lambda\rangle_{L^2(\mathbb{R}^n)}\varphi_Y^\lambda. \tag{3.1.4}$$

Let a be in $L^\infty(\mathbb{R}^{2n})$. The Wick($\lambda$) quantization of a is defined as

$$a^{\mathrm{Wick}(\lambda)} = \int_{\mathbb{R}^{2n}} a(Y)\Sigma_Y^\lambda dY. \tag{3.1.5}$$

To check (3.1.3), starting from (3.1.4) is an easy exercise on the Weyl quantization left to the reader.

Proposition 3.1.2. *Let λ be a positive number and a be in $L^\infty(\mathbb{R}^{2n})$. Then*

$$a^{\mathrm{Wick}(\lambda)} = W_\lambda^* a^\mu W_\lambda, \quad 1^{\mathrm{Wick}(\lambda)} = \mathrm{Id}_{L^2(\mathbb{R}^n)} \tag{3.1.6}$$

where W_λ is the isometric mapping from $L^2(\mathbb{R}^n)$ to $L^2(\mathbb{R}^{2n})$ given in (1.3.2), and a^μ the operator of multiplication by a in $L^2(\mathbb{R}^{2n})$. The operator $\pi_{H_\lambda} = W_\lambda W_\lambda^*$ is the orthogonal projection on a closed proper subspace H_λ of $L^2(\mathbb{R}^{2n})$. Moreover, we have

$$\|a^{\mathrm{Wick}(\lambda)}\|_{\mathcal{L}(L^2(\mathbb{R}^n))} \le \|a\|_{L^\infty(\mathbb{R}^{2n})}, \tag{3.1.7}$$

$$a(X) \ge 0 \Longrightarrow a^{\mathrm{Wick}(\lambda)} \ge 0, \tag{3.1.8}$$

$$\left\|\Sigma_Y^\lambda \Sigma_Z^\lambda\right\|_{\mathcal{L}(L^2(\mathbb{R}^n))} \le 2^n e^{-\frac{\pi}{2}\Gamma_\lambda(Y-Z)}. \tag{3.1.9}$$

Moreover the kernel of $\pi_H = \pi_H^1$ is $e^{-\frac{\pi}{2}\|X-Y\|^2} e^{-i\pi[X,Y]}$, $\quad [X,Y] = \langle \sigma X, Y\rangle$.

Proof. Here we assume that $\lambda = 1$ and omit the indexation by λ. The calculations are analogous for other positive values of λ. The first properties and (3.1.8) are immediate consequences of lemma 1.3.1. The operator π_H is an orthogonal projection on its range, which is the same as the range of W and the latter is closed since W is isometric. On the other hand, π_H is not onto, otherwise π_H would be the identity of $L^2(\mathbb{R}^{2n})$ and for all $u \in \mathcal{S}(\mathbb{R}^n)$, we would have

$$\begin{aligned}\|u\|^2_{L^2(\mathbb{R}^n)} &= 2\,\mathrm{Re}\langle D_{x_1}u, ix_1u\rangle_{L^2(\mathbb{R}^n)} = 2\,\mathrm{Re}\langle \xi_1^{\mathrm{Wick}}u, ix_1^{\mathrm{Wick}}u\rangle_{L^2(\mathbb{R}^n)} \\ &= 2\,\mathrm{Re}\langle \xi_1 Wu, i\pi_H x_1 Wu\rangle_{L^2(\mathbb{R}^{2n})} = 2\,\mathrm{Re}\langle \xi_1 Wu, ix_1 Wu\rangle_{L^2(\mathbb{R}^{2n})} = 0.\end{aligned}$$

Now, with $L^2(\mathbb{R}^n)$ dot-products, we have

$$\begin{aligned}|\langle a^{\mathrm{Wick}}u, v\rangle| &= \left|\int_{\mathbb{R}^{2n}} a(Y)\langle \Sigma_Y u, v\rangle dY\right| = \left|\int_{\mathbb{R}^{2n}} a(Y)\langle \Sigma_Y u, \Sigma_Y v\rangle dY\right| \\ &\le \|a\|_{L^\infty(\mathbb{R}^{2n})} \int_{\mathbb{R}^{2n}} \|\Sigma_Y u\|_{L^2(\mathbb{R}^n)} \|\Sigma_Y v\|_{L^2(\mathbb{R}^n)}\, dY \\ &\le \|a\|_{L^\infty(\mathbb{R}^{2n})} \left(\int_{\mathbb{R}^{2n}} \|\Sigma_Y u\|^2_{L^2(\mathbb{R}^n)}\, dY\right)^{1/2} \left(\int_{\mathbb{R}^{2n}} \|\Sigma_Y v\|^2_{L^2(\mathbb{R}^n)}\, dY\right)^{1/2} \\ &= \|a\|_{L^\infty(\mathbb{R}^{2n})} \|u\|_{L^2(\mathbb{R}^n)} \|v\|_{L^2(\mathbb{R}^n)},\end{aligned}$$

yielding (3.1.7). For $Y, Z \in \mathbb{R}^{2n}$ a straightforward computation shows that the Weyl symbol of $\Sigma_Y \Sigma_Z$ is, as a function of the variable $X \in \mathbb{R}^{2n}$, setting $\Gamma_1(T) = |T|^2$

$$e^{-\frac{\pi}{2}|Y-Z|^2} e^{-2i\pi[X-Y, X-Z]} 2^n e^{-2\pi|X-\frac{Y+Z}{2}|^2}.$$

Since for the Weyl quantization, one has $\|a^w\|_{\mathcal{L}(L^2(\mathbb{R}^n))} \le 2^n \|a\|_{L^1(\mathbb{R}^{2n})}$, we get the result (3.1.9). The very last assertion is left as an (easy) exercise for the reader. □

Remark. The positivity property (3.1.8) is not satisfied for the Weyl quantization since the Wigner function $\mathcal{H}(u,u)$ (see (2.1.4)) is not always non-negative, although it is actually positive if u is a Gaussian function. We leave to the reader the computation of

$$\mathcal{H}(u_1,u_1)(x,\xi) = 4\pi 2^n e^{-2\pi(|x|^2+|\xi|^2)}(\xi_1^2+x_1^2-\frac{1}{4\pi}), \quad u_1(x)=2^{n/4}x_1 2\pi^{1/2}e^{-\pi|x|^2}.$$

which is negative in a neighborhood V of the origin. Now, choosing a non-negative $a(x,\xi)\in C_c^\infty(V)$ and using (2.1.3) we get $\langle a^w u_1, u_1\rangle < 0$. On the other hand we have the familiar

$$\mathcal{H}(u_0,u_0)(x,\xi) = 2^n e^{-2\pi(|x|^2+|\xi|^2)}, \quad u_0(x)=2^{n/4}e^{-\pi|x|^2}.$$

3.2. The Gårding inequality with gain of one derivative.

Proposition 3.2.1. *Let m be a real number and $p(x,\xi,\lambda)$ be a symbol in S^m_{scl} (see (2.3.3)). Then*

$$p^{\text{Wick}(\lambda)} = p^w + r(p)^w, \tag{3.2.1}$$

with $r(p)\in S^{m-1}_{\text{scl}}$ so that the mapping $p\mapsto r(p)$ is continuous. Moreover, $r(p)=0$ if p is a linear form or a constant.

Proof. From the definition 3.1.1, one has $p^{\text{Wick}(\lambda)} = \widetilde{p}^w$, with

$$\begin{aligned}\widetilde{p}(X) &= \int_{\mathbb{R}^{2n}} p(X+Y)\ e^{-2\pi\Gamma_\lambda(Y)}2^n dY \\ &= p(X) + \underbrace{\int_0^1\int_{\mathbb{R}^{2n}}(1-\theta)p''(X+\theta Y)Y^2 e^{-2\pi\Gamma_\lambda(Y)}2^n dY d\theta}_{r(p)(X)}.\end{aligned} \tag{3.2.2}$$

We note now that the estimates (2.3.3) of S^m_{scl} on p are equivalent to

$$|p^{(k)}(X)T^k| \le C_k\lambda^{m-\frac{k}{2}}\Gamma_\lambda(T)^{\frac{k}{2}} \quad \text{or} \quad |p^{(k)}(X)|_{\Gamma_\lambda} \le C_k\lambda^{m-\frac{k}{2}}.$$

Thus we get

$$|r(p)^{(k)}(X)|_{\Gamma_\lambda} \le C_{k+2}\lambda^{m-\frac{k+2}{2}}\int_{\mathbb{R}^{2n}}\Gamma_\lambda(Y)e^{-2\pi\Gamma_\lambda(Y)}2^{n-1}dY,$$

and since $\det(\Gamma_\lambda)=1$, the integral above is a constant and this implies that $r\in S^{m-1}_{\text{scl}}$. The last point in the proposition follows from the formula (3.2.2) showing that $r(p)$ depends linearly on p''. $\square$

Remark. For further understanding of our results, it would be better to use symbol classes defined by a metric in the phase space, as introduced in the chapter 18 of [H2]. As we have seen above,

$$S^m_{\text{scl}} = S(\lambda^m, \lambda^{-1}\Gamma_\lambda),$$

that is symbols such that

$$|a^{(k)}(X)T^k| \le \gamma_k(a)\lambda^{m-\frac{k}{2}}\Gamma_\lambda(T)^{\frac{k}{2}},$$

or more accurately, for all $k \in \mathbb{N}$,

$$\gamma_k(a) = \sup_{\substack{X\in\mathbb{R}^{2n},\lambda\ge 1,\\ T\in\mathbb{R}^{2n},\Gamma_\lambda(T)=1}} |a^{(k)}(X)T^k|\lambda^{-m+\frac{k}{2}} < +\infty.$$

The following theorem was proven in 1966 by L.Hörmander and a generalization to systems was given the same year by P.Lax & L.Nirenberg. The reader can check the theorem 18.6.14 in [H2] for a (much) wider generalization of this statement. The name given to this inequality by the aforementioned authors was "Sharp Gårding inequality", a terminology that may look inappropriate nowadays since, in the scalar case, a drastic improvement of that sharpness was given in 1978 by C.Fefferman & D.H.Phong in [FP] (see our section 5 below). However, in the vector-valued case, the Fefferman-Phong inequality is not true in general, as observed in [Br]; a class of counterexamples were studied more systematically in [Pa].

