CARLEMAN INEQUALITIES

Nicolas Lerner
Université Pierre et Marie Curie (Paris 6)

March 14, 2016

Contents

Chapter 1

Basic tools for Carleman estimates techniques

1.1 Preliminaries

In 1939, the Swedish mathematician Torsten Carleman introduced in [5] a new method to prove uniqueness properties for $2D$ elliptic equations.

T. Carleman, 1892–1949

These inequalities have found many different applications in various branches of mathematical analysis, from uniqueness properties to control theory. Carleman's arguments are based upon some weighted inequalities and can be used with very little regularity assumptions on the operator under scope, a sharp contrast with Holmgren's uniqueness theorems which require analyticity.

The very first question raised and solved by Carleman was the following. Let Ω be a connected open subset of $\mathbb{R}^2$ and let assume that u is a solution of the elliptic PDE

$$(\partial_x^2 + \partial_y^2)u = V(x,y)u, \quad V \in L^\infty(\Omega),$$

such that u vanishes on a non-empty open subset ω of Ω. Then u is vanishing all over Ω. When V is an analytic function, thanks to the ellipticity of the constant coefficient Laplace operator, the function u is analytic and cannot vanish on ω without being identically 0 on the connected component Ω. However, even with a smooth (C^∞) function V, nothing better than C^∞ regularity can be achieved for u and C^∞ functions can vanish on open sets without being identically 0. So the result of [5] was really entering uncharted territory since most uniqueness results were using either hyperbolicity or, when hyperbolicity was not satisfied (such as for an elliptic operator), Cauchy-Kovalevskaya and Holmgren's theorem, requiring strong analyticity structure of the operator, were at the core of the arguments.

Instead of providing right away some elements on Carleman's method, it seems better to review the most standard Cauchy uniqueness results for strictly hyperbolic operators, such as the wave operator.

1.1.1 Hyperbolicity, energy method, well-posedness

We consider the following Cauchy problem ($t \in \mathbb{R}$ is the time-variable, $x \in \mathbb{R}^d$ are the space variables), $c > 0$ (speed of propagation),

$$\begin{cases} c^{-2}\partial_t^2 u - \Delta_x u = Vu + f, \\ u(0,x) = v_0(x), \\ (\partial_t u)(0,x) = v_1(x), \end{cases} \tag{1.1.1}$$

and we want to prove uniqueness: let u_1, u_2 be two solutions of (1.1.1) with the same initial data v_0, v_1; then by linearity the function $w = u_1 - u_2$ satisfies

$$\begin{cases} c^{-2}\partial_t^2 w - \Delta_x w = Vw, \\ w(0,x) = 0, \\ (\partial_t w)(0,x) = 0. \end{cases} \tag{1.1.2}$$

We calculate for $v \in C^2(\mathbb{R}, C_c^2(\mathbb{R}^d))$, with dot-products and norms in $L^2(\mathbb{R}^d)$, using the notation $v(t)(x) = v(t,x)$,

$$2\langle c^{-2}\partial_t^2 v - \Delta_x v, \partial_t v\rangle_{L^2(\mathbb{R}^d)} = \frac{d}{dt}\Bigl(c^{-2}\|\dot v(t)\|^2 + \|(\nabla_x v)(t)\|^2\Bigr),$$

so that, with $\Box_c = c^{-2}\partial_t^2 - \Delta$,

$$c^{-2}\|\dot v(t)\|^2 + \|(\nabla_x v)(t)\|^2 = c^{-2}\|\dot v(0)\|^2 + \|(\nabla_x v)(0)\|^2 + 2\int_0^t \langle(\Box_c v)(s), \dot v(s)\rangle ds.$$

This equality is true as well for functions in $C^2(\mathbb{R}_+, H^1(\mathbb{R}^d))$ and assuming that regularity for w we define the energy $E(t)$ of w by

$$E(t) = c^{-2}\|\dot w(t)\|^2 + \|(\nabla_x w)(t)\|^2,$$

and we have for $t \ge 0$, $E(t) = E(0) + 2\int_0^t \langle(\Box_c w)(s), \dot w(s)\rangle ds$. Using the equation satisfied by w, this gives

$$E(t) \le E(0) + 2\int_0^t \|V(s)w(s)\|\|\dot w(s)\| ds.$$

We have the Sobolev injection (for $d > 2$)

$$\dot{H}^1(\mathbb{R}^d) = \dot{W}^{1,2}(\mathbb{R}^d) \hookrightarrow W^{0,p}(\mathbb{R}^d), \quad \frac{1-0}{d} = \frac{1}{2} - \frac{1}{p}, \quad p = \frac{2d}{d-2}.$$

We infer that

$$\|V(s)w(s)\|^2_{L^2(\mathbb{R}^d)} \le \|V(s)^2\|_{L^{\frac{d}{2}}(\mathbb{R}^d)} \|w(s)^2\|_{L^{\frac{d}{d-2}}(\mathbb{R}^d)} = \|V(s)\|^2_{L^d} \|w\|^2_{L^{\frac{2d}{d-2}}},$$

so that $\|V(s)w(s)\|_{L^2(\mathbb{R}^d)} \le \|V(s)\|_{L^d} \|w\|_{L^{\frac{2d}{d-2}}} \le \kappa_d \|V(s)\|_{L^d} \|\nabla_x w\|_{L^2}$ and

$$E(t) \le E(0) + 2\int_0^t c\kappa_d \|V(s)\|_{L^d} \|\nabla_x w\|_{L^2} \|\dot{w}(s)\| c^{-1} ds$$
$$\le E(0) + c\kappa_d \int_0^t \|V(s)\|_{L^d} E(s) ds = R(t).$$

We obtain $\dot{R} = c\kappa_d \|V(t)\|_{L^d} E(t) \le c\kappa_d \|V(t)\|_{L^d} R(t)$, which implies

$$E(t) \le R(t) \le R(0) \exp c\kappa_d \int_0^t \|V(s)\|_{L^d} ds$$

and thus

$$0 \le E(t) \le E(0) e^{c\kappa_d \int_0^t \|V(s)\|_{L^d} ds} \tag{1.1.3}$$

with a finite rhs if we assume $V \in L^1_{loc}(\mathbb{R}_+, L^d(\mathbb{R}^d))$. Of course, Inequality (1.1.3) is providing uniqueness since $E(0) = 0$ implies $E(t) = 0$ for all positive times, but it contains a much stronger information, the so-called well-posedness of the Cauchy problem for the wave equation: the solution at time t has an energy controlled by the energy at initial time via a simple inequality of type

$$E(t) \le E(0)C(t),$$

where C is a known function depending on explicit given quantities (here the potential and the speed of propagation). That notion of *well-posedness* was introduced by the French mathematician Jacques Hadamard (1865–1963).

We would like to go beyond the global calculation (in the x variables) and provide a local uniqueness argument by a simple modification of the *energy method* displayed above. Let Ω be a C^1 open subset of $\mathbb{R}^d$: it means that there exists a C^1 function $\rho : \mathbb{R}^d \to \mathbb{R}$ such that

$$\Omega = \{x \in \mathbb{R}^d, \rho(x) < 0\}, \quad \rho(x) = 0 \Longrightarrow d\rho(x) \ne 0. \tag{1.1.4}$$

Lemma 1.1.1 (A consequence of Green's formula). *Let Ω be a C^1 open subset of $\mathbb{R}^d$ and let u, v be C^2 real-valued functions on $\overline{\Omega}$. Then we have*

$$\langle \Delta u, v\rangle_{L^2(\Omega)} + \langle \nabla u, \nabla v\rangle_{L^2(\Omega)} = \int_{\partial\Omega} v\frac{\partial u}{\partial \nu} d\sigma, \tag{1.1.5}$$

$$\langle \Delta u, v\rangle_{L^2(\Omega)} - \langle u, \Delta v\rangle_{L^2(\Omega)} = \int_{\partial\Omega} \big(v\frac{\partial u}{\partial \nu} - u\frac{\partial v}{\partial \nu}\big) d\sigma. \tag{1.1.6}$$

Proof. We have

$$\langle \Delta u, v\rangle_{L^2(\Omega)} = \int_\Omega v \operatorname{div}(\nabla u) dx = \int_\Omega \Big(\operatorname{div}(v\nabla u) - \nabla u\cdot\nabla v\Big) dx$$
$$= \int_{\partial\Omega} v\frac{\partial u}{\partial \nu} d\sigma - \langle \nabla u, \nabla v\rangle_{L^2(\Omega)},$$

proving (1.1.5), and implying (1.1.6) by switching u with v. □

We calculate then, using (1.1.5) with $u = w, v = \dot{w}$,

$$-2\operatorname{Re}\langle \Delta w, \partial_t w\rangle_{L^2(\Omega)} = -2\int_{\partial\Omega} \dot{w}\frac{\partial w}{\partial \nu} d\sigma + 2\langle \nabla w, \nabla \dot{w}\rangle_{L^2(\Omega)},$$

and this gives

$$\langle \Box_c w, 2\partial_t w\rangle_{L^2(\Omega)} + 2\int_{\partial\Omega} \dot{w}\frac{\partial w}{\partial \nu} d\sigma = \frac{d}{dt}\big(c^{-2}\|\dot{w}(t)\|^2_{L^2(\Omega)} + \|\nabla w\|^2_{L^2(\Omega)}\big). \tag{1.1.7}$$

We define now for $T_0 > 0$, $0 \le t \le T_0$, $R_0 = cT_0$,

$$F(t) = \int_{B(x_0, R_0 - ct)} \big(c^{-2}|\dot{w}(t,x)|^2 + |\nabla w(t,x)|^2\big) dx.$$

We have with $\Omega(t) = B(x_0, R_0 - ct)$

$$\dot{F}(t) = \langle \Box_c w, 2\partial_t w\rangle_{L^2(\Omega(t))} + 2\int_{\partial\Omega(t)} \dot{w}\frac{\partial w}{\partial \nu} d\sigma$$
$$+ \int \underbrace{\frac{d}{dt}\big(\mathbf{1}_{|x-x_0|\le R_0 - ct}\big)}_{-\delta(R_0 - ct - |x-x_0|)c} \big(c^{-2}|\dot{w}|^2 + |\nabla w|^2\big) dx,$$

and thus

$$\dot{F}(t) = \langle \Box_c w, 2\partial_t w\rangle_{L^2(\Omega(t))} + \int_{\partial\Omega(t)} \underbrace{\big(2\dot{w}\frac{\partial w}{\partial \nu} - c^{-1}|\dot{w}|^2 - c|\nabla w|^2\big)}_{\le 0} d\sigma.$$

We have thus for

$$\begin{cases} c^{-2}\partial_t^2 w - \Delta_x w = Vw, & 0 \le t \le T_0, |x - x_0| \le R_0 = cT_0 \\ w(0,x) = 0, & |x - x_0| \le R_0, \\ (\partial_t w)(0,x) = 0, & |x - x_0| \le R_0, \end{cases} \tag{1.1.8}$$

assuming for simplicity $V \in L^\infty_{loc}$,

$$\dot{F}(t) \le 2\|Vw\|_{L^2(\Omega(t))}\|\dot{w}\|_{L^2(\Omega(t))} \le 2\sigma\|w\|_{L^2(\Omega(t))}\|\dot{w}\|_{L^2(\Omega(t))} \le \sigma F(t).$$

We get from Gronwall's inequality for $0 \le t \le T_0$,

$$0 \le F(t) \le e^{\sigma t} F(0)$$

and thus a local uniqueness property for the wave equation with a bounded measurable potential. The same method provides a much more precise result: for w such that

$$\begin{cases} c^{-2}\partial_t^2 w - \Delta_x w = Vw + f, & 0 \le t \le T_0, |x - x_0| \le R_0 = cT_0 \\ w(0,x) = w_0(x), & |x - x_0| \le R_0, \\ (\partial_t w)(0,x) = w_1(x), & |x - x_0| \le R_0, \end{cases} \tag{1.1.9}$$

we find

$$\dot{F}(t) \le \sigma F(t) + 2\|f(t)\|_{L^2(\Omega(t))}\|\dot{w}\|_{L^2(\Omega(t))} \le (\sigma+1)F(t) + c^2\|f(t)\|^2_{L^2(\Omega(t))}$$

entailing

$$\|\dot{w}(t)\|^2_{\Omega(t)} + \|\nabla w(t)\|^2_{\Omega(t)} = F(t) \le e^{(\sigma+1)t}\big(\|w_1\|^2_{\Omega(0)} + \|\nabla w_0\|^2_{\Omega(0)}\big) + \int_0^t e^{(\sigma+1)(t-s)} c^2 \|f(s)\|^2_{L^2(\Omega(s))} ds.$$

These inequalities are interesting since for instance with a null source f, assuming that the initial data w_0, w_1 are vanishing on $B(x_0, R_0)$, we obtain nonetheless that the solution w of (1.1.9) vanishes near x_0 for a small positive time, but much more, that is w vanishes on the cone

$$\cup_{0 \le t \le T_0} B(x_0, R_0 - ct) = \{(t,x) \in [0,T_0] \times \mathbb{R}^d,\ |x - x_0| + ct \le R_0 = cT_0\}.$$

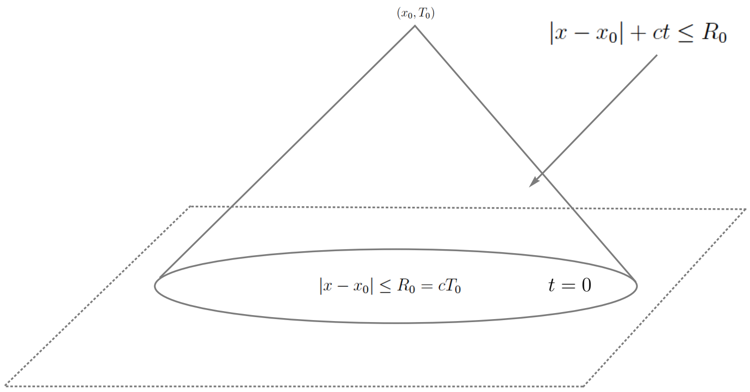


Figure 1.1: Initial data vanishing on $|x - x_0| \le R_0 \Longrightarrow$ the solution is vanishing on the cone $|x - x_0| + ct \le R_0 = cT_0$

Also, we see that the values of w at time $T > 0$ on the ball $B(x_0, R)$ will depend on the values of w at initial time $t = 0$ on the ball $B(x_0, R + cT)$.

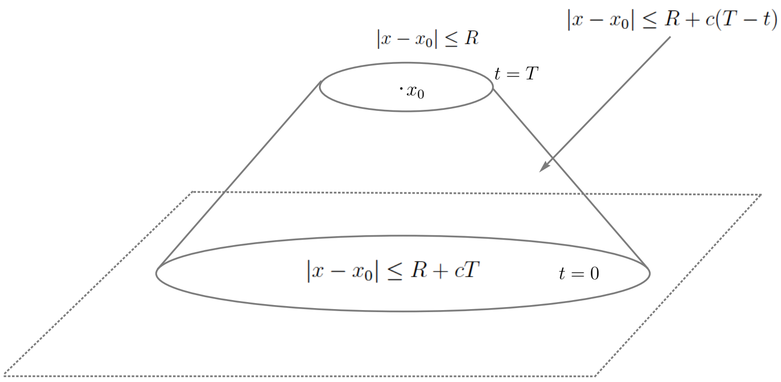


Figure 1.2: Domain of dependence for the wave equation

Remark 1.1.2 (Local and global uniqueness)**.** The reader may wonder why we made two different discussions above about global uniqueness and local uniqueness. We were able to prove that both global and local uniqueness hold for the wave equation. We may point out here that local uniqueness is a much stronger property than global uniqueness. In particular, if we study the heat equation

$$\frac{\partial}{\partial t} - \Delta_x, \quad t \in \mathbb{R}_+, x \in \mathbb{R}^d, \tag{1.1.10}$$

a global uniqueness result is not difficult to obtain, say for $C^1(\mathbb{R}_+, \mathscr{S}'(\mathbb{R}^d))$ solutions $u(t)$ of

$$\frac{\partial u}{\partial t} - \Delta_x u = f, \quad u(0,x) = u_0(x), \quad u_0 \in \mathscr{S}'(\mathbb{R}^d).$$

Using the Fourier transformation[1] we get that $\hat{u}(t,\xi) = e^{-4\pi^2 t|\xi|^2}\widehat{u_0}(\xi)$ so that if u_0 is vanishing, we have that u is vanishing, settling the global uniqueness property: if the Cauchy data vanishes globally in the space variables x, then the solution vanishes as well.

On the other hand, we know that a fundamental solution of the heat equation is

$$E(t,x) = H(t)(4\pi t)^{-d/2} e^{-\frac{|x|^2}{4t}} : \quad \frac{\partial E}{\partial t} - \Delta_x E = \delta(t) \otimes \delta(x).$$

[1]We define the Fourier transformation of a function u in the Schwartz space $\mathscr{S}(\mathbb{R}^d)$ by

$$\hat{u}(\xi) = \int_{\mathbb{R}^d} e^{-2i\pi x\cdot\xi} u(x)dx$$

and we get that $u(x) = \int_{\mathbb{R}^d} e^{2i\pi x\cdot\xi}\hat{u}(\xi)d\xi$. For u in the topological dual $\mathscr{S}'(\mathbb{R}^d)$ of $\mathscr{S}(\mathbb{R}^d)$ we define $\langle \hat{u}, \phi\rangle_{\mathscr{S}'(\mathbb{R}^d),\mathscr{S}(\mathbb{R}^d)} = \langle u, \hat{\phi}\rangle_{\mathscr{S}'(\mathbb{R}^d),\mathscr{S}(\mathbb{R}^d)}$. The same inversion formula holds. See Section 4.1 in our Appendix for more details on the Fourier transformation.

We know also that the C^∞ singular support of E is reduced to $0_{\mathbb{R}_t\times\mathbb{R}^d_x}$ and that the support of E is equal to the half-space $\{t \ge 0\}$. As a result for any point $x_0 \in \mathbb{R}^d\backslash\{0\}$, the C^∞ function $E(t,x)$ satisfies

$$\frac{\partial E}{\partial t} - \Delta_x E = 0 \quad \text{on } \mathbb{R}_t \times B(x_0, |x_0|), \quad E_{|\{t\le 0\}\times B(x_0,|x_0|)} = 0,$$

violating the local uniqueness property. This simple example is certainly a useful *caveat* about a global uniqueness property which turns out to be quite weak and very far from a local uniqueness property.

1.1.2 Lax-Mizohata Theorems

We have seen above that for the wave equation, a very satisfactory uniqueness theorem can be proven, going much beyond the uniqueness property: we were in fact able to prove a *well-posedness* result. We showed that some precise inequalities are controlling the size of the solution at time t by the size of the data at initial time. It turns out that this property is also true for strictly hyperbolic equations and not only for the wave equation.

Let us define the notion of strict hyperbolicity for a linear operator of order m. We are given on some open set U of $\mathbb{R}^n$ a linear scalar operator with smooth coefficients

$$P(x,D) = \sum_{|\alpha|\le m} a_\alpha(x) D_x^\alpha,$$

and a C^∞ hypersurface

$$\Sigma = \{x \in U, \rho(x) = 0\}, \ \rho \in C_c^1(U;\mathbb{R}), \ d\rho(x) \ne 0 \text{ at } \Sigma.$$

We define the principal symbol p_m of P as

$$p_m(x,\xi) = \sum_{|\alpha|=m} a_\alpha(x)\xi^\alpha, \quad (x,\xi) \in U \times \mathbb{R}^n.$$

We shall say that P is strictly hyperbolic with respect to Σ if for $x \in \Sigma$ and $\xi \in \mathbb{R}^n$ such that $\xi \wedge d\rho(x) \ne 0$, the polynomial in the variable σ given by

$$p_m(x, \xi + \sigma d\rho(x)) \text{ has simple real roots and } p_m(x, d\rho(x)) \ne 0.$$

Choosing local coordinates such that $\Sigma = \{x \in U, x_n = 0\}$, we have $d\rho = \vec{e}_n$ and we consider for $\xi' \ne 0$, $q(\sigma) = p_m(x', 0, \xi', \xi_n + \sigma)$. We require that the polynomial in τ of degree m given by

$$Q(\tau) = p_m(x', 0, \xi', \tau)$$

has simple real roots and $p_m(x', 0, 0, 1) \ne 0$. Of course, if it occurs at some point $x' = x'_0$ for all $\xi' \in \mathbb{S}^{n-2}$, the same property is true for the polynomial

$$\tau \mapsto p_m(x', x_n, \xi', \tau)$$

for (x', x_n) in a small enough neighborhood of $(x'_0, 0)$ in $\mathbb{R}^n$ and for $\xi' \neq 0$ in $\mathbb{R}^{n-1}$. In fact if we know that for $|\xi'_0| = 1$,

$$p_m(x'_0, 0, \xi'_0, \tau_0) = 0 \Longrightarrow \tau_0 \in \mathbb{R}, \quad \partial_\tau p_m(x'_0, 0, \xi', \tau_0) \neq 0,$$

we can apply the implicit function theorem for the function $p_m(x', x_n, \xi', \tau)$ and we find a neighborhood of $(x'_0, 0, \xi'_0, \tau_0)$ such that

$$p_m(x', x_n, \xi', \tau) = 0 \Longleftrightarrow \tau = \lambda(x', x_n, \xi'),$$

where λ is a smooth function homogeneous of degree 1 with respect to ξ'. Eventually we find m distinct real roots

$$\big(\lambda_j(x', x_n, \xi')\big)_{1\le j\le m},$$

for the polynomial $p_m(x', x_n, \xi', \tau)$ of the variable τ and we have

$$p_m(x', x_n, \xi', \tau) = e(x', x_n) \prod_{1\le j\le m} \big(\tau - \lambda_j(x', x_n, \xi')\big), \tag{1.1.11}$$

where the function e is not vanishing near the point $(x'_0, 0)$.

Of course the wave equation with propagation speed c (a positive parameter)

$$\frac{1}{c^2}\frac{\partial^2}{\partial t^2} - \Delta_x$$

is strictly hyperbolic with respect to any *spacelike* hypersurface, i.e. an hypersurface Σ of $\mathbb{R}_t \times \mathbb{R}^d_x$ with a conormal vector $\nu = (\tau, \xi) \in \mathbb{R}^{1+d}$ such that

$$-c^{-2}\tau^2 + |\xi|^2 < 0.$$

If a spacelike Σ is given by an equation $\rho(t, x) = 0$ with $d\rho \neq 0$ at Σ, we have $\nu = (\partial_t\rho, \partial_x\rho)$ and $|\partial_t\rho| > c|\partial_x\rho|$. We have thus

$$\Sigma \equiv ct = \alpha(x), \quad |\nabla\alpha| < 1.$$

So the wave equation is strictly hyperbolic with respect to any hyperplane with equation

$$ct = \langle \xi, x\rangle, \quad \xi \in \mathbb{R}^d, \quad \text{provided } |\xi| < 1,$$

so that this hyperplane does not intersect the light cone $c|t| = |x|$, except at $t = 0, x = 0$.

More generally, considering a symmetric matrix

$$\big(g_{jk}(x)\big)_{1\le j,k\le n}$$

with signature $(1, n-1)$ (1 positive eigenvalue, $n-1$ negative eigenvalues), we may consider its inverse matrix $\bigl(g^{jk}(x)\bigr)_{1\le j,k\le n}$ and setting $|\det(g_{jk})| = |g|$, we define the wave operator attached to g by

$$\Box_g = |g|^{-1/2} \sum_{1\le j,k\le n} \frac{\partial}{\partial x_j} |g|^{1/2} g^{jk}(x) \frac{\partial}{\partial x_k}.$$

We note that, for $u, v \in C_c^2$, we have

$$\begin{aligned}
\langle \Box_g u, v\rangle_{L^2(g)} &= \int (\Box_g u)(x)\overline{v(x)}|g(x)|^{1/2}dx \\
&= \sum_{1\le j,k\le n} \int (\partial_j g^{jk}|g|^{1/2}\partial_k u)(x)\overline{v(x)}dx \\
&= \sum_{1\le j,k\le n} \int u\ \overline{\partial_k |g|^{1/2} g^{jk}\partial_j v} dx \\
&= \int u\ \overline{|g|^{-1/2} \sum_{1\le j,k\le n} \partial_k |g|^{1/2} g^{jk}\partial_j v} |g|^{1/2} dx = \langle u, \Box_g v\rangle_{L^2(g)}.
\end{aligned}$$

The principal symbol of this wave equation is

$$p(x,\xi) = -\sum_{1\le j,k\le n} g^{jk}(x)\xi_j\xi_k = -\langle g^{-1}(x)\xi, \xi\rangle_{T_x(M), T_x^*(M)}.$$

The dual wave cone C_x at x is defined as

$$C_x = \{\xi \in T_x^*(M), \langle g^{-1}(x)\xi,\xi\rangle_{T_x(M),T_x^*(M)} > 0\}$$

and an hypersurface Σ with equation $\rho(x) = 0$ ($d\rho \ne 0$ at $\rho = 0$) will be said *spacelike* whenever

$$\langle g^{-1}(x)d\rho(x), d\rho(x)\rangle_{T_x(M),T_x^*(M)} > 0, \quad \text{i.e. } d\rho(x) \in C_x.$$

Since the symmetric matrix $g^{-1}(x)$ has signature $(1, n-1)$, we may assume, by rotation and rescaling that it is a diagonal matrix with $n-1$ eigenvalues equal to -1 and one eigenvalue equal to 1, i.e reduce our problem to the wave equation with speed 1. We have to deal with $\rho(t,x) = t - \alpha(x)$, $\|\nabla\alpha\| < 1$ and

$$q(\sigma) = -(\tau+\sigma)^2 + \|\xi + \sigma y\|^2, \quad \|y\| < 1, \quad (\tau,\xi)\wedge(1,y) \ne 0.$$

We have

$$q(\sigma) = \sigma^2(-1+\|y\|^2) + 2\sigma(-\tau + y\cdot\xi) - \tau^2 + \|\xi\|^2,$$

a real second-degree polynomial in the variable σ whose discriminant is

$$\begin{aligned}
\Delta = (-\tau + y\cdot\xi)^2 - (-\tau^2 + \|\xi\|^2)(-1+\|y\|^2) \\
= (y\cdot\xi)^2 - 2\tau y\cdot\xi + \tau^2\|y\|^2 + \|\xi\|^2(1-\|y\|^2).
\end{aligned}$$

If $y = 0$, we have $\Delta = \|\xi\|^2 > 0$ since $(\tau, \xi) \wedge (1, y) \neq 0$. If $y \neq 0$, we may assume that $y = \theta \mathbf{e}_1, 0 < \theta < 1$. We find

$$\Delta = \theta^2 \xi_1^2 - 2\theta\tau\xi_1 + \tau^2\theta^2 + \xi_1^2(1-\theta^2) + |\xi'|^2(1-\theta^2) = (\xi_1 - \theta\tau)^2 + |\xi'|^2(1-\theta^2),$$

so that $\Delta \geq 0$; if $\Delta = 0$ we get

$$\xi' = 0, \xi_1 = \theta\tau \Longrightarrow (\tau, \xi) = (\tau, \theta\tau, 0), (1, y) = (1, \theta, 0),$$

which is incompatible with $(\tau, \xi) \wedge (1, y) \neq 0$. As a result, the discriminant is positive and the roots are real and distinct.

