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Preface

This volume is a textbook on Integration Theory, supplemented by 160 exercises provided with detailed answers. There are already many excellent texts on this topic and it is legitimate to ask whether it is worth while to add a new entry in an already long list of books on Measure Theory.

Nevertheless, the author's teaching experience has shown that many of these books were too difficult for a student exposed to integration theory for the first time. We have tried here to keep a rather elementary level, at least in the way of exposing our arguments and proofs, which are hopefully complete, detailed, sometimes at the cost of a lack of concision. Moreover, we hope that the many exercises (with answers) included at the end of each chapter will represent a key asset for the present book.

Another trend present in the contemporary textbook literature on integration theory is simply to omit the not-so-easy construction of Lebesgue measure. We are strongly opposed to this tendency, and we have made all efforts in our redaction to provide a complete construction of the mathematical objects used in the book, first and foremost for the construction of Lebesgue measure. Our point of view here is not exclusive of some compromises in the reading order which can be used by the reader trying to learn this material: the chapters of this book are of course ordered logically (chapter $n+1$ is using chapters $1, \dots, n$ and never chapter $n+2, \dots$), but some "construction" chapters, such as Chapter 2, parts of Chapters 4, 5, could be bypassed at first reading. We expect that a mathematically curious reader will feel the need of a construction after experiencing some of the most efficient (or more computational) parts of the theory and then will go back to these construction chapters.

Last but not least, we hope that this book could serve as a reasonable "entrance gate" to Integration Theory for scientists and mathematicians non-experts in measure theory. Another fact of mathematical life, say in the last thirty years, is that it is more and more difficult to learn some mathematics not directly connected with your professional area. Where is it possible for an Analyst to learn the algebraic properties of Theta functions? Where to find a text on Fourier Analysis accessible to an Algebraic Geometer? Although both questions above have (manifolds) answers, it remains difficult to find a way to enter a domain with which you are not a priori conversant. It is the author's opinion that accessibility is now a rarefied good in the literature, and we hope that the present book will provide its share of that good.

Integration Theories

The initial goal of integration theory, founded more that two millennia ago (the Greek scientist Archimedes of Syracuse, who lived in the third century B.C, was able to provide a quadrature of the parabola) is to compute areas, volumes of various objects. A somewhat simplified mathematical version of this question is to consider a function $f : [0,1] \longrightarrow \mathbb{R}_+$ and try to evaluate the area A between the x-axis and the curve $y = f(x)$. The standard notations are

$$A = \int_0^1 f(x)dx.$$

Of course some assumptions should be made on the function f for this area to make sense.

Riemann's integral

Greek mathematicians of the third century B.C. were aware of volumes of spheres, cones, polyhedra, and of many classical geometric objects. Later, in the early eighteenth century, Gottfried Wilhelm LEIBNIZ (1646–1716) introduced the *Infinitesimal Calculus*, whereas Isaac Newton (1642–1727) defined the *Calculus of Fluxions*, both inventions (close to each other) closely linked with a notion of integral calculus. However the first systematic attempt to define the integral of a function is due to the German mathematician Bernhard RIEMANN (1826–1866): cutting the source space (here $[0,1]$) into tiny pieces,

$$[0 = x_0, x_1], \dots [x_k, x_{k+1}], \dots, [x_{N-1}, x_N = 1], \quad x_j \uparrow,$$

you expect A to be close to

$$S_N = \sum_{0 \le k < N} (x_{k+1} - x_k) f(m_k), \quad \text{where } m_k \in [x_k, x_{k+1}],$$

since the area A should resemble the sum of the areas of the vertical rectangles with base (x_k, x_{k+1}) and height $f(m_k)$. In fact, assuming for instance f uniform limit of step functions (a step function is a finite linear combination of characteristic functions of intervals), you obtain that S_N has a limit when