Theorem 3.2.2. *Let $a(x, \xi, \lambda)$ be a symbol in S^1_{scl} (cf. (2.3.3)), taking nonnegative values. Then the operator a^w is semi-bounded from below in $L^2(\mathbb{R}^n)$, and more precisely, there exists a semi-norm $\gamma_{\alpha\beta}(a)$ of a and a constant C_n, depending only on the dimension such that*

$$a^w + C_n\gamma_{\alpha\beta}(a) \ge 0, \quad \textit{as an operator.} \tag{3.2.3}$$

Proof. It appears as an immediate consequence of the propositions 3.2.1 and of (3.1.8): we have from (3.2.1)

$$a^w = a^{\text{Wick}(\lambda)} - r(a)^w, \qquad r(a) \in S^0_{\text{scl}} \text{ thus } -r(a)^w \in \mathcal{L}(L^2(\mathbb{R}^n)) ,$$

and from (3.1.8) $a^{\text{Wick}(\lambda)} \ge 0$, yielding the result. □

3.3. Variations. In this section, we show that using the method of proof of the theorem 18.1.14 in [H2], we can in fact obtain a stronger result than the theorem 3.2.2.

Let $\phi(x, \xi)$ an even L^2 function on $\mathbb{R}^{2n}$ with L^2 norm 1. We define, using (2.3.2), $\Psi = \phi^* \circ \phi$, with ϕ^* standing for the standard symbol of the adjoint. We note that Ψ is even and that

$$\iint \Psi(x, \xi)dx = 1. \tag{3.3.1}$$

In fact, we have

$$\begin{aligned}\Psi(x,\xi) &= \iint \phi^*(x,\xi+\eta)\phi(x+y,\xi)e^{-2i\pi y\eta}dyd\eta\\ &= \iiiint \bar\phi(x+z,\zeta+\xi+\eta)e^{-2i\pi z\zeta}\phi(x+y,\xi)e^{-2i\pi y\eta}dzdyd\zeta d\eta\\ &= \iiiint \bar\phi(x+z,\xi+\eta)e^{-2i\pi z\zeta}\phi(x+y,\xi)e^{-2i\pi y(\eta-\zeta)}dzdyd\zeta d\eta\\ &= \iiiint \bar\phi(z,\eta)e^{-2i\pi(z-x)\zeta}\phi(y,\xi)e^{-2i\pi(y-x)(\eta-\zeta-\xi)}dzdyd\zeta d\eta\\ &= \iint \bar\phi(y,\eta)\phi(y,\xi)e^{-2i\pi(y-x)(\eta-\xi)}dyd\eta,\end{aligned}$$

and since ϕ is even we obtain

$$\begin{aligned}\Psi(-x,-\xi) &= \iint \bar\phi(y,\eta)\phi(y,-\xi)e^{-2i\pi(y+x)(\eta+\xi)}dyd\eta\\ &= \iint \bar\phi(-y,-\eta)\phi(-y,-\xi)e^{-2i\pi(-y+x)(-\eta+\xi)}dyd\eta\\ &= \iint \bar\phi(y,\eta)\phi(y,\xi)e^{-2i\pi(y-x)(\eta-\xi)}dyd\eta = \Psi(x,\xi).\end{aligned}$$

Moreover we have

$$\begin{aligned}\iint \Psi(x,\xi)dxd\xi = \iiiint \bar\phi(y,\eta)\phi(y,\xi)e^{-2i\pi(y-x)(\eta-\xi)}dydxd\eta d\xi\\ = \iint \bar\phi(y,\xi)\phi(y,\xi)dyd\xi = 1.\end{aligned}$$

We consider now the symbol $a^\flat$ defined by

(3.3.2)
$$a^\flat(x,\xi) = \iint a(x+y,\xi+\eta)\Psi(y,\eta)dyd\eta = \iint a(y,\eta)\Psi(x-y,\xi-\eta)dyd\eta.$$

Recalling the definition of section 2.2, we use the phase translation

$$(\tau_{y,\eta}u)(x) = u(x-y)e^{2i\pi(x-\frac{y}{2})\eta}, \quad \text{so that} \quad \tau^*_{y,\eta} = \tau_{-y,-\eta}$$

and we get

(3.3.3)
$$\begin{aligned}0 \le \tau_{y,\eta}\mathrm{Op}(\phi^*(x,\xi))\mathrm{Op}(\phi(x,\xi))\tau^*_{y,\eta} = \tau_{y,\eta}\mathrm{Op}(\Psi(x,\xi))\tau^*_{y,\eta}\\ = \mathrm{Op}((x,\xi)\mapsto\Psi(x-y,\xi-\eta)),\end{aligned}$$

since

$$\begin{aligned}(\tau_{y,\eta}\mathrm{Op}(\Psi(x,\xi))\tau_{y,\eta}^*u)(x) &= e^{2i\pi(x-\frac{y}{2})\eta}\int \Psi(x-y,\xi)e^{2i\pi(x-y)\xi}\widehat{\tau_{y,\eta}^* u}(\xi)d\xi\\
&= e^{2i\pi(x-\frac{y}{2})\eta}\iint \Psi(x-y,\xi)e^{2i\pi(x-y-z)\xi}(\tau_{y,\eta}^*u)(z)dzd\xi\\
&= e^{2i\pi(x-\frac{y}{2})\eta}\iint \Psi(x-y,\xi)e^{2i\pi(x-y-z)\xi}u(z+y)e^{-2i\pi(z+\frac{y}{2})\eta}dzd\xi\\
&= e^{2i\pi(x-\frac{y}{2})\eta}\iint \Psi(x-y,\xi)e^{2i\pi(x-z)\xi}u(z)e^{-2i\pi(z-\frac{y}{2})\eta}dzd\xi\\
&= \iint \Psi(x-y,\xi)e^{2i\pi(x-z)(\xi+\eta)}u(z)dzd\xi\\
&= \iint \Psi(x-y,\xi-\eta)e^{2i\pi(x-z)\xi}u(z)dzd\xi\\
&= \int \Psi(x-y,\xi-\eta)e^{2i\pi x\xi}\hat{u}(\xi)d\xi\\
&= \big(\mathrm{Op}((x,\xi)\mapsto \Psi(x-y,\xi-\eta))u\big)(x).\end{aligned}$$

From (3.3.2) and $a\ge 0$, we get

$$\mathrm{Op}(a^\flat)=\iint a(y,\eta)\mathrm{Op}(\Psi(x-y,\xi-\eta))dyd\eta\ge 0,\quad \text{as an operator.} \tag{3.3.4}$$

Lemma 3.3.1. *Let a be a function defined on $\mathbb{R}^{2n}$ such that $a''\in S^0_{0,0}$ (smooth bounded functions as well as all their derivatives). Then with $a^\flat$ defined in* (3.3.2)*, $a-a^\flat$ belongs to $S^0_{0,0}$.*

Proof. We have

$$\begin{aligned}2a^\flat(x,\xi)-2a(x,\xi) &= \iint \big(a(x+y,\xi+\eta)+a(x-y,\xi-\eta)-2a(x,\xi)\big)\Psi(y,\eta)dyd\eta\\
&= \iint a^{[2]}_{y,\eta}(x,\xi)\Psi(y,\eta)dyd\eta,\end{aligned}$$

with

$$a^{[2]}_Y(X)=\int_{-1}^1(1-|\theta|)a''(X+\theta Y)Y^2d\theta,\qquad Y=(y,\eta),\ X=(x,\xi),$$

a symbol which belongs to $S^0_{0,0}$ with semi-norms controlled by semi-norms of $a''\times|Y|^2$. We have

$$2a^\flat(X)-2a(X)=\int_{-1}^1\int_{\mathbb{R}^{2n}}(1-|\theta|)a''(X+\theta Y)Y^2\Psi(Y)dYd\theta.$$

We may also assume that Ψ belongs to $L^1(\mathbb{R}^{2n})$ and is rapidly decreasing, entailing that for all semi-norms γ in $S^0_{0,0}$,

$$\gamma(a^\flat-a)\le\iint\gamma(a^{[2]}_{y,\eta})|\Psi(y,\eta)|dyd\eta<\infty.\quad\square$$

As a consequence, we get the

Theorem 3.3.2. *Let a be a nonnegative function defined on $\mathbb{R}^{2n}$ such that the Hessian $a'' \in S^0_{0,0}$. Then the operators a^w and $\operatorname{Re}(a(x,D))$ are semi-bounded from below.*

Proof. We have from the lemma 3.3.1 and the lemma 2.3.2

$$\operatorname{Op}(a) \in \operatorname{Op}(a^\flat) + \operatorname{Op}(S^0_{0,0}) \subset \operatorname{Op}(a^\flat) + \mathcal{L}(L^2(\mathbb{R}^n)),$$

so that using (3.3.4), we get that $\operatorname{Re}\operatorname{Op}(a)$ is semi-bounded from below. We can also change the quantization and consider a symbol a satisfying the assumptions of the theorem 3.3.2: with $a^w = \operatorname{Op}(J^{1/2}a)$, we see from our section 2.1 that $J^{1/2}a \in a + i\pi D_x \cdot D_\xi a + S^0_{0,0}$ so that

$$a^w \in \operatorname{Op}(a) + \operatorname{Op}(i\pi D_x \cdot D_\xi a) + \mathcal{L}(L^2(\mathbb{R}^n))$$

and taking real parts of the operators we have

$$a^w \in \operatorname{Re}\operatorname{Op}(a) + \operatorname{Re}\operatorname{Op}(i\pi D_x \cdot D_\xi a) + \mathcal{L}(L^2(\mathbb{R}^n)). \tag{3.3.5}$$