Remark 1.1.3. We note that it is meaningless to say that an operator is hyperbolic: what makes sense is to say that an operator is hyperbolic with respect to some hypersurface. For instance the wave equation $c^{-2}\partial_t^2 - \Delta_x$ is shown above to be hyperbolic with respect to any (*spacelike*) hypersurface with equation $\rho(t, x) = 0$ with

$$c^{-2}(\partial_t \rho)^2 > \|\nabla_x \rho\|^2, \quad (\text{e.g. } t = 0),$$

but the wave equation is not hyperbolic with respect to a *characteristic* hypersurface (i.e. such that $c^{-2}(\partial_t \rho)^2 = \|\nabla_x \rho\|^2$) or a *timelike* hypersurface (i.e. such that $c^{-2}(\partial_t \rho)^2 < \|\nabla_x \rho\|^2$). To check the latter statement, we see only that for the hyperplane $x_1 + at = 0, c^2 > a^2$ we have $\nu = (a, 1, 0)$

$$\begin{aligned} q(\sigma) = -c^{-2}(\tau + \sigma a)^2 + (\xi_1 + \sigma)^2 + \|\xi'\|^2 \\ = \sigma^2(-a^2c^{-2} + 1) + 2\sigma(-a\tau c^{-2} + \xi_1) - c^{-2}\tau^2 + \|\xi\|^2, \end{aligned}$$

and for $\|\xi'\| = 1, \tau = 0, \xi_1 = 0$ (so that $(\tau, \xi) \wedge \nu \neq 0$) we have

$$q(\sigma) = \sigma^2 \underbrace{(-a^2c^{-2} + 1)}_{>0} + 1,$$

whose roots are purely imaginary.

For strictly hyperbolic operators, we can apply a variant of the energy method described in (the previous) Section 1.1.1 and prove some well-posedness inequalities for such evolution equations. We want now to show that without hyperbolicity, no well-posedness could be expected. In fact, we shall see that

$$\text{Strict Hyperbolicity} \Longrightarrow \text{Well-posedness} \Longrightarrow \text{Hyperbolicity}.$$

The first implication is proven in Section 1.1.1 for the wave equation and the second implication is known by the generic name of Lax-Mizohata Theorem[2]. Hyperbolicity will mean here for a scalar operator that the roots in (1.1.11) are real-valued but not

[2] Peter Lax is a Hungarian-born (1926) American mathematician. Shigeru Mizohata (1924–2002) is a Japanese mathematician.

necessarily distinct. To start with a simple example, closely linked with Carleman's interests, we shall consider the following evolution equation

$$\partial_t^2 u + \partial_x^2 u = 0, \quad u(0,x) = u_0(x), \quad (t,x) \in \mathbb{R}^2. \tag{1.1.12}$$

The operator $\partial_t^2 + \partial_x^2$ is simply the (elliptic) Laplace operator, which is not hyperbolic (with respect to any hypersurface): the roots of $\tau \mapsto \tau^2 + \xi^2 = 0$ are purely imaginary. We cannot expect a control of the solution of (1.1.12) at a positive time t by the initial datum: we cannot have, say for N large integer, K, L relatively compact open subsets of $\mathbb{R}^2$

$$\|u(t)\|_{H^{-N}(K)} \le C_{N,K,L} \|u(0)\|_{H^N(L)}.$$

Taking for instance $u(0,x) = \cos(\lambda x)$, we find that $u(t,x) = e^{\lambda t}\cos(\lambda x)$ solves (1.1.12). At time $t=0$, we have $\|u(0)\|_{H^N(L)} \le C_L \lambda^N$ and for $t>0$, $\|u(t)\|_{H^{-N}(K)} \ge e^{\lambda t} c_K \lambda^{-N}$, $c_K > 0$. The inequality above would imply for some $t>0$ and any positive λ

$$e^{\lambda t} c_K \lambda^{-N} \le \|u(t)\|_{H^{-N}(K)} \le C_{N,K,L} \|u(0)\|_{H^N(L)} \le C_{N,K,L} C_L \lambda^N,$$

and we would have for some $t>0$ $\limsup_{\lambda \to +\infty} e^{\lambda t} \lambda^{-2N} < +\infty$, which is absurd. We may rephrase this by saying that the Cauchy problem for the Laplace equation is *ill-posed*: strong oscillations in the initial data ($\cos \lambda x$) keep that data bounded, but trigger an exponential increase in time ($e^{\lambda t} \cos \lambda x$).

The paper [17] by P. Lax and the article [27] by S. Mizohata provided a more general statement, proving that a well-posed problem must be hyperbolic. Further developments were given by the Ivrii-Petkov article [16]. We reproduce here their arguments in a more specialized framework. We consider a $N \times N$ system of PDE with constant coefficients in one space dimension: for a $N \times N$ real-valued matrix A, our evolution equation is

$$\partial_t \begin{pmatrix} u_1 \\ \vdots \\ u_N \end{pmatrix} - \mathbf{A}\partial_x \begin{pmatrix} u_1 \\ \vdots \\ u_N \end{pmatrix} = 0, \qquad \begin{pmatrix} u_1(0,x) \\ \vdots \\ u_N(0,x) \end{pmatrix} = \begin{pmatrix} \nu_1(x) \\ \vdots \\ \nu_N(x) \end{pmatrix}.$$

Let us assume that this system is not hyperbolic, i.e. the matrix A has a non-real eigenvalue $\lambda + i\mu, \mu \ne 0$. We have for a non-zero vector $X + iY$ in $\mathbb{C}^N$,

$$A(X+iY) = (\lambda + i\mu)(X+iY)$$

and since A is real-valued, we get $AX = \lambda X - \mu Y, \quad AY = \mu X + \lambda Y$. We note that $X \wedge Y \ne 0$, otherwise if $X \ne 0$ (resp. $Y \ne 0$), we have $Y = \alpha X$ (resp. $X = \alpha Y$) and $X + iY = (1+i\alpha)X$ (resp. $X + iY = (\alpha + i)Y$) and thus

$$(1+i\alpha)AX = A(X+iY) = (\lambda+i\mu)(X+iY) = (\lambda+i\mu)(1+i\alpha)X,$$

$$(\text{resp.} \quad (\alpha+i)AY = A(X+iY) = (\lambda+i\mu)(X+iY) = (\lambda+i\mu)(\alpha+i)Y),$$

implying $AX = (\lambda + i\mu)X$ (resp. $AY = (\lambda + i\mu)Y$) which is not possible since AX is real-valued and $\mu X \neq 0$ (resp. $\mu Y \neq 0$). We calculate then

$$(A-\lambda)^2 Y = (A-\lambda)\mu X = -\mu^2 Y \Longrightarrow (A-\lambda)^{2k} Y = (-1)^k \mu^{2k} Y.$$

As a result for $\tau \in \mathbb{R}$, we have

$$\begin{aligned} e^{i\tau A} Y &= e^{i\tau\lambda} e^{i\tau(A-\lambda)} Y \\ &= e^{i\tau\lambda}\Big(\sum_{k\ge 0} \frac{i^{2k}\tau^{2k}}{(2k)!}(-1)^k \mu^{2k} Y + \sum_{k\ge 0} \frac{i^{2k+1}\tau^{2k+1}}{(2k+1)!}(-1)^k \mu^{2k}(A-\lambda)Y\Big) \\ &= e^{i\tau\lambda}\big(\cosh(\mu\tau)Y + i\sinh(\mu\tau)X\big). \end{aligned}$$

Assuming that we have a solution valued in $C^1([0,T]; \mathscr{S}'(\mathbb{R};\mathbb{R}^N))$ for some $T > 0$ and for an initial data in the Schwartz space $\mathscr{S}(\mathbb{R};\mathbb{R}^N)$, considering $v(t,\xi)$ the Fourier transform with respect to x of u, we get

$$\dot v(t,\xi) = i2\pi\xi A\, v(t,\xi), \quad v(0,\xi) = \widehat{\nu}(\xi).$$

Let $\chi \in C_c^\infty(\mathbb{R})$, equal to 1 in a neighborhood of 0 and let $\epsilon > 0$ be given. We consider the compactly supported distribution $\chi(\epsilon\xi)v(t,\xi)$ and we have

$$\begin{aligned} &\frac{d}{dt}\langle \big(e^{-2i\pi t\xi A}\chi(\epsilon\xi)v(t,\xi)\big), \phi(\xi)\rangle \\ &\quad = -\langle 2i\pi\xi A e^{-2i\pi t\xi A}\chi(\epsilon\xi)v(t,\xi), \phi(\xi)\rangle + \langle e^{-2i\pi t\xi A}\chi(\epsilon\xi)2i\pi\xi A v(t,\xi), \phi(\xi)\rangle = 0, \end{aligned}$$

so that $e^{-2i\pi t\xi A}\chi(\epsilon\xi)v(t,\xi) = \chi(\epsilon\xi)\hat\nu(\xi)$ and thus

$$\chi(\epsilon\xi)v(t,\xi) = \chi(\epsilon\xi)e^{2i\pi t\xi A}\hat\nu(\xi).$$

Choosing $\widehat{\nu}(\xi) = \omega(\xi)Y = e^{-(1+\xi^2)^{1/4}}Y$ (a vector in the Schwartz space), we find

$$\chi(\epsilon\xi)v(t,\xi) = \chi(\epsilon\xi)\omega(\xi)e^{i2\pi t\xi\lambda}\big(\cosh(\mu 2\pi t\xi)Y + i\sinh(\mu 2\pi t\xi)X\big).$$

The weak limit of the lhs is $v(t,\xi)$ and testing on $\phi \in C_c^\infty(\mathbb{R})$, the equality above implies that

$$\begin{aligned} &\langle v(t,\xi), \phi(\xi)\rangle_{\mathscr{S}'(\mathbb{R};\mathbb{R}^N),\mathscr{S}(\mathbb{R})} \\ &\qquad = \int \phi(\xi)e^{i2\pi t\xi\lambda}\big(\cosh(\mu 2\pi t\xi)Y + i\sinh(\mu 2\pi t\xi)X\big)\omega(\xi)d\xi. \end{aligned}$$

In particular this implies that the linear form defined for $\phi \in C_c^\infty(\mathbb{R})$ by the lhs is a tempered distribution: this is not the case since for

$$\phi(\xi) = e^{-2i\pi t\xi\lambda}e^{-(1+\xi^2)^{1/4}}\chi(\epsilon\xi)\kappa(\xi)$$

$(\kappa \in C^\infty(\mathbb{R};\mathbb{R}_+), \operatorname{supp}\kappa = [0,+\infty), \kappa = 1 \text{ on } [1,+\infty))$, that would imply that

$$I_\epsilon = \int_0^{+\infty} \chi(\epsilon\xi)\kappa(\xi)\cosh(\mu 2\pi t\xi)e^{-2(1+\xi^2)^{1/4}}d\xi,$$

has a finite limit when ϵ goes to 0 which is not the case: by Fatou's lemma

$$\liminf I_\epsilon \geq \int_0^{+\infty} \kappa(\xi) \cosh(\mu 2\pi t\xi) e^{-2(1+\xi^2)^{1/4}} d\xi = +\infty.$$

As a result our very mild assumption of well-posedness, i.e. for an initial data in the Schwartz space, there exists a solution in $C^1([0,T];\mathscr{S}'(\mathbb{R};\mathbb{R}^N))$ for some $T>0$, cannot hold and the problem is ill-posed in that sense.

1.1.3 Holmgren's Uniqueness Theorems

Theorem 1.1.4 (Holmgren's Uniqueness Theorem).[3] *Let*

$$P(x,D) = \sum_{|\alpha|\leq m} a_\alpha(x) D_x^\alpha,$$

be a linear operator with analytic coefficients on some open subset Ω of $\mathbb{R}^n$ and let Σ be a non-characteristic C^1 hypersurface[4]*, so that we have a partition $\Omega = \Omega_- \cup \Sigma \cup \Omega_+, \Omega_\pm$ open. Let u be a distribution on Ω such that $u_{|\Omega_-} = 0$. Then $u = 0$ in an open neighborhood of Σ.*

For a proof of this result, see for instance *Theorem 8.6.5* in [12] or Section 21 in [36]. Note that this result implies that the Laplace equation $\Delta u = Vu$ with V analytic has the Cauchy uniqueness with respect to any hypersurface (the ellipticity implies that any hypersurface is non-characteristic). However, it leaves wide open the Cauchy uniqueness for the same problem when V is not analytic.

1.1.4 Carleman's idea

Let us choose a model problem, simple enough to get an easy exposition of Carleman's main initial ideas. We are interested in proving that for $u \in C^1(\mathbb{R}^2_{t,x};\mathbb{C})$, $a \in L^\infty(\mathbb{R}^2)$,

$$\left.\begin{aligned} \partial_t u + i\partial_x u = a(t,x)u, \\ u(t,x) = 0 \text{ for } t<0, \end{aligned}\right\} \Longrightarrow u = 0. \tag{1.1.13}$$

Although Carleman's question was concerned by the Laplace operator, the problem above is dealing with the $\bar\partial$ equation, still an elliptic operator (but with complex coefficients). As already mentioned, this result is **not** a consequence of Holmgren's Theorem since the function a fails to be analytic.

Carleman's idea dealt with proving some weighted estimate, say for smooth compactly supported functions w, a real-valued function ϕ and a large parameter λ: there exists $C>0$ such that for all $w \in C_c^1(\mathbb{R}^2)$ and all $\lambda \geq 1$,

$$C\|e^{-\lambda\phi}(\partial_t + i\partial_x)w\|_{L^2(\mathbb{R}^2)} \geq \lambda^{1/2}\|e^{-\lambda\phi} w\|_{L^2(\mathbb{R}^2)}. \tag{1.1.14}$$

[3] Erik Holmgren (1872–1943) is a Swedish mathematician who proved a special case of this theorem. The German-born American Mathematician Fritz John (1910–1994) proved this result for classical solutions. This result fails to generalize to non-linear equations as proven by [25].

[4] $\Sigma = \{x \in \Omega, \rho(x) = 0\}, \rho \in C^1(\Omega;\mathbb{R}), d\rho \neq 0$ at $\rho = 0$, $p_m(x, d\rho(x)) \neq 0, p_m(x,\xi) = \sum_{|\alpha|=m} a_\alpha(x)\xi^\alpha$.

Before embarking on the proof of such an inequality, let us show that a good choice of the function ϕ will lead to a proof of the sought uniqueness property. Let us take

$$\phi(t,x) = t + \alpha x^2 - \frac{t^2}{2}, \quad \alpha > 0 \text{ to be chosen later.}$$

and let us apply (1.1.14) to $w = \chi u$, where χ is a smooth cutoff function,

$$\chi = 1 \text{ on } t^2 + x^2 \leq 1 \text{ and supported in } t^2 + x^2 \leq \beta^2,\ \beta > 1 \text{ to be chosen later.}$$

We find with L^2 norms, since $(\partial_t + i\partial_x)u = au$

$$\begin{aligned}\lambda^{1/2}\|e^{-\lambda\phi}\chi u\| &\leq C\|e^{-\lambda\phi}[\partial_t + i\partial_x, \chi]u\| + C\|e^{-\lambda\phi}\chi a u\| \\ &\leq C\|e^{-\lambda\phi}(\chi'_t + i\chi'_x)u\| + C\|a\|_{L^\infty(\operatorname{supp}\chi)}\|e^{-\lambda\phi}\chi u\|.\end{aligned}$$

We note that $\operatorname{supp}(\chi'_t + i\chi'_x)u \subset \{1 \leq t^2 + x^2 \leq \beta^2, t \geq 0\} = K$ since $\nabla\chi$ is supported in the ring $1 \leq t^2 + x^2 \leq \beta^2$ and $\operatorname{supp} u \subset \{t \geq 0\}$. As a result, on the support of $(\chi'_t + i\chi'_x)u$, we have

$$\phi = t - \frac{t^2}{2} + \alpha x^2 \geq t - \frac{t^2}{2} + \alpha(1-t^2) \geq t - t^2(\frac{1}{2} + \alpha) + \alpha \geq t - t\beta(\frac{1}{2} + \alpha) + \alpha \geq \alpha > 0$$

if we choose

$$1 < \beta \leq \frac{1}{\frac{1}{2} + \alpha} = \frac{2}{1 + 2\alpha}, \quad \text{i.e. } 0 < \alpha < 1/2.$$

As a result we have

$$\big(\lambda^{1/2} - C\|a\|_{L^\infty(\operatorname{supp}\chi)}\big)\|e^{-\lambda\phi}\chi u\| \leq C\|\nabla\chi\|_{L^\infty} e^{-\lambda\alpha}\|u\|_{L^2(K)},$$

so that for $\lambda^{1/2} \geq 2C\|a\|_{L^\infty(\operatorname{supp}\chi)}$

$$\lambda^{1/2}\|e^{-\lambda\phi}\chi u\| \leq 2C\|\nabla\chi\|_{L^\infty} e^{-\alpha\lambda}\|u\|_{L^2(K)}.$$

On the other hand, on $t^2 + x^2 \leq \alpha^4$, we have

$$\phi(t,x) \leq t + x^2 - \frac{t^2}{2} \leq \alpha^2 + \alpha^4$$

and this implies for λ large enough,

$$\begin{aligned}\lambda^{1/2} e^{-\lambda(\alpha^2+\alpha^4)}\|u\|_{L^2(t^2+x^2\leq\alpha^4)} \leq \lambda^{1/2}\|e^{-\lambda\phi}u\|_{L^2(t^2+x^2\leq\alpha^4)} &\leq \lambda^{1/2}\|e^{-\lambda\phi}\chi u\| \\ &\leq 2C\|\nabla\chi\|_{L^\infty} e^{-\lambda\alpha}\|u\|_{L^2(K)},\end{aligned}$$

implying that u vanishes on $t^2 + x^2 \leq \alpha^4$, since for α small enough

$$\alpha > \alpha^2 + \alpha^4, \quad \text{true e.g. when } 0 < \alpha < 1/2.$$

Since the problem is translation invariant with respect to x, we get that u vanishes on $t \leq \alpha^2$ and by a connexity argument that u vanishes on $\mathbb{R}^2$.

We are left with the proof of the estimate (1.1.14). Defining $v = e^{-\lambda\phi}w$, it amounts to prove

$$C\|e^{-\lambda\phi}(\partial_t + i\partial_x)e^{\lambda\phi}v\|_{L^2(\mathbb{R}^2)} = C\|(\partial_t + i\lambda\phi'_x + i\partial_x + \lambda\phi'_t)v\|_{L^2(\mathbb{R}^2)} \geq \lambda^{1/2}\|v\|_{L^2(\mathbb{R}^2)}.$$

We note that

$$\begin{aligned}\|(\partial_t + i\lambda\phi'_x + i\partial_x + \lambda\phi'_t)v\|^2 &= \|(\partial_t + i\lambda\phi'_x)v\|^2 + \|(i\partial_x + \lambda\phi'_t)v\|^2 \\ &\qquad + 2\,\mathrm{Re}\langle(\partial_t + i\lambda\phi'_x)v, (i\partial_x + \lambda\phi'_t)v\rangle.\end{aligned}$$

We have

$$\begin{aligned}&2\,\mathrm{Re}\langle(\partial_t + i\lambda\phi'_x)v, (i\partial_x + \lambda\phi'_t)v\rangle \\ &\qquad = \langle(\partial_t + i\lambda\phi'_x)v, (i\partial_x + \lambda\phi'_t)v\rangle + \langle(i\partial_x + \lambda\phi'_t)v, (\partial_t + i\lambda\phi'_x)v\rangle \\ &\qquad\qquad = \langle -(\partial_t + i\lambda\phi'_x)(i\partial_x + \lambda\phi'_t)v + (i\partial_x + \lambda\phi'_t)(\partial_t + i\lambda\phi'_x)v, v\rangle.\end{aligned}$$

We need to calculate the commutator

$$[i\partial_x + \lambda\phi'_t, \partial_t + i\lambda\phi'_x] = -\lambda\phi''_{xx} - \lambda\phi''_{tt} = \lambda(1 - 2\alpha).$$

As a result, for $0 < \alpha \leq 1/4$, we have

$$\|(\partial_t + i\lambda\phi'_x + i\partial_x + \lambda\phi'_t)v\|^2 \geq \frac{\lambda}{2}\|v\|^2,$$

providing (1.1.14).

The hypersurface Σ separates the reference open set Ω in disjoint open subsets Ω_+(above Σ) and Ω_- (below Σ). In our picture, Σ is the hyperplane $x_n = 0$, which can always be achieved by a C^1 changes of variables. Although it is tempting to choose the weight ϕ to be equal to x_n, it is not a good idea and some convexification should be performed: In particular the level set $\phi = 0$ should contain a point of Σ, be included in $\overline{\Omega}_-$ and such that $\phi > 0$ on $\operatorname{supp}\chi \cap \overline{\Omega}_+$.

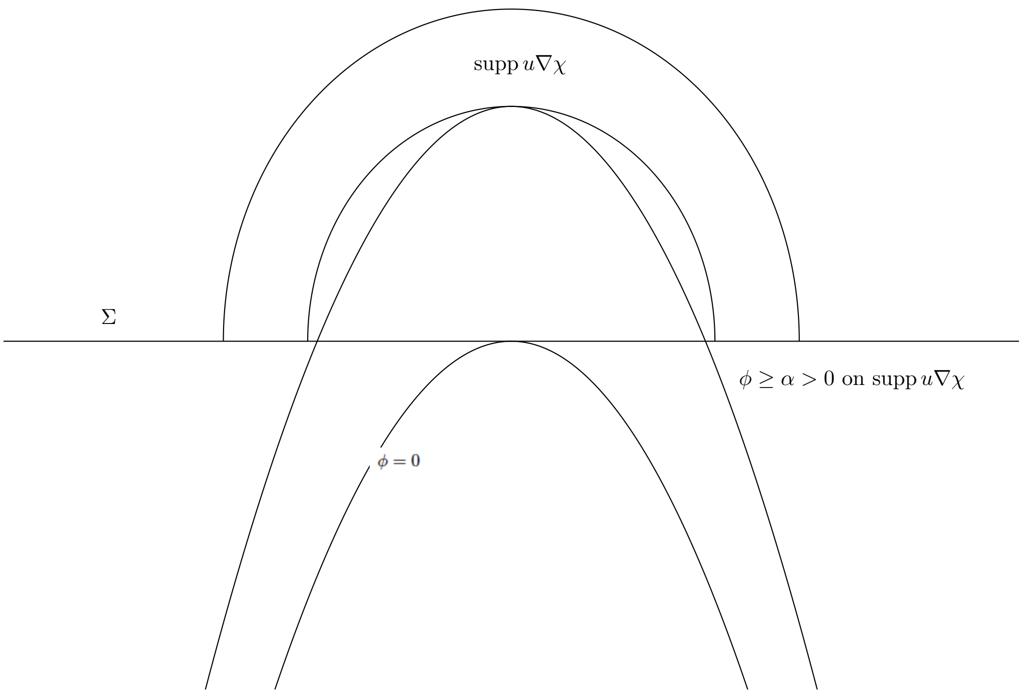


Figure 1.3: CONVEXIFICATION.

1.2 Conjugation identities

1.2.1 Conjugation

We have several things to understand in the previous calculations. A first task is understand the method by which we were able to prove the estimate (1.1.14). Let us start with a differential operator

$$P_m(x, D) = \sum_{|\alpha|=m} a_\alpha(x) D_x^\alpha, \tag{1.2.1}$$

where a_α are smooth functions defined in an open subset Ω of $\mathbb{R}^n$. We do not care so much about lower order terms, since we shall be interested in differential inequalities of type

$$|(P_m u)(x)| \le \sum_{0 \le j < m} V_j(x) |\nabla^j u(x)|, \tag{1.2.2}$$

say with V_j non-negative locally bounded. We want to know if the hypothesis (1.2.2) along with the vanishing of u in some open subset of Ω could imply that u vanishes all over Ω. We have seen that a well-chosen real-valued weight ϕ and an inequality of type

$$\exists C > 0, \exists \lambda_0 \ge 1, \forall \lambda \ge \lambda_0, \forall w \in C_c^\infty(\Omega),$$
$$C \|e^{-\lambda\phi} P_m w\|_{L^2} \ge \sum_{0 \le j < m} \lambda^{m - \frac12 - j} \|e^{-\lambda\phi} \nabla^j w\|_{L^2}, \tag{1.2.3}$$

will be enough to tackle our unique continuation problem.

Lemma 1.2.1. *Let us assume that the function ϕ is smooth, bounded on Ω as well as all its derivatives of order less than m. Property (1.2.3) is equivalent to*

$$\exists C > 0, \exists \lambda_0 \ge 1, \forall \lambda \ge \lambda_0, \forall w \in C_c^\infty(\Omega),$$
$$C \|P_m(x, D - i\lambda d\phi) v\|_{L^2} \ge \sum_{0 \le j < m} \lambda^{m - \frac12 - j} \|v\|_{H^j}. \tag{1.2.4}$$

Proof. We assume that (1.2.4) holds and for $w \in C_c^\infty(\Omega)$, we define $v = e^{-\lambda\phi} w$. We note that

$$e^{-\lambda\phi} D_j e^{\lambda\phi} = D_j - i\lambda \frac{\partial \phi}{\partial x_j} \Longrightarrow e^{-\lambda\phi} D_x e^{\lambda\phi} = D_x - i\lambda d\phi(x),$$

so that

$$e^{-\lambda\phi} P_m(x, D) w = \sum_{|\alpha|=m} a_\alpha e^{-\lambda\phi} D_x^\alpha e^{\lambda\phi} v = P_m(x, D - i\lambda d\phi) v. \tag{1.2.5}$$

We calculate now

$$e^{-\lambda\phi} \nabla^j w = e^{-\lambda\phi} \nabla^j e^{\lambda\phi} v = (\nabla + \lambda d\phi)^j v.$$

This implies that

$$\sum_{0\le j\le m-1} \lambda^{m-j}\|e^{-\lambda\phi}\nabla^j w\|_{L^2} = \sum_{0\le j\le m-1} \lambda^{m-j}\|(\nabla+\lambda d\phi)^j v\|_{L^2}$$

$$\le C\sum_{0\le j\le m-1}\lambda^{m-j}\sum_{j_1+j_2\le j}\lambda^{j_1}\|v\|_{H^{j_2}} \le C\sum_{0\le j\le m-1}\lambda^{m-j}\sum_{j_1\le j}\lambda^{j_1}\|v\|_{H^{j-j_1}}$$

$$\le C\sum_{0\le j\le m-1}\sum_{j_1\le j}\lambda^{m-(j-j_1)}\|v\|_{H^{j-j_1}} \le C'\sum_{0\le k\le m-1}\lambda^{m-k}\|v\|_{H^k}.$$

Using (1.2.4) and (1.2.5), we find that

$$\sum_{0\le j\le m-1}\lambda^{m-j}\|e^{-\lambda\phi}\nabla^j w\|_{L^2} \le C''\lambda^{1/2}\|e^{-\lambda\phi}P_m \underbrace{e^{\lambda\phi}v}_{=w}\|,$$

which is (1.2.3).