$$\sup_{0 \le k < N} (x_{k+1} - x_k) \text{ goes to zero},$$

and you define that limit as $\int_0^1 f(x)dx$. It is indeed a simple matter to show directly that this procedure works for a continuous function on $[0,1]$. That theory is quite elementary but has several downsides. The very first one is a terrible instability with respect to small perturbations: in particular, if you modify the function f (say f continuous) on a rather small set such as the rational numbers $\mathbb{Q}$, you may ruin the integrability in the above sense. The rational numbers should be considered as "small" since it is a countable set $\{x_n\}_{n \in \mathbb{N}}$ and thus, for any $\epsilon > 0$,

$$\mathbb{Q} \subset \cup_{n \in \mathbb{N}} (x_n - \frac{\epsilon}{2^{n+2}}, x_n + \frac{\epsilon}{2^{n+2}})$$

and thus the "length" ℓ of $\mathbb{Q}$ is such that for any $\epsilon > 0$,

$$\ell \le \epsilon \sum_{n \in \mathbb{N}} 2^{-n-1} = \epsilon \Longrightarrow \ell = 0.$$

In particular, it is easy to show that the integral of $\mathbf{1}_{\mathbb{Q}\cap[0,1]}$ (a small perturbation of 0) cannot be defined by the procedure sketched above. Although the latter function may appear to be quite pathological, the effects of this instability are disturbing. Other difficulties are occurring with the Riemann integral, with complications to integrate unbounded functions and also to develop a comprehensive theory of multidimensional integrals.

The Lebesgue perspective

A key point in Lebesgue theory of integration (see e.g. [8]) is that to calculate the integral of $f : X \longrightarrow \mathbb{R}$, one should not cut into small pieces the source space X (for instance in small subintervals if X is an interval of $\mathbb{R}$) but that the *target space* should be cut into pieces depending on the function f itself. It is easy to illustrate this in the (very) simple case where

$$f : X = \{x_1, \dots, x_m\} \longrightarrow \{y_1, \dots, y_n\} = Y \subset \mathbb{R}.$$

We can evaluate $\sum_{x_j \in X} f(x_j)$ by sorting out the values taken by f:

$$\sum_{x_j \in X} f(x_j) = \sum_{y_k \in Y} y_k \operatorname{card}\bigl(\{x \in X, f(x) = y_k\}\bigr).$$

Also, playing around freely with the notations, say for f non-negative on $\mathbb{R}$, $H = \mathbf{1}_{\mathbb{R}_+}$,

$$\begin{aligned}\int_{\mathbb{R}} f(x)dx = \iint H(f(x)-y)H(y)dydx = \int\Bigl(\int H(f(x)-y)dx\Bigr)H(y)dy \\ = \int H(y)\ \texttt{measure}\bigl(\{x \in \mathbb{R}, f(x) > y\}\bigr)dy.\end{aligned}$$

If we can "measure" the sets $\{x \in \mathbb{R}, f(x) > y\}$, it is thus quite natural to take as a definition for the integral of f the last expression. Note that this expression is very simple if f is taking a finite number of values $y_1, \dots, y_N$: we have in that case

$$\int f(x)dx = \sum_{1 \le k \le N} y_k\ \texttt{measure}\bigl(\{x \in \mathbb{R}, f(x) = y_k\}\bigr).$$

The set $\{x \in \mathbb{R}, f(x) = y_k\}$ could be quite complicated and we shall see that many functions could be well approximated by *simple functions*, i.e. finite linear combinations of characteristic functions. To overcome the difficulties linked to the integration of unbounded functions, we may consider $f(x) = \frac{1}{2}x^{-1/2}\mathbf{1}_{(0,1)}(x)$ (integral 1); we get according to the previous computation,

$$\begin{aligned}\int_0^1 \frac{1}{2\sqrt{x}}dx &= \int_0^{+\infty} \texttt{measure}\bigl(\{x \in (0,1), \frac{1}{2\sqrt{x}} > y\}\bigr)dy \\ &= \int_0^{+\infty} \min(1, \frac{1}{4y^2})dy = \int_0^{1/2} dy + \int_{1/2}^{+\infty} \frac{1}{4y^2}dy = \frac{1}{2} + \frac{1}{4\frac{1}{2}} = 1,\end{aligned}$$