We check now, using that a is real-valued (entailing $i\pi D_x \cdot D_\xi a \in i\mathbb{R}$),

$$2\operatorname{Re}\operatorname{Op}(i\pi D_x \cdot D_\xi a) = \operatorname{Op}(i\pi D_x \cdot D_\xi a + J\overline{i\pi D_x \cdot D_\xi a}) = \operatorname{Op}((\operatorname{Id} - J) i\pi D_x \cdot D_\xi a)$$

which belongs to $\operatorname{Op}(S^0_{0,0})$. As a consequence, (3.3.5) is giving that a^w is semi-bounded from below since $\operatorname{Re}\operatorname{Op}(a)$ is already proven so. □

It is interesting to see that the non-asymptotic result of theorem 3.3.2 implies the asymptotic statement of theorem 3.2.2; as a matter of fact, if we consider a nonnegative symbol $a(x,\xi,\lambda)$ in S^1_{scl}, the operator $a(x,\xi,\lambda)^w$ is unitarily equivalent to $a(x\lambda^{-1/2}, \xi\lambda^{1/2}, \lambda)^w$ and from the estimates (2.3.3), we get that the nonnegative symbol

$$b(x,\xi) = a(x\lambda^{-1/2}, \xi\lambda^{1/2}, \lambda)$$

satisfies indeed the assumptions of the theorem. As a result, the operator b^w (and thus the unitarily equivalent a^w) is semi-bounded from below.

As said above, that proof of the sharp Gårding inequality is borrowed from [H2]. This is a general idea of mollifying the symbol by a normalized function of type $\phi^* \circ \phi$. Nothing else at this stage is really needed and the classical so-called *coherent states method* is simply dealing with a Gaussian function ϕ, with the only but no crucial advantage that the computations of $\phi^* \circ \phi$ can be made explicitely with other Gaussians. This point of view is precisely the most synthetic and seems suitable to tackle a group situation.

Our last remark is dealing with the standard classes of pseudodifferential operators and with the asymptotic point of view, which plays an important role for PDE. The standard classes of symbols $S^m_{\rho,\delta}$ on $\mathbb{R}^n$ are well-known. Sticking for simplicity with the case $\rho =$

$1, \delta = 0$ and calling $S^m = S^m_{1,0}$, one can get standard continuity and composition results. However the statement of the sharp Gårding inequality is: let a be a nonnegative symbol in S^1, then

$$\operatorname{Re}\langle \operatorname{Op}(a)u, u\rangle + C\,\|u\|^2_{L^2} \geq 0. \tag{3.3.6}$$

This result is in fact a consequence of Theorem 3.3.2. Let us sketch the proof of the non-semi-classical (3.3.6), using Theorem 3.3.2.

(1) Using a Littlewood-Paley decomposition, one writes

$$a(x,\xi) = \sum_{\nu \in \mathbb{N}} \varphi_\nu(\xi) a(x,\xi), \quad \operatorname{Op}(a) = \sum_{\nu \in \mathbb{N}} \psi_\nu(D) \operatorname{Op}(\varphi_\nu a) \psi_\nu(D) + R$$

with $\psi_\nu = 1$ on the support of φ_ν; R belongs to $S^{-\infty}$ (it is not completely obvious because of the summation in ν).

(2) We are reduced to prove a semiclassical statement, since the conditions on the derivatives of $\varphi_\nu a$ are expressed in terms of the frequency $2^\nu = 1/h$. Essentially, the statement to be proven is: for b smooth nonnegative bounded with bounded derivatives the operator $h^{-1}\operatorname{Op}(b(x, h\xi))$ is semi-bounded from below (uniformly in $h \in]0,1]$).

(3) We note that the previous operator is unitarily equivalent to

$$h^{-1}\operatorname{Op}(b(h^{1/2}x, h^{1/2}\xi))$$

and that the seminorms in $S^0_{0,0}$ of A'' where

$$A(x,\xi) = h^{-1} b(h^{1/2}x, h^{1/2}\xi)$$

are bounded for $h \in]0,1]$. It is indeed the case and we can use Theorem 3.3.2 to conclude.

4. Energy estimates via the Wick quantization

4.1. Subelliptic operators satisfying condition (P). We intend to illustrate in this section the usefulness of Wick quantization to prove energy estimates. We want to give a simple proof of a well-known theorem on subellipticity for differential operators. In this case, the proof of the theorem conjectured in the papers of Egorov [Eg] is known since 1971 with the work of Treves [Tr]. Our method here falls short of giving a proof in the general case (i.e. for pseudo-differential operators) . Hörmander gave a complete proof in 1979 of the general theorem on subellipticity which can be found in chapter 27 of [H2]. We will go a little beyond the differential case, proving the theorem when the zero set of the imaginary part is included in its critical set (see theorem 4.1.3) below. Moreover, we believe that this elementary proof, reducing actually the problem to simple ordinary differential equations, is a good example of the wave-packet technique.

Let P be a principal type pseudodifferential operator on a manifold $\mathcal{M}$ whose principal symbol p satisfies condition $(\overline{\psi})$:

$$(4.1.1)\qquad (\overline{\psi})\qquad \begin{cases} \forall z\in\mathbb{C}, \operatorname{Im}(zp) \text{ does not change sign from } + \text{ to } - \\ \text{along the oriented bicharacteristic curves of } \operatorname{Re}(zp). \end{cases}$$

Assume also that for some $z\in\mathbb{C}$ and in some conic open set $\Omega\subset T^*(\mathcal{M})\backslash 0$ and some integer k

$$(4.1.2)\qquad \mathrm{H}^k_{\mathbb{R}e(zp)}(\operatorname{Im}(zp))\neq 0.$$

The following condition is equivalent to $(\overline{\psi})$ for differential operators

$$(4.1.3)\qquad (P)\qquad \begin{cases} \forall z\in\mathbb{C}, \operatorname{Im}(zp) \text{ does not change sign} \\ \text{along the oriented bicharacteristic curves of } \operatorname{Re}(zp). \end{cases}$$

Theorem 4.1.1. *Let P be a properly supported principal type pseudo-differential operator of order m on a manifold $\mathcal{M}$ satisfying* $(4.1.2-4.1.3)$*. Then P is subelliptic on Ω with loss of $k/(k+1)$ derivatives: for any real s*

$$(4.1.4)\qquad u\in\mathcal{D}'(\mathcal{M}),\quad Pu\in H^s(\Omega)\Longrightarrow u\in H^{s+m-\frac{k}{k+1}}(\Omega)\ .$$

Here, k is necessarily an even integer.

After classical reductions, theorem 4.1.1 in $n+1$ dimensions will follow from theorem 4.1.3 below. We need first to state a

Definition 4.1.2. Let n be an integer and $A_\Lambda(t,x,\xi)$ be a family of smooth functions on $\mathbb{R}_t\times\mathbb{R}^n_x\times\mathbb{R}^n_\xi$, depending on a parameter $\Lambda\geq 1$. Let m be a real number. We shall say that $(A_\Lambda)\in S^m_n$ if for each $(l,\alpha,\beta)\in\mathbb{N}\times\mathbb{N}^n\times\mathbb{N}^n$,

$$(4.1.5)\qquad \sup_{\Lambda\geq 1,(t,x,\xi)\in\mathbb{R}^{2n+1}} |(D^l_t D^\alpha_x D^\beta_\xi A_\Lambda)(t,x,\xi)|\Lambda^{-m+\frac{|\alpha|+|\beta|}{2}} = C_{l\alpha\beta}<\infty,$$

$$(4.1.6)\qquad \operatorname{supp} A_\Lambda\subset[-1,1]\times\{x\in\mathbb{R}^n, |x|\leq\Lambda^{1/2}\}\times\{\xi\in\mathbb{R}^n, |\xi|\leq\Lambda^{1/2}\}.$$

The semi-norms of the family (A_Λ) are defined as the constants $C_{l\alpha\beta}$ in (4.1.5). Note that for each t, the function $(x,\xi)\mapsto A_\Lambda(t,\Lambda^{1/2}x,\Lambda^{-1/2}\xi)$ belongs to S^m_{scl} as defined by (2.3.3).

We can now state the

Theorem 4.1.3. *Let n be an integer and $Q_\Lambda(t,x,\xi)$ be a family of smooth functions such that $(Q_\Lambda) \in S^1_n$. Assume moreover that*

$$Q_\Lambda(t,x,\xi) = 0 \implies d_{x,\xi} Q_\Lambda(t,x,\xi) = 0. \tag{4.1.7}$$

$$\inf_{\substack{\Lambda\ge 1, |2t|\le 1,\\ |2x|\le \Lambda^{1/2}, |2\xi|\le \Lambda^{1/2}}} |D_t^k Q_\Lambda(t,x,\xi)|\Lambda^{-1} = \delta_0 > 0. \tag{4.1.8}$$

Let $\chi_\Lambda(t,x,\xi)$ be a family of smooth functions such that the family $(\chi_\Lambda(\frac{t}{2},\frac{x}{2},\frac{\xi}{2}))_\Lambda \in S^0_n$. There exists a positive constant δ_1, such that, for any $u(t,x) \in C^\infty_c((-\frac12,\frac12)_t, \mathcal{S}(\mathbb{R}^n_x))$,

$$\|D_t u + iQ_\Lambda(t,x,D_x)u\|_{L^2(\mathbb{R}^{n+1})} + \|u\|_{L^2(\mathbb{R}^{n+1})} \ge \delta_1 \Lambda^{\frac{1}{k+1}} \|\chi_\Lambda(t,x,D_x)u\|_{L^2(\mathbb{R}^{n+1})}. \tag{4.1.9}$$

Note that δ_1 depends only on the dimension n, δ_0 and the semi-norms of the families $(Q_\Lambda), (\chi_\Lambda)$.