Conversely, let us assume that (1.2.3) holds. Let $v \in C_c^\infty(\Omega)$ and let us apply (1.2.3) to $w = e^{\lambda\phi}v$: we obtain

$$C\|\underbrace{e^{-\lambda\phi}P_m e^{\lambda\phi}v}_{\substack{=P_m(x,D-i\lambda d\phi)v\\ \text{from (1.2.5)}}}\|_{L^2} \ge \sum_{0\le j<m}\lambda^{m-\frac12-j}\|\underbrace{e^{-\lambda\phi}\nabla^j e^{\lambda\phi}v}_{(\nabla+\lambda d\phi)^j v}\|_{L^2}.$$

We have also $\nabla^j = (\nabla+\lambda d\phi-\lambda d\phi)^j$ which implies that

$$\|\lambda^{m-j}\nabla^j v\|_{L^2} \le C\sum_{j'+j''=j}\|\lambda^{m-j+j''}(\nabla+\lambda d\phi)^{j'}v\|_{L^2} + \|\underbrace{r_{m-1}(\lambda,\nabla)}_{\substack{\text{polynomial}\\ \text{with degree } m-1}}v\|_{L^2}$$

$$\le C\sum_{j'\le j}\lambda^{m-j'}\|(\nabla+\lambda d\phi)^{j'}v\| + C_1\sum_{0\le k\le l\le m-1}\lambda^{m-1-l}\|\nabla^k v\|.$$

We have thus for $\lambda \ge 1$,

$$\sum_{0\le j\le m-1}\lambda^{m-j-\frac12}\|\nabla^j v\|$$

$$\le C_2\sum_{0\le j\le m-1}\lambda^{m-j-\frac12}\|(\nabla+\lambda d\phi)^j v\| + C_3\sum_{0\le k\le l\le m-1}\lambda^{m-\frac32-l}\|\nabla^k v\|$$

$$(\text{since } -l\le -k)\qquad \le C_4\|e^{-\lambda\phi}P_m e^{\lambda\phi}v\| + C_5\lambda^{-1}\sum_{0\le j\le m-1}\lambda^{m-j-\frac12}\|\nabla^j v\|,$$

which gives (1.2.4) for λ large enough. We note also that since there exist positive constant c_m, C_m such that

$$c_m(\lambda^2+|\xi|^2)^{m-1} \le \sum_{0\le j\le m-1}\lambda^{2(m-1-j)}|\xi|^{2j} \le C_m(\lambda^2+|\xi|^2)^{m-1},$$

we can replace for $\lambda \ge 1$ the rhs of (1.2.4) by $\lambda^{1/2}\|v\|_{\mathcal{H}^{m-1}_\lambda}$ with

$$\|v\|_{\mathcal{H}^k_\lambda} = \|(\lambda+|D|)^k v\|_{L^2}. \tag{1.2.6}$$

□

We are then left with the study of the *conjugate operator* $P_m(x, D - i\lambda d\phi)$ which is a polynomial of degree m in D, λ, whose symbol is

$$p_m(x, \xi - i\lambda d\phi(x)) + r_{m-1}(x, \xi, \lambda),$$

where r_{m-1} is a polynomial in (ξ, λ) (with coefficients depending on x) with degree $\leq m-1$. Since we expect to proving an estimate

$$C\|P_m(x, D - i\lambda d\phi)v\| \geq \lambda^{1/2}\|v\|_{\mathcal{H}_\lambda^{m-1}},$$

the term r_{m-1} is unimportant since the rhs of the above inequality will absorb this for λ large enough, thanks to the following lemma.

Lemma 1.2.2.
(1) *Let $q(x, \xi, \lambda)$ be a polynomial of degree μ in the variables $(\xi, \lambda) \in \mathbb{R}^n \times [1, +\infty)$ with coefficients smooth functions of $x \in \Omega$ open subset of $\mathbb{R}^n$. Then for any compact subset K of Ω, there exists a constant C_K such that, for all $v \in C_K^\infty(\Omega)$,*

$$\|q(x, D_x, \lambda)v\|_{L^2} \leq C_K\|v\|_{\mathcal{H}_\lambda^\mu},$$

where $\mathcal{H}_\lambda^\mu$ is defined in (1.2.6).
(2) *Let $Q(x, \xi, \lambda)$ be a polynomial of degree 2μ in the variables $(\xi, \lambda) \in \mathbb{R}^n \times [1, +\infty)$ with coefficients smooth functions of $x \in \Omega$ open subset of $\mathbb{R}^n$. Then for any compact subset K of Ω, there exists a constant C_K such that, for all $v \in C_K^\infty(\Omega)$,*

$$|\langle Q(x, D_x, \lambda)v, v\rangle| \leq C_K\|v\|^2_{\mathcal{H}_\lambda^\mu}.$$

Proof. (1) We have $q(x, D_x, \lambda) = \sum_{|\alpha|+k\leq\mu} a_\alpha(x)\lambda^k D_x^\alpha$ and thus

$$\|q(x, D_x, \lambda)v\|_{L^2} \leq \sum_{|\alpha|+k\leq\mu} \|a_\alpha\|_{L^\infty(K)}\lambda^k\|D_x^\alpha v\|_{L^2} \leq C_K\|v\|_{\mathcal{H}^\mu},$$

since, for $|\alpha| + k \leq \mu$, the Fourier multiplier $\lambda^k\xi^\alpha$ has an absolute value smaller than

$$\lambda^k|\xi|^{|\alpha|} \leq (\lambda + |\xi|)^\mu.$$

(2) We have $\langle Q(x, D_x, \lambda)v, v\rangle = \sum_{|\alpha|+k\leq 2\mu} \lambda^k\langle a_\alpha(x)D_x^\alpha v, v\rangle$ and with $\chi_K \in C_c^\infty(\Omega)$ equal to 1 on K,

$$\langle a_\alpha(x)D_x^\alpha v, v\rangle = \langle\langle D_x\rangle^{-\mu}\chi_K(x)a_\alpha(x)D_x^\alpha\langle D_x\rangle^{-\mu}\langle D_x\rangle^\mu v, \langle D_x\rangle^\mu v\rangle.$$

It is thus enough to prove that the operator $\langle D_x\rangle^{-\mu}b(x)D_x^\alpha\langle D_x\rangle^{-\mu}$ is bounded on $L^2(\mathbb{R}^n)$ for $|\alpha| \leq 2\mu$ and $b \in C_c^\infty(\mathbb{R}^n)$: we write

$$\langle D_x\rangle^{-\mu}b(x)D_x^\alpha\langle D_x\rangle^{-\mu} = \langle D_x\rangle^{-\mu}b(x)\langle D_x\rangle^\mu \underbrace{\langle D_x\rangle^{-\mu}D_x^\alpha\langle D_x\rangle^{-\mu}}_{\text{bounded on } L^2(\mathbb{R}^n)}$$

so that it is enough[5] to prove that $\langle D_x\rangle^{-\mu}b(x)\langle D_x\rangle^{\mu}$ is bounded on L^2. We have with $u, v \in \mathscr{S}(\mathbb{R}^n)$,

$$\langle\langle D_x\rangle^{-\mu}b(x)\langle D_x\rangle^{\mu}u, v\rangle = \iiint b(x)e^{2i\pi x\cdot(\xi-\eta)}\langle\xi\rangle^{\mu}\hat{u}(\xi)\langle\eta\rangle^{-\mu}\overline{\hat{v}(\eta)}d\eta d\xi dx$$
$$= \iint \hat{b}(\eta-\xi)\langle\xi\rangle^{\mu}\hat{u}(\xi)\langle\eta\rangle^{-\mu}\overline{\hat{v}(\eta)}d\eta d\xi.$$

Since $\hat{b}$ belongs to the Schwartz class, the kernel $\kappa(\xi,\eta) = \hat{b}(\eta-\xi)\langle\xi\rangle^{\mu}\langle\eta\rangle^{-\mu}$ is such that for $N \ge 2\max(\mu, n+1)$,

$$\int |\kappa(\xi,\eta)|d\eta \le C_N\langle\xi\rangle^{\mu}\int(1+|\xi-\eta|)^{-N}(1+|\eta|)^{-\mu}d\eta$$
$$\le C_N\langle\xi\rangle^{\mu}\int(1+|\xi-\eta|)^{-N/2}(1+|\eta|+|\xi-\eta|)^{-\mu}d\eta \le C_N'$$

and a similar estimate holds for $\sup_\eta \int |\kappa(\xi,\eta)|d\xi$. The Schur criterion gives thus the L^2 boundedness. □

1.2.2 Symbol of the conjugate

We may somehow concentrate our attention on the symbol

$$a(x,\xi,\lambda) = p_m(x, \xi - i\lambda d\phi(x)) \tag{1.2.7}$$

which is an homogeneous polynomial of degree m in ξ, λ. Proving a Carleman estimate for P amounts to proving an a priori estimate for the operator with symbol a under the condition that $\lambda \ge \lambda_0 \ge 1$.

Let $Q(x,D) = \sum_{|\alpha|\le m} c_\alpha(x)D_x^\alpha$ be a differential operator on $\mathbb{R}^n$ with smooth coefficients. We define the adjoint operator $Q^*(x,D)$ by the identity

$$\forall u, v \in C_c^\infty(\mathbb{R}^n), \langle Q(x,D)^*u, v\rangle_{L^2} = \langle u, Q(x,D)v\rangle_{L^2}.$$

We see at once that

$$Q(x,D)^* = \sum_{|\alpha|\le m} D_x^\alpha \overline{c_\alpha(x)}. \tag{1.2.8}$$

Lemma 1.2.3. *Let $Q(x,D) = \sum_{|\alpha|\le m} c_\alpha(x)D_x^\alpha$ be a differential operator on $\mathbb{R}^n$ with smooth coefficients. We define*

$$J = \frac{1}{2}(Q+Q^*) \text{ (self-adjoint part)}, \qquad K = \frac{1}{2}(Q-Q^*) \text{ (anti-adjoint part)}.$$

[5] For $k + |\alpha| \le 2\mu$, we write as above

$$\lambda^k\langle a_\alpha(x)D_x^\alpha v, v\rangle = \lambda^k\langle\underbrace{\langle D_x\rangle^{-\frac{|\alpha|}{2}}a_\alpha(x)D_x^\alpha\langle D_x\rangle^{-\frac{|\alpha|}{2}}}_{L^2 \text{ bounded}}\langle D_x\rangle^{\frac{|\alpha|}{2}}v, \langle D_x\rangle^{\frac{|\alpha|}{2}}v\rangle$$

so that $|\lambda^k\langle a_\alpha(x)D_x^\alpha v, v\rangle| \le C\|\lambda^{k/2}\langle D_x\rangle^{\frac{|\alpha|}{2}}v\|^2 \le C\|v\|^2_{\mathcal{H}^\mu_\lambda}$

We have $J = J^*, K^* = -K$ and for $u \in C_c^\infty(\mathbb{R}^n)$,

$$\|Qv\|_{L^2}^2 = \|Jv\|_{L^2}^2 + \|Kv\|_{L^2}^2 + \langle [J,K]v, v\rangle_{L^2}.$$

In particular, we have always $\|Qv\|_{L^2}^2 \ge \langle [J,K]v, v\rangle_{L^2}$.

Proof. We have $Q^{**} = Q$ so that the properties of J, K are obvious. Moreover, we have for $v \in C_c^\infty(\mathbb{R}^n)$,

$$\|Qv\|_{L^2}^2 = \|Jv\|^2 + \|Kv\|_{L^2}^2 + \underbrace{\langle Jv, Kv\rangle + \langle Kv, Jv\rangle}_{\langle K^*Jv,v\rangle + \langle J^*Kv,v\rangle}$$

so that

$$2\operatorname{Re}\langle Jv, Kv\rangle = \langle (-KJ + JK)v, v\rangle = \langle [J,K]v, v\rangle,$$

which is the sought result. □

Remark 1.2.4. Note that the differential operator $[J,K]$ is self-adjoint since

$$[J,K]^* = (JK)^* - (KJ)^* = K^*J^* - J^*K^* = -KJ + JK = [J,K].$$

Definition 1.2.5. Let Ω be an open subset of $\mathbb{R}^n$ and let m be a non-negative integer. $\Sigma^m(\Omega)$ is defined as the set of polynomials of degree $\le m$ in the variables $(\xi, \lambda) \in \mathbb{R}^n \times \mathbb{R}$ with coefficients smooth functions of $x \in \Omega$.

Theorem 1.2.6. *Let $p_j(x,\xi,\lambda) \in \Sigma^{m_j}$, $j = 1, 2$. We define $P_j = p_j(x, D_x, \lambda)$ and we shall say that the polynomial p is the symbol of the operator P_j. We shall write as well $P_j = op(p_j)$. Then $P_1P_2 = q(x, D_x, \lambda)$ where $q \in \Sigma^{m_1+m_2}$ and more precisely*

$$q(x,\xi,\lambda) = p_1(x,\xi,\lambda)p_2(x,\xi,\lambda) + \frac{1}{i}(\frac{\partial p_1}{\partial \xi} \cdot \frac{\partial p_2}{\partial x})(x,\xi,\lambda) + r_{m_1+m_2-2}(x,\xi,\lambda), \quad (1.2.9)$$

with $r_{m_1+m_2-2} \in \Sigma^{m_1+m_2-2}$. We have also $[P_1, P_2] = T$ with $T = t(x, D_x, \lambda)$ and

$$t = \frac{1}{i}\{p_1, p_2\} + s_{m_1+m_2-2}, \quad s_{m_1+m_2-2} \in \Sigma^{m_1+m_2-2}, \quad (1.2.10)$$

and where the Poisson bracket $\{p_1, p_2\}$ is defined as

$$\begin{aligned}\{p_1, p_2\}(x,\xi,\lambda) &= (\frac{\partial p_1}{\partial \xi} \cdot \frac{\partial p_2}{\partial x})(x,\xi,\lambda) - (\frac{\partial p_1}{\partial x} \cdot \frac{\partial p_2}{\partial \xi})(x,\xi,\lambda) \\ &= \sum_{1\le j\le n}\Big(\frac{\partial p_1}{\partial \xi_j}\frac{\partial p_2}{\partial x_j} - \frac{\partial p_1}{\partial x_j}\frac{\partial p_2}{\partial \xi_j}\Big).\end{aligned} \quad (1.2.11)$$

The symbol of P_1^* is equal to

$$p_1^*(x,\xi,\lambda) = \overline{p_1(x,\xi,\lambda)} + \frac{1}{i}\sum_{1\le j\le n}\overline{\frac{\partial^2 p_1}{\partial \xi_j \partial x_j}(x,\xi,\lambda)}, \quad \textit{modulo } \Sigma^{m-2}. \quad (1.2.12)$$

Proof. This is a standard consequence of elementary identities for pseudodifferential operators, but it is certainly simpler to derive here a direct proof. We note that (1.2.10) follows from (1.2.9). By linearity, it is enough to consider

$$\begin{aligned}
&a_1(x)D_x^{\alpha_1}\lambda^{k_1}a_2(x)D_x^{\alpha_2}\lambda^{k_2}\\
&= \lambda^{k_1+k_2}a_1(x)i^{-|\alpha_1|}\sum_{\beta+\gamma=\alpha_1}(\partial_x^\beta a_2)\partial_x^\gamma D_x^{\alpha_2}\frac{\alpha_1!}{\beta!\gamma!}\\
&= \lambda^{k_1+k_2}a_1(x)a_2(x)D_x^{\alpha_1+\alpha_2} + \sum_{1\le j\le n}\lambda^{k_1+k_2}a_1(x)i^{-|\alpha_1|}\partial_{x_j}a_2\partial_x^{\alpha_1-e_j}D_x^{\alpha_2}\frac{\alpha_1!}{1!(\alpha_1-e_j)!}\\
&\qquad\qquad + \Sigma^{m_1+m_2-2}\\
&= \lambda^{k_1+k_2}a_1(x)a_2(x)D_x^{\alpha_1+\alpha_2} + \frac{1}{i}\sum_{1\le j\le n}\lambda^{k_1+k_2}a_1(x)\partial_{x_j}a_2\alpha_{1,j}D_x^{\alpha_1-e_j}D_x^{\alpha_2}\\
&\qquad\qquad + r_{m_1+m_2-2}\in\Sigma^{m_1+m_2-2},
\end{aligned}$$

an operator whose symbol is $p_1p_2+\frac{1}{i}\sum_{1\le j\le n}\frac{\partial p_1}{\partial\xi_j}\frac{\partial p_2}{\partial x_j}$ modulo $\Sigma^{m_1+m_2-2}$, which is the sought formula. The last assertion follows from the fact that with

$$Q=\sum_{|\alpha|+k\le m}c_\alpha(x)\lambda^k D_x^\alpha,\quad q(x,\xi,\lambda)=\sum_{|\alpha|+k\le m}c_\alpha(x)\lambda^k\xi^\alpha$$

and from (1.2.8), we see that $Q^*=\sum_{|\alpha|+k\le m}\lambda^k D_x^\alpha\overline{c_\alpha(x)}$ whose symbol is, modulo Σ^{m-2},

$$\sum_{|\alpha|+k\le m}\overline{c_\alpha(x)}\lambda^k\xi^\alpha+\sum_{\substack{|\alpha|+k\le m\\|\alpha|\ge1}}\lambda^k\frac{1}{i}\sum_{1\le j\le n}\frac{\partial\overline{c_\alpha}}{\partial x_j}\partial_{\xi_j}(\xi^\alpha)=\overline{q}+\frac{1}{i}\sum_{1\le j\le n}\frac{\partial^2\overline{q}}{\partial\xi_j\partial x_j},$$

completing the proof. □

Lemma 1.2.7. *Let $q\in\Sigma^\mu$ be a real-valued symbol. Then*

$$q^*(x,D_x,\lambda)q(x,D_x,\lambda)=q(x,\xi,\lambda)^2+r_{2\mu-1}\quad \textit{modulo } \Sigma^{2\mu-2},$$

with $r_{2\mu-1}\in\Sigma^{2\mu-1}$ is purely imaginary. We have also for $v\in C_K^\infty(\Omega)$,

$$\|q(x,D_x,\lambda)v\|^2=\operatorname{Re}\langle q^2(x,D_x,\lambda)v,v\rangle+O(\|v\|^2_{\mathcal{H}_\lambda^{\mu-1}}),\tag{1.2.13}$$

where $\mathcal{H}^\mu$ is defined in (1.2.6).

Proof. Since $q^*=\overline{q}+\frac{1}{i}\frac{\partial^2\overline{q}}{\partial x\cdot\partial\xi}+\Sigma^{\mu-2}$, the symbol of $q^*(x,D_x,\lambda)q(x,D_x,\lambda)$ is

$$(\overline{q}+\frac{1}{i}\frac{\partial^2\overline{q}}{\partial x\cdot\partial\xi})q+\frac{1}{i}\frac{\partial\big(\overline{q}+\frac{1}{i}\frac{\partial^2\overline{q}}{\partial x\cdot\partial\xi}\big)}{\partial\xi}\cdot\frac{\partial q}{\partial x}\quad\text{modulo }\Sigma^{2\mu-2},$$

that is, since q is real-valued, $q^2+\frac{1}{i}\frac{\partial^2 q}{\partial x\cdot\partial\xi}q+\frac{1}{i}\frac{\partial q}{\partial\xi}\cdot\frac{\partial q}{\partial x}$ modulo $\Sigma^{2\mu-2}$, providing the first formula. We note than that

$$r^*_{2\mu-1}=-r_{2\mu-1}\quad\text{modulo }\Sigma^{2\mu-2},$$

and since

$$\|q(x,D_x,\lambda)v\|^2 = \operatorname{Re}\langle \operatorname{op}(q^2)v,v\rangle + \underbrace{\operatorname{Re}\langle \operatorname{op}(r_{2\mu-1})v,v\rangle}_{=\frac12\langle(\operatorname{op}(r_{2\mu-1})+\operatorname{op}(r^*_{2\mu-1}))v,v\rangle},$$

Lemma 1.2.2 gives the answer. ☐

Proposition 1.2.8. *Let a be given by (1.2.7), where $p_m(x,\xi)$ is an homogeneous polynomial in the variables $\xi\in\mathbb{R}^n$ with smooth coefficients of $x\in\Omega$ where Ω is an open subset of $\mathbb{R}^n$. We define*

$$c_{2m-1,\phi}(x,\xi,\lambda) = \operatorname{Im}\left(\overline{\frac{\partial p_m}{\partial\xi}(x,\zeta)}\cdot\frac{\partial p_m}{\partial x}(x,\zeta)\right) - \lambda\phi''(x)\overline{\frac{\partial p_m}{\partial\xi}(x,\zeta)}\frac{\partial p_m}{\partial\xi}(x,\zeta), \quad (1.2.14)$$

with $\zeta=\xi-i\lambda d\phi(x)$. The notations above stand for

$$\overline{\frac{\partial p_m}{\partial\xi}(x,\zeta)}\cdot\frac{\partial p_m}{\partial x}(x,\zeta) = \sum_{1\le j\le n}\overline{\frac{\partial p_m}{\partial\xi_j}(x,\zeta)}\cdot\frac{\partial p_m}{\partial x_j}(x,\zeta),$$

$$\phi''(x)\overline{\frac{\partial p_m}{\partial\xi}(x,\zeta)}\frac{\partial p_m}{\partial\xi}(x,\zeta) = \sum_{1\le j,k\le n}\frac{\partial^2\phi}{\partial x_j\partial x_k}(x)\overline{\frac{\partial p_m}{\partial\xi_k}(x,\zeta)}\frac{\partial p_m}{\partial\xi_j}(x,\zeta).$$

For every compact subset K of Ω, there exists a constant C such that for all $v\in C^\infty_K(\Omega)$

$$C\|v\|^2_{\mathcal{H}^{m-1}_\lambda} + \|P_m(x,D-i\lambda d\phi)v\|^2_{L^2} \ge \|\operatorname{op}(\operatorname{Re}a_m)v\|^2_{L^2} + \|\operatorname{op}(\operatorname{Im}a_m)v\|^2_{L^2} + \operatorname{Re}\langle\operatorname{op}(c_{2m-1,\phi})v,v\rangle.$$

N.B. We may notice that the latter quantity in (1.2.14) is real-valued whenever ϕ is real-valued since its complex-conjugate is

$$\sum_{1\le j,k\le n}\frac{\partial^2\phi}{\partial x_j\partial x_k}(x)\overline{\frac{\partial p_m}{\partial\xi_j}(x,\zeta)}\frac{\partial p_m}{\partial\xi_k}(x,\zeta) = \sum_{1\le j,k\le n}\frac{\partial^2\phi}{\partial x_j\partial x_k}(x)\overline{\frac{\partial p_m}{\partial\xi_k}(x,\zeta)}\frac{\partial p_m}{\partial\xi_j}(x,\zeta),$$

by symmetry of the matrix $\phi''(x)$.

Proof. The proof is a direct application of Theorem 1.2.6. The symbol of the operator $P_m(x,D-i\lambda d\phi)$ is $p_m(x,\xi-i\lambda d\phi)+r_{m-1}$ with $r_{m-1}\in\Sigma^{m-1}$. As a result for $v\in C^\infty_K(\Omega)$, we have,

$$C\|v\|^2_{\mathcal{H}^{m-1}_\lambda} + \|P_m(x,D-i\lambda d\phi)v\|^2_{L^2} \ge \|\operatorname{op}(p_m(x,\zeta))v\|^2_{L^2}.$$

Now, according to the last statement in Theorem 1.2.6 and to Lemmas 1.2.3 and 1.2.2, we have, with $a_m(x,\xi,\lambda)=p_m(x,\zeta)$, $r_{m-1},s_{m-1}\in\Sigma^{m-1}, r_{2m-2}\in\Sigma^{2m-2}$,

$$\|\operatorname{op}(a_m)v\|^2_{L^2} = \|\operatorname{op}(\operatorname{Re}a_m+r_{m-1})v\|^2_{L^2} + \|\operatorname{op}(\operatorname{Im}a_m+s_{m-1})v\|^2_{L^2} + \operatorname{Re}\langle[\operatorname{op}(\operatorname{Re}a_m),i\operatorname{op}(\operatorname{Im}a_m)]v,v\rangle + \langle\operatorname{op}(r_{2m-2})v,v\rangle,$$

so that

$$C_1\|v\|^2_{\mathcal{H}^{m-1}_\lambda} + \|P_m(x, D - i\lambda d\phi)v\|^2_{L^2} \geq \|\mathrm{op}(\mathrm{Re}\, a_m)v\|^2_{L^2} + \|\mathrm{op}(\mathrm{Im}\, a_m)v\|^2_{L^2} + \mathrm{Re}\langle[\mathrm{op}(\mathrm{Re}\, a_m), i\mathrm{op}(\mathrm{Im}\, a_m)]v, v\rangle.$$

Using now (1.2.13) and Theorem 1.2.6, we obtain

$$C_2\|v\|^2_{\mathcal{H}^{m-1}_\lambda} + \|P_m(x, D - i\lambda d\phi)v\|^2_{L^2} \geq \|\mathrm{op}(\mathrm{Re}\, a_m)v\|^2_{L^2} + \|\mathrm{op}(\mathrm{Im}\, a_m)v\|^2_{L^2} + \mathrm{Re}\langle \mathrm{op}(\{\mathrm{Re}\, a_m, \mathrm{Im}\, a_m\})v, v\rangle.$$

We note also that

$$2i\,\mathrm{Im}\Big(\frac{\partial \overline{a_m}}{\partial \xi}\cdot\frac{\partial a_m}{\partial x}\Big) = \frac{\partial \overline{a_m}}{\partial \xi}\cdot\frac{\partial a_m}{\partial x} - \frac{\partial a_m}{\partial \xi}\cdot\frac{\partial \overline{a_m}}{\partial x} = \{\overline{a_m}, a_m\} = \{\mathrm{Re}\, a_m - i\,\mathrm{Im}\, a_m, \mathrm{Re}\, a_m + i\,\mathrm{Im}\, a_m\} = 2i\,\{\mathrm{Re}\, a_m, \mathrm{Im}\, a_m\},$$

so that

$$\{\mathrm{Re}\, a_m, \mathrm{Im}\, a_m\} = \mathrm{Im}\Big(\frac{\partial \overline{a_m}}{\partial \xi}\cdot\frac{\partial a_m}{\partial x}\Big),$$

entailing the sought result since

$$\frac{\partial \overline{a_m}}{\partial \xi}\cdot\frac{\partial a_m}{\partial x} = \overline{\frac{\partial p_m}{\partial \xi}(x,\zeta)}\cdot\Big(\frac{\partial p_m}{\partial x}(x,\zeta) - \lambda\phi''(x)\overline{\frac{\partial p_m}{\partial \xi}(x,\zeta)}\Big).$$

□

Lemma 1.2.9. *Let ρ be a C^∞ real-valued function defined on an open set Ω of $\mathbb{R}^n$ such that $d\rho(x) \neq 0$ at $\rho(x) = 0$. Let $x_0 \in \Omega$ such that $\rho(x_0) = 0$. We define for $\mu > 0$*

$$\phi_{\rho,\mu}(x) = \rho(x) - \mu\rho(x)^2 + \frac{1}{2\mu}|x - x_0|^2. \tag{1.2.15}$$

Then there exists a neighborhood V of x_0 in Ω such that

$$\{x \in V, \phi_{\rho,\mu}(x) = 0\}\backslash\{x_0\} \subset \{x \in V, \rho(x) < 0\}.$$

For any compact subset K of $V\backslash\{x_0\}$, we have $\inf_{x\in K,\rho(x)\geq 0}\phi_{\rho,\mu}(x) > 0$. In particular, if we consider $\chi \in C^\infty_c(V)$, which is equal to 1 on a neighborhood of x_0, we have

$$\inf_{x\in \mathrm{supp}\,\nabla\chi, \rho(x)\geq 0}\phi_{\rho,\mu}(x) > 0.$$

Proof. If $x \in \Omega$ is such that $\rho(x) - \mu\rho(x)^2 + \frac{1}{2\mu}|x - x_0|^2 = 0, x \neq x_0$, this implies that

$$\rho(x)\big(1 - \mu\rho(x)\big) < 0.$$

Defining $V = \{x \in \Omega, \rho(x) < 1/\mu\}$ (which is a neighborhood of x_0), we obtain the sought inclusion. We have also for $x \in V$ and $\rho(x) \geq 0$,

$$\rho(x) - \mu\rho(x)^2 + \frac{1}{2\mu}|x - x_0|^2 = \rho(x)\big(1 - \mu\rho(x)\big) + \frac{1}{2\mu}|x - x_0|^2 \geq \frac{1}{2\mu}|x - x_0|^2,$$

so that $\{x \in V, x \neq x_0, \rho(x) \geq 0\} \subset \{x \in V, \phi_{\rho,\mu}(x) > 0\}$ and for any compact subset K of $\{x \in V, x \neq x_0\}$, we have $\inf_{x\in K,\rho(x)\geq 0} \phi_{\rho,\mu}(x) > 0$. The last statement follows from the choice $K = \operatorname{supp} \nabla\chi$. □