and many other examples involving unbounded functions can be dealt with. If we go back to our stability problem, we may consider the function $q = \mathbf{1}_{\mathbb{Q}}, f : \mathbb{R} \to \mathbb{R}_+$, then the integral of f is equal to the integral of $f + q$:

$$\int_{\mathbb{R}} (f+q)(x)dx = \int_0^{+\infty} \texttt{measure}\big(\{x \in \mathbb{R}, f(x) + q(x) > y\}\big)dy$$
$$= \int_0^{+\infty} \texttt{measure}\ \big(\{x \in \mathbb{R}, f(x) > y\}\big)dy = \int f(x)dx,$$

since the function q vanishes except on a set with measure 0. Since the reader may feel skeptical about the perturbation by this function q, let us give a finite version of it, illustrating the instability occurring with the Riemann approach, an instability which is not present with the Lebesgue simple method outlined above. We consider the interval $[0, 1]$ and for some large integer N the function

$$f(x) = \sum_{0 \le k < N} \mathbf{1}_{[\frac{k}{N}, \frac{k+2^{-N}}{N}]}(x).$$

Applying the Riemann method, using the sequence $x_k = k/N, 0 \le k < N$, we deal with

$$S = \sum_{0 \le k < N} (\frac{k+1}{N} - \frac{k}{N}) f(m_k), \quad m_k \in [\frac{k}{N}, \frac{k+1}{N}].$$

We may for instance choose $m_k = x_k = k/N$, so that $f(m_k) = 1$ and $S = 1$. On the other hand, Lebesgue's method uses the fact the the function f is taking two values $0, 1$, and the evaluation of the integral by this method gives

$$I = \texttt{measure}\{x \in [0,1], f(x) = 1\} = \sum_{0 \le k < N} 2^{-N}/N = 2^{-N}.$$

Nonetheless this value turns out to be the exact value of the integral, but also it goes to 0 when N goes to infinity whereas S is stuck at 1, very far from the true value I. It is of course a scaling problem, since choosing the sequence (x_k) such that $\sup_k |x_{k+1} - x_k| \le 2^{-N}$ will provide a more accurate value for S. Nevertheless this scaling phenomenon is a good illustration of the fact that a perturbation f with a small integral but with a large sup norm could trigger huge variations of S, although the Lebesgue calculation remains stable.

There is much more to say in favour of Lebesgue's point of view and in particular the fact that we can define a Banach space (complete normed vector space) of integrable functions, the space $L^1(\mathbb{R}^n)$, and also spaces $L^p(\mathbb{R}^n)$, $1 \le p \le +\infty$, other Banach spaces (L^2 is a Hilbert space), is of considerable interest and well-tuned to the developments of functional analysis. Moreover, Lebesgue's theory provides its user with a remarkably simple convergence theorem, the so-called Lebesgue's dominated convergence theorem. The problem at hand is to decide whether $\int f_n(x)dx$ is converging with limit $\int f(x)dx$ when we have already a (weak) pointwise information, i.e. $\lim_n f_n(x) = f(x)$ for all x. A precise statement can be found in Chapter 1 (Theorem 1.6.8), but let just say here that a domination condition

$$\sup_n |f_n(x)| = g(x) \quad \text{is such that} \quad \int |g(x)| dx < +\infty,$$

will ensure nonetheless the sought convergence of integrals but also convergence of the sequence of functions $(f_n)_{n\in\mathbb{N}}$ in the functional space L^1.

Is there a downside to Lebesgue's integration theory[1]? Mathematically speaking, the answer is no, and that theory has been widely used, polished and sometimes generalized to many different situations. However, it is true that Lebesgue's theory of integration is not elementary and that its actual construction requires a significant effort. On the other hand the *Instruction Manual* for Lebesgue Integration is indeed quite simple and one should encourage the reader to enjoy the simplicity and efficiency of that theory before going back to the more austere construction aspects.