Let us note right now that condition (4.1.7) is satisfied by nonnegative (and nonpositive) functions. Properties (4.1.2-4.1.3) imply that, $q \ge 0$ or $q \le 0$ in some conic open neighborhood of a point where (4.1.2) is satisfied.

4.2. Polynomial behaviour of some functions.

Lemma 4.2.1. *Let $k \in \mathbb{N}^*, \delta > 0$ and $C > 0$ be given. Let I be an interval of $\mathbb{R}$ and $q : I \longrightarrow \mathbb{R}$ be a C^k function satisfying*

$$\inf_{t\in I} |q^{(k)}(t)| \ge \delta. \tag{4.2.1}$$

Then, for any $h > 0$, the set

$$\{t \in I, |q(t)| \le Ch^k\} \subset \cup_{1\le l\le k} J_l, \tag{4.2.2}$$

where J_l is an interval with length $h(\alpha_k C\delta^{-1})^{1/k}, \alpha_k = 2^{2k}k!$. As a consequence the Lebesgue measure of $\{t \in I, |q(t)| \le Ch^k\}$ is smaller than

$$h(\frac{C}{\delta})^{1/k} 4k(k!)^{1/k} \le h\left(\frac{C}{\delta}\right)^{1/k} 4k^2.$$

Proof. Let $k \in \mathbb{N}^*$, h a positive number and set $E_k(h,C,q) = \{t \in I, |q(t)| \le Ch^k\}$. Let us first assume $k = 1$. Assume that $t, t_0 \in E_1(h,C,q)$; then the mean value theorem and (4.2.1) imply

$$2Ch \ge |q(t) - q(t_0)| \ge \delta|t - t_0|$$

so that $E_1(h,C,q)\cap\{t,|t-t_0|>h2C/\delta\}=\emptyset$: otherwise we would have $2Ch>h\delta 2C/\delta$. As a result for any $t_0,t\in E_1(h,C,q)$, $|t-t_0|\le h2C/\delta$. Either $E_1(h,C,q)$ is empty or it is not empty and then included in an interval with length $\le h4C/\delta$.

Let us now assume that $k\ge 2$. If $E_k(h,C,q)=\emptyset$, (4.2.2) is true. We assume that there exists $t_0\in E_k(h,C,q)$, and we write for $t\in I$,

$$q(t)=q(t_0)+\underbrace{\int_0^1 q'\bigl(t_0+\theta(t-t_0)\bigr)d\theta}_{Q(t)}(t-t_0). \tag{4.2.3}$$

Then, if $t\in E_k(h,C,q)$, we have $2Ch^k\ge|Q(t)(t-t_0)|$. Now, for a given $\omega>0$, either $|t-t_0|\le\omega h/2$ and $t\in[t_0-\omega h/2,t_0+\omega h/2]$, or $|t-t_0|>\omega h/2$ and from the previous inequality we infer $|Q(t)|\le\omega^{-1}4Ch^{k-1}$, i.e. we get that

$$E_k(h,C,q)\subset[t_0-\omega h/2,t_0+\omega h/2]\cup E_{k-1}(h,\omega^{-1}4C,Q). \tag{4.2.4}$$

But the function Q satisfies the assumptions of the lemma with $k-1$, δ replaced by δ/k: in fact for $t\in I$, $Q^{(k-1)}(t)=\int_0^1 q^{(k)}\bigl(t_0+\theta(t-t_0)\bigr)\theta^{k-1}d\theta$, and if $q^{(k)}(t)\ge\delta$ on I, we get $Q^{(k-1)}(t)\ge\delta/k$. By induction on k and using (4.2.4), we get that

$$E_k(h,C,q)\subset[t_0-\omega h/2,t_0+\omega h/2]\cup_{1\le l\le k-1}J_l,\quad |J_l|\le h(4C\omega^{-1}k\delta^{-1}\alpha_{k-1})^{1/(k-1)}. \tag{4.2.5}$$

We choose now $\omega=(4C\omega^{-1}k\delta^{-1}\alpha_{k-1})^{1/(k-1)}$, i.e. $\omega^k=4C\delta^{-1}k\alpha_{k-1}$, that is

$$\omega=(C\delta^{-1}4k\alpha_{k-1})^{1/k}$$

yielding the result if $\alpha_k=4k\alpha_{k-1}$, i.e. $\alpha_k=(4k)(4(k-1))\dots(4\times 2)\alpha_1=4^{k-1}k!2^2=2^{2k}k!$. The proof of the lemma is complete. $\square$

Lemma 4.2.2. *Let $f:\mathbb{R}\mapsto[0,+\infty)$ be a C^1 function so that (distribution derivative) $f''\in L^\infty(\mathbb{R})$. Then for all $x\in\mathbb{R}$,*

$$f'(x)^2\le 2f(x)\,\|f''\|_{L^\infty(\mathbb{R})}\,. \tag{4.2.6}$$

Proof. The following formula is true for f distribution , $h\in\mathbb{R}$ given :

$$f(x+h)=f(x)+f'(x)h+\int_0^1(1-\theta)f''(x+\theta h)d\theta h^2.$$

Then, since $f\in C^1$, for all $h\in\mathbb{R}$, we get $0\le f(x)+f'(x)h+\frac12\|f''\|_{L^\infty(\mathbb{R})}h^2$. The nonpositivity of the discriminant of this polynomial is given by (4.2.6). $\square$

It is easy to extend this lemma to functions whose zero set is included in the critical set :

Remark 4.2.3. If $f : \mathbb{R} \mapsto \mathbb{R}$ is twice differentiable, $f'' \in L^\infty$ and $f'(x) = 0$ when $f(x) = 0$, then

$$f'(x)^2 \leq 2|f(x)|\, \|f''\|_{L^\infty(\mathbb{R})}\,. \tag{4.2.7}$$

In fact, for $f \in C^1$, $F(x) = |f(x)|$ is such that $F'(x) = f'(x)f(x)/|f(x)|$ on $f(x) \neq 0$. Moreover, if $f(x) = 0$ then $f'(x) = 0$ so that

$$F(x+h) - F(x) = |f(x+h)| = |f(x) + f'(x)h + o(h)| = o(h),$$

so that $F'(x) = 0$ there. We get then that F is C^1 with

$$F'(x) = s(x)f'(x), \quad s(x) = \frac{f(x)}{|f(x)|} \text{ for } f(x) \neq 0, s(x) = 0 \text{ elsewhere.}$$

Moreover, if $f(x) = 0$,

$$F'(x+h) - F'(x) = s(x+h)[f'(x) + f''(x)h + o(h)] = s(x+h)[f''(x)h + o(h)]. \tag{4.2.8}$$

If $f''(x) \neq 0$, $s(x+h)$ =sign$[f(x) + f'(x)h + f''(x)h^2/2 + o(h^2)] = \text{sign}(f''(x))$ for h small enough, we get $F''(x) = |f''(x)|$ there. If $f''(x) = 0$, we get from (4.2.8) that $F''(x) = 0$. We can apply the lemma 4.2.2 to F and obtain (4.2.7).

Lemma 4.2.3. *Let q be a smooth real-valued function defined on $(-1, 1)$ such that*

$$q(t) > 0 \text{ and } s > t \implies q(s) \geq 0. \tag{4.2.9}$$

Let $\Phi \in C_c^\infty((-1, 1))$ be given . There exists a function $S : (-1, 1) \mapsto \{\pm\frac{1}{2}, \pm\frac{3}{2}\}$ such that for any $\rho > 0$ and $\Lambda \geq 1$,

$$2\,\mathrm{Re}\langle \rho D_t\Phi + i\Lambda q\Phi, iS\Phi\rangle_{L^2(\mathbb{R})} \geq \ \rho\,\|\Phi\|^2_{L^\infty(\mathbb{R})} + \int \Lambda|q(t)||\Phi(t)|^2 dt. \tag{4.2.10}$$

If in addition q satisfies (4.2.1)

$$[\frac{\gamma(k,\delta)}{\rho} + 2]\ 2\mathbb{R}e\langle \rho D_t\Phi + i\Lambda q\Phi, iS\Phi\rangle_{L^2(\mathbb{R})} \geq \ \int [\ \Lambda|q(t)| + \Lambda^{\frac{1}{k+1}}\ ]\ |\Phi(t)|^2 dt\ , \tag{4.2.11}$$

where $\gamma(k, \delta)$ is a positive constant depending only on k, δ. As a consequence we have

$$2[\frac{\gamma(k,\delta)}{\rho} + 2]\ \ \|\rho D_t\Phi + i\Lambda q\Phi\|_{L^2(\mathbb{R})}\, \|\Phi\|_{L^2(\mathbb{R})} \geq \ \int [\ \Lambda|q(t)| + \Lambda^{\frac{1}{k+1}}\ ]\ |\Phi(t)|^2 dt\ . \tag{4.2.12}$$

Proof. We define $\theta = \sup\{t \in (-1, 1), q(t) < 0\}$ and $\theta = -1$ if this set is empty. The condition (4.2.9) implies readily

$$q(t)\,\mathrm{sign}(t - \theta) = |q(t)|.$$

We compute then, with a given $T \in (-1,1)$, H the characteristic function of $\mathbb{R}_+$,

(4.2.12)

$$2\operatorname{Re}\langle \rho D_t\Phi + i\Lambda q\Phi, i[H(t-T)H(T-\theta) - H(T-t)H(\theta-T) + \frac{1}{2}\operatorname{sign}(t-\theta)]\Phi\rangle_{L^2(\mathbb{R})}$$
$$= \rho|\Phi(T)|^2 + 2H(T-\theta)\int_T^1 \Lambda|q(t)|\ |\Phi(t)|^2 dt + 2H(\theta-T)\int_{-1}^T \Lambda|q(t)|\ |\Phi(t)|^2 dt$$
$$+ \int_{-1}^1 \Lambda|q(t)|\ |\Phi(t)|^2 dt + \rho|\Phi(\theta)|^2.$$