Lemma 1.2.10. *Let ρ be a C^∞ real-valued function defined on an open set Ω of $\mathbb{R}^n$ such that $d\rho(x) \neq 0$ at $\rho(x) = 0$. Let $x_0 \in \Omega$ such that $\rho(x_0) = 0$. We define for $\mu > 0$*

$$\Phi_{\rho,\mu}(x) = \rho'(x_0)(x - x_0) + \frac{1}{2}\rho''(x_0)(x - x_0)^2 - \mu\big(\rho'(x_0)(x - x_0)\big)^2 + \frac{1}{2\mu}|x - x_0|^2. \tag{1.2.16}$$

Then there exists a neighborhood V of x_0 in Ω such that

$$\{x \in V, \Phi_{\rho,\mu}(x) = 0\}\backslash\{x_0\} \subset \{x \in V, \rho(x) < 0\}.$$

For any compact subset K of $V\backslash\{x_0\}$, we have $\inf_{x\in K,\rho(x)\geq 0} \Phi_{\rho,\mu}(x) > 0$. In particular, if we consider $\chi \in C_c^\infty(V)$, which is equal to 1 on a neighborhood of x_0, we have

$$\inf_{x\in \operatorname{supp}\nabla\chi,\rho(x)\geq 0} \Phi_{\rho,\mu}(x) > 0.$$

Proof. If $x \in \Omega$ is such that

$$\rho'(x_0)(x - x_0) + \frac{1}{2}\rho''(x_0)(x - x_0)^2 - \mu\big(\rho'(x_0)(x - x_0)\big)^2 + \frac{1}{2\mu}|x - x_0|^2 = 0,$$

this implies, with σ_j bounded in a fixed neighborhood of x_0,

$$\rho(x) + \sigma_3(x)(x - x_0)^3 - \mu\big(\rho(x) + \sigma_2(x)(x - x_0)^2\big)^2 = -\frac{1}{2\mu}|x - x_0|^2,$$

so that with C_j positive constants independent of μ,

$$\begin{aligned}\rho(x) - \mu\rho(x)^2 &\leq -\frac{1}{2\mu}|x - x_0|^2 + (C_0 + C_1\mu)|x - x_0|^3 \\ &= -\frac{1}{2\mu}|x - x_0|^2\Big(1 - 2\mu(C_0 + C_1\mu)|x - x_0|\Big).\end{aligned}$$

We may assume that $2\mu(C_0 + C_1\mu)|x - x_0| \leq 1/2$, and this implies that

$$\rho(x)\big(1 - \mu\rho(x)\big) < 0.$$

Defining

$$V = \{x \in \Omega, \rho(x) < 1/\mu, 2\mu(C_0 + C_1\mu)|x - x_0| < 1/2\}$$

(which is a neighborhood of x_0), we obtain the sought inclusion. We have also for $x \in V$ and $\rho(x) \geq 0$,

$$\Phi_{\rho,\mu}(x) = \rho(x) + \sigma_3(x)(x - x_0)^3 - \mu\big(\rho(x) + \sigma_2(x)(x - x_0)^2\big)^2 + \frac{1}{2\mu}|x - x_0|^2,$$

so that $\Phi_{\rho,\mu}(x) \geq \rho(x) - \mu\rho(x)^2 + \frac{1}{2\mu}|x-x_0|^2 - (C_0 + C_1\mu)|x-x_0|^3$ and

$$\Phi_{\rho,\mu}(x) \geq \rho(x) - \mu\rho(x)^2 + \frac{1}{2\mu}|x-x_0|^2\big(1 - 2\mu(C_0+C_1\mu)|x-x_0|\big).$$

As a result $\{x \in V, x \neq x_0, \rho(x) \geq 0\} \subset \{x \in V, \Phi_{\rho,\mu}(x) > 0\}$ and for any compact subset K of $\{x \in V, x \neq x_0\}$, we have $\inf_{x\in K,\rho(x)\geq 0} \Phi_{\rho,\mu}(x) > 0$. The last statement follows from the choice $K = \operatorname{supp} \nabla\chi$. □

1.2.3 Simple characteristics

Let us now discuss the case of simple characteristics. We may assume that our oriented hypersurface is given by the equation $t = 0$ near $0 \in \mathbb{R}^1_t \times \mathbb{R}^d_x$ and that our differential operator has the principal symbol (a polynomial with degree m in ξ, τ),

$$p_m(t, x; \tau, \xi).$$

Let $\phi(t,x) = t - \frac{\mu t^2}{2} + \frac{|x|^2}{2\mu}$ be a real-valued weight function ($\mu > 0$). We calculate $c_{2m-1,\phi}$ as given by (1.2.14) and we obtain with

$$\zeta = \big(\tau - i\lambda(1-\mu t), \xi - i\lambda\frac{x}{\mu}\big) \in \mathbb{C}^{1+d},$$

$$\begin{aligned} c_{2m-1,\phi}(t,x,\tau,\xi,\lambda) = \\ \operatorname{Im}\left(\overline{\frac{\partial p_m}{\partial\tau}(t,x,\zeta)} \cdot \frac{\partial p_m}{\partial t}(t,x,\zeta)\right) + \operatorname{Im}\left(\overline{\frac{\partial p_m}{\partial\xi}(t,x,\zeta)} \cdot \frac{\partial p_m}{\partial x}(t,x,\zeta)\right) \\ + \lambda\mu\left|\frac{\partial p_m}{\partial\tau}(t,x,\zeta)\right|^2 - \frac{\lambda}{\mu}\left|\frac{\partial p_m}{\partial\xi}(t,x,\zeta)\right|^2. \end{aligned} \tag{1.2.17}$$

We see that the dominant term in that symbol is $\lambda\mu\left|\frac{\partial p_m}{\partial\tau}(t,x,\zeta)\right|^2$. We shall assume that the characteristics are simple: for $(\tau,\xi,\lambda) \in (\mathbb{R}\times\mathbb{R}^d\times\mathbb{R}_+)\backslash\{0\}$

$$p_m(0,0,\tau - i\lambda, \xi) = 0 \Longrightarrow \frac{\partial p_m}{\partial\tau}(0,0,\tau-i\lambda,\xi) \neq 0. \tag{1.2.18}$$

Note that above hypothesis implies that the hypersurface with equation $t = 0$ is non-characteristic for P, i.e. $p_m(0,0;1,0) \neq 0$, otherwise, the polynomial $p_m(0,0;\tau,0)$ would be the zero polynomial. That hypothesis means simply that the m roots of the (complex-valued) polynomial $\tau \mapsto p_m(0,0;\tau,\xi)$ are simple for $\xi \in \mathbb{R}^d\backslash\{0\}$: if $\xi = 0$, we have

$$p_m(0,0;\tau - i\lambda, 0) = p_m(0,0;1,0)(\tau - i\lambda)^m$$

which is not zero for $\tau - i\lambda \neq 0$.

Lemma 1.2.11. *Let $p_m(x,t,\xi,\tau)$ be a polynomial of degree m with real coefficients such that (1.2.18) holds. There exists a constant $\mu > 0$ such that for $(t,x) \in W_\mu = \{(t,x) \in \mathbb{R}^{1+d}, |t| + |x| \leq \mu^{-2}\}$, for $(\tau,\xi,\lambda) \in \mathbb{R}\times\mathbb{R}^d\times(0,+\infty)$,*

$$\mu|p_m(t,x,\zeta)|^2(\lambda^2+\tau^2+|\xi|^2)^{-1/2} + c_{2m-1,\phi}(t,x,\tau,\xi,\lambda) \geq \mu^{-1}\lambda(\lambda^2+\tau^2+|\xi|^2)^{m-1}.$$

Proof. Since both sides of the inequality are homogeneous with degree $2m-1$ with respect to (τ, ξ, λ), it is enough to prove that on the half-sphere $\lambda^2 + \tau^2 + |\xi|^2 = 1, \lambda > 0$. By *reductio ad absurdum*, a violation of the previous inequality would mean that there exists a sequence $(t_k, x_k) \in W_k$ and a sequence $(\lambda_k, \tau_k, \xi_k)$ on the half-sphere such that

$$k|p_m(t_k, x_k, \tau_k - i\lambda_k(1 - kt_k), \xi_k - i\lambda_k k^{-1} x_k)|^2 + c_{2m-1,\phi}(t_k, x_k, \tau_k, \xi_k, \lambda_k) < \frac{\lambda_k}{k}. \tag{1.2.19}$$

By compactness of the closure of the half-sphere, we may assume that $(\lambda_k, \tau_k, \xi_k)$ is converging to $(\lambda_0, \tau_0, \xi_0)$ on the closure of half-sphere. Since kt_k goes to 0, multiplying the previous inequality by k^{-1} provides

$$p_m(0, 0; \tau_0 - i\lambda_0, \xi_0) = 0. \tag{1.2.20}$$

We assume first that $\lambda_0 > 0$. We have

$$c_{2m-1,\phi}(t_k, x_k, \tau_k, \xi_k, \lambda_k) = \lambda_k k \left| \frac{\partial p_m}{\partial \tau}(t_k, x_k, \zeta_k) \right|^2 + O(1)$$

and multiplying Inequality (1.2.19) by $1/k\lambda_k$ ($k\lambda_k$ goes to $+\infty$ with k), we get

$$\frac{\partial p_m}{\partial \tau}(0, 0, \tau_0 - i\lambda_0, \xi_0) = 0.$$

From our assumption (1.2.18), this is impossible since $(\tau_0, \xi_0, \lambda_0) \neq 0$.
We assume now that $\lambda_0 = 0$, so that $\tau_0^2 + \xi_0^2 = 1$. We have

$$\begin{aligned} &c_{2m-1,\phi}(t_k, x_k, \tau_k, \xi_k, \lambda_k) \\ &\qquad = \lambda_k k \left| \frac{\partial p_m}{\partial \tau}(t_k, x_k, \zeta_k) \right|^2 + O(\lambda_k) + \frac{1}{2i} \left\{ \overline{p_m}, p_m \right\} (t_k, x_k; \tau_k, \xi_k). \end{aligned}$$

We have assumed that p_m has real coefficients, so that $\{\overline{p_m}, p_m\}(t_k, x_k; \tau_k, \xi_k)$ is identically 0. Multiplying (1.2.19) by $1/k\lambda_k$ (λ_k is positive), we get

$$\left| \frac{\partial p_m}{\partial \tau}(t_k, x_k, \zeta_k) \right|^2 + O(1/k) < \frac{1}{k^2}$$

implying that $\frac{\partial p_m}{\partial \tau}(0, 0, \tau_0, \xi_0) = 0$. Since we have already $p_m(0, 0, \tau_0, \xi_0) = 0$ from (1.2.20) and $\lambda_0 = 0$, this is impossible since $\tau_0^2 + \xi_0^2 = 1$. The proof of the lemma is complete. □

Using now the last inequality in Proposition 1.2.8, we get for $\lambda \geq \mu$ that

$$\begin{aligned} C\|v\|^2_{\mathcal{H}^{m-1}_\lambda} + \|P_m(x, D - i\lambda d\phi)v\|^2_{L^2} \geq \mu \|\mathrm{op}(\operatorname{Re} a_m)v\|^2_{\mathcal{H}^{-1/2}} + \mu \|\mathrm{op}(\operatorname{Im} a_m)v\|^2_{\mathcal{H}^{-1/2}} \\ + \operatorname{Re}\langle \mathrm{op}(c_{2m-1,\phi})v, v\rangle. \end{aligned}$$

and Lemma 1.2.11 , (1.2.13) (with $\mu = m-1$), along with Gårding's inequality (see our Appendix, Section 4.2) implies

$$C_1\|v\|^2_{\mathcal{H}^{m-1}_\lambda} + \|P_m(x, D - i\lambda d\phi)v\|^2_{L^2} \geq \frac{\lambda}{\mu}\|v\|^2_{\mathcal{H}^{m-1}_\lambda},$$

entailing (1.2.4) and thus (1.2.3). Applying the convexity property of Lemma 1.2.10, we obtain the following uniqueness result, essentially due to A. Calderón [4].

Theorem 1.2.12. *Let P be a differential operator of order m with C^∞ real-valued coefficients in the principal part, L^∞ complex-valued for lower order terms, in some open subset Ω of $\mathbb{R}^n$. Let Σ be a C^1 hypersurface of Ω given by an equation $\rho(x) = 0$, with $d\rho \neq 0$ at Σ. Let $x_0 \in \Sigma$; we assume that for $(\xi, \lambda) \in (\mathbb{R}^n \times \mathbb{R}_+)\backslash\{0\}$*

$$p_m(x_0, \xi - i\lambda d\rho(x_0)) = 0 \Longrightarrow \{p_m, \rho\}(x_0, \xi - i\lambda d\rho(x_0)) \neq 0. \tag{1.2.21}$$

If u is an H^m_{loc} function, supported in $\{x \in W, \rho(x) \geq 0\}$ where W is a neighborhood of x_0, is such that

$$|(Pu)(x)| \leq \sum_{0\leq j<m} V_j(x)|\nabla^j u(x)|, \qquad V_j \in L^\infty_{loc},$$

then u is vanishing in a neighborhood of x_0.

N.B. Using a more specific Gårding's inequality as in Section 8.3 and 8.4 of [8], it is possible to reduce the regularity requirements for the principal part in the above theorem to C^1. Some refinements of these methods, taking into account that the operators involved are differential (and not general pseudodifferential operators), are presented in [33] and allow a version of the previous theorem for Lipschitz regularity in the principal part. This is in some sense optimal, as far as regularity is concerned, since some counterexamples are available for operators with Hölder continuous coefficients of any order < 1.

1.3 Pseudo-convexity

1.3.1 Checking the symbol of the conjugate operator

Let us now discuss the next case, when the characteristics may fail to be simple. We may assume that our oriented hypersurface is given by the equation $t = 0$ near $0 \in \mathbb{R}^1_t \times \mathbb{R}^d_x$ and that our differential operator has the principal symbol (a polynomial with degree m in ξ, τ),

$$p_m(t, x; \tau, \xi).$$

Let $\phi(t, x) = t - \frac{\mu t^2}{2} + \frac{|x|^2}{2\mu}$ be a real-valued weight function ($\mu > 0$). We calculate $c_{2m-1,\phi}$ as given by (1.2.14) and we obtain with

$$\zeta = \big(\tau - i\lambda(1 - \mu t), \xi - i\lambda\frac{x}{\mu}\big) \in \mathbb{C}^{1+d},$$

$$c_{2m-1,\phi}(t,x,\tau,\xi,\lambda) = \operatorname{Im}\left(\overline{\frac{\partial p_m}{\partial \tau}(t,x,\zeta)}\cdot\frac{\partial p_m}{\partial t}(t,x,\zeta)\right) + \operatorname{Im}\left(\overline{\frac{\partial p_m}{\partial \xi}(t,x,\zeta)}\cdot\frac{\partial p_m}{\partial x}(t,x,\zeta)\right) + \lambda\mu\left|\frac{\partial p_m}{\partial \tau}(t,x,\zeta)\right|^2 - \frac{\lambda}{\mu}\left|\frac{\partial p_m}{\partial \xi}(t,x,\zeta)\right|^2. \quad (1.3.1)$$

When the characteristics are simple, we have seen that that the dominant term in that symbol is $\lambda\mu\left|\frac{\partial p_m}{\partial \tau}(t,x,\zeta)\right|^2$. However, we want to deal with some cases when this term is actually vanishing. We may have for some $(\tau,\xi,\lambda)\in(\mathbb{R}\times\mathbb{R}^d\times\mathbb{R}_+)\backslash\{0\}$

$$p_m(0,0,\tau-i\lambda,\xi)=0 \quad \textbf{and} \quad \frac{\partial p_m}{\partial \tau}(0,0,\tau-i\lambda,\xi)=0. \quad (1.3.2)$$

We have then to focus our attention on the second term of (1.3.1) (the first is vanishing and the very last one will be proven unimportant, thanks to the occurrence of the large term μ in the denominator). We note first that

$$c_{2m-1,\phi}(t,x,\tau,\xi,0)=0. \quad (1.3.3)$$

since, as p_m has real-valued coefficients,

$$\operatorname{Im}\left(\overline{\frac{\partial p_m}{\partial \tau}(t,x,\tau,\xi)}\cdot\frac{\partial p_m}{\partial t}(t,x,\tau,\xi)\right) + \operatorname{Im}\left(\overline{\frac{\partial p_m}{\partial \xi}(t,x,\tau,\xi)}\cdot\frac{\partial p_m}{\partial x}(t,x,\tau,\xi)\right) = \frac{1}{2i}\left\{\overline{p_m},p_m\right\}\equiv 0.$$

We shall assume that for $\lambda\geq 0, (\tau,\xi,\lambda)\neq(0,0,0)$,

$$p_m(0,0,\tau-i\lambda,\xi)=\frac{\partial p_m}{\partial \tau}(0,0,\tau-i\lambda,\xi)=0 \Longrightarrow \lim_{\epsilon\to 0_+}\frac{1}{\lambda+\epsilon}\left(\operatorname{Im}\Big(\overline{\frac{\partial p_m}{\partial \tau}}\cdot\frac{\partial p_m}{\partial t}\Big)(0,0,\tau-i(\lambda+\epsilon),\xi) + \operatorname{Im}\Big(\overline{\frac{\partial p_m}{\partial \xi}}\cdot\frac{\partial p_m}{\partial x}\Big)(0,0,\tau-i(\lambda+\epsilon),\xi)\right)>0. \quad (1.3.4)$$

Note that for $\lambda>0$ the limit is useless and for $\lambda=0$, this assumption means only that for $(\tau,\xi)\neq(0,0)$,

$$\frac{\partial\sigma_{2m-1,\phi}}{\partial\lambda}(0,0,\tau,\xi,\lambda)_{|\lambda=0}>0 \quad \text{at} \quad p_m(0,0,\tau,\xi)=\frac{\partial p_m}{\partial\tau}(0,0,\tau,\xi)=0,$$

with

$$\sigma_{2m-1,\phi}(t,x,\tau,\xi,\lambda) = \operatorname{Im}\left(\overline{\frac{\partial p_m}{\partial \xi}(t,x,\tau-i\lambda\partial_t\phi,\xi-i\lambda\partial_x\phi)}\cdot\frac{\partial p_m}{\partial x}(t,x,\tau-i\lambda\partial_t\phi,\xi-i\lambda\partial_x\phi)\right) + \operatorname{Im}\left(\overline{\frac{\partial p_m}{\partial \xi}(t,x,\tau-i\lambda\partial_t\phi,\xi-i\lambda\partial_x\phi)}\cdot\frac{\partial p_m}{\partial x}(t,x,\tau-i\lambda\partial_t\phi,\xi-i\lambda\partial_x\phi)\right).$$

Lemma 1.3.1. *Let $p_m(x,t,\xi,\tau)$ be a polynomial of degree m with real coefficients such that (1.3.4) holds. There exists a constant $\mu > 0$ such that for $(t,x) \in W_\mu = \{(t,x) \in \mathbb{R}^{1+d}, |t|+|x| \le \mu^{-2}\}$, for $(\tau,\xi,\lambda) \in \mathbb{R}\times\mathbb{R}^d\times(0,+\infty)$,*

$$\mu|p_m(t,x,\zeta)|^2(\lambda^2+\tau^2+|\xi|^2)^{-1/2} + c_{2m-1,\phi}(t,x,\tau,\xi,\lambda) \ge \mu^{-1}\lambda(\lambda^2+\tau^2+|\xi|^2)^{m-1}.$$

Proof. Since both sides of the inequality are homogeneous with degree $2m-1$ with respect to (τ,ξ,λ), it is enough to prove that on the half-sphere $\lambda^2+\tau^2+|\xi|^2 = 1, \lambda > 0$. By *reductio ad absurdum*, a violation of the previous inequality would mean that there exists a sequence $(t_k,x_k) \in W_k$ and a sequence (λ_k,τ_k,ξ_k) on the half-sphere such that

$$k|p_m(t_k,x_k,\tau_k - i\lambda_k(1-kt_k),\xi_k - i\lambda_k k^{-1}x_k)|^2 + c_{2m-1,\phi}(t_k,x_k,\tau_k,\xi_k,\lambda_k) < \frac{\lambda_k}{k}. \quad (1.3.5)$$

By compactness of the closure of the half-sphere, we may assume that (λ_k,τ_k,ξ_k) is converging to (λ_0,τ_0,ξ_0) on the closure of half-sphere. Since kt_k goes to 0, multiplying the previous inequality by k^{-1} provides

$$p_m(0,0;\tau_0 - i\lambda_0,\xi_0) = 0. \quad (1.3.6)$$

We assume first that $\lambda_0 > 0$. We have

$$c_{2m-1,\phi}(t_k,x_k,\tau_k,\xi_k,\lambda_k) = \lambda_k k \left|\frac{\partial p_m}{\partial\tau}(t_k,x_k,\zeta_k)\right|^2 + O(1)$$

and multiplying Inequality (1.3.5) by $1/k\lambda_k$ ($k\lambda_k$ goes to $+\infty$ with k), we get

$$\frac{\partial p_m}{\partial\tau}(0,0,\tau_0 - i\lambda_0,\xi_0) = 0.$$

From (1.3.5), we obtain that

$$\frac{1}{\lambda_k}\operatorname{Im}\left(\overline{\frac{\partial p_m}{\partial\tau}(t_k,x_k,\zeta_k)}\cdot\frac{\partial p_m}{\partial t}(t_k,x_k,\zeta_k)\right) + \frac{1}{\lambda_k}\operatorname{Im}\left(\overline{\frac{\partial p_m}{\partial\xi}(t_k,x_k,\zeta_k)}\cdot\frac{\partial p_m}{\partial x}(t_k,x_k,\zeta_k)\right) - \frac{1}{k}\left|\frac{\partial p_m}{\partial\xi}(t_k,x_k,\zeta_k)\right|^2 < \frac{1}{k},$$

which is incompatible with (1.3.4).

We assume now that $\lambda_0 = 0$, so that $\tau_0^2+\xi_0^2 = 1$. We have

$$c_{2m-1,\phi}(t_k,x_k,\tau_k,\xi_k,\lambda_k) = \lambda_k k\left|\frac{\partial p_m}{\partial\tau}(t_k,x_k,\zeta_k)\right|^2 + O(\lambda_k) + \frac{1}{2i}\left\{\overline{p_m},p_m\right\}(t_k,x_k;\tau_k,\xi_k). \quad (1.3.7)$$

We have assumed that p_m has real coefficients, so that $\{\overline{p_m}, p_m\}(t_k, x_k; \tau_k, \xi_k)$ is identically 0. Multiplying (1.3.5) by $1/k\lambda_k$ (λ_k is positive), we get

$$\left|\frac{\partial p_m}{\partial \tau}(t_k, x_k, \zeta_k)\right|^2 + O(1/k) < \frac{1}{k^2}$$

implying that $\frac{\partial p_m}{\partial \tau}(0, 0, \tau_0, \xi_0) = 0$. From (1.3.5), we obtain that (note that $\lambda_k > 0$ with limit 0)

$$\begin{aligned}\frac{1}{\lambda_k} \operatorname{Im}\left(\overline{\frac{\partial p_m}{\partial \tau}(t_k, x_k, \zeta_k)} \cdot \frac{\partial p_m}{\partial t}(t_k, x_k, \zeta_k)\right) \\ + \frac{1}{\lambda_k} \operatorname{Im}\left(\overline{\frac{\partial p_m}{\partial \xi}(t_k, x_k, \zeta_k)} \cdot \frac{\partial p_m}{\partial x}(t_k, x_k, \zeta_k)\right) \\ + k\left|\frac{\partial p_m}{\partial \tau}(t_k, x_k, \zeta_k)\right|^2 \\ < \frac{1}{k}\left|\frac{\partial p_m}{\partial \xi}(t_k, x_k, \zeta_k)\right|^2 + \frac{1}{k}.\end{aligned}$$

We have thus

$$\begin{aligned}\frac{1}{\lambda_k} \operatorname{Im}\left(\overline{\frac{\partial p_m}{\partial \tau}(t_k, x_k, \zeta_k)} \cdot \frac{\partial p_m}{\partial t}(t_k, x_k, \zeta_k)\right) \\ + \frac{1}{\lambda_k} \operatorname{Im}\left(\overline{\frac{\partial p_m}{\partial \xi}(t_k, x_k, \zeta_k)} \cdot \frac{\partial p_m}{\partial x}(t_k, x_k, \zeta_k)\right) \\ = \frac{\widetilde{\sigma}_{2m-1}(t_k, x_k, \zeta_k) - \widetilde{\sigma}_{2m-1}(t_k, x_k, \xi_k)}{\lambda_k} \le O(1/k).\end{aligned}$$

Thanks to the hypothesis (1.3.4), the lhs has the positive limit

$$\frac{\partial \sigma_{2m-1,\phi}}{\partial \lambda}(0, 0, \tau_0, \xi_0, 0)$$

which is incompatible with the previous inequality. The proof of the lemma is complete. □

Using now the last inequality in Proposition 1.2.8, we get for $\lambda \ge \mu$ that

$$\begin{aligned}C\|v\|^2_{\mathcal{H}^{m-1}_\lambda} + \|P_m(x, D - i\lambda d\phi)v\|^2_{L^2} \ge \mu\|\operatorname{op}(\operatorname{Re} a_m)v\|^2_{\mathcal{H}^{-1/2}} + \|\operatorname{op}(\operatorname{Im} a_m)v\|^2_{\mathcal{H}^{-1/2}} \\ + \operatorname{Re}\langle \operatorname{op}(c_{2m-1,\phi})v, v\rangle.\end{aligned}$$

and Lemma 1.2.11 , (1.2.13) along with Gårding's inequality (see our Appendix, Section 4.2) implies

$$C_1\|v\|^2_{\mathcal{H}^{m-1}_\lambda} + \|P_m(x, D - i\lambda d\phi)v\|^2_{L^2} \ge \frac{\lambda}{\mu}\|v\|^2_{\mathcal{H}^{m-1}_\lambda},$$

entailing (1.2.4) and thus (1.2.3). Applying the convexity property of Lemma 1.2.10, we obtain the following uniqueness result, due to L. Hörmander [9], Chapter 28.

Theorem 1.3.2. *Let P be a differential operator of order m with C^∞ real-valued coefficients in the principal part, L^∞ complex-valued for lower order terms, in some open subset Ω of $\mathbb{R}^n$. Let Σ be a C^2 hypersurface of Ω given by an equation $\rho(x) = 0$, with $d\rho \neq 0$ at Σ. Let $x_0 \in \Sigma$; we assume that for $(\xi, \lambda) \in (\mathbb{R}^n \times \mathbb{R}_+)\backslash\{0\}$*

$$p_m(x_0, \xi - i\lambda d\rho(x_0)) = \{p_m, \rho\}(x_0, \xi - i\lambda d\rho(x_0)) = 0 \Longrightarrow$$
$$\lim_{\substack{\epsilon \to 0_+ \\ \epsilon > 0}} \frac{1}{\lambda + \epsilon} \operatorname{Im}\Big(\overline{\frac{\partial p_m}{\partial \xi}(x_0, \zeta)} \cdot \frac{\partial p_m}{\partial x}(x_0, \zeta)\Big) - \rho''(x_0)\overline{\frac{\partial p_m}{\partial \xi}(x_0, \zeta)}\frac{\partial p_m}{\partial \xi}(x_0, \zeta) > 0, \tag{1.3.8}$$

with $\zeta = \xi - i(\lambda + \epsilon)d\rho(x_0)$. If u is an H^m_{loc} function, supported in $\{x \in W, \rho(x) \geq 0\}$ where W is a neighborhood of x_0, is such that

$$|(Pu)(x)| \leq \sum_{0 \leq j < m} V_j(x)|\nabla^j u(x)|, \qquad V_j \in L^\infty_{loc},$$

then u is vanishing in a neighborhood of x_0.