We may draw a comparison with the construction and use of the real numbers: the real line $\mathbb{R}$ is widely used in Calculus and elsewhere as a basic mathematical object, but few students actually went through a construction of $\mathbb{R}$. In fact, $\mathbb{R}$ is also a very complicated object, as could be seen through the many examples of the present book (cardinality questions, non-measurable subsets, Cantor ternary set, Cantor sets with positive measure, category and measure,...), but nobody (?) is suggesting that getting some familiarity with the real line should not be a part of a standard mathematical curriculum.

Description of the contents of the book

Chapter 1, entitled *General Theory of Integration*, is presenting the basic framework for integration theory, with the notion of measure space. We obtain rather easily the three classical convergence theorems (Beppo Levi, Fatou, Lebesgue's dominated convergence) and we can define the space of integrable functions $L^1(\mu)$. This abstract presentation of integration is not difficult to follow, but there is obviously a shortage of significant examples of measure spaces at this stage of the exposition.

The main examples are constructed in Chapter 2, *Actual construction of measure spaces*; a first route is following the Riesz-Markov representation Theorem via linear forms on continuous compactly supported functions. We present as well the more set-theoretic Carathéodory approach. At the end of this chapter, we

[1] An utterly pragmatic point of view was defended by Richard W. HAMMING (1915–1998), a computer scientist and mathematician: *" Does anyone believe that the difference between the Lebesgue and Riemann integrals can have physical significance, and that whether say, an airplane would or would not fly could depend on this difference? If such were claimed, I should not care to fly in that plane." In N. Rose Mathematical Maxims and Minims, Raleigh NC: Rome Press Inc., 1988.* That criticism is surprising, since the norms of the functional spaces provided by Lebesgue theory are actually used in the numerical approximations and their stability is expressed by inequalities involving those norms.

introduce the notion of Hausdorff measure. Among the statements in the exercises, one may single out the construction of a non-measurable set, using the axiom of choice. The parts dealing with the construction of the Lebesgue measure are quite technical, and while using some earlier version of these notes for teaching a one-semester course, we always postponed the exposition of the details of the construction of Lebesgue measure to the very last week of class, after the students have acquired some familiarity with the scope and means of that integration theory.

Chapter 3 deals with *Spaces of integrable functions.* The important convexity inequalities (Jensen, Hölder, Minkowski) are studied and the definition of $L^p(\mu)$ spaces ($1 \le p \le \infty$) are given along with their main properties, most notably the fact that they are Banach spaces. We study as well integrals depending on a parameter, with continuity and differentiability properties; this part is of course related to many practical examples such as the Gamma function, Zeta function and many integrals or series depending on a parameter. Riemann-Lebesgue Lemma, Egoroff' and Lusin's theorems are proven. The last section provides a survey of various notions of convergence encountered in the text. Some exercises are related to various explicit computations, others to more abstract questions, such as examples of non-separable spaces.

The fourth chapter, *Integration on a product space,* is constructing integrals on product spaces, and contains statements and proofs of Tonelli and Fubini theorems. Some exercises are purely computational (e.g. computation of the volumes of the Euclidean balls in $\mathbb{R}^n$), others are more abstract, for instance with the study of the notion of monotone class.

Chapter 5 is entitled *Diffeomorphisms of open subsets of* $\mathbb{R}^n$ *and integration.* We deal there with the change-of-variable formula and give some classical examples, such as polar coordinates. We also define the integration on a smooth hypersurface of the Euclidean $\mathbb{R}^n$, using implicitly a distribution approach to the construction of the simple layer. The last part of this chapter goes back to the notion of Hausdorff measures introduced in Chapter 2 and to the construction of Cantor sets. We give many details on the construction of the classical Cantor ternary set, along with the computation of its Hausdorff dimension and with the study of the Cantor function (a.k.a. as the devil's staircase). We study also Cantor sets with positive measure and compare the (unrelated) notions of category and measure. We calculate the cardinalities of the Borel and Lebesgue σ-algebras on $\mathbb{R}^n$: this requires some effort related to the introduction of cardinals and ordinals and we have devoted a lengthy appendix to these topics.