This implies (4.2.10). To get (4.2.12), we notice first, applying lemma 4.2.1 ($\mathcal{L}$ stands for the Lebesgue measure) and (4.2.10), that the following inequalities hold:

$$\Lambda^{\frac{1}{k+1}} \int |\Phi(t)|^2 dt = \Lambda^{\frac{1}{k+1}} \int_{\{|\Lambda q(t)|\ <\ \Lambda^{\frac{1}{k+1}}\}} |\Phi(t)|^2 dt + \Lambda^{\frac{1}{k+1}} \int_{\{|\Lambda q(t)| \geq \Lambda^{\frac{1}{k+1}}\}} |\Phi(t)|^2 dt$$
$$\leq \Lambda^{\frac{1}{k+1}} \|\Phi\|^2_{L^\infty(\mathbb{R})} \times \mathcal{L}[\{t \in (-1,1), |q(t)| \leq \Lambda^{-\frac{k}{k+1}}\}] + \int |\Lambda q(t)|\ |\Phi(t)|^2 dt$$
$$\leq \gamma(k,\delta)\|\Phi\|^2_{L^\infty(\mathbb{R})} + 2\mathbb{R}e\langle \rho D_t\Phi + i\Lambda q\Phi, iS\Phi\rangle_{L^2(\mathbb{R})}$$
$$\leq [\frac{2\gamma(k,\delta)}{\rho} + 2]\operatorname{Re}\langle \rho D_t\Phi + i\Lambda q\Phi, iS\Phi\rangle_{L^2(\mathbb{R})},$$

which gives (4.2.12). The proof of the lemma 4.2.3 is complete. □

Lemma 4.2.4. *Let (Q_Λ) be a family of smooth functions in S^1_n(def.4.1.2). Assume that* (4.1.8) *is satisfied as well as* (4.2.9) *for each $q(t) = Q_\Lambda(t,x,\xi)$. There exists a constant C, such that for any $\Phi \in C^\infty_c(\mathbb{R}_t \times \mathbb{R}^n_x \times \mathbb{R}^n_\xi)$ supported in*

$$\max\{|2t|, |2x\Lambda^{-1/2}|, |2\xi\Lambda^{-1/2}|\} \leq 1,$$

the following inequality holds (here $\mathbb{R}^d_X = \mathbb{R}^n_x \times \mathbb{R}^n_\xi$ and the norms are $L^2(\mathbb{R}_t \times \mathbb{R}^d_X)$)

$$C\|D_t\Phi + iQ_\Lambda\Phi\|\ \ \|\Phi\| \geq \Lambda^{\frac{1}{k+1}}\|\Phi\|^2 + \iint |Q_\Lambda(t,X)||\Phi(t,X)|^2 dtdX. \tag{4.2.13}$$

Moreover, for $\Phi \in C^\infty_c((-1,1), \mathcal{S}(\mathbb{R}^{2n}))$

$$C\|D_t\Phi + iQ_\Lambda\Phi\|\,\|\Phi\| \geq \iint |Q_\Lambda(t,X)||\Phi(t,X)|^2 dtdX. \tag{4.2.14}$$

Note that C depends only on the dimension, the semi-norms of the family (Q_Λ) and δ_0 in (4.1.8).

Proof. We want to apply the lemma 4.2.3 to

$$q(t) = \Lambda^{-1}Q_\Lambda(\frac{t}{2}, X),\ \text{whenever}\ X = (x,\xi)\ \text{ such that }\ \max\{|2x|, |2\xi|\} \leq \Lambda^{1/2} \tag{4.2.15}$$

with δ in (4.2.1) given by $2^{-k}\delta_0$, δ_0 defined in (4.1.8). From (4.2.12), we get, for each $X=(x,\xi)$ and $\Psi(t,X)$ smooth supported in

$$|t|<1\times\{|2x\Lambda^{-1/2}|\le 1\}\times\{|2\xi\Lambda^{-1/2}|\le 1\},$$

the inequality

$$2[\frac{\gamma(k,\delta)}{\rho}+2]\left\|\rho D_t\Psi(t,X)+iQ_{\Lambda}(\frac{t}{2},X)\Psi(t,X)\right\|_{L^2(\mathbb{R}_t)}\|\Psi(t,X)\|_{L^2(\mathbb{R}_t)}\ge \int[|Q_\Lambda(\frac{t}{2},X)|+\Lambda^{\frac{1}{k+1}}]|\Psi(t,X)|^2dt. \tag{4.2.16}$$

Integrating (4.2.16) with respect to X, the Cauchy-Schwarz inequality gives the result in (4.2.13), with $\Psi(t,X)=\Phi(\frac{t}{2},X)$ and $\rho=2$. To get (4.2.14), we only need to use (4.2.10) and integrate with respect to X. □

4.3. Energy identities.

Lemma 4.3.1. *Let (Q_Λ) be a family of smooth functions satisfying the assumptions of the lemma 4.2.4. We write Q_{Λ}^{μ} for the operator of multiplication by $Q_\Lambda(t,X)$ on $L^2(\mathbb{R}^{2n})$. If π_H is defined in proposition 3.1.2 (for $\lambda=1$), we have the following estimate for the commutator: for any $\Psi\in L^2(\mathbb{R}^{2n})$*

$$\|[\pi_H,Q_\Lambda^\mu]\Psi\|^2_{L^2(\mathbb{R}^{2n})}\le c_nC\int|Q_\Lambda(t,X)||\Psi(X)|^2dX+c_nC\,\|\Psi\|^2_{L^2(\mathbb{R}^{2n})}, \tag{4.3.1}$$

where c_n depends only on the dimension and $C=\max_{|\alpha|+|\beta|=2}C_{0\alpha\beta}$ (these constants are the semi-norms of the family (Q_Λ) defined in (4.1.5)).

Proof. From the proposition 3.1.2, the kernel of the commutator is

$$e^{-\frac{\pi}{2}|X-Y|^2}\;e^{-i\pi[X,Y]}\big[Q_\Lambda(t,Y)-Q_{\Lambda}(t,X)\big] \tag{4.3.2}$$

$$=e^{-\frac{\pi}{2}|X-Y|^2}\;e^{-i\pi[X,Y]}\;\big[d_XQ_{\Lambda}(t,Y)(Y-X)-\int_0^1Q_{\Lambda}''(t,Y+\theta(X-Y))d\theta(Y-X)^2\big].$$

The second term in the bracket gives rise to a bounded operator, thanks to (4.1.5). Since the multiplication by $d_XQ(t,X)$ can be estimated by (4.2.7), we get the result of the lemma. □

Proof of Theorem 4.1.3. We apply now the lemma 4.2.4 to a function

$$\Phi(t,X)=\chi(X)(Wu)(t,X)$$

where W is given in the proposition 3.1.2, $u(t,x)$ is in $C_c^\infty((-1,1),\mathcal{S}(\mathbb{R}^n))$, $\chi_{\Lambda}(X)$ satisfies (4.1.5) with $m=0$ and

$$\operatorname{supp}\chi_{\Lambda}\subset\{X=(x,\xi),2\Lambda^{-1/2}|x|\le 1,2\Lambda^{-1/2}|\xi|\le 1\}. \tag{4.3.3}$$

We get, with $L^2(\mathbb{R}_t \times \mathbb{R}^{2n}_{x,\xi})$ norms

$$C \|D_t \chi_{\Lambda} W u + i Q_{\Lambda} \chi_{\Lambda} W u\| \|\chi_{\Lambda} W u\| \geq \Lambda^{\frac{1}{k+1}} \|\chi_{\Lambda} W u\|^2 + \iint |Q_{\Lambda}(t,X)| \, |\chi_{\Lambda} W u(t,X)|^2 dt dX, \tag{4.3.4}$$

which implies, with $\pi_K = \operatorname{Id} - \pi_H$,

$$C\Big(\|\pi_H[D_t W u + i Q_{\Lambda} W u]\|^2 + \|\pi_K[D_t W u + i Q_{\Lambda} W u]\|^2 \Big)^{1/2} \|W u\| \geq \Lambda^{\frac{1}{k+1}} \|\chi_{\Lambda} W u\|^2 + \textstyle\iint |Q_{\Lambda}(t,X)| \, |\chi_{\Lambda} W u(t,X)|^2 dt dX \ . \tag{4.3.5}$$

Moreover, from the lemma 4.2.4 we have

$$C \|D_t W u + i Q_{\Lambda} W u\| \|W u\| \geq \iint |Q_{\Lambda}(t,X)| |W u(t,X)|^2 dt dX, \tag{4.3.6}$$

We obtain

$$C \|W[D_t u + i W^* Q_{\Lambda} W u]\| \|W u\| + C \|[\pi_H, Q_{\Lambda}] W u]\| \|W u\| \geq \Lambda^{\frac{1}{k+1}} \|\chi_{\Lambda} W u\|^2 + \textstyle\iint |Q_{\Lambda}(t,X)| |W u(t,X)|^2 dt dX. \tag{4.3.7}$$

Using now the lemma 4.3.1, we estimate the bracket in (4.3.7) :

$$C \|W[D_t u + i W^* Q_{\Lambda} W u]\| \|W u\| + C_1 \varepsilon \textstyle\iint |Q_{\Lambda}(t,X)| |W u(t,X)|^2 dt dX + C_1 \varepsilon^{-1} \|W u\|^2 \geq \Lambda^{\frac{1}{k+1}} \|\chi_{\Lambda} W u\|^2 + \displaystyle\iint |Q_{\Lambda}(t,X)| |W u(t,X)|^2 dt dX. \tag{4.3.8}$$

We get (4.1.9), choosing ε small enough, using the fact that W is isometric (proposition 3.1.2), and that

$$W^* Q_{\Lambda} W - Q_{\Lambda}(t, x, D_x)$$

is uniformly bounded on $L^2(\mathbb{R}^n)$(proposition 3.1.2). The proof of Theorem 4.1.3 is complete.