N.B. Using a more specific Gårding's inequality as in Section 8.3 and 8.4 of [8], it is possible to reduce the regularity requirements for the principal part in the above theorem to C^2. Some refinements of these methods, taking into account that the operators involved are differential (and not general pseudodifferential operators), are presented in [33] and seem to allow a version of the previous theorem for Lipschitz regularity in the principal part; however, the pseudo-convexity assumption would have to be modified to be meaningful for the derivative of a Lipschitz-continuous function, which is not defined pointwise but only as a bounded measurable function.

1.3.2 Comments

(a) Invariance of the assumptions by change of coordinates

We consider a reference open subset Ω of $\mathbb{R}^n$ and Σ an oriented C^1 hypersurface of Ω:

$$\Sigma = \{x \in \Omega, \rho(x) = 0\}$$

where $\rho : \Omega \to \mathbb{R}$ is a C^1 function such that $\rho(x) = 0$ implies $d\rho(x) \neq 0$. Note that, applying the implicit function theorem, this implies that, for any $x_0 \in \Sigma$, there exists a neighborhood $V_0 = W_0 \times J_0$ of x_0, where W_0 is an open ball in $\mathbb{R}^{n-1}$ and J_0 is an open interval of $\mathbb{R}$, a C^1 function α defined on W_0 such that

$$R(\Sigma \cap V_0) = \{(x', x_n) \in W_0 \times J_0, x_n = \alpha(x')\},$$

where $R \in O(n)$. We have the following partition of Ω:

$$\Omega = \Omega_- \cup \Sigma \cup \Omega_+, \quad \Omega_\pm = \{x \in \Omega, \pm\rho(x) > 0\}$$

and the open sets $\Omega_\pm$ have closure $\Omega_\pm \cup \Sigma$ in Ω: the inclusion of the closure of Ω_+ into $\Omega_+ \cup \Sigma$ is obvious, since a limit point x of a sequence (x_k) such that $\rho(x_k) > 0$ should satisfy $\rho(x) \ge 0$ and conversely if x_0 is such that $\rho(x_0) = 0$, we find a system of (linear) coordinates on a neighborhood V_0 of x_0, centered at x_0, such that Σ appears as the graph of a C^1 function α as above, with $\alpha(0) = 0$. Then $0_{\mathbb{R}^n}$ is the limit of $(0_{\mathbb{R}^{n-1}}, \epsilon)$ which belong to Ω_+.[6] An oriented hypersurface of Ω does not have a unique equation ρ, but if we are given two C^1 equations ρ_1, ρ_2 such that $\rho_j(x) = 0 \Longrightarrow d\rho_j(x) \ne 0$, with $\Sigma = \{x \in \Omega, \rho_j(x) = 0\}, \Omega_\pm = \{\pm\rho_j > 0\}, j = 1, 2,$ by the implicit function reasoning displayed above, we get near a distinguished point x_0 a system of linear coordinates such that

$$\rho_j(x) = e_j(x)\big(x_n - \alpha(x')\big), \quad e_j > 0 \text{ continuous.}$$

As a result the conormal bundle $\Sigma^\perp$ of Σ is well-defined, by

$$\Sigma^\perp = \{(x, \xi) \in \Omega \times \mathbb{R}^n, \rho(x) = 0, \xi \wedge d\rho(x) = 0\}. \tag{1.3.9}$$

Note that $\rho = 0$ is one constraint, and since $d\rho(x) \ne 0$ at $\rho(x) = 0$, $\xi \wedge d\rho(x) = 0$ means that ξ is proportional to $d\rho(x)$, that is $n-1$ constraints: the conormal bundle is n dimensional. To take into account the orientation of the hypersurface Σ, we may also define the *positive conormal bundle*

$$\Sigma^\perp_+ = \{(x, \xi) \in \Omega \times \mathbb{R}^n, \rho(x) = 0, \xi \in \mathbb{R}_+ d\rho(x)\}, \tag{1.3.10}$$

which is a closed subset of the conormal bundle. Note that these objects are intrinsically defined, nonetheless independently of a choice of coordinates, but also are not dependent of a choice of a defining function ρ for the oriented hypersurface Σ.

We consider now a differential operator $P = \sum_{|\alpha| \le m} a_\alpha(x) D_x^\alpha$ whose coefficients a_α are $C^2(\Omega)$ functions for $|\alpha| = m$ and $L^\infty(\Omega)$ for $|\alpha| < m$. If we perform a C^∞ change of coordinates $U \ni y \mapsto \kappa(y) = x \in \Omega, \kappa^{-1} = \nu$, we have for $u \in C^\infty(\Omega), v = u \circ \kappa \in C^\infty(U)$

$$(Pu)(x) = (Pu)(\kappa(y)) = \sum_{|\alpha| \le m} a_\alpha(\kappa(y)) \prod_{1 \le j \le n} \Big(\sum_{1 \le k \le n} \frac{\partial y_k}{\partial x_j} D_{y_k} \Big)^{\alpha_j} v.$$

Considering the mapping

$$\mathbb{R}^n \ni \eta = (\eta_1, \dots, \eta_n) \mapsto \Big(\sum_{1 \le k \le n} \frac{\partial y_k}{\partial x_j} \eta_k \Big)_{1 \le j \le n} = {}^t\nu'(x)\eta = {}^t\nu'(\kappa(y))\eta = {}^t\kappa'^{-1}(y)\eta$$

[6] Same story for Ω_-. Note that, for a smooth function ρ, the closure of the set $\{\rho > 0\}$ is included in but not always equal to $\{\rho \ge 0\}$: take for instance a function ρ defined on $\mathbb{R}$ such that ρ vanishes on $[-1, 1]$ and is positive outside this interval: we have then

$$\overline{\{\rho > 0\}} = (-\infty, -1] \cup [1, +\infty), \quad \{\rho \ge 0\} = \mathbb{R}.$$

Of course that function has critical points at $\rho = 0$, which is excluded by our assumption.

we may write that

$$(Pu)(x) = \sum_{|\alpha|=m} a_\alpha(\kappa(y))\big({}^t\kappa'(y)^{-1}D_y\big)^\alpha v + \text{lower order terms.}$$

Defining the principal symbol of the operator P on $\Omega \times \mathbb{R}^n = T^*(\Omega)$ (cotangent bundle of Ω) by

$$p_m(x,\xi) = \sum_{|\alpha|=m} a_\alpha(x)\xi^\alpha,$$

we obtain that the operator $\kappa^* P \kappa_*$ is also a differential operator of order m: here we used the notation κ^* for the pullback by κ and κ_* for the pushforward (i.e. the pullback by κ^{-1}). We have for $v \in C^\infty(U)$, $\kappa_* v = v \circ \nu = u$,

$$(\kappa^* P \kappa_* v)(y) = \big(P\kappa_* v\big)(\kappa(y)) = \big(P(v\circ\nu)\big)(\kappa(y)).$$

Moreover the principal symbol of the differential operator $\kappa^* P \kappa_*$ is

$$q_m(y,\eta) = p_m\big(\kappa(y), {}^t\kappa'(y)^{-1}\eta\big). \tag{1.3.11}$$

We have proven that the principal symbol of the differential operator P is invariantly defined on the cotangent bundle of Ω: for $(x,\xi) \in T^*(\Omega)$, and κ as above, we have the canonical mapping

$$T^*(U) \ni (y,\eta) \mapsto \big(\kappa(y), {}^t\kappa'(y)^{-1}\eta\big) \in T^*(\Omega)$$

and Formula (1.3.11) proves the invariance by diffeomorphism of the principal symbol of a differential operator.

Let us go back to our assumptions in Theorem 1.3.2. We are given an oriented hypersurface Σ, a differential operator P and a distinguished point x_0 in Σ. The operator P has the principal symbol p, ρ is a defining function for Σ. We require for $\zeta = \xi - i\lambda d\rho(x_0)$ with $\lambda > 0$, whenever $p(x_0,\zeta) = \{p,\rho\}(x_0,\zeta) = 0$

$$\begin{cases} \text{if } \lambda > 0 & \frac{1}{\lambda}\operatorname{Im}\Big(\overline{\frac{\partial p_m}{\partial\xi}(x_0,\zeta)}\cdot\frac{\partial p_m}{\partial x}(x_0,\zeta)\Big) - \rho''(x_0)\overline{\frac{\partial p_m}{\partial\xi}(x,\zeta)}\frac{\partial p_m}{\partial\xi}(x,\zeta) > 0,\\ \text{if } \lambda = 0 & \frac{\partial}{\partial\lambda}\operatorname{Im}\Big(\overline{\frac{\partial p_m}{\partial\xi}(x_0,\zeta)}\cdot\frac{\partial p_m}{\partial x}(x_0,\zeta)\Big)_{|\lambda=0} - \rho''(x_0)\overline{\frac{\partial p_m}{\partial\xi}(x,\xi)}\frac{\partial p_m}{\partial\xi}(x,\xi) > 0. \end{cases}$$

We note that the points ζ belong to the complexified cotangent space at x_0 and also that $-\operatorname{Im}\zeta \in \Sigma_+^\perp(x_0) : 0 \ne \zeta \in T^*_{x_0}(\Omega) - i\Sigma_+^\perp(x_0)$.

Poisson bracket. We consider an open subset Ω of $\mathbb{R}^n$ and the cotangent bundle $T^*(\Omega) = \Omega \times \mathbb{R}^n$. Let a, b be C^1 functions defined on $T^*(\Omega)$. We define the Poisson bracket $\{a,b\}$ by

$$\{a,b\} = \frac{\partial a}{\partial\xi}\cdot\frac{\partial b}{\partial x} - \frac{\partial a}{\partial x}\cdot\frac{\partial b}{\partial\xi}. \tag{1.3.12}$$

With the two-form σ defined by

$$\sigma = \sum_{1\le j\le n} d\xi_j \wedge dx_j, \tag{1.3.13}$$

we define the Hamiltonian vector field H_a of a by

$$\sigma \lrcorner\, H_a = -da \Longrightarrow \langle da, X\rangle = X(a) = \sigma(X, H_a) \text{ for a vector field } X,$$

i.e. $p \cdot \partial_x a + q \cdot \partial_\xi a = q \cdot \partial_\xi a - p \cdot (-\partial_x a), H_a = \partial_\xi a \partial_x - \partial_x a \partial_\xi$ and

$$H_a(b) = \{a, b\} = \sigma(H_a, H_b). \tag{1.3.14}$$

If we perform a C^∞ change of coordinates $U \ni y \mapsto \kappa(y) = x \in \Omega, \kappa^{-1} = \nu$, this induces the diffeomorphism $\tilde\kappa$

$$T^*(U) \ni (y, \eta) \mapsto \big(\kappa(y), {}^t\kappa'(y)^{-1}\eta\big) \in T^*(\Omega),$$

and the pullback of σ by $\tilde\kappa$ is, written with Einstein's convention,

$$\tilde\kappa^*(\sigma) = \tilde\kappa^*(d\xi_j \wedge dx_j) = \frac{\partial \nu_k}{\partial x_j} d\eta_k \wedge \frac{\partial \kappa_j}{\partial y_l} dy_l = d\eta_k \wedge dy_k.$$

We infer that, for a, b smooth on $T^*(\Omega)$,

$$\{a \circ \tilde\kappa, b \circ \tilde\kappa\} = \tilde\kappa^*(\sigma)\big(H_{a\circ\tilde\kappa}, H_{b\circ\tilde\kappa}\big) = \sigma(T(\tilde\kappa)H_{a\circ\tilde\kappa}, T(\tilde\kappa)H_{b\circ\tilde\kappa}) = \sigma(H_a, H_b) \circ \tilde\kappa.$$

We have

$$\{p, \rho\} = \frac{\partial p}{\partial \xi} \cdot \frac{\partial \rho}{\partial x}.$$

We note as well that

$$\begin{aligned}
\operatorname{Im}\Big(\overline{\frac{\partial p_m}{\partial \xi}(x,\zeta)} \cdot \frac{\partial p_m}{\partial x}(x,\zeta)\Big) &= \frac{1}{2i}\Big(\overline{\frac{\partial p_m}{\partial \xi}(x,\zeta)} \cdot \frac{\partial p_m}{\partial x}(x,\zeta) - \overline{\frac{\partial p_m}{\partial x}(x,\zeta)} \cdot \frac{\partial p_m}{\partial \xi}(x,\zeta)\Big) \\
&= \frac{1}{2i}\Big\{\overline{p_m(x, \xi + i\eta)}, p_m(x, \xi + i\eta)\Big\}_{|\eta = \operatorname{Im}\zeta}.
\end{aligned} \tag{1.3.15}$$

Let us calculate, with $\zeta = \xi - i\lambda d\rho(x_0)$,

$$\begin{aligned}
&\frac{\partial}{\partial \lambda} \operatorname{Im}\Big(\overline{\frac{\partial p_m}{\partial \xi}(x_0,\zeta)} \cdot \frac{\partial p_m}{\partial x}(x_0,\zeta)\Big) \\
&= \operatorname{Im}\Big(\overline{\frac{\partial^2 p_m}{\partial \xi^2}(x_0,\zeta)(-id\rho(x_0))} \cdot \frac{\partial p_m}{\partial x}(x_0,\zeta) + \overline{\frac{\partial p_m}{\partial \xi}(x_0,\zeta)} \cdot \frac{\partial^2 p_m}{\partial x \partial \xi}(x_0,\zeta)(-id\rho(x_0))\Big) \\
&= \operatorname{Re}\Big(\overline{\frac{\partial^2 p_m}{\partial \xi^2}(x_0,\zeta)d\rho(x_0)} \cdot \frac{\partial p_m}{\partial x}(x_0,\zeta) - \overline{\frac{\partial p_m}{\partial \xi}(x_0,\zeta)} \cdot \frac{\partial^2 p_m}{\partial x \partial \xi}(x_0,\zeta)d\rho(x_0)\Big),
\end{aligned}$$

so that

$$\begin{aligned}
&\frac{\partial}{\partial \lambda} \operatorname{Im}\Big(\overline{\frac{\partial p_m}{\partial \xi}(x_0,\zeta)} \cdot \frac{\partial p_m}{\partial x}(x_0,\zeta)\Big)_{|\lambda=0} - \rho''(x_0)\overline{\frac{\partial p_m}{\partial \xi}(x,\xi)}\frac{\partial p_m}{\partial \xi}(x,\xi) \\
&= \operatorname{Re}\Big(\overline{\frac{\partial^2 p_m}{\partial \xi^2}(x_0,\xi)d\rho(x_0)} \cdot \frac{\partial p_m}{\partial x}(x_0,\xi) - \overline{\frac{\partial p_m}{\partial \xi}(x_0,\xi)} \cdot \frac{\partial^2 p_m}{\partial x \partial \xi}(x_0,\xi)d\rho(x_0)\Big) \\
&\qquad - \rho''(x_0)\overline{\frac{\partial p_m}{\partial \xi}(x,\xi)}\frac{\partial p_m}{\partial \xi}(x,\xi).
\end{aligned}$$

We note as well that

$$\begin{aligned}\{\overline{p},\{p,\rho\}\} &= \partial_\xi \overline{p}\cdot\partial_x\big(\partial_\xi p\cdot\partial_x\rho\big)-\partial_x\overline{p}\cdot\partial_\xi\big(\partial_\xi p\cdot\partial_x\rho\big)\\ &= \partial_\xi\overline{p}\cdot\partial_x\partial_\xi p\cdot\partial_x\rho+\partial_\xi\overline{p}\cdot\partial_\xi p\cdot\partial_x^2\rho-\partial_x\overline{p}\cdot\partial_\xi^2 p\cdot\partial_x\rho,\end{aligned}$$

entailing

$$\begin{aligned}&-\operatorname{Re}\big(\{\overline{p_m},\{p_m,\rho\}\}\,(x,\xi)\big)\\ &\qquad=\frac{\partial}{\partial\lambda}\operatorname{Im}\Big(\overline{\frac{\partial p_m}{\partial\xi}(x_0,\zeta)}\cdot\frac{\partial p_m}{\partial x}(x_0,\zeta)\Big)_{|\lambda=0}-\rho''(x_0)\overline{\frac{\partial p_m}{\partial\xi}(x,\xi)}\frac{\partial p_m}{\partial\xi}(x,\xi).\end{aligned}$$

We have thus proven the following

Lemma 1.3.3. *With the above notations, the pseudo-convexity hypothesis is: for $\zeta=\xi-i\lambda d\rho(x_0)$ with $\lambda\ge 0$, whenever $p_m(x_0,\zeta)=\{p_m,\rho\}(x_0,\zeta)=0$ and $(\xi,\lambda)\in(\mathbb{R}^n\times\mathbb{R}_+)\backslash\{0,0)\}$,*

$$\begin{cases}\text{if } \lambda>0 & \frac{1}{\lambda}\operatorname{Im}\overline{\partial_\xi p_m(x_0,\zeta)}\cdot\partial_x p_m(x_0,\zeta)-\rho''(x_0)\overline{\frac{\partial p_m}{\partial\xi}(x,\zeta)}\frac{\partial p_m}{\partial\xi}(x_0,\zeta)>0,\\ \text{if } \lambda=0 & \operatorname{Re}\big(\{\overline{p_m},\{p_m,\rho\}\}\,(x_0,\xi)\big)<0.\end{cases}$$

(b) Invariance of the assumptions by change of defining function for Σ

We have defined in (1.3.10) the positive conormal bundle and we see that $0\neq\zeta\in T^*_{x_0}(\Omega)-i\Sigma_+^{\perp}(x_0)$: the condition $\{p_m,\rho\}(x_0,\zeta)=0$ means that the vector $\partial_\xi p_m(x_0,\zeta)$ is in the kernel of the covector $\partial_x\rho(x_0)$, i.e. in the kernel of all covectors in $\Sigma_+^{\perp}(x_0)$. Now, if a vector T is such that the bracket of duality $\langle d\rho(x_0),T\rangle=0$, the quantity

$$\langle\rho''(x_0)T,T\rangle=\frac{1}{2}\frac{d^2}{dt^2}\big(\rho(x_0+tT)\big)_{|t=0},$$

so that, replacing ρ by $e\rho$, with a positive function e near x_0 such that $\rho(x_0)=0$, we obtain

$$\begin{aligned}(e\rho)''(x_0)T^2&=\frac{1}{2}\frac{d^2}{dt^2}\big((e\rho)(x_0+tT)\big)_{|t=0}\\ &=\frac{1}{2}\frac{d^2}{dt^2}e(x_0+tT)\underbrace{\rho(x_0)}_{=0}+\frac{d}{dt}(e(x_0+tT)\underbrace{\langle d\rho(x_0),T\rangle}_{=0}+e(x_0)\langle\rho''(x_0)T,T\rangle.\end{aligned}$$

As a result with $\tilde\zeta=\xi-i\lambda e(x_0)d\rho(x_0)$ and $e(x_0)>0, p_m(x_0,\tilde\zeta)=\{p_m,e\rho\}\,(x_0,\tilde\zeta)=0$, we have with

$$\begin{aligned}&\frac{1}{\lambda}\operatorname{Im}\overline{\partial_\xi p_m(x_0,\tilde\zeta)}\cdot\partial_x p_m(x_0,\tilde\zeta)-e(x_0)\rho''(x_0)\overline{\frac{\partial p_m}{\partial\xi}(x,\tilde\zeta)}\frac{\partial p_m}{\partial\xi}(x_0,\tilde\zeta)\\ &\quad=e(x_0)\Big(\frac{1}{\lambda}\operatorname{Im}\overline{\partial_\xi p_m(x_0,\tilde\zeta)}\cdot\partial_x p_m(x_0,\tilde\zeta)-\rho''(x_0)\overline{\frac{\partial p_m}{\partial\xi}(x,\tilde\zeta)}\frac{\partial p_m}{\partial\xi}(x_0,\tilde\zeta)\Big)>0\end{aligned}$$

from the assumption of pseudo-convexity.

1.3.3 Examples

Simple characteristics

A reminder of a discussion above. We may assume that our oriented hypersurface is given by the equation $t = 0$ near $0 \in \mathbb{R}^1_t \times \mathbb{R}^d_x$ and that our differential operator has a **real-valued** principal symbol (a polynomial with degree m in ξ, τ),

$$p_m(t, x; \tau, \xi).$$

We assume for $(\tau, \xi, \lambda) \in \mathbb{R} \times \mathbb{R}^{n-1} \times \mathbb{R}_+ \backslash \{(0, 0, 0)\}$,

$$p_m(0, 0; \tau - i\lambda, \xi) = 0 \Longrightarrow \partial_\tau p_m(0, 0; \tau - i\lambda, \xi) \neq 0.$$

Note that above hypothesis implies that the hypersurface with equation $t = 0$ is non-characteristic for P, i.e. $p_m(0, 0; 1, 0) \neq 0$, otherwise, the polynomial $p_m(0, 0; \tau, 0)$ would be the zero polynomial. That hypothesis means simply that the m roots of the polynomial $\tau \mapsto p_m(0, 0; \tau, \xi)$ are simple for $\xi \in \mathbb{R}^d \backslash \{0\}$: if $\xi = 0$, we have

$$p_m(0, 0; \tau - i\lambda, 0) = p_m(0, 0; 1, 0)(\tau - i\lambda)^m$$

which is not zero for $\tau - i\lambda \neq 0$. We may note as well that, since p_m is a polynomial with real coefficients, the roots go by conjugate pairs, so that requiring that a non-real root with negative imaginary part is simple entails that the conjugate root is simple as well.

Lemma 1.3.4. *A second-order elliptic operator P with real smooth coefficients in the principal part, L^∞_{loc} for the lower order terms, such as the Laplace operator, is such that any hypersurface is pseudo-convex and thus has unique continuation from any open subset: if Ω is a connected open set of $\mathbb{R}^n$ and if $\Omega_0 \neq \emptyset$ is open $\subset \Omega$,*

$$Pu = 0 \text{ in } \Omega, \quad u_{|\Omega_0} = 0, \quad u \in H^2_{loc}(\Omega),$$

this implies that $u = 0$ in Ω.

Proof. In particular let $p_2(x, \xi)$ be a second-order elliptic polynomial with real coefficients and let ρ be a (real-valued) function such that $\rho = 0 \Longrightarrow d\rho \neq 0$. Let us assume that for some $(\xi, \lambda) \neq (0, 0)$

$$p_2(x, \xi - i\lambda d\rho(x)) = 0.$$

Then from the ellipticity property, we have $\lambda \neq 0$ and we cannot have

$$\partial_\xi p_2(x, \xi - i\lambda d\rho(x)) \cdot d\rho(x) = 0,$$

otherwise since p_2 has real coefficients,

$$p_2(x, \xi \pm i\lambda d\rho(x)) = \frac{d}{d\lambda}\big(p_2(x, \xi \pm i\lambda d\rho(x))\big) = 0,$$

and the polynomial $\sigma \mapsto p_2(x, \xi - i\sigma d\rho(x))$ will have two distinct double zeroes at $\sigma = \pm\lambda$, which is not possible since it is a polynomial of degree 2 with leading coefficient $-p_2(x, d\rho(x)) \neq 0$ from the ellipticity. To prove unique continuation from the non-empty open set Ω_0, we need to use the connexity of Ω as follows.

(i) If $\partial(\operatorname{supp} u) = \emptyset$, then $\operatorname{supp} u = \operatorname{interior}(\operatorname{supp} u)$ and thus is closed and open. Since $(\operatorname{supp} u)^c$ is non-empty, we must have $\operatorname{supp} u = \emptyset$.

(ii) We assume now that $\partial(\operatorname{supp} u) \neq \emptyset$ and we consider a point $x_0 \in \partial(\operatorname{supp} u)$. Since $x_0 \in \Omega$, there exists $r_0 > 0$ such that $B(x_0, r_0) \subset \Omega$. We also know that x_0 is not an interior point of $\operatorname{supp} u$, so that $B(x_0, r_0/4) \cap (\operatorname{supp} u)^c \neq \emptyset$. Let $x_1 \in B(x_0, r_0/4) \cap (\operatorname{supp} u)^c$. We note that

$$B(x_1, r_0/4) \subset B(x_0, r_0/2)(\subset \Omega) \text{ since } |y - x_1| < r_0/4 \Longrightarrow |y - x_0| < r_0/2.$$

We consider

$$R = \{r \in (0, +\infty), B(x_1, r) \subset (\operatorname{supp} u)^c\}.$$

The set R is not empty since x_1 belongs to the open set $(\operatorname{supp} u)^c$. On the other hand, an upper bound for R is $r_0/4$ since $|x_1 - x_0| < r_0/4$ and $x_0 \in \operatorname{supp} u$. Let $r_1 = \sup R :$ we have $0 < r_1 \leq r_0/4$ and the ball $B(x_1, r_1) \subset (\operatorname{supp} u)^c$ since

$$B(x_1, r_1) = \cup_{k \in \mathbb{N}} B(x_1, r_1 - \epsilon_k), \epsilon_k > 0, \lim_k \epsilon_k = 0.$$

Using that $B(x_1, r_1) \subset B(x_0, r_0/2) \subset \Omega$, and that the sphere $|x - x_1| = r_1$ is a smooth hypersurface of Ω, Cauchy uniqueness with respect to that sphere shows that u must vanish in a neighborhood of the compact set $\bar{B}(x_1, r_1)$, in particular in $B(x_1, r)$ with $r > r_1$, contradicting the supremum property of r_1. The hypothesis (ii) is absurd. □

Lemma 1.3.5. *Let P be an operator of order m with smooth real-valued coefficients and let Σ be a smooth hypersurface with a defining function ρ. Let us assume that P is strictly hyperbolic with respect to Σ, i.e. for $\xi \wedge d\rho(x) \neq 0$,*

the roots of $\sigma \mapsto p_m(x, \xi + \sigma d\rho(x))$ are real and simple and $p_m(x, d\rho(x)) \neq 0$.

Then Σ is strictly pseudo-convex with respect to P and P has unique continuation from $\Sigma_\pm$.

Proof. We may assume that ρ is a coordinate t and for $\mathbb{R}^{n-1} \ni \xi \neq 0$

$$p_m(t, x, 1, 0) \neq 0, \quad \tau \mapsto p_m(t, x, \tau, \xi) \text{ has simple real roots.}$$

Then the assumption of simple characteristics holds since for $\xi \neq 0$

$$p_m(t, x, \tau - i\lambda, \xi) = 0 \Longrightarrow \lambda = 0, \partial_\tau p_m \neq 0$$

and if $\xi = 0$, $p_m(t, x, \tau - i\lambda, 0) = (\tau - i\lambda)^m \underbrace{p_m(x, t, 1, 0)}_{\neq 0} \neq 0$ if $(\tau, \lambda) \neq (0, 0)$. □

More generally, we have

Lemma 1.3.6. *Let P be an operator of order m with real coefficients and let Σ be a non-characteristic hypersurface with a defining function ρ such that for $\xi \wedge d\rho(x) \neq 0$*

$$\text{the roots of } \sigma \mapsto p_m(x, \xi + \sigma d\rho(x)) \text{ are simple.}$$

Then Σ is pseudo-convex with respect to P and P has unique continuation from $\Sigma_\pm$.

We may notice that all these simple characteristics results are already encompassing many interesting cases, including the strictly hyperbolic cases and the elliptic operators with degree two and real coefficients. Also of course for these problems, the orientation of the hypersurface does not matter, a kind of reversibility property. We shall see that it is a sharp contrast with the pseudo convexity assumptions where the orientation of Σ plays an important rôle.