Convolution is the topic of Chapter 6, in which the Banach algebra $L^1(\mathbb{R}^n)$ is studied, as well as the classical Young's inequality. Weak L^p spaces are introduced and we give a proof of the Hardy-Littlewood-Sobolev inequality, following an explicit argument due to E. Lieb and M. Loss [43]. In the exercises, the reader will find various computations related to the heat equation and to the Laplace operator. We give also a study of Lorentz spaces and of the notion of decreasing rearrangement.

Chapter 7 is entitled *Complex measures* and is essentially devoted to the proof of the classical Radon-Nikodym theorem, as well as to the expression of the dual of $L^p(\mu)$ for $1 \le p < \infty$. We give several examples with the spaces c_0, ℓ^p, and study various possible behaviours of weakly convergent sequences. The decomposition in absolutely continuous, pure point, singular continuous parts for a Borel measure on the real line is studied as well as the notion of polar decomposition of a vector-valued measure.

Basic Harmonic Analysis on $\mathbb{R}^n$ is the topic of Chapter 8. Here we have chosen to follow Laurent Schwartz' presentation of Fourier transformation, first via the space $\mathscr{S}(\mathbb{R}^n)$ of rapidly decreasing functions, for which it is truly easy to prove the Fourier inversion formula. Introducing the space $\mathscr{S}'(\mathbb{R}^n)$ of tempered distributions as the topological dual space of the Fréchet space $\mathscr{S}(\mathbb{R}^n)$ was impossible to resist, since the Fourier inversion formula follows almost immediately on the huge space $\mathscr{S}'(\mathbb{R}^n)$, by a trivial abstract nonsense argument. We took advantage of the fact that tempered distributions are much easier to understand than general distributions, essentially because the space $\mathscr{S}(\mathbb{R}^n)$ is simply a Fréchet space, whose topology is defined by a countable family of semi-norms. Understanding general distributions is complicated by the fact that the space of test functions is not metrizable. Anyhow, we recover easily the standard properties of the Fourier transformation as well as basic properties of periodic distributions. Along the way, we provide a proof of the Poisson summation formula using a Gabor's wavelet method (coherent states method).

The last chapter is the ninth, *Classical inequalities*, which begins with Hadamard's three-lines theorem and Riesz-Thorin interpolation. Although this technique is useful to provide natural generalizations of Young's inequality, it falls short of dealing with natural operators such that the Hilbert transform: for that purpose, we give a proof of the Marcinkiewicz Theorem. We introduce the notion of maximal function, and prove the Lebesgue differentiation theorem. In order to study Sobolev spaces, we start with a classical inequality due to Gagliardo and Nirenberg. It turns out that this inequality is a perfect tool to handle Sobolev embedding theorems. We would have liked to expand that chapter to study Fourier multipliers and Hörmander-Mikhlin theorems as well as more general Sobolev spaces, including the homogeneous ones. The best way to do this would have been to introduce various tools of harmonic analysis, such as Calderón-Zygmund operators and pseudodifferential techniques: this would have been obviously too much and we refer the reader to [5] for these developments.

Let us go through our *Appendix*, essentially intended to reach a reasonable self-containedness for the present book. The first section is concerned with set theory, cardinals, ordinals: these notions are important for the understanding of many problems related to measure theory, and we have chosen a rather lengthy and elementary presentation of this topic. Section 2 deals with various topological questions, including the notion of filter, useful for Tychonoff theorem. A proof of Baire theorem is given and some classical consequences are recalled (Banach-Steinhaus, open mapping theorem): these questions are important for the understanding of duality, which is also related to measure theory and L^p spaces. The last three sections of the appendix are concerned with basic formulas and classical computations related to integration. Although it might seem preposterous to provide again this widely available material in such a book, the author would like to point out in the first place that some of these formulas are not so easy to derive, but above all, it seems that the true absurdity would be to teach Lebesgue measure to people ignoring basic formulas of integral calculus. These elementary computational aspects are here as a gentle reminder that Mathematics is also about computation, and that refined concepts and tools are often finding their motivations in intricate calculations.