5. The Fefferman-Phong inequality

5.1. The semi-classical inequality. We consider a function $a \in C^\infty(\mathbb{R}^{2n})$ bounded as well as all its derivatives. The (semi-classical) Fefferman-Phong inequality states that, if a is a nonnegative function, there exists C such that, for all $u \in L^2(\mathbb{R}^n)$ and all $h \in (0,1)$

$$\operatorname{Re}\langle a(x, hD)u, u\rangle_{L^2(\mathbb{R}^n)} + C h^2 \|u\|^2_{L^2} \geq 0,$$

or equivalently (with an a priori different constant C)

$$a(x, h\xi)^w + C h^2 \geq 0.$$

The constants C above depend only a finite number of derivatives of a. Let us ask our first question:

Q1: *How many derivatives of a are needed to control C?*

From the proof by Fefferman and Phong ([FP]), it is clear that the number N of derivatives of a needed to control C should be

$$N = 4 + \nu(n).$$

Since the proof is using an induction on the dimension, it is not completely obvious to answer to our question with a reasonably simple ν. We remark that, with a unitary equivalence,

$$h^{-2}a(x, h\xi)^w \equiv h^{-2}a(xh^{1/2}, h^{1/2}\xi)^w.$$

Defining $A(x,\xi) = h^{-2}a(xh^{1/2}, h^{1/2}\xi)$, we see that the following property holds:

$$(\sharp) \qquad A(x,\xi) \geq 0, \qquad A^{(k)} \text{ is bounded for } k \geq 4.$$

Bony proved in 1998 ([Bo1]) that

$$(\sharp) \Longrightarrow A^w + C \geq 0.$$

Naturally, from the above identities, this implies the Fefferman-Phong inequality. This result shows a twofold phenomenon:

- Only derivatives with order ≥ 4 are needed.
- The control of these derivatives is quite weak, of type $S^0_{0,0}$. In particular, the derivatives of large order do not get small (the class $S^0_{0,0}$ does not have an asymptotic calculus).

Our second question is

Q2: *Is it possible to relax ($\sharp$) by asking only $A^{(4)} \in \mathcal{A}$,*

where $\mathcal{A}$ is a suitable Banach algebra containing $S^0_{0,0}$? We shall in fact prove a result involving a Wiener-type algebra introduced by Sjöstrand in [S1]. To formulate this, we need first to introduce that algebra.

5.2. The Sjöstrand algebra. Let $\mathbb{Z}^{2n}$ be the standard lattice in $\mathbb{R}^{2n}_X$ and let $1 = \sum_{j\in\mathbb{Z}^{2n}} \chi_0(X-j), \chi_0 \in C^\infty_{\mathrm{c}}(\mathbb{R}^{2n})$, be a partition of unity. We note $\chi_j(X) = \chi_0(X-j)$.

Definition 5.2.1. Let $a \in \mathcal{S}'(\mathbb{R}^{2n})$. We shall say that a belongs to $\mathcal{A}$ whenever $\omega_a \in L^1(\mathbb{R}^{2n})$, with $\omega_a(\Xi) = \sup_{j\in\mathbb{Z}^{2n}} |\mathcal{F}(\chi_j a)(\Xi)|$. $\mathcal{A}$ is a Banach algebra for the multiplication with the norm $\|a\|_{\mathcal{A}} = \|\omega_a\|_{L^1(\mathbb{R}^{2n})}$.

The next three lemmas are propositions 1.2.1, 1.2.3 and lemma A.2.1 in [LM].

Lemma 5.2.2. *We have $S^0_{0,0} \subset S^0_{0,0;2n+1} \subset \mathcal{A} \subset C^0(\mathbb{R}^{2n}) \cap L^\infty(\mathbb{R}^{2n})$, where $S_{0,0;2n+1}$ is the set of functions defined on $\mathbb{R}^{2n}$ such that $|(\partial_\xi^\alpha \partial_x^\beta a)(x,\xi)| \le C_{\alpha\beta}$ for $|\alpha| + |\beta| \le 2n+1$. The algebra $\mathcal{A}$ is stable by change of quantization, i.e. for all t real, $a \in \mathcal{A} \Longleftrightarrow J^t a = \exp(2i\pi t D_x \cdot D_\xi) a \in \mathcal{A}$.*

We recall that $(a_1 \sharp a_2)^w = a_1^w a_2^w$ with

$$(a_1 \sharp a_2)(X) = 2^{2n} \iint_{\mathbb{R}^{2n} \times \mathbb{R}^{2n}} a_1(Y_1) a_2(Y_2) e^{-4i\pi[X-Y_1, X-Y_2]} dY_1 dY_2.$$

Lemma 5.2.3. *The bilinear map $a_1, a_2 \mapsto a_1 \sharp a_2$ is defined on $\mathcal{A} \times \mathcal{A}$ and continuous valued in $\mathcal{A}$, which is a (noncommutative) Banach algebra for $\sharp$. The maps $a \mapsto a^w, a(x,D)$ are continuous from $\mathcal{A}$ to $\mathcal{L}(L^2(\mathbb{R}^n))$.*

Lemma 5.2.4. *Let b be a function in $\mathcal{A}$ and $T \in \mathbb{R}^{2n}, t \in \mathbb{R}$. Then the functions $\tau_T b$, b_t defined by $\tau_T b(X) = b(X-T), b_t(X) = b(tX)$ belong to $\mathcal{A}$ and*

$$\sup_{T \in \mathbb{R}^{2n}} \|\tau_T b\|_{\mathcal{A}} \le C \|b\|_{\mathcal{A}}, \qquad \|b_t\|_{\mathcal{A}} \le (1+|t|)^{2n} C \|b\|_{\mathcal{A}}.$$

Comments on the Wiener Lemma. The standard Wiener's lemma states that if $a \in \ell^1(\mathbb{Z}^d)$ is such that $u \mapsto a * u = C_a u$ is invertible as an operator on $\ell^2(\mathbb{Z}^d)$, then the inverse operator is of the form C_b for some $b \in \ell^1(\mathbb{Z}^d)$. Sjöstrand has proven several types of Wiener lemmas for $\mathcal{A}$ ([S2]). First a commutative version, saying that if $a \in \mathcal{A}$ and $1/a$ is a bounded function, then $1/a$ belongs to $\mathcal{A}$. Next, a noncommutative version of the Wiener lemma for the algebra $\mathcal{A}$: if an operator a^w with $a \in \mathcal{A}$ is invertible as a continuous operator on L^2, then the inverse operator is b^w with $b \in \mathcal{A}$. In a paper by Gröchenig and Leinert ([GL]), the authors prove several versions of the noncommutative Wiener lemma, and their definition of the twisted convolution is indeed very close to (a discrete version of) the composition formula above.

The main result of this chapter is the following

Theorem 5.2.5. *There exists a constant C such that, for all nonnegative functions a defined on $\mathbb{R}^{2n}$ satisfying $a^{(4)} \in \mathcal{A}$, the operator a^w is semi-bounded from below and, more precisely, satisfies*

$$a^w + C\|a^{(4)}\|_{\mathcal{A}} \ge 0.$$

The constant C depends only on the dimension n.

Note that this answers positively to our question (about relaxing the assumption on $a^{(4)}$), and as a byproduct gives the answer $4+2n+\epsilon$ for the number of derivatives needed to control

C in the Fefferman-Phong inequality[6]. Some results of this type were proven by Sjöstrand in [S2], namely the standard Gårding inequality with gain of one derivative for his class, $a \ge 0, a'' \in \mathcal{A} \Longrightarrow a(x, h\xi)^w + Ch \ge 0$. A version of the Hörmander-Melin inequality with gain of 6/5 of derivatives (see [H1]) was given by Hérau ([Hé]) who used a limited regularity on the symbol a, only such that $a^{(3)} \in \mathcal{A}$.

The chapter 2 implies readily the improvement of the Gårding inequality with gain of one derivative. Take $a \ge 0$ such that $a'' \in \mathcal{A}$: then $a^w = a^{\text{Wick}} - r(a)^w \ge -r(a)^w$, with $r(a)(X) = \int_0^1 \int_{\mathbb{R}^{2n}} (1-\theta) a''(X + \theta Y) Y^2 e^{-2\pi|Y|^2} 2^n dY d\theta$. Since $\mathcal{A}$ is stable by translation (see the lemma 5.2.4), we see that $r(a) \in \mathcal{A}$ and thus $r(a)^w$ is bounded on $L^2(\mathbb{R}^n)$ from the lemma 5.2.3.

5.3. Composition formulas. The next three lemmas are lemmas 2.2.1, 2.3.1, 2.3.3 in [LM].

Lemma 5.3.1. *Let a be a function defined on $\mathbb{R}^{2n}$ such that the fourth derivatives $a^{(4)}$ belong to $\mathcal{A}$. Then we have*

$$a^w = \Big(a - \frac{1}{8\pi}\operatorname{trace} a''\Big)^{\text{Wick}} + \rho_0(a^{(4)})^w,$$

with $\rho_0(a^{(4)}) \in \mathcal{A}$: more precisely $\|\rho_0(a^{(4)})\|_{\mathcal{A}} \le C_n \|a^{(4)}\|_{\mathcal{A}}$.

One should not expect the quantity $a - \frac{1}{8\pi}\operatorname{trace} a''$ to be nonnegative: this quantity will take negative values even in the simplest case $a(x,\xi) = x^2 + \xi^2$, so that the positivity of the quantization expressed by the lemma 4 is far from enough to get our result.