Pseudoconvexity

We may assume that our oriented hypersurface is given by the equation $t = 0$ near $0 \in \mathbb{R}^1_t \times \mathbb{R}^d_x$ and that our differential operator has a **real-valued** principal symbol (a polynomial with degree m in ξ, τ),

$$p_m(t, x; \tau, \xi).$$

We shall assume that for $\lambda \geq 0, (\tau, \xi, \lambda) \neq (0, 0, 0)$,

$$p_m(0,0,\tau - i\lambda, \xi) = \frac{\partial p_m}{\partial \tau}(0,0,\tau - i\lambda, \xi) = 0 \Longrightarrow \begin{cases} \text{for } \lambda > 0, & \frac{1}{\lambda} \operatorname{Im}\left(\overline{\frac{\partial p_m}{\partial \xi}} \cdot \frac{\partial p_m}{\partial x}\right)(0,0,\tau - i\lambda, \xi) > 0, \\ \text{for } \lambda = 0, & H^2_{p_m}(t)(0,0,\tau,\xi) < 0. \end{cases} \tag{1.3.16}$$

In particular, if the non-real roots are simple, i.e. for $\lambda > 0$,

$$p_m(0,0,\tau - i\lambda, \xi) = 0 \Longrightarrow \frac{\partial p_m}{\partial \tau}(0,0,\tau - i\lambda, \xi) \neq 0, \tag{1.3.17}$$

then we need only to check that for $(\tau, \xi) \neq (0, 0)$, at $(0, 0, \tau, \xi)$,

$$p_m = H_{p_m}(t) = 0 \Longrightarrow H^2_{p_m}(t) < 0. \tag{1.3.18}$$

Going back to general coordinates and equation for Σ, we see that this condition is $p_m = H_{p_m}(\rho) = 0 \Longrightarrow H^2_{p_m}(\rho) < 0$ and can be illustrated geometrically. To say that $H_{p_m}(\rho) = 0$ means that the Hamiltonian vector field of p_m is tangent to the hypersurface Σ, viewed as a hypersurface of $T^*(\Omega)$. So Condition (1.3.18) expresses the fact the bicharacteristic curves of p_m, i.e. the integral curves of H_{p_m}, whenever they are tangent to Σ, must have the concavity property $H^2_{p_m}(\rho) < 0$. This appears clearly in Picture 1.3.

In other words, Σ is "above" the tangential characteristics, a sort of convexity assumption. The integral curves of H_p in the phase space are the bicharacteristic curves and the characteristic curves are simply their first projection. The bicharacteristics are defined by

$$\dot{x}(t) = \frac{\partial p}{\partial \xi}(x(t), \xi(t)), \quad \dot{\xi}(t) = -\frac{\partial p}{\partial x}(x(t), \xi(t))$$

so that, calculating

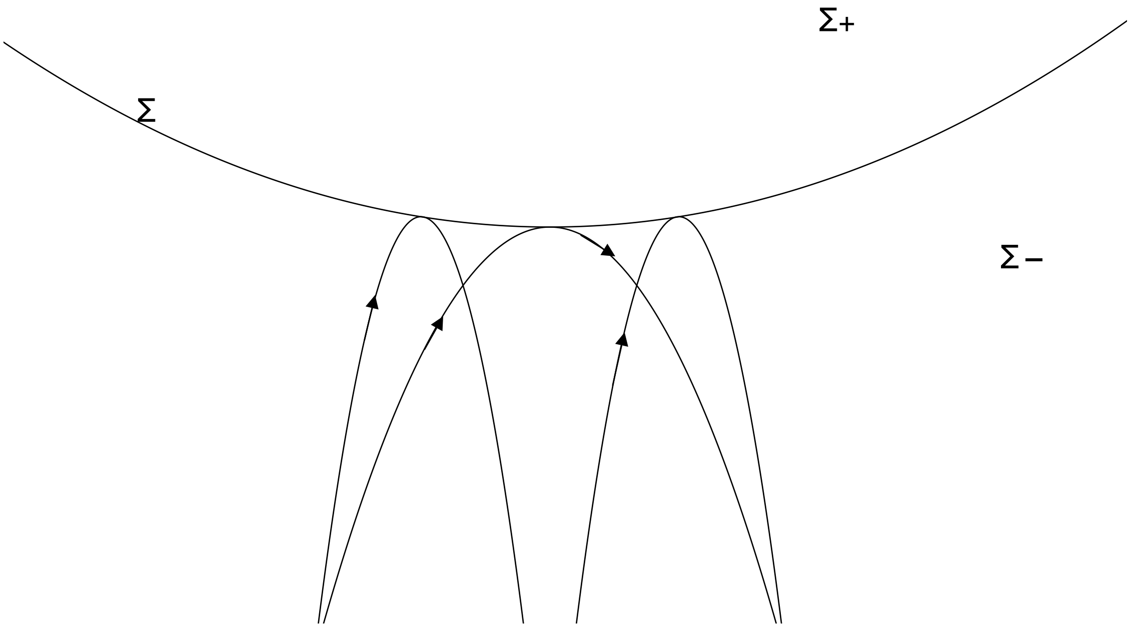


Figure 1.4: Pseudoconvex Σ with respect to the characteristic curves of P

$$\frac{d}{dt}\big(\rho(x(t))\big) = H_p(\rho)(x(t), \xi(t)), \quad \frac{d^2}{dt^2}\big(\rho(x(t))\big) = H_p^2(\rho)(x(t), \xi(t))$$

and with $\rho(x_0) = H_p(\rho)(x_0, \xi_0) = 0$, the pseudo-convexity condition is indeed $H_p^2(\rho)(x_0, \xi_0) < 0$.

Pseudoconvexity for real second order operators

In that case, we shall assume that our oriented hypersurface is non-characteristic and given by the equation $t = 0$ near $0 \in \mathbb{R}_t^1 \times \mathbb{R}_x^d$ and that our differential operator has a **real-valued** principal symbol (a polynomial with degree 2 in ξ, τ),

$$p_2(t, x; \tau, \xi).$$

We recall that our pseudo-convexity assumption is given by (1.3.8); we note also that if $\lambda \neq 0, (\tau, \xi) \in \mathbb{R} \times \mathbb{R}^{n-1}$,

$$p_2(0,0;\tau - i\lambda, \xi) = \partial_\tau p_2(0,0;\tau - i\lambda, \xi) = 0$$

implies by conjugation

$$p_2(0,0;\tau + i\lambda, \xi) = \partial_\tau p_2(0,0;\tau + i\lambda, \xi) = 0,$$

which gives that the polynomial $\tau \mapsto p_2(0,0;\tau,\xi)$ has two double-roots $\tau \pm i\lambda$, which is impossible. If P is a second order operator with real coefficients in the principal part the pseudo-convexity hypothesis with respect to a non-characteristic hypersurface means that for $\mathbb{R}^n \ni \xi \neq 0$

$$p(x_0, \xi) = \{p, \rho\}(x_0, \xi) = 0 \Longrightarrow \{p, \{p, \rho\}\}(x_0, \xi) < 0. \tag{1.3.19}$$

In fact non-real roots cannot be double since they occur in conjugate pair.

Tricomi operator

We consider the Tricomi operator

$$\mathcal{T} = D_n^2 + x_n |D'|^2, \quad \Sigma_+ \equiv x_n > 0. \tag{1.3.20}$$

This is a second-order operator with real coefficients and Σ is non-characteristic. $\xi_n = 0$ is a double root of the equation $p(0;\xi', \xi_n) = 0$ and

$$\{p, \rho\} = 2\xi_n, \quad \{p, \{p, \rho\}\} = 2\left\{\xi_n^2 + x_n |\xi'|^2, \xi_n\right\} = -2|\xi'|^2 < 0,$$

so that Σ is strongly pseudo-convex at Σ with respect to $\mathcal{T}$. Looking at the bicharacteristic curves starting at $(x_0', 0; \xi_0', 0)$ with $|\xi_0'| = 1$, we have

$$\frac{dx_n}{dt} = 2\xi_n, \quad \frac{dx'}{dt} = 2x_n \xi', \quad \frac{d\xi_n}{dt} = -|\xi'|^2, \quad \frac{d\xi'}{dt} = 0,$$

so that $\xi' = \xi_0', \quad \xi_n = -t, \quad x_n = -t^2, \quad x' = x_0' - \frac{2t^3}{3}\xi_0'$.

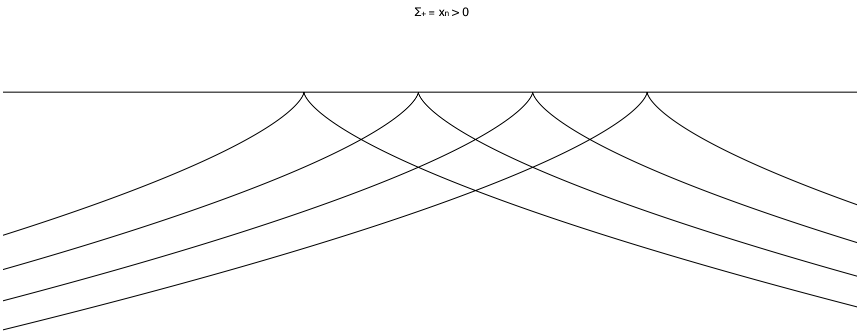


Figure 1.5: Projection of the bicharacteristic curves of the Tricomi operator $\mathcal{T}$.

Constant coefficients

• When P has constant coefficients and Σ non-characteristic with respect to P is given by the equation $x_n = f(x')$ with $f(0) = 0, f'(0) = 0$, the pseudoconvexity condition is

$$\forall(\xi', \xi_n, \tau) \in \mathbb{R}^{n-1} \times \mathbb{R} \times \mathbb{R}_+ \backslash\{0\},\ p(\xi', \xi_n - i\tau) = \frac{\partial p}{\partial \xi_n}(\xi', \xi_n - i\tau) = 0$$
$$\Longrightarrow f''(0)\overline{\frac{\partial p}{\partial \xi'}(\xi', \xi_n - i\tau)}\frac{\partial p}{\partial \xi'}(\xi', \xi_n - i\tau) > 0$$

and for principal type operators, this follows from the convexity of f, i.e. of Σ_+.

• Note however that for a constant coefficients operator such as

$$\Box_c = c^{-2}\partial_t^2 - \Delta_x,$$

an oriented hypersurface Σ can be pseudo-convex and Σ_+ may fail to be convex: let us consider a one-sheet hyperboloid $\mathcal{H}_\sigma$ with equation

$$|x|^2 = 1 + \sigma^2 t^2.$$

Then with $\rho = |x|^2 - 1 - \sigma^2 t^2, \sigma > c > 0, \quad p = -c^{-2}\tau^2 + |\xi|^2$, we have

$$d\rho = 2x \cdot dx - 2\sigma^2 t dt \neq 0, \quad \text{at } \rho = 0\ (t = 0 \Longrightarrow |x| = 1),$$
$$p(d\rho) = -c^{-2}(-2\sigma^2 t)^2 + 4|x|^2 = 4\big(|x|^2 - \sigma^4 c^{-2} t^2\big)$$
$$= 4\big(1 + \sigma^2 t^2 - \sigma^4 c^{-2} t^2\big) > 0 \text{ at } t = 0,$$

so that $\mathcal{H}_\sigma$ is time-like (and in particular non-characteristic) at $t = 0$. Assuming $p = \{p, \rho\} = 0 = \rho$, means

$$\tau^2 = c^2|\xi|^2, \quad |x|^2 = 1 + \sigma^2 t^2, \quad -2c^{-2}\tau(-2t\sigma^2) + 2\xi \cdot 2x = 0,$$

so that we may assume $|\xi| = 1$, and at $t = 0$ we have $|x| = 1$,

$$|\tau| = c, \quad \xi \cdot x = 0.$$

We have then

$$\begin{aligned}\frac{1}{4}H_p^2(\rho) &= \partial_\tau p \partial_t\{\xi \cdot x + \tau t\sigma^2 c^{-2}\} + \partial_\xi p \partial_x\{\xi \cdot x + \tau t\sigma^2 c^{-2}\} \\ &= -2\tau c^{-2}\tau\sigma^2 c^{-2} + 2\xi \cdot \xi = 2 - 2\tau^2 c^{-4}\sigma^2 = 2 - 2\, c^{-2}\sigma^2 < 0,\end{aligned}$$

since $\sigma > c > 0$, proving pseudo-convexity for $\mathcal{H}_\sigma$ with respect to $\Box_c$ at $t = 0$. However, Σ_+ fails to be convex: for $x', x'' \in \mathbb{S}^{n-1}, x' \neq x''$, we have $(x', t = 0), (x'', t = 0) \in \Sigma_+$ and

$$\left(\frac{1}{2}(x' + x''), t = 0\right) \in \mathring{\Sigma}_-$$

since $\frac{1}{4}|x' + x''|^2 = \frac{1}{2}(1 + \langle x', x''\rangle) < 1$: we have indeed from Cauchy-Schwarz inequality and $x', x'' \in \mathbb{S}^{n-1}, x' \neq x''$, that $\langle x', x''\rangle < 1$ (Cauchy-Schwarz provides the large inequality $|\langle x', x''\rangle| \leq 1$ and the equality would imply $x' \wedge x'' = 0$, i.e. $x' = x''$ (excluded) or $x' = -x''$ inducing $\langle x', x''\rangle = -1$).

A short summary

Let $P(x, D_x) = \sum_{|\alpha| \leq m} a_\alpha(x) D_x^\alpha$ be a differential operator of order m on an open subset Ω of $\mathbb{R}^n$ with coefficients in L^∞_{loc}, with a **smooth principal part with real coefficients**. Let Σ be a C^2 hypersurface, given by $\{x \in \Omega, \rho(x) = 0\}$ where ρ is C^1 with $d\rho \neq 0$ at $\rho = 0$. We note

$$p_m(x, \xi) = \sum_{|\alpha| = m} a_\alpha(x)\xi^\alpha.$$

Let $x_0 \in \Sigma$. The simple characteristics assumption is

$$\begin{aligned}&\forall(\xi, \lambda) \in (\mathbb{R}^n \times \mathbb{R}_+)\backslash\{(0, 0)\}, \\ &\qquad p_m(x_0, \xi - i\lambda d\rho(x_0)) = 0 \Longrightarrow \frac{\partial p_m}{\partial \xi}(x_0, \xi - i\lambda d\rho(x_0)) \cdot d\rho(x_0) \neq 0. \quad (1.3.21)\end{aligned}$$

Carleman estimate. *Assumption* (1.3.21) *implies a Carleman inequality: there exists $\lambda_0 \geq 0$ and a neighborhood V_0 of x_0 such that for all $v \in C_c^\infty(V_0)$,*

$$C\|P_{\lambda,\phi} v\|_{L^2} \geq \lambda^{1/2}\|v\|_{\mathcal{H}_\lambda^{m-1}}, \quad \|v\|^2_{\mathcal{H}_\lambda^{m-1}} = \int |\hat{v}(\xi)|^2(|\xi|^2 + \lambda^2)^{m-1} d\xi, \qquad (1.3.22)$$

where ϕ is a real-valued C^∞ function with a non-vanishing gradient, "convexifying" i.e. such that

$$\{x \in V_0, \phi(x) = 0\}\backslash\{x_0\} \subset \{\rho < 0\}.$$

For elliptic operators, the natural Carleman estimate is

$$C\|P_{\lambda,\phi}v\|_{L^2} \geq \lambda^{-1/2}\|v\|_{\mathcal{H}^m_\lambda}. \tag{1.3.23}$$

The pseudo-convexity assumption is $\forall(\xi, \lambda) \in (\mathbb{R}^n \times \mathbb{R}_+)\backslash\{(0,0)\}$

$$p_m(x_0, \overbrace{\xi - i\lambda d\rho(x_0)}^{\zeta}) = \frac{\partial p_m}{\partial \xi}(x_0, \xi - i\lambda d\rho(x_0)) \cdot d\rho(x_0) = 0 \Longrightarrow$$
$$\begin{cases} \text{if } \lambda > 0, & \lambda^{-1}\operatorname{Im}\big(\overline{\partial_\xi p_m(x_0,\zeta)} \cdot \partial_x p_m(x_0,\zeta)\big) - \rho''(x_0)\overline{\partial_\xi p_m(x_0,\zeta)}\partial_\xi p_m(x_0,\zeta) > 0, \\ \text{if } \lambda = 0, & H^2_{p_m}(\rho)(x_0,\zeta) < 0. \end{cases} \tag{1.3.24}$$

That hypothesis implies as well the Carleman estimate (1.3.22) and (1.3.23) in the elliptic case.

1.4 Complex coefficients and principal normality

The reader may have noticed that we have assumed so far that the coefficients of our operators are real-valued. It turns out that it is possible to extend the pseudo-convexity hypothesis to some complex-valued operators and to retain the conclusions about Carleman inequalities and uniqueness. The fact that the coefficients were real was technically helpful in controlling the term $\{\overline{p_m}, p_m\}$ in (1.3.7). Here we shall only review quickly some elements related to these questions.

1.4.1 Principal normality

To control the above Poisson bracket, we shall introduce the following definition.

Definition 1.4.1 (see Definition 28.2.4 in [9])**.** Let P be a differential operator of degree m on an open subset of $\mathbb{R}^n$ with principal symbol p_m. We shall say that P is *principally normal* whenever for each $(x_0, \xi_0) \in \Omega \times (\mathbb{R}^n\backslash\{0\})$ such that $p_m(x_0, \xi_0) = 0$, there exists a neighborhood V of $(x_0, \xi_0/|\xi_0|)$ in $\Omega \times \mathbb{S}^{n-1}$ and $C > 0$ such that for all $(x, \xi) \in V$,

$$|\{\overline{p_m}, p_m\}(x,\xi)| \leq C|p_m(x,\xi)|. \tag{1.4.1}$$

In particular differential operators with real coefficients are principally normal.

The terminology *principally normal* comes from the fact the principal symbol of the adjoint of P is $\overline{p_m}$, so that the principal symbol of the commutator $[P^*, P]$ is

$$\frac{1}{i}\{\overline{p_m}, p_m\}$$

so that the vanishing of this Poisson bracket at $p_m = 0$ appears as a mild commutation property of P with its adjoint, i.e. a commutation at the level of the principal symbols. The following properties of Poisson brackets will be useful for the forthcoming calculations.

Lemma 1.4.2. *For a C^1 complex-valued symbol p, we have*

$$\{\overline{p}, p\} = 2i\,\{\operatorname{Re} p, \operatorname{Im} p\} \in i\mathbb{R}, \tag{1.4.2}$$

$$\operatorname{Im}\{\overline{p}, p\} = \frac{1}{i}\{\overline{p}, p\} = 2\,\{\operatorname{Re} p, \operatorname{Im} p\}\,. \tag{1.4.3}$$

Proof. It is enough to prove (1.4.2): we have with $a = \operatorname{Re} p, b = \operatorname{Im} p$,

$$\{\overline{p}, p\} = \{a - ib, a + ib\} = \{a, ib\} + \{-ib, a\} = 2i\,\{a, b\}\,.$$

□

Some examples

The operator with (complex) symbol

$$p = \tau^2 + t(\xi_1^2 + \xi_2^2) - ie^t\xi_1^2 \tag{1.4.4}$$

is principally normal since for $\tau^2 + \xi_1^2 + \xi_2^2 = 1$,

$$-\frac{1}{2i}\{\bar{p}, p\} = \left\{\tau^2 + t(\xi_1^2 + \xi_2^2), e^t\xi_1^2\right\} = 2\tau e^t\xi_1^2 \Longrightarrow |\,\{\bar{p}, p\}\,| \le 4|\tau||\operatorname{Im} p| \le 4|\operatorname{Im} p|.$$

We may note also that Condition (1.4.1) may be replaced by the apparently weaker

$$(\sharp) \qquad \operatorname{Im}\{\overline{p_m}, p_m\}(x, \xi) \ge -C|p_m(x, \xi)||\xi|^{m-1},$$

but since $\operatorname{Im}\{\overline{p_m}, p_m\}$ is a polynomial with degree $2m-1$ in the ξ variable, $(\sharp)$ implies

$$\begin{aligned}\operatorname{Im}\{\overline{p_m}, p_m\}(x, -\xi) &\ge -C|p_m(x, -\xi)||\xi|^{m-1}\\ &\Longrightarrow \operatorname{Im}\{\overline{p_m}, p_m\}(x, \xi) \le C|p_m(x, \xi)||\xi|^{m-1}\\ &\quad\Longrightarrow |\operatorname{Im}\{\overline{p_m}, p_m\}(x, \xi)| \le C|p_m(x, \xi)||\xi|^{m-1} \quad \text{i.e. (1.4.1)}.\end{aligned}$$

Carleman estimates and local solvability

We note as well that a Carleman estimate of type (1.2.3) would imply local solvability for P so that the Nirenberg-Treves' condition (P) should be satisfied for an operator satisfying a Carleman estimate: for a complex-valued (homogeneous) symbol p of principal type with $d_\xi \operatorname{Re} p \ne 0$, condition (P) requires that the imaginary part of p does not change sign along the bicharacteristic curves of the real part; in particular if $\gamma_0 = (x_0, \xi_0)$ is a characteristic point for p and if

$$\dot{\gamma}(t) = H_{\operatorname{Re} p}(\gamma), \gamma(0) = \gamma_0, \quad t \mapsto (\operatorname{Im} p)(\gamma(t)) \quad \text{does not change sign.}$$

If it stays non-negative, this implies that

$$(\operatorname{Im} p)(\gamma(0)) = 0, \quad H_{\operatorname{Re} p}(\operatorname{Im} p)(\gamma(0)) = 0, \quad \text{i.e.} \quad \{\operatorname{Re} p, \operatorname{Im} p\}(\gamma(0)) = 0.$$

We know that for a principal type operator

$$\text{Carleman estimate} \Longrightarrow \text{Local solvability} \underbrace{\Longleftrightarrow}_{\text{see [3]}} \text{Condition } (P),$$

and since we expect pseudo-convexity and principal normality to imply a Carleman estimate, we can check that (1.4.1) implies condition (P): with the above notations, we have near γ_0 that $|\{\operatorname{Re} p, \operatorname{Im} p\}| \le C|p||\xi|^{m-1}$ which implies, since $\operatorname{Re} p(\gamma(t)) \equiv 0$,

$$|\frac{d}{dt} \operatorname{Im} p(\gamma(t))| \le C|p(\gamma(t))||\xi(t)|^{m-1} = C|\operatorname{Im} p(\gamma(t))||\xi(t)|^{m-1}$$

and thus, by Gronwall's inequality $\operatorname{Im} p(\gamma(t)) \equiv 0$, a very strong form of Condition (P).

The operator

$$D_t + it^2 D_x \tag{1.4.5}$$

is not principally normal since $\{\tau, t^2\xi\} = 2t\xi$ so that the imaginary part does not vanish identically as principal normality would imply. However, it satisfies Condition (P) since the function $t \mapsto t^2\xi$ is either always non-negative or always non-positive. It is a known result that for (non-vanishing) complex vector field, Condition (P) ensures uniqueness for the Cauchy problem with respect to a non-characteristic hypersurface: a very complete study on complex vector fields is given in the paper [32] by X. Saint Raymond. The book [37] by C. Zuily is providing a proof of a Carleman estimate for vector fields satisfying Condition (P).

1.4.2 Fefferman-Phong inequality, Weyl quantization

Theorem 1.4.3 (Fefferman-Phong inequality)**.** *Let Ω be an open subset of $\mathbb{R}^n$ and let m be an integer ≥ 2. Let $a \in \Sigma^m(\Omega)$ (see Definition 1.2.5) be a nonnegative symbol and let K be a compact subset of Ω. Then there exists a constant C such that for all $u \in C_K^\infty(\Omega)$,$\forall \lambda \ge 1$,*

$$\operatorname{Re}\langle a(x, D_x, \lambda)u, u\rangle + C\|u\|^2_{\mathcal{H}_\lambda^{\frac{m-2}{2}}} \ge 0, \tag{1.4.6}$$

where the Sobolev space $\mathcal{H}_\lambda^s$ is defined in (1.2.6).

We give some references and comments on this inequality in our Appendix 4.4. We shall use this inequality as a tool to handle our Carleman estimates for operators with complex symbols in a rather similar way as we have used Gårding's inequality. However, the reader must be aware that the proof of the Fefferman-Phong inequality is much more involved than the proof of the lowerbound given by Gårding's inequality, so that the technical apparatus used for principally normal operators increases dramatically with that inequality entering the game. On the other hand, a drawback of this tool is that it is greedy with derivatives and it is not so easy to determine the number of derivatives needed for the method to work: for this reason, we shall always assume some smoothness in these cases.

Another tool which shall simplify our discussion is Weyl quantization. We refer the reader to our Appendix 4.3 and to the book [24].

1.4.3 Pseudo-convexity for principally normal operators

We may then go back to our discussion of the symbol of the conjugate operators: we have to deal with

$$a(x, \xi, \lambda) = p_m(x, \xi - i\lambda d\phi(x))$$

and we may calculate the composition

$$\bar{a}\sharp a \equiv |a|^2 + \frac{1}{2i}\{\bar{a}, a\} \equiv |p_m(x, \xi - i\lambda d\phi(x))|^2 + c_{2m-1,\phi}(x, \xi, \lambda), \qquad \mod \Sigma^{2m-2}$$

where $c_{2m-1,\phi}$ is given by (1.2.14). Choosing ϕ as in (1.2.15) we shall prove

Lemma 1.4.4. *Let $p_m(x, \xi)$ be a principally normal symbol such that the pseudo-convexity hypothesis (1.3.8) holds for a function ρ at a point x_0. There exists a constant $\mu > 0$ such that for $x \in W_\mu = \{x \in \mathbb{R}^n, |x - x_0| \le \mu^{-2}\}$, for $(\xi, \lambda) \in \mathbb{R}^n \times (\mu^2, +\infty)$,*

$$|p_m(x, \xi - i\lambda d\phi(x))|^2 + c_{2m-1,\phi}(x, \xi, \lambda) \ge \mu^{-1}\lambda(\lambda^2 + |\xi|^2)^{m-1},$$

with the quadratic weight

$$\phi(x) = \rho'(x_0) \cdot (x - x_0) + \frac{1}{2}\rho''(x_0)(x - x_0)^2 - \frac{\mu}{2}(\rho'(x_0) \cdot (x - x_0))^2 + \frac{|x - x_0|^2}{2\mu^2}.$$

Proof. The discussion follows the same lines as in the proof of Lemma 1.3.1 and we shall use here as well *a reductio ad absurdum*. We find sequences (x_k, ξ_k, λ_k), assuming as we may that $x_0 = 0, |d\rho(0)| = 1$, with

$$|x_k| \le k^{-2}, \quad (\xi_k, \lambda_k) = (\lambda_k^2 + |\xi_k|^2)^{1/2}(\Xi_k, \Lambda_k), \quad \lim_k(\Xi_k, \Lambda_k) = (\Xi_0, \Lambda_0),$$

with

$$\Lambda_k > 0, \Lambda_0 \ge 0, \quad \Lambda_k^2 + |\Xi_k|^2 = 1 = \Lambda_0^2 + |\Xi_0|^2,$$

so that, with $\zeta_k = \xi_k - i\lambda_k d\phi_k(x_k)$, $\lambda_k \ge k^2$,

$$|p_m(x_k, \zeta_k)|^2 + c_{2m-1,\phi}(x_k, \xi_k, \lambda_k) < k^{-1}\lambda_k(\lambda_k^2 + |\xi_k|^2)^{m-1}. \tag{1.4.7}$$

We note that, since $|x_k| \le k^{-2}$,

$$d\phi_k(x_k) = \rho'(0) + \rho''(0)x_k - k\rho'(0)x_k\rho'(0) + k^{-2}x_k \Longrightarrow \lim_k d\phi_k(x_k) = d\rho(0) \ne 0.$$

We have thus

$$|\zeta_k|^2 = |\xi_k|^2 + \lambda_k^2 |d\phi_k(x_k)|^2, \quad \lim_k \frac{|\zeta_k|^2}{|\xi_k|^2 + \lambda_k^2} = |\Xi_0|^2 + |d\rho(0)|^2 \Lambda_0^2 = 1.$$

Dividing both sides by $|\zeta_k|^{2m}$, we obtain with $Z_k = \zeta_k / |\zeta_k|$

$$\lim_k Z_k = \Xi_0 - i\Lambda_0 d\phi(0) = Z_0, \quad |p_m(x_k, Z_k)|^2 \le O(|\zeta_k|^{-1}) \Longrightarrow p_m(x_0, Z_0) = 0.$$

We have also

$$c_{2m-1,\phi} = \operatorname{Im}\Bigl(\overline{\partial_\xi p_m(x,\zeta)} \cdot \partial_x p_m(x,\zeta)\Bigr) - \lambda \phi''(0) \overline{(\partial_\xi p_m)(x,\zeta)} (\partial_\xi p_m)(x,\zeta)$$

with $\phi''(0) = \rho''(0) - \mu\rho'(0)^2 + \mu^{-2}$ so that (1.4.7) implies

$$|p_m(x_k,\zeta_k)|^2 + k\lambda_k |\rho'(0)\cdot(\partial_\xi p_m)(x_k,\zeta_k)|^2 + \operatorname{Im}\bigl(\overline{\partial_\xi p_m} \cdot \partial_x p_m\bigr)(x_k,\xi_k) \le \lambda_k O(|\zeta_k|^{2m-2}). \tag{1.4.8}$$

Previously the term $\operatorname{Im}\bigl(\overline{\partial_\xi p_m} \cdot \partial_x p_m\bigr)(x_k,\xi_k)$ was identically 0 since we had supposed our operator with real coefficients (true as well for a constant coefficient operator); we (badly) need to control that term in the case where $\Lambda_0 = 0$. Let us start by checking the

Case $\Lambda_0 > 0$. Then we have

$$\lambda_k |\zeta_k|^{2m-2} = \frac{\lambda_k}{|\zeta_k|} |\zeta_k|^{2m-1} \ge \frac{\Lambda_0}{2} |\zeta_k|^{2m-1}, \quad \text{for } k \text{ large enough,}$$

and thus dividing (1.4.8) by $k\lambda_k|\zeta_k|^{2m-2}$ (note that $\lambda_k > 0$), we find that

$$\lim_k \left|\operatorname{Im}\bigl(\overline{\partial_\xi p_m} \cdot \partial_x p_m\bigr)(x_k,\xi_k)\right| (k\lambda_k|\zeta_k|^{2m-2})^{-1} \le \lim_k \frac{|\zeta_k|^{2m-1}}{k\frac{\Lambda_0}{2}|\zeta_k|^{2m-1}} = 0.$$

and this proves

$$0 = \lim_k |\rho'(0)\cdot(\partial_\xi p_m)(x_k, Z_k)|^2 = \{p_m, \rho\}(0, Z_0). \tag{1.4.9}$$

Case $\Lambda_0 = 0$. Here we shall use the principal normality assumption: we know that

$$\operatorname{Im}\bigl(\overline{\partial_\xi p_m} \cdot \partial_x p_m\bigr)(x_k,\xi_k) \ge -C_0|\xi_k|^{m-1}|p_m(x_k,\xi_k)|$$

so that we infer from (1.4.8)

$$\begin{aligned}
|p_m(x_k,\zeta_k)|^2 &+ k\lambda_k|\rho'(0)\cdot(\partial_\xi p_m)(x_k,\zeta_k)|^2 \\
&\le C_0|\xi_k|^{m-1}|p_m(x_k,\xi_k)| + \lambda_k O(|\zeta_k|^{2m-2}) \\
&\le |p_m(x_k,\zeta_k)||\zeta_k|^{m-1}C_0 + \lambda_k O(|\zeta_k|^{2m-2}) \\
&\le \frac{1}{2}|p_m(x_k,\zeta_k)|^2 + \frac{1}{2}C_0^2|\zeta_k|^{2m-2} + \lambda_k O(|\zeta_k|^{2m-2}),
\end{aligned}$$

an inequality that we can divide by $k\lambda_k|\zeta_k|^{2m-2}$ to obtain

$$|\rho'(0)\cdot(\partial_\xi p_m)(x_k,Z_k)|^2 \le \frac{C_0^2}{2k\lambda_k} + O(k^{-1})$$

and since $k\lambda_k \ge k$, we obtain in that case as well $\{p_m,\rho\}(0,Z_0)=0$. In both cases we have found a point Z_0 where

$$p_m(0,Z_0) = \{p_m,\rho\}(0,Z_0) = 0.$$

We may now apply the pseudo-convexity hypothesis. Developing the expression of (1.4.8), we get

$$\begin{aligned}|p_m(x_k,\zeta_k)|^2 + k\lambda_k|\rho'(0)\cdot(\partial_\xi p_m)(x_k,\zeta_k)|^2 - \lambda_k\rho''(0)\overline{\partial_\xi p_m(x_k,\zeta_k)}\partial_\xi p_m(x_k,\zeta_k)\\ + \operatorname{Im}\big(\overline{\partial_\xi p_m(x_k,\zeta_k)}\cdot\partial_x p_m(x_k,\zeta_k)\big) \le k^{-1}\lambda_k O(|\zeta_k|^{2m-2}).\end{aligned}$$

If $\Lambda_0>0$ we may divide this inequality by $\lambda_k|\zeta_k|^{2m-2}$ to reach a contradiction.

Let us assume now that $\Lambda_0=0$. We have

$$\begin{aligned}|p_m(x_k,\zeta_k)|^2 + k\lambda_k|\rho'(0)\cdot(\partial_\xi p_m)(x_k,\zeta_k)|^2 - \lambda_k\rho''(0)\overline{\partial_\xi p_m(x_k,\zeta_k)}\partial_\xi p_m(x_k,\zeta_k)\\ + \lambda_k\frac{\partial}{\partial\lambda}\Big(\operatorname{Im}\big(\overline{\partial_\xi p_m(x_k,\xi_k-i\lambda d\phi_k(x_k))}\cdot\partial_x p_m(x_k,\xi_k-i\lambda d\phi(x_k))\big)\Big)_{|\lambda=0}\\ + \operatorname{Im}\big(\overline{\partial_\xi p_m}\cdot\partial_x p_m\big)(x_k,\xi_k) \le k^{-1}\lambda_k O(|\zeta_k|^{2m-2}) + O(\lambda_k^2|\zeta_k|^{2m-3}),\end{aligned}$$

which gives, using principal normality,

$$\begin{aligned}&|p_m(x_k,\zeta_k)|^2 + k\lambda_k|\rho'(0)\cdot(\partial_\xi p_m)(x_k,\zeta_k)|^2 - \lambda_k\rho''(0)\overline{\partial_\xi p_m(x_k,\zeta_k)}\partial_\xi p_m(x_k,\zeta_k)\\ &+ \lambda_k\frac{\partial}{\partial\lambda}\Big(\operatorname{Im}\big(\overline{\partial_\xi p_m(x_k,\xi_k-i\lambda d\phi_k(x_k))}\cdot\partial_x p_m(x_k,\xi_k-i\lambda d\phi(x_k))\big)\Big)_{|\lambda=0}\\ &\le C_0|\xi_k|^{m-1}|p_m(x_k,\xi_k)| + k^{-1}\lambda_k O(|\zeta_k|^{2m-2}) + O(\lambda_k^2|\zeta_k|^{2m-3})\\ &\le C_0|\zeta_k|^{m-1}|p_m(x_k,\zeta_k)| + o(\lambda_k|\zeta_k|^{2m-2})\\ &\qquad + k^{-1}\lambda_k O(|\zeta_k|^{2m-2}) + O(\lambda_k^2|\zeta_k|^{2m-3})\\ &\le \frac{1}{2}|p_m(x_k,\zeta_k)|^2 + O(|\zeta_k|^{2m-2}) + o(\lambda_k|\zeta_k|^{2m-2}) + k^{-1}\lambda_k O(|\zeta_k|^{2m-2})\\ &\qquad + O(\lambda_k^2|\zeta_k|^{2m-3}).\end{aligned}$$

We divide this inequality by $\lambda_k|\zeta_k|^{2m-2}$ to obtain (note that $\lambda_k \ge k^2$)

$$\begin{aligned}-\rho''(0)\overline{\partial_\xi p_m(x_k,Z_k)}\partial_\xi p_m(x_k,Z_k) + \frac{\partial}{\partial\lambda}\operatorname{Im}\big(\overline{\partial_\xi p_m(x_k,Z_k)}\cdot\partial_x p_m(x_k,Z_k)\big)\\ \le o(1) + O(\Lambda_k).\end{aligned}$$

This is incompatible with (1.3.8) for $\Lambda_0=0$. $\square$

Theorem 1.4.5. *Let P be a principally normal differential operator of order m with C^∞ coefficients in the principal part, L^∞ complex-valued for lower order terms, in some open subset Ω of $\mathbb{R}^n$. Let Σ be a C^1 hypersurface of Ω given by an equation $\rho(x) = 0$, with $d\rho \neq 0$ at Σ. Let $x_0 \in \Sigma$; we assume that for $(\xi, \lambda) \in (\mathbb{R}^n \times \mathbb{R}_+)\backslash\{0\}$*

$$p_m(x_0, \xi - i\lambda d\rho(x_0)) = \{p_m, \rho\}(x_0, \xi - i\lambda d\rho(x_0)) = 0 \Longrightarrow$$
$$\lim_{\epsilon\to 0_+} \frac{1}{\lambda+\epsilon} \operatorname{Im}\Big(\overline{\frac{\partial p_m}{\partial \xi}(x_0,\zeta)} \cdot \frac{\partial p_m}{\partial x}(x_0,\zeta)\Big) - \rho''(x_0)\overline{\frac{\partial p_m}{\partial \xi}(x,\zeta)}\frac{\partial p_m}{\partial \xi}(x,\zeta) > 0, \quad (1.4.10)$$

with $\zeta = \xi - i(\lambda+\epsilon)d\phi(x_0)$. If u is an H^m_{loc} function, supported in $\{x \in W, \rho(x) \geq 0\}$ where W is a neighborhood of x_0, is such that

$$|(Pu)(x)| \leq \sum_{0\leq j<m} V_j(x)|\nabla^j u(x)|, \qquad V_j \in L^\infty_{loc},$$

then u is vanishing in a neighborhood of x_0.

Proof. An immediate consequence of Lemma 1.4.4 and of Fefferman-Phong inequality. □

We note that the proof, although technically more complicated is conceptually quite simple: we start with the symbol

$$p_m(x, \xi - i\lambda d\phi(x)) = a(x, \xi, \lambda),$$

we calculate $c = \overline{a}\sharp a$, prove that $c(x,\xi,\lambda) \geq \mu^{-1}\lambda(\lambda^2 + |\xi|^2)^{m-1}$, apply Fefferman-Phong inequality to obtain

$$\|P_\lambda u\|^2_{L^2} - \mu^{-1}\lambda\|u\|^2_{\mathcal{H}^{m-1}_\lambda} + C_0\|u\|^2_{\mathcal{H}^{m-1}_\lambda} \geq 0,$$

providing the following Carleman estimate for λ large enough

$$2\|P_\lambda u\|_{L^2} \geq \mu^{-1/2}\lambda^{1/2}\|u\|_{\mathcal{H}^{m-1}_\lambda}.$$

Chapter 2

Inequalities for elliptic operators with jumps at an interface

2.1 Introduction

This chapter is based upon the joint paper [18] of the author with Jérôme Le Rousseau, which appeared in *Analysis & PDE.*

2.1.1 Preliminaries

We have seen in the first chapter that a Carleman estimate could be proven for second-order elliptic operators, say with real-valued coefficients which are regular enough. Inspecting our proofs, it can be established that Lipschitz continuity is enough to handle uniqueness properties for second-order elliptic operators with real-valued coefficients.

Furthermore, it was shown by A. Pliś [29] that Hölder continuity is not enough to get unique continuation: this author constructed a real homogeneous linear differential equation of second order and of elliptic type on $\mathbb{R}^3$ without the unique continuation property although the coefficients are Hölder-continuous with any exponent less than one. The constructions by K. Miller in [26], and later by N. Filonov in [7], showed that Hölder continuity is not sufficient to obtain unique continuation for second-order elliptic operators, even in divergence form.

Reminders on pseudo-convexity

Let $P(x, D_x) = \sum_{|\alpha|\leq m} a_\alpha(x) D_x^\alpha$ be a differential operator of order m on an open subset Ω of $\mathbb{R}^n$ with coefficients in L^∞_{loc}, with a smooth principal **part with real coefficients**. Let Σ be a C^2 hypersurface, given by $\{x \in \Omega, \rho(x) = 0\}$ where ρ is C^1 with $d\rho \neq 0$ at $\rho = 0$. We note

$$p_m(x, \xi) = \sum_{|\alpha|=m} a_\alpha(x)\xi^\alpha.$$

Let $x_0 \in \Sigma$. The simple characteristics assumption is

$$\forall(\xi,\lambda)\in(\mathbb{R}^n\times\mathbb{R}_+)\backslash\{(0,0)\},$$
$$p_m(x_0,\xi-i\lambda d\rho(x_0))=0 \Longrightarrow \frac{\partial p_m}{\partial \xi}(x_0,\xi-i\lambda d\rho(x_0))\cdot d\rho(x_0)\neq 0. \quad (2.1.1)$$

Carleman estimate. *Assumption* (2.1.1) *implies a Carleman inequality: there exists $\lambda_0 \geq 0$ and a neighborhood V_0 of x_0 such that for all $v \in C_c^\infty(V_0)$,*

$$C\|P_{\lambda,\phi}v\|_{L^2}\geq \lambda^{1/2}\|v\|_{\mathcal{H}_\lambda^{m-1}}, \quad \|v\|^2_{\mathcal{H}_\lambda^{m-1}}=\int|\hat v(\xi)|^2(|\xi|^2+\lambda^2)^{m-1}d\xi, \qquad (2.1.2)$$

where ϕ is a real-valued C^∞ function with a non-vanishing gradient, "convexifying" i.e. such that

$$\{x\in V_0,\phi(x)=0\}\backslash\{x_0\}\subset\{\rho<0\}.$$

For elliptic operators, the natural Carleman estimate is

$$C\|P_{\lambda,\phi}v\|_{L^2}\geq\lambda^{-1/2}\|v\|_{\mathcal{H}_\lambda^m}. \qquad (2.1.3)$$

The pseudo-convexity assumption is $\forall(\xi,\lambda)\in(\mathbb{R}^n\times\mathbb{R}_+)\backslash\{(0,0)\}$

$$p_m(x_0,\overbrace{\xi-i\lambda d\rho(x_0)}^{\zeta})=\frac{\partial p_m}{\partial\xi}(x_0,\xi-i\lambda d\rho(x_0))\cdot d\rho(x_0)=0\Longrightarrow$$
$$\begin{cases}\text{if } \lambda>0, & \lambda^{-1}\operatorname{Im}\big(\overline{\partial_\xi p_m(x_0,\zeta)}\cdot\partial_x p_m(x_0,\zeta)\big)-\rho''(x_0)\overline{\partial_\xi p_m(x_0,\zeta)}\partial_\xi p_m(x_0,\zeta)>0,\\ \text{if } \lambda=0, & H^2_{p_m}(\rho)(x_0,\zeta)<0.\end{cases} \qquad (2.1.4)$$

That hypothesis implies as well the Carleman estimate (2.1.2) and (2.1.3) in the elliptic case.

Second order elliptic operators with real coefficients. We consider an operator defined on an open set Ω of $\mathbb{R}^n$

$$P=\sum_{1\leq j,k\leq n}a_{jk}(x)D_jD_k,\quad A(x)=\big(a_{jk}(x)\big)_{\substack{1\leq j\leq n\\1\leq k\leq n}},\ \text{symmetric}\gg 0, \qquad (2.1.5)$$
$$A(x)\ \text{smooth real-valued},\exists c_0>0,\forall(x,\xi)\in\Omega\times\mathbb{R}^n,\ \langle A(x)\xi,\xi\rangle\geq c_0|\xi|^2. \qquad (2.1.6)$$

Lemma 2.1.1. *Let P,Ω be as above. Let ρ be a C^1 real-valued function in Ω with a non-vanishing gradient. Then the simple characteristics assumption* (2.1.1) *holds at every point of Ω. As a result, unique continuation holds for P across any C^1 hypersurface.*

Proof. If $p_m(x,\zeta)=\langle A(x)(\xi-i\lambda d\rho(x)),(\xi-i\lambda d\rho(x))\rangle=0$, since A is real-valued we get

$$\begin{cases}\langle A(x)\xi,\xi\rangle=\lambda^2\langle A(x)d\rho(x),d\rho(x)\rangle\\ \lambda\langle A(x)\xi,d\rho(x)\rangle=0.\end{cases} \qquad (2.1.7)$$

If we have moreover $\partial_\xi p_m(x,\zeta)\cdot d\rho(x)=0$, this means

$$\langle A(x)(\xi - i\lambda d\rho(x), d\rho(x)\rangle = 0$$

and thus $\langle A(x)\xi, d\rho(x)\rangle = \lambda\langle A(x)d\rho(x), d\rho(x)\rangle = 0$. As a result, we get $\langle A(x)\xi,\xi\rangle = 0$ and thus $\xi = 0$ by ellipticity of A. Moreover, since $d\rho(x)$ is non-vanishing, we find $\lambda = 0$, proving the lemma. □

2.1.2 Jump discontinuities

Although the situation seems to be almost completely clarified by the previous results, with a minimal and somewhat necessary condition on Lipschitz continuity, we are interested in the following second-order elliptic operator $\mathscr{L}$,

$$\begin{gathered}\mathscr{L}w = -\operatorname{div}(A(x)\nabla w), \\ A(x) = (a_{jk}(x))_{1\le j,k\le n} = A^T(x), \quad \inf_{\|\xi\|_{\mathbb{R}^n}=1}\langle A(x)\xi,\xi\rangle > 0,\end{gathered} \tag{2.1.8}$$

in which the matrix A has a jump discontinuity across a smooth hypersurface. However we shall impose some stringent – yet natural – restrictions on the domain of functions w, which will be required to satisfy some homogeneous *transmission conditions*, detailed in the next sections. Roughly speaking, it means that w must belong to the domain of the operator, with continuity at the interface, so that ∇w remains bounded and continuity of the flux across the interface, so that $\operatorname{div}(A\nabla w)$ remains bounded, avoiding in particular the occurrence of a simple or multiple layer at the interface. The article [6] by A. Doubova, A. Osses, and J.-P. Puel tackled that problem, in the isotropic case (the matrix A is scalar $c\,\mathrm{Id}$) with a monotonicity assumption: the observation takes place in the region where the diffusion coefficient c is the 'lowest'. The case of an arbitrary dimension without any monotonicity condition in the elliptic case was solved by J. Le Rousseau and L. Robbiano in [19]: there the isotropic case is treated as well as a particular case of anisotropic medium.

We want here to show that a Carleman estimate can be proven for any operator of type (2.1.8) without an isotropy assumption: $A(x)$ is a symmetric positive-definite matrix with a jump discontinuity across a smooth hypersurface. We also provide conditions on the Carleman weight function that are rather simple to handle and can be proven to be sharp.

The approach we follow differs from that of [19] where the authors base their analysis on the usual Carleman method for certain microlocal regions and on Calderón projectors for others. The regions they introduce are determined by the ellipticity or non-ellipticity of the conjugated operator. Here, our approach is somewhat closer to A. Calderón's original work on unique continuation [4]: the conjugated operator is factored out in first-order (pseudo-differential) operators for which estimates are derived. Naturally, the quality of these estimates depends on their elliptic or non-elliptic nature; we thus recover microlocal regions that correspond to those of [19].

2.1.3 Framework

Let Ω be an open subset of $\mathbb{R}^n$ and Σ be a C^∞ oriented hypersurface of Ω: we have the partition

$$\Omega = \Omega_+ \cup \Sigma \cup \Omega_-, \quad \overline{\Omega_\pm} = \Omega_\pm \cup \Sigma, \quad \Omega_\pm \text{ open subsets of } \mathbb{R}^n, \tag{2.1.9}$$

and we introduce the following Heaviside-type functions

$$H_\pm = \mathbf{1}_{\Omega_\pm}. \tag{2.1.10}$$

We consider the elliptic second-order operator

$$\mathscr{L} = D \cdot AD = -\operatorname{div}(A(x)\nabla), \qquad (D = -i\nabla), \tag{2.1.11}$$

where $A(x)$ is a symmetric positive-definite $n \times n$ matrix, such that

$$A = H_- A_- + H_+ A_+, \quad A_\pm \in C^\infty(\Omega). \tag{2.1.12}$$

We shall consider functions w of the following type:

$$w = H_- w_- + H_+ w_+, \quad w_\pm \in C^\infty(\Omega). \tag{2.1.13}$$

We have

$$dw = H_- dw_- + H_+ dw_+ + (w_+ - w_-)\delta_\Sigma \nu,$$

where δ_Σ is the Euclidean hypersurface measure on Σ and ν is the unit conormal vector field to Σ pointing into Ω_+. To remove the singular term, we assume

$$w_+ = w_- \quad \text{at } \Sigma, \tag{2.1.14}$$

so that $Adw = H_- A_- dw_- + H_+ A_+ dw_+$ and we claim that

$$\operatorname{div}(Adw) = H_- \operatorname{div}(A_- dw_-) + H_+ \operatorname{div}(A_+ dw_+) + \langle A_+ dw_+ - A_- dw_-, \nu\rangle \delta_\Sigma. \tag{2.1.15}$$

In fact to get the latter formula, we may assume that $\Omega_\pm = \{x, \pm\rho(x) > 0\}$ with ρ a C^1 function such that $d\rho \neq 0$ at $\rho = 0$. We have then $H_\pm = H(\pm\rho(x))$, where H is the Heaviside function (characteristic function of $\mathbb{R}_+$). We have, using Einstein's convention on summation of repeated indices,

$$\begin{aligned}
\frac{\partial}{\partial x_j}\big(H(\rho(x)) a^+_{jk}(x) \frac{\partial w^+}{\partial x_k}\big) &= \delta_0(\rho)\frac{\partial \rho}{\partial x_j} a^+_{jk}(x)\frac{\partial w^+}{\partial x_k} + H(\rho(x))\frac{\partial}{\partial x_j}\big(a^+_{jk}(x)\frac{\partial w^+}{\partial x_k}\big) \\
&= \delta_\Sigma \underbrace{\nu_j a^+_{jk}(x)\frac{\partial w^+}{\partial x_k}}_{\langle A_+ dw_+, \nu\rangle} + \underbrace{H(\rho(x))\frac{\partial}{\partial x_j}\big(a^+_{jk}(x)\frac{\partial w^+}{\partial x_k}\big)}_{H_+ \operatorname{div}(A_+ dw_+)}
\end{aligned}$$

$$\begin{aligned}
\frac{\partial}{\partial x_j}\big(H(-\rho(x)) a^-_{jk}(x) \frac{\partial w^-}{\partial x_k}\big) &= -\delta_0(\rho)\frac{\partial \rho}{\partial x_j} a^-_{jk}(x)\frac{\partial w^-}{\partial x_k} + H(-\rho(x))\frac{\partial}{\partial x_j}\big(a^-_{jk}(x)\frac{\partial w^-}{\partial x_k}\big) \\
&= -\delta_\Sigma \underbrace{\nu_j a^-_{jk}(x)\frac{\partial w^-}{\partial x_k}}_{\langle A_- dw_-, \nu\rangle} + \underbrace{H(-\rho(x))\frac{\partial}{\partial x_j}\big(a^-_{jk}(x)\frac{\partial w^-}{\partial x_k}\big)}_{H_- \operatorname{div}(A_- dw_-)},
\end{aligned}$$

and thus

$$\operatorname{div}(Adw) - H_- \operatorname{div}(A_-dw_-) - H_+ \operatorname{div}(A_+dw_+) = \delta_\Sigma\langle A_+dw_+, \nu\rangle - \delta_\Sigma\langle A_-dw_-, \nu\rangle,$$

which is (2.1.15). Moreover, we shall assume that

$$\langle A_+dw_+ - A_-dw_-, \nu\rangle = 0 \quad \text{at } \Sigma, \text{ i.e. } \langle dw_+, A_+\nu\rangle = \langle dw_-, A_-\nu\rangle, \tag{2.1.16}$$

so that

$$\operatorname{div}(Adw) = H_- \operatorname{div}(A_-dw_-) + H_+ \operatorname{div}(A_+dw_+). \tag{2.1.17}$$

Conditions (2.1.14)-(2.1.16) will be called *transmission conditions* on the function w and we define the vector space

$$\mathcal{W} = \{H_-w_- + H_+w_+\}_{w_\pm \in C^\infty(\Omega) \text{ satisfying } (2.1.14)\text{-}(2.1.16)}. \tag{2.1.18}$$

Note that (2.1.14) is a continuity condition of w across Σ and (2.1.16) is concerned with the continuity of $\langle Adw, \nu\rangle$ across Σ, i.e. the continuity of the flux of the vector field Adw across Σ. A weight function "suitable for observation from Ω_-" is defined as a Lipschitz continuous function φ on Ω such that

$$\varphi = H_-\varphi_- + H_+\varphi_+, \quad \varphi_\pm \in C^\infty(\Omega), \quad \varphi_+ = \varphi_-, \quad \langle d\varphi_\pm, X\rangle > 0 \quad \text{at } \Sigma, \tag{2.1.19}$$

for any positively transverse vector field X to Σ (i.e. $\langle \nu, X\rangle > 0$).

2.2 Carleman estimate

2.2.1 Theorem

Theorem 2.2.1. *Let $\Omega, \Sigma, \mathscr{L}, \mathcal{W}$ be as in (2.1.9), (2.1.11) and (2.1.18). Then for any compact subset K of Ω, there exist a weight function φ satisfying (2.1.19) and positive constants C, λ_1 such that for all $\lambda \geq \lambda_1$ and all $w \in \mathcal{W}$ with* $\operatorname{supp} w \subset K$,

$$\begin{aligned} C\|e^{-\lambda\varphi}\mathcal{L}w\|_{L^2(\mathbb{R}^n)} \geq \\ \lambda^{3/2}\|e^{-\lambda\varphi}w\|_{L^2(\mathbb{R}^n)} + \lambda^{1/2}\|H_+e^{-\lambda\varphi}\nabla w_+\|_{L^2(\mathbb{R}^n)} + \lambda^{1/2}\|H_-e^{-\lambda\varphi}\nabla w_-\|_{L^2(\mathbb{R}^n)} \\ + \lambda^{3/2}|(e^{-\lambda\varphi}w)_{|\Sigma}|_{L^2(\Sigma)} + \lambda^{1/2}|(e^{-\lambda\varphi}\nabla w_+)_{|\Sigma}|_{L^2(\Sigma)} + \lambda^{1/2}|(e^{-\lambda\varphi}\nabla w_-)_{|\Sigma}|_{L^2(\Sigma)}. \end{aligned} \tag{2.2.1}$$

2.2.2 Comments

Remark 2.2.2. It is important to notice that whenever a true discontinuity occurs for the vector field $A\nu$, then the space $\mathcal{W}$ does *not* contain $C^\infty(\Omega)$: the inclusion $C^\infty(\Omega) \subset \mathcal{W}$ implies from (2.1.16) that for all $w \in C^\infty(\Omega)$, $\langle dw, A_+\nu - A_-\nu\rangle = 0$ at Σ so that $A_+\nu = A_-\nu$ at Σ, that is continuity for $A\nu$. The Carleman estimate which is proven in the present paper takes naturally into account these transmission conditions on the function w and it is important to keep in mind that the occurrence of a jump is excluding many smooth functions from the space $\mathcal{W}$. On the other hand, we have $\mathcal{W} \subset \operatorname{Lip}(\Omega)$.