Remark. We note that, from the lemma 5.3.1 and the L^2 boundedness of operators with symbols in $\mathcal{A}$, the theorem is reduced to proving

$$a \ge 0, a^{(4)} \in \mathcal{A} \Longrightarrow \Big(a - \frac{1}{8\pi}\operatorname{trace} a''\Big)^{\text{Wick}} + C \ge 0.$$

Lemma 5.3.2. *For $p, q \in L^\infty(\mathbb{R}^{2n})$ real-valued with $p'' \in L^\infty(\mathbb{R}^{2n})$, we have*

$$\operatorname{Re}\big(p^{\text{Wick}} q^{\text{Wick}}\big) = \Big(pq - \frac{1}{4\pi}\nabla p \cdot \nabla q\Big)^{\text{Wick}} + R, \text{ with } \|R\|_{\mathcal{L}(L^2(\mathbb{R}^n))} \le C(n)\, \|p''\|_{L^\infty} \|q\|_{L^\infty}.$$

Lemma 5.3.3. *For p measurable real-valued function such that $p'', (p'p'')', (pp'')'' \in L^\infty$, we have*

$$p^{\text{Wick}} p^{\text{Wick}} = \int \Big[p(Z)^2 - \frac{1}{4\pi}|\nabla p(Z)|^2\Big] \Sigma_Z dZ + S,$$
$$\|S\|_{\mathcal{L}(L^2(\mathbb{R}^n))} \le C(n) \left(\|p''\|^2_{L^\infty} + \|(p''p')'\|_{L^\infty} + \|(pp'')''\|_{L^\infty}\right).$$

[6] This threshold was improved recently by A.Boulkhemair [B3] who proved that only $4 + n + \epsilon$ derivatives were needed.

Remark 5.3.4. Further reduction. To get our theorem, we shall prove

$$a \geq 0, a^{(4)} \in L^\infty(\mathbb{R}^{2n}) \Longrightarrow \Big(a - \frac{1}{8\pi}\operatorname{trace} a''\Big)^{\mathrm{Wick}} + C \geq 0.$$

We leave now the arguments of harmonic analysis and we will use a structure theorem on nonnegative $C^{3,1}$ functions as sum of squares of $C^{1,1}$ functions to write the operator $\Big(a - \frac{1}{8\pi}\operatorname{trace} a''\Big)^{\mathrm{Wick}}$ as a sum of squares of operators, up to L^2-bounded operators, thanks to the last two lemmas.

5.4. Sketching the proof. Our main argument relies on a decomposition theorem for nonnegative functions as sum of squares.

Theorem 5.4.1. *Let $m \in \mathbb{N}$. There exists an integer N and a positive constant C such that the following property holds. Let a be a nonnegative $C^{3,1}$ function defined on $\mathbb{R}^m$ such that $a^{(4)} \in L^\infty$; then we can write*

$$a = \sum_{1 \leq j \leq N} b_j^2$$

where the b_j are $C^{1,1}$ functions such that $b_j'', (b_j' b_j'')', (b_j b_j'')'' \in L^\infty$. More precisely, we have

$$\|b_j''\|_{L^\infty}^2 + \|(b_j' b_j'')'\|_{L^\infty} + \|(b_j b_j'')''\|_{L^\infty} \leq C \|a^{(4)}\|_{L^\infty}.$$

Note that this implies that each function b_j is such that b_j^2 is $C^{3,1}$ and that N and C depend only on the dimension m.

Part of this theorem is a consequence of the classical proof of the Fefferman-Phong inequality in [FP] and of the more refined analysis of Bony ([Bo1]) (see also the papers by Guan [Gu] and Tataru [Ta]). However the control of the L^∞ norm of the quantities $(b_j' b_j'')', (b_j b_j'')''$ seems to be new and is important for us.

Sketching the proof. We use a Calderón-Zygmund method and define

$$\rho(x) = \big(|a(x)| + |a''(x)|^2\big)^{1/4}, \quad \Omega = \{x, \rho(x) > 0\},$$

assuming as we may $\|a^{(4)}\|_{L^\infty} \leq 1$. Note that, since ρ is continuous, the set Ω is open. The metric $|dx|^2/\rho(x)^2$ is slowly varying in Ω: $\exists r_0 > 0, C_0 \geq 1$ such that

$$x \in \Omega, |y - x| \leq r_0 \rho(x) \Longrightarrow y \in \Omega, C_0^{-1} \leq \frac{\rho(x)}{\rho(y)} \leq C_0.$$

The constants r_0, C_0 can be chosen as "universal" constants, thanks to the normalization on $a^{(4)}$ above. Moreover the nonnegativity of a implies with $\gamma_j = 1$ for $j = 0, 2, 4$, $\gamma_1 = 3, \gamma_3 = 4$,

$$|a^{(j)}(x)| \leq \gamma_j \rho(x)^{4-j}, \quad 1 \leq j \leq 4.$$

Remark. We shall use the following notation: let A be a symmetric k-linear form on real normed vector space V. We define the norm of A by

$$\|A\| = \sup_{\|T\|=1} |AT^k|.$$

Since the symmetrized products of $T_1 \otimes \cdots \otimes T_k$ can be written as a linear combination of k-th powers, that norm is equivalent to the natural norm

$$|||A||| = \sup_{\substack{\|T_j\|=1,\\ 1\le j\le k}} |AT_1 \ldots T_k|$$

and in fact, when V is Euclidean, we have the equality $\|A\| = |||A|||$ (see [Ke]). For an arbitrary normed space, the best estimate is $|||A||| \le \frac{k^k}{k!}\,\|A\|$ (see the remark 3.1.2 in [LM]).

The basic properties of slowly varying metrics are summarized in the following lemma (see e.g. section 1.4 in [H2]).

Lemma 5.4.2. *Let a, ρ, Ω, r_0 be as above. There exists a positive number $r_0' \le r_0$, such that for all $r \in]0, r_0']$, there exists a sequence $(x_\nu)_{\nu\in\mathbb{N}}$ of points in Ω and a positive number M_r, such that the following properties are satisfied. We define $U_\nu, U_\nu^*, U_\nu^{**}$ as the closed Euclidean balls with center x_ν and radius $r\rho_\nu, 2r\rho_\nu, 4r\rho_\nu$ with $\rho_\nu = \rho(x_\nu)$. There exist two families of nonnegative smooth functions on $\mathbb{R}^m$, $(\varphi_\nu)_{\nu\in\mathbb{N}}$, $(\psi_\nu)_{\nu\in\mathbb{N}}$ such that*

$$\sum_\nu \varphi_\nu^2(x) = \mathbf{1}_\Omega(x),\ \operatorname{supp}\varphi_\nu \subset U_\nu, \psi_\nu \equiv 1 \quad \text{on } U_\nu^*,$$

$\operatorname{supp}\psi_\nu \subset U_\nu^{**} \subset \Omega$. *Moreover, for all integers l, we have*

$$\sup_{x\in\Omega,\nu\in\mathbb{N}} \|\varphi_\nu^{(l)}(x)\|\rho_\nu^l + \sup_{x\in\Omega,\nu\in\mathbb{N}} \|\psi_\nu^{(l)}(x)\|\rho_\nu^l < \infty.$$

The overlap of the balls U_ν^{**} is bounded, i.e.

$$\bigcap_{\nu\in\mathcal{N}} U_\nu^{**} \ne \emptyset \quad \Longrightarrow \quad \#\mathcal{N} \le M_r.$$

Moreover, $\rho(x) \sim \rho_\nu$ all over U_ν^{**} (i.e. the ratios $\rho(x)/\rho_\nu$ are bounded above and below by a fixed constant, provided that $x \in U_\nu^{**}$).

Since a is vanishing on Ω^c, we obtain

$$a(x) = \sum_{\nu\in\mathbb{N}} a(x)\varphi_\nu^2(x).$$

Definition 5.4.3. Let a, ρ, Ω be as above. Let θ be a positive number $\le \theta_0$, where $\theta_0 < 1/2$ is a fixed constant. A point $x \in \Omega$ is said to be

(i) θ-elliptic whenever $a(x) \ge \theta\rho(x)^4$,

(ii) θ-nondegenerate whenever $a(x) < \theta\rho(x)^4$: we have then $\|a''(x)\|^2 \ge \rho(x)^4/2$.

Let us first consider the "elliptic" indices ν such that x_ν is θ-elliptic. For $x \in U_\nu^{**}$, we have $a(x) \sim \rho_\nu^4$, so that with

$$b_\nu(x) = a(x)^{1/2}\psi_\nu(x), \quad b_\nu^2 = a\psi_\nu^2, \quad \varphi_\nu^2 b_\nu^2 = a\varphi_\nu^2$$

and on $\operatorname{supp}\varphi_\nu$ (where $\psi_\nu \equiv 1$),

$$\begin{cases} b'_\nu &= 2^{-1}a^{-1/2}a', \\ b''_\nu &= -2^{-2}a^{-3/2}a'^2 + 2^{-1}a^{-1/2}a'', \\ b'''_\nu &= 3\times 2^{-3}a^{-5/2}a'^3 - \frac{3}{4}a^{-3/2}a'a'' + 2^{-1}a^{-1/2}a''', \\ b_\nu^{(4)} &= -\frac{15}{16}a^{-7/2}a'^4 + \frac{9}{4}a^{-5/2}a'^2a'' - \frac{3}{4}a^{-3/2}a''^2 \\ &\qquad -a^{-3/2}a'a''' + \frac{1}{2}a^{-1/2}a^{(4)}, \end{cases}$$

yielding easily the result. The whole difficulty is concentrated on the next case.

The nondegenerate indices ν are those for which x_ν is θ-nondegenerate. Since a'' is large, according to our scaling, we may choose the coordinates on U_ν such that

$$\partial_1^2 a(x) \ge \rho_\nu^2/2 \text{ for } |x - x_\nu| \lesssim \rho_\nu.$$

Since we know also that a is small at some point in U_ν (if the constant θ_0 is suitably chosen, cf. the lemma A.1.5 in [LM]), we get that $\partial_1 a$ vanishes somewhere in U_ν. From the implicit function theorem, there exists α such that $\partial_1 a(\alpha(x'), x') = 0$ and thus, with $\beta = x_1 - \alpha(x'), R = \Bigl(\int_0^1 (1-t)\partial_1^2 a\bigl(\alpha(x') + t(x_1 - \alpha(x')), x'\bigr)dt\Bigr)^{1/2}$,

$$\begin{aligned} a(x) = a(x_1, x') &= R^2\beta^2 + a(\alpha(x'), x') \\ &= \int_0^1 (1-t)\partial_1^2 a\bigl(\alpha(x') + t(x_1 - \alpha(x')), x'\bigr)dt\bigl(x_1 - \alpha(x')\bigr)^2 + a(\alpha(x'), x'). \end{aligned}$$

We find easily $|\alpha(x') - x_{\nu 1}| \lesssim \rho_\nu$, $\quad |\alpha'(x')| \lesssim 1,\ |\alpha''(x')| \lesssim \rho_\nu^{-1},\ |\alpha'''(x')| \lesssim \rho_\nu^{-2}$. Following Bony's argument, we compute the derivatives of

$$x' \mapsto a(\alpha(x'), x') = c(x').$$

We have, denoting by ∂_2 the x'-partial derivative,

$$c' = \alpha'\partial_1 a + \partial_2 a = \partial_2 a,$$

(here we have used the identity $\partial_1 \mathbf{a}(\alpha(\mathbf{x}'), \mathbf{x}') \equiv \mathbf{0}$),

$$\begin{aligned} c'' &= \alpha'\partial_1\partial_2 a + \partial_2^2 a, \\ c''' &= \alpha''\partial_1\partial_2 a + {\alpha'}^2\partial_1^2\partial_2 a + 2\alpha'\partial_1\partial_2^2 a + \partial_2^3 a, \\ c'''' &= \alpha'''\partial_1\partial_2 a + 3\alpha''\alpha'\partial_1^2\partial_2 a + 3\alpha''\partial_1\partial_2^2 a \\ &\quad + {\alpha'}^3\partial_1^3\partial_2 a + 3{\alpha'}^2\partial_1^2\partial_2^2 a + 3\alpha'\partial_1\partial_2^3 a + \partial_2^4 a, \end{aligned}$$

$$\text{so that}\quad |c'| \lesssim \rho^3, \quad |c''| \lesssim \rho^2, \quad |c'''| \lesssim \rho, \quad |c''''| \lesssim 1.$$

This forces the function $B(x) = R(x)^2(x_1 - \alpha)^2$ to be $C^{3,1}$ with a j-th derivative bounded above by ρ_ν^{4-j} $(0 \le j \le 4)$, since it is the case for a and c. Defining $b(x) = R(x)\big(x_1 - \alpha(x')\big)$ we see that

$$a = b^2 + c, \quad |(b^2)^{(j)}| = |B^{(j)}| \lesssim \rho_\nu^{4-j},\ 0 \le j \le 4.$$

As a consequence, we have

$$R^2\beta^2 = \overbrace{B(\alpha(x'), x')}^{=0} + \overbrace{\int_0^1 \partial_1 B(\alpha(x') + \theta(x_1 - \alpha(x')), x') d\theta\, \beta}^{\in C^{2,1}},$$
$$|\beta^{(j)}| \lesssim \rho^{1-j}, 0 \le j \le 3,$$

and since the open set $\{\beta \neq 0\}$ is dense,

$$R^2\beta = \int_0^1 \partial_1 B(\alpha(x') + \theta(x_1 - \alpha(x')), x') d\theta \in C^{2,1},$$
$$|(R^2\beta)^{(j)}| \lesssim \rho_\nu^{3-j},\ 0 \le j \le 3.$$

Also we have $0 < R^2 = \omega \in C^{1,1}, \omega \sim \rho_\nu^2$ and

$$|\omega^{(j)}| \lesssim \rho_\nu^{2-j}, 0 \le j \le 2,$$

entailing that with $R = \omega^{1/2}$,

$$|R' = \frac{1}{2}\omega^{-1/2}\omega'| \lesssim 1,\ |R'' = -\frac{1}{4}\omega^{-3/2}\omega'^2 + \frac{1}{2}\omega^{-1/2}\omega''| \lesssim \rho_\nu^{-1}.$$

Using Leibniz' formula, we get

$$(R^2\beta)''' = (\omega\beta)''' = \omega'''\beta + 3\omega''\beta' + 3\omega'\beta'' + \omega\beta''',$$

which makes sense since ω''' is a distribution of order 1 and β is $C^{2,1}$. We know that $(\omega\beta)'''$ is L^∞, and since it is also the case of $\omega''\beta', \omega'\beta'', \omega\beta'''$, we get that $\omega'''\beta$ is bounded. On the other hand we have, since $\omega = R^2$,

$$\omega''' = 6R'R'' + 2\underbrace{R}_{C^{1,1}}\ \underbrace{R'''}_{\substack{\text{distribution}\\ \text{of order 1}}}$$

entailing that $\beta(6R'R'' + 2RR''')$ is L^∞ and since it is the case of $\beta R'R''$, we get that $\beta RR'''$ is L^∞ . With $b = R\beta$, we get $b'b'' = (R'\beta + R\beta')(R''\beta + 2R'\beta' + R\beta'')$ and to check that

$(b'b'')'$ is in L^∞, it is enough to check the derivatives of $R''\beta R'\beta$, $R''\beta R\beta'$ which are, up to bounded terms,

$$R'''\beta R'\beta = R'''\beta RR'\frac{\beta}{R}, \quad R'''\beta R\beta'$$

which are bounded according to the estimates above. Note that b'' is bounded. We want also to verify that $(bb'')''$ is bounded. We use that $(b^2)^{(4)}$ is bounded and since we have

$$\underbrace{(b^2)''''}_{\text{bounded}} = 2(b'\otimes b' + bb'')'' = 2\underbrace{(b'\otimes b'' + b''\otimes b')'}_{\text{bounded}} + 2(bb'')'',$$

we obtain the boundedness of $(bb'')''$. We can conclude by using an induction on the dimension (c is defined on $\mathbb{R}^{m-1}$) and a standard argument due to Guan ([Gu]) on slowly varying metrics.

Lemma 5.4.4. *Let a be a nonnegative function defined on $\mathbb{R}^{2n}$ such that $a^{(4)}$ belongs to $L^\infty(\mathbb{R}^{2n})$. We have from the theorem 5.4.1 the identity $a = \sum_{1\le j\le N} b_j^2$ along with some estimates on each b_j and its derivatives. Then we have*

$$\Big(a - \frac{1}{8\pi}\operatorname{trace} a''\Big)^{\text{Wick}} = \sum_{1\le j\le N}\left[\Big(b_j - \frac{1}{8\pi}\operatorname{trace} b_j''\Big)^{\text{Wick}}\right]^2 + R$$

where R is a L^2-bounded operator such that $\|R\|_{\mathcal{L}(L^2(\mathbb{R}^n))} \le C\|a^{(4)}\|_{L^\infty(\mathbb{R}^{2n})}$, C depending only on the dimension n.

This lemma is Lemma 3.2.1 in [LM] and is a direct consequence of section 5.3 and of the theorem 5.4.1. It allows us to obtain the reduction of remark 5.3.4 and to get the proof of the theorem 5.2.5.

A final comment. One may ask the following question: why did we not apply the induction argument on the Sjöstrand algebra $\mathcal{A}$ directly, and avoid that complicated detour with the Wick calculus? The answer to that interrogation is simple: as seen above the Fefferman-Phong induction procedure requires a cutting process (this is the metric $dX^2/\rho(X)^2$) and also a bending of the phase space (the function α is not linear). Although the cutting part may respect $\mathcal{A}$, it is not very likely that the rigid affine structure of $\mathcal{A}$ would survive the bending. We were somehow forced to push the induction procedure in some other corner, far away from the quantization business, and our theorem on nonnegative functions, although proven by induction on the dimension, is collecting all the information on lower dimensions.

6. Appendix

6.1. Cotlar's lemma. We recall the statement of the celebrated Cotlar's lemma in a version given in the paper [BL](lemme 4.2.3′) (see also [H1],[Un]).

Lemma 6.1.1 (Cotlar's lemma). *Let $(\Omega, \mathcal{M}, \mu)$ be a σ-finite measured space where μ is a positive σ-finite measure and let $\mathbf{H}$ be a Hilbert space. Let $\omega \mapsto A_\omega$ be a weakly measurable mapping from Ω into $\mathcal{L}(\mathbf{H})$. We assume that*

$$M = \max\left(\sup_{\omega\in\Omega}\int_\Omega \|A_\omega^* A_{\omega'}\|^{1/2} d\mu(\omega'),\ \sup_{\omega\in\Omega}\int_\Omega \|A_\omega A_{\omega'}^*\|^{1/2} d\mu(\omega')\right) < +\infty.$$

Then the operator $A = \int_\Omega A_\omega d\mu(\omega)$ is bounded on $\mathbf{H}$ with norm less than M.

Lemma 6.1.2. *Let ω be a measurable function defined on $\mathbb{R}^{2n} \times \mathbb{R}^{2n}$ such that*

$$|\omega(Y, Z)| \le \gamma_0 \bigl(1 + |Y - Z|\bigr)^{N_0}.$$

Then the operator $\iint \omega(Y, Z)\Sigma_Y \Sigma_Z dY dZ$ is bounded on $L^2(\mathbb{R}^n)$ with $\mathcal{L}(L^2(\mathbb{R}^n))$ norm bounded above by a constant depending on γ_0, N_0.

Proof. Writing

$$\Sigma_Y \Sigma_Z \Sigma_{Y'} \Sigma_{Z'} = \Sigma_Y \Sigma_Z\ \Sigma_Z \Sigma_{Y'}\ \Sigma_{Y'} \Sigma_{Z'}$$

we see that it is an immediate consequence of the lemma 6.1.1 and of the formula (3.1.9). □

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