Remark 2.2.3. We can also point out the geometric content of our assumptions, which do not depend on the choice of a coordinate system. For each $x \in \Omega$, the matrix $A(x)$ is a positive-definite symmetric mapping from $T_x(\Omega)^*$ onto $T_x(\Omega)$ so that $A(x)dw(x)$ belongs indeed to $T_x(\Omega)$ and Adw is a vector field with a L^2 divergence (Inequality (2.2.1) yields the L^2 bound by density).

If we were to consider a more general framework in which the matrix $A(x)$, symmetric, positive-definite belongs to $BV(\Omega) \cap L^\infty(\Omega)$, and w is a Lipschitz continuous function on Ω the vector field Adw is in $L^\infty(\Omega)$: the second transmission condition reads in that framework $\operatorname{div}(Adw) \in L^\infty(\Omega)$. Proving a Carleman estimate in such a case is a wide open question.

2.3 Proof for a model case

We provide in this subsection an outline of the main arguments used in our proof. To avoid technicalities, we somewhat simplify the geometric data and the weight function, keeping of course the anisotropy. We consider the operator

$$\mathscr{L}_0 = \sum_{1 \le j \le n} D_j c_j D_j, \tag{2.3.1}$$

$$c_j(x) = H_+ c_j^+ + H_- c_j^-,\ c_j^\pm > 0 \text{ constants},\ H_\pm = \mathbf{1}_{\{\pm x_n > 0\}},$$

with $D_j = \frac{\partial}{i\partial x_j}$, and the vector space $\mathcal{W}_0$ of functions $H_+ w_+ + H_- w_-$, $w_\pm \in C_c^\infty(\mathbb{R}^n)$, such that

$$\text{at } x_n = 0,\ w_+ = w_-, \quad c_n^+ \partial_n w_+ = c_n^- \partial_n w_-, \tag{2.3.2}$$
$$\text{(transmission conditions across } x_n = 0).$$

As a result, for $w \in \mathcal{W}_0$, we have $D_n w = H_+ D_n w_+ + H_- D_n w_-$ and

$$\mathscr{L}_0 w = \sum_j (H_+ c_j^+ D_j^2 w_+ + H_- c_j^- D_j^2 w_-). \tag{2.3.3}$$

We also consider a weight function[1]

$$\varphi = \underbrace{(\alpha_+ x_n - \beta x_n^2/2)}_{\varphi_+} H_+ + \underbrace{(\alpha_- x_n - \beta x_n^2/2)}_{\varphi_-} H_-, \quad \alpha_\pm > 0, \quad \beta > 0, \tag{2.3.4}$$

a positive parameter λ and the vector space $\mathcal{W}_\lambda$ of functions $H_+ v_+ + H_- v_-$, $v_\pm \in C_c^\infty(\mathbb{R}^n)$, such that at $x_n = 0$,

$$v_+ = v_-, \tag{2.3.5}$$
$$c_n^+(D_n v_+ - i\lambda\alpha_+ v_+) = c_n^-(D_n v_- - i\lambda\alpha_- v_-). \tag{2.3.6}$$

[1] We shall introduce later some minimal requirements on the weight function and suggest other possible choices.

Observe that $w \in \mathcal{W}_0$ is equivalent to $v = e^{-\lambda\varphi} w \in W_\lambda$. We have

$$e^{-\lambda\varphi}\mathscr{L}_0 w = \underbrace{e^{-\lambda\varphi}\mathscr{L}_0 e^{\lambda\varphi}}_{\mathscr{L}_\lambda}(e^{-\lambda\varphi}w)$$

so that proving a weighted a priori estimate $\|e^{-\lambda\varphi}\mathscr{L}_0 w\|_{L^2(\mathbb{R}^n)} \gtrsim \|e^{-\lambda\varphi}w\|_{L^2(\mathbb{R}^n)}$ for $w \in \mathcal{W}_0$ amounts to getting $\|\mathscr{L}_\lambda v\|_{L^2(\mathbb{R}^n)} \gtrsim \|v\|_{L^2(\mathbb{R}^n)}$ for $v \in \mathcal{W}_\lambda$.

2.3.1 Pseudo-differential factorization

Using Einstein convention on repeated indices $j \in \{1, \dots, n-1\}$, we have

$$\mathscr{L}_\lambda = (D_n - i\lambda\varphi')c_n(D_n - i\lambda\varphi') + D_j c_j D_j$$

and for $v \in \mathcal{W}_\lambda$, from (2.3.3), with

$$m_\pm = m_\pm(D') = (c_n^\pm)^{-1/2}(c_j^\pm D_j^2)^{1/2}, \tag{2.3.7}$$

$$\mathscr{L}_\lambda v = H_+ c_n^+\big((D_n - i\lambda\varphi'_+)^2 + m_+^2\big)v_+ + H_- c_n^-\big((D_n - i\lambda\varphi'_-)^2 + m_-^2\big)v_-,$$

so that

$$\begin{aligned}\mathscr{L}_\lambda v = H_+ c_n^+\big(D_n - i(\overbrace{\lambda\varphi'_+ - m_+}^{f_+})\big)\big(D_n - i(\overbrace{\lambda\varphi'_+ + m_+}^{e_+})\big)v_+ \\ + H_- c_n^-\big(D_n - i(\underbrace{\lambda\varphi'_- + m_-}_{e_-})\big)\big(D_n - i(\underbrace{\lambda\varphi'_- - m_-}_{f_-})\big)v_-.\end{aligned} \tag{2.3.8}$$

Note that e_- is elliptic positive in the sense that

$$e_- = \lambda\varphi'_- + m_- = \lambda\alpha_- + m_- - \lambda\beta x_n \gtrsim \lambda + |D'|, \quad \text{since } x_n \le 0.$$

Moreover e_+ is elliptic positive at $x_n = 0$ since

$$e_+ = \lambda\varphi'_+ + m_+ = \lambda\alpha_+ + m_+ - \lambda\beta x_n \gtrsim \lambda + |D'|, \quad \text{at } x_n = 0.$$

We want at this point to use some natural estimates for these first-order factors on the half-lines $\mathbb{R}_\pm$.

Lemma 2.3.1 (Half-line estimate, type e_-). *Let μ, γ be non-negative parameters. Then for $\omega \in C_c^1(\mathbb{R})$, we have*

$$\sqrt{2}\|D_t\omega - i(\mu - \gamma t)\omega\|_{L^2(\mathbb{R}_-)} \ge \|\mu\omega\|_{L^2(\mathbb{R}_-)} + \mu^{1/2}|\omega(0)|.$$

Proof. We have

$$\begin{aligned}&\|D_t\omega - i(\mu - \gamma t)\omega\|^2_{L^2(\mathbb{R}_-)} \\ &\quad= \|D_t\omega\|^2_{L^2(\mathbb{R}_-)} + \|(\mu - \gamma t)\omega\|^2_{L^2(\mathbb{R}_-)} + 2\operatorname{Re}\langle D_t\omega, -iH(-t)(\mu - \gamma t)\omega\rangle \\ &\quad\ge \int_{-\infty}^0 \big((\mu - \gamma t)^2 + \gamma\big)|\omega(t)|^2 dt + \mu|\omega(0)|^2 \ge \|\mu\omega\|^2_{L^2(\mathbb{R}_-)} + \mu|\omega(0)|^2,\end{aligned} \tag{2.3.9}$$

which is somehow a perfect estimate of elliptic type, suggesting that the first-order factor containing e_- should be easy to handle. □

Lemma 2.3.2 (Half-line estimate, type e_+)**.** *Let μ be a real parameter and let γ be a non-negative parameter. Then for $\omega \in C_c^1(\mathbb{R})$, we have*

$$\|D_t\omega - i(\mu - \gamma t)\omega\|^2_{L^2(\mathbb{R}_+)} \geq \gamma\|\omega\|^2_{L^2(\mathbb{R}_+)} - \mu|\omega(0)|^2.$$

Proof. We have

$$\begin{aligned} &\|D_t\omega - i(\mu - \gamma t)\omega\|^2_{L^2(\mathbb{R}_+)} \\ &\quad = \|D_t\omega\|^2_{L^2(\mathbb{R}_+)} + \|(\mu - \gamma t)\omega\|^2_{L^2(\mathbb{R}_+)} + 2\operatorname{Re}\langle D_t\omega, -iH(t)(\mu - \gamma t)\omega\rangle \\ &\quad \geq \int_0^{+\infty} \big((\mu - \gamma t)^2 + \gamma\big)|\omega(t)|^2 dt - \mu|\omega(0)|^2 \geq \gamma\|\omega\|^2_{L^2(\mathbb{R}_+)} - \mu|\omega(0)|^2, \end{aligned} \tag{2.3.10}$$

an estimate of lesser quality, because we need to secure a control of $\omega(0)$ to handle this type of factor when $\mu > 0$. When $\mu \leq 0$, the estimate is similar to Lemma's 2.3.1 result. □

2.3.2 Sign discussion

From Lemma 2.3.1 and (2.3.8), we see that the factor containing e_- should be easier to handle. We have another factor

$$f_- = \underbrace{\lambda\alpha_- - m_-}_{\mu = f_-(0)} - \underbrace{\lambda\beta}_{\gamma} x_n,$$

and we note that $f_-(0) \geq 0 \Longrightarrow f_- \geq 0$ on $\mathbb{R}_-$ since f_- is decreasing with x_n.

Assuming $f_-(0) \geq 0$, we may apply twice Lemma 2.3.1:

$$\begin{aligned} 2c_n^-\|(D_n - ie_-)(D_n - if_-)v_-\|_{L^2(\mathbb{R}_-)} &\geq c_n^-(\lambda\alpha_- + m_-)\sqrt{2}\|(D_n - if_-)v_-\|_{L^2(\mathbb{R}_-)} \\ &\geq c_n^-(\lambda\alpha_- + m_-)\Big(f_-(0)\|v_-\|_{L^2(\mathbb{R}_-)} + f_-(0)^{1/2}|v_-(0)|\Big). \end{aligned} \tag{2.3.11}$$

We check now the case $f_-(0) < 0$.

Applying Lemma 2.3.1, we get

$$\begin{aligned} &2c_n^-\|(D_n - ie_-)(D_n - if_-)v_-\|_{L^2(\mathbb{R}_-)} \\ &\qquad \geq c_n^-(\lambda\alpha_- + m_-)^{1/2}\sqrt{2}|(D_n - if_-)v_-(0)| \\ &\qquad \geq (\lambda\alpha_- + m_-)^{1/2}\sqrt{2}\left|c_n^-(D_nv_- - i\lambda\alpha_-v_-)(0) + ic_n^-m_-v_-(0)\right|. \end{aligned} \tag{2.3.12}$$

Our key assumption will be that

$$\boxed{f_-(0) < 0 \Longrightarrow f_+(0) \leq 0.} \tag{2.3.13}$$

We shall explain this assumption later; let us go on with collecting our estimates. We note in particular that since f_+ is a decreasing function of the variable x_n, this implies that $f_+(x_n) \le 0$ on $x_n \ge 0$. Applying Lemma 2.3.2, we get

$$\begin{aligned} 2c_n^+ \|(D_n - if_+)(D_n - ie_+)v_+\|_{L^2(\mathbb{R}_+)} & \\ \ge c_n^+(m_+ - \lambda\alpha_+)^{1/2}\sqrt{2}|(D_n - ie_+)v_+(0)| & \\ \ge (m_+ - \lambda\alpha_+)^{1/2}\sqrt{2}\left|c_n^+(D_n v_+ - i\lambda\alpha_+ v_+)(0) - ic_n^+ m_+ v_+(0)\right|. & \quad (2.3.14) \end{aligned}$$

As a result, defining

$$\mathcal{N}_- = c_n^-(D_n v_- - i\lambda\alpha_- v_-)(0) \underbrace{=}_{\text{from (2.3.6)}} \mathcal{N}_+ = c_n^+(D_n v_+ - i\lambda\alpha_+ v_+)(0), \quad (2.3.15)$$

we find that, with $\mathcal{N} = \mathcal{N}_- = \mathcal{N}_+$,

$$2\|\mathscr{L}_\lambda v\| \ge (\lambda\alpha_- + m_-)^{1/2}\left|\mathcal{N} + ic_n^- m_- v_-(0)\right| + (m_+ - \lambda\alpha_+)^{1/2}\left|\mathcal{N} - ic_n^+ m_+ v_-(0)\right|.$$

so that

$$2\|\mathscr{L}_\lambda v\| \ge \min\big((\lambda\alpha_- + m_-)^{1/2}, (m_+ - \lambda\alpha_+)^{1/2}\big)|ic_n^- m_- v_-(0) + ic_n^+ m_+ v_+(0)|. \quad (2.3.16)$$

We note then that from (2.3.5), $v_-(0) = v_+(0) = v(0)$ and also that

$$c_n^- m_- + c_n^+ m_+ \ge \sigma_0|\xi'|, \quad \text{for some positive } \sigma_0. \quad (2.3.17)$$

2.3.3 Back to the Carleman estimate

With (2.3.8), we have

$$\begin{aligned} \|\mathscr{L}_\lambda v\|^2_{L^2(\mathbb{R})} = \|c_n^-\big(D_n - ie_-)(D_n - if_-)v_-\big)\|^2_{L^2(\mathbb{R}_-)} & \\ + \|c_n^+\big(D_n - if_+)(D_n - ie_+)v_+\big)\|^2_{L^2(\mathbb{R}_+)}. & \end{aligned}$$

Let $\kappa > 0$ to be chosen later.

If $f_-(0) = \lambda\alpha_- - m_-(\xi') \ge \kappa(\lambda + |\xi'|)$,

we get from (2.3.11),

$$\begin{aligned} 2\|c_n^-\big(D_n - ie_-)(D_n - if_-)v_-\big)\|_{L^2(\mathbb{R}_-)} & \\ \ge c_n^-(\lambda\alpha_- + m_-)\Big(\kappa(\lambda + |\xi'|)\|v_-\|_{L^2(\mathbb{R}_-)} + \kappa^{1/2}(\lambda + |\xi'|)^{1/2}|v_-(0)|\Big), & \quad (2.3.18) \end{aligned}$$

a satisfactory estimate. Note in particular that we get the surface term estimate

$$\|\mathscr{L}_\lambda v\|_{L^2(\mathbb{R})} \gtrsim \lambda^{3/2}|v(0)|. \quad (2.3.19)$$

If $f_-(0) = \lambda\alpha_- - m_-(\xi') < \kappa(\lambda + |\xi'|)$, then we assume that $f_+(0) = \lambda\alpha_+ - m_+(\xi') \le -\kappa(\lambda + |\xi'|)$.

We obtain from (2.3.14), (2.3.16) and (2.3.17),

$$2\|c_n^+\big(D_n - if_+)(D_n - ie_+)v_-\big)\|_{L^2(\mathbb{R}_-)} \ge \kappa^{1/2}(\lambda + |\xi'|)^{1/2}\sigma_0|\xi'||v(0)|. \tag{2.3.20}$$

We note also that

$$\lambda\alpha_+ - m_+(\xi') \le -\kappa(\lambda + |\xi'|) \Longrightarrow C|\xi'| \ge m_+(\xi') \ge (\kappa + \alpha_+)\lambda + \kappa|\xi'|,$$

and for κ small enough, this gives $|\xi'| \gtrsim \lambda \gtrsim |\xi'|$. As a result we get with a fixed constant C_0

$$C_0\|\mathscr{L}_\lambda v\|_{L^2(\mathbb{R})} \ge \kappa^{1/2}(\lambda + |\xi'|)^{3/2}|v(0)|, \tag{2.3.21}$$

which implies the surface term estimate

$$\|\mathscr{L}_\lambda v\|_{L^2(\mathbb{R})} \gtrsim \lambda^{3/2}|v(0)|. \tag{2.3.22}$$

We have now to prove that it is indeed possible to choose a small positive κ such that

$$\lambda\alpha_- - m_-(\xi') < \kappa(\lambda + |\xi'|) \Longrightarrow \lambda\alpha_+ - m_+(\xi') \le -\kappa(\lambda + |\xi'|).$$

By homogeneity, it means that we have to find $\kappa > 0$ such that on the half-sphere $\lambda^2 + |\xi'|^2 = 1, \lambda \ge 0$,

$$\lambda\alpha_- - m_-(\xi') < \kappa \Longrightarrow \kappa \le m_+(\xi') - \lambda\alpha_+.$$

Lemma 2.3.3. *Let $m_\pm$ be continuous positive and positively homogeneous functions of degree 1 on $\mathbb{R}^{n-1}\backslash\{0\}$. We choose*

$$\alpha_- = 1, \quad \alpha_+ = \frac{1}{2}\inf_{\mathbb{S}^{n-2}} \frac{m_+(\eta)}{m_-(\eta)}, \tag{2.3.23}$$

$$0 < \kappa \le 1/2, \quad \kappa \le \frac{\inf_{\mathbb{S}^{n-2}} m_+(\eta)}{4\alpha_+ + 2}. \tag{2.3.24}$$

Then for $(\xi', \lambda) \in \mathbb{R}^{n-1} \times [0, 1]$ such that $\lambda^2 + |\xi'|^2 = 1$,

$$\lambda\alpha_- - m_-(\xi') < \kappa \Longrightarrow \kappa \le m_+(\xi') - \lambda\alpha_+. \tag{2.3.25}$$

Proof. (1) We assume first that $\xi' = 0$ so that $\lambda = 1$ and $m_\pm(0) = 0$. The implication (2.3.25) holds true since $1 < \kappa$ does not occur.
(2) We assume $\lambda^2 + |\xi'|^2 = 1, 0 \le \lambda \le \inf_{\mathbb{S}^{n-2}} m_+/2\alpha_+$. The implication (2.3.25) holds true since its conclusion is verified:

$$\kappa + \lambda\alpha_+ \le \frac{1}{2}m_+(\xi') + \frac{1}{2}m_+(\xi') = m_+(\xi')$$

(3) We assume $\lambda^2 + |\xi'|^2 = 1, \lambda \geq \inf_{\mathbb{S}^{n-2}} m_+/2\alpha_+$. Then if $\lambda\alpha_- - m_-(\xi') < \kappa$, we have $\lambda\alpha_- - \kappa < m_-(\xi')$ and thus

$$m_+(\xi') \geq 2m_-(\xi')\alpha_+ \geq 2(\lambda\alpha_- - \kappa)\alpha_+ = 2(\lambda - \kappa)\alpha_+,$$

so that

$$\begin{aligned} m_+(\xi') - \lambda\alpha_+ - \kappa \geq \lambda\alpha_+ - 2\kappa\alpha_+ - \kappa \geq \lambda\alpha_+ - (1 + 2\alpha_+)\frac{\inf_{\mathbb{S}^{n-2}} m_+(\xi')}{4\alpha_+ + 2} \\ = \lambda\alpha_+ - \frac{1}{2}\inf_{\mathbb{S}^{n-2}} m_+(\xi') \geq 0, \end{aligned}$$

completing the proof of the lemma. □

We have proven above the following

Proposition 2.3.4. *Let $\mathscr{L}_\lambda$ be given by (2.3.8), $m_\pm(\xi')$ by (2.3.7) (elliptic positive homogeneous with degree 1). Let $\varphi_\pm$ be given by (2.3.4) such that the assumption (2.3.23) holds. Then there exists a constant C such that for all*

$$v = H(x_n)v_+(x', x_n) + H(-x_n)v_-(x', x_n)$$

with $v_\pm \in \mathscr{S}(\mathbb{R}^n)$ satisfying (2.3.5) and (2.3.6), we have

$$C\|\mathscr{L}_\lambda v\|_{L^2(\mathbb{R}^n)} \geq \lambda^{3/2}|v(0,\cdot)|_{L^2(\mathbb{R}^{n-1})}. \tag{2.3.26}$$

This provides the fourth term in (2.2.1), which is a "surface term" and we have to show now that we may obtain the other terms using the key estimate above.

2.3.4 Carleman estimate, continued

We shall start with rewriting the lemmas above.

Lemma 2.3.5. *Let μ, γ be non-negative parameters. Then for $\omega \in \mathscr{S}(\mathbb{R})$, we have*

$$\|D_t\omega - i(\mu - \gamma t)\omega\|^2_{L^2(\mathbb{R}_-)} \geq \|D_t\omega\|^2_{L^2(\mathbb{R}_-)} + \|\mu\omega\|^2_{L^2(\mathbb{R}_-)} + \mu|\omega(0)|^2.$$

Proof. We have

$$\begin{aligned} &\|D_t\omega - i(\mu - \gamma t)\omega\|^2_{L^2(\mathbb{R}_-)} \\ &\quad = \|D_t\omega\|^2_{L^2(\mathbb{R}_-)} + \|(\mu - \gamma t)\omega\|^2_{L^2(\mathbb{R}_-)} + 2\operatorname{Re}\langle D_t\omega, -iH(-t)(\mu - \gamma t)\omega\rangle \\ &\quad = \|D_t\omega\|^2_{L^2(\mathbb{R}_-)} + \int_{-\infty}^0 \big((\mu - \gamma t)^2 + \gamma\big)|\omega(t)|^2 dt + \mu|\omega(0)|^2 \\ &\quad \geq \|D_t\omega\|^2_{L^2(\mathbb{R}_-)} + \|\mu\omega\|^2_{L^2(\mathbb{R}_-)} + \mu|\omega(0)|^2, \end{aligned}$$

completing the proof. □

Lemma 2.3.6.
(i) *Let μ, γ be non-negative parameters. Then for $\omega \in \mathscr{S}(\mathbb{R})$, we have*

$$\|D_t\omega - i(\mu - \gamma t)\omega\|^2_{L^2(\mathbb{R}_+)} + \mu|\omega(0)|^2 \geq \|D_t\omega\|^2_{L^2(\mathbb{R}_+)} + \gamma\|\omega\|^2_{L^2(\mathbb{R}_+)} + \|(\mu - \gamma t)\omega\|^2_{L^2(\mathbb{R}_+)}.$$

(ii) *Let μ be a negative parameter and let γ be a non-negative parameter. Then for $\omega \in \mathscr{S}(\mathbb{R})$, we have*

$$\|D_t\omega - i(\mu - \gamma t)\omega\|^2_{L^2(\mathbb{R}_+)} \geq \|D_t\omega\|^2_{L^2(\mathbb{R}_+)} + \|(\mu - \gamma t)\omega\|^2_{L^2(\mathbb{R}_+)} + \gamma\|\omega\|^2_{L^2(\mathbb{R}_+)} + |\mu||\omega(0)|^2.$$

Proof. We have

$$\begin{aligned}
&\|D_t\omega - i(\mu - \gamma t)\omega\|^2_{L^2(\mathbb{R}_+)} \\
&\quad= \|D_t\omega\|^2_{L^2(\mathbb{R}_+)} + \|(\mu - \gamma t)\omega\|^2_{L^2(\mathbb{R}_+)} + 2\,\mathrm{Re}\langle D_t\omega, -iH(t)(\mu - \gamma t)\omega\rangle \\
&\quad\geq \|D_t\omega\|^2_{L^2(\mathbb{R}_+)} + \int_0^{+\infty} \big((\mu - \gamma t)^2 + \gamma\big)|\omega(t)|^2 dt - \mu|\omega(0)|^2 \\
&\quad\geq \|D_t\omega\|^2_{L^2(\mathbb{R}_+)} + \gamma\|\omega\|^2_{L^2(\mathbb{R}_-)} - \mu|\omega(0)|^2,
\end{aligned}$$

proving the lemma. □

Using Proposition 2.3.4, we want now to prove the estimate of Theorem 2.2.1. We have from (2.3.8),

$$\|\mathscr{L}_\lambda v\|^2_{L^2(\mathbb{R})} = (c_n^+)^2\|(D_n - if_+)(D_n - ie_+)v_+\|^2_{L^2(\mathbb{R}_+)} + (c_n^-)^2\|(D_n - ie_-)(D_n - if_-)v_-\|^2_{L^2(\mathbb{R}_-)}$$

with

$$f_\pm = \lambda\alpha_\pm - m_\pm(\xi') - \lambda\beta x_n, \quad e_\pm = \lambda\alpha_\pm + m_\pm(\xi') - \lambda\beta x_n, \tag{2.3.27}$$

where β is a non-negative parameter and $\alpha_\pm$ are determined by (2.3.23). Since the coefficients $c_n^\pm$ are positive and bounded away from 0, we find a constant C_0 such that

$$C_0\|\mathscr{L}_\lambda v\|^2_{L^2(\mathbb{R})} \geq \|(D_n - if_+)(D_n - ie_+)v_+\|^2_{L^2(\mathbb{R}_+)} + \|(D_n - ie_-)(D_n - if_-)v_-\|^2_{L^2(\mathbb{R}_-)}. \tag{2.3.28}$$

Let $\epsilon_0 \in (0, 1]$ to be chosen later.
[1] **We assume first that** $1 + |\xi'| \leq \epsilon_0\lambda$**.** Then we have

$$f_-(0) = \lambda\alpha_- - m_-(\xi') \geq \lambda\alpha_- - |\xi'|\|m_-\|_{L^\infty(\mathbb{S}^{n-2})} \geq \lambda\big(\alpha_- - \epsilon_0\|m_-\|_{L^\infty(\mathbb{S}^{n-2})}\big) \geq \lambda\alpha_-/2,$$

provided

$$2\epsilon_0\|m_-\|_{L^\infty(\mathbb{S}^{n-2})} \le \alpha_-. \tag{2.3.29}$$

Under that condition, we may apply Lemma 2.3.5 to get

$$\begin{aligned}
&\|(D_n - ie_-)(D_n - if_-)v_-\|^2_{L^2(\mathbb{R}_-)}\\
&\ge e_-(0)^2\|(D_n - if_-)v_-\|^2_{L^2(\mathbb{R}_-)} + e_-(0)|D_n v_-(0) - if_-(0)v_-(0)|^2\\
&\ge \lambda^2\alpha_-^2\|(D_n - if_-)v_-\|^2_{L^2(\mathbb{R}_-)} + \lambda\alpha_-|D_n v_-(0) - if_-(0)v_-(0)|^2\\
&\ge \lambda^2\alpha_-^2\|D_n v_-\|^2_{L^2(\mathbb{R}_-)} + \frac{1}{4}\lambda^4\alpha_-^4\|v_-\|^2_{L^2(\mathbb{R}_-)} + \lambda\alpha_-|D_n v_-(0) - if_-(0)v_-(0)|^2.
\end{aligned}$$

This implies with a fixed constant C_1 that

$$C_1\|\mathscr{L}_\lambda v\|^2_{L^2(\mathbb{R})} \ge \lambda^2\|D_n v_-\|^2_{L^2(\mathbb{R}_-)} + \lambda^4\|v_-\|^2_{L^2(\mathbb{R}_-)} + \lambda|D_n v_-(0) - if_-(0)v_-(0)|^2.$$

Using (2.3.26) and $|f_-(0)| \le \lambda\alpha_-$, we obtain with a fixed constant C_2 that

$$C_2\|\mathscr{L}_\lambda v\|^2_{L^2(\mathbb{R})} \ge \lambda^2\|D_n v_-\|^2_{L^2(\mathbb{R}_-)} + \lambda^4\|v_-\|^2_{L^2(\mathbb{R}_-)} + \lambda|D_n v_-(0)|^2, \tag{2.3.30}$$

a better estimate than what is required. We need now to handle the positive half-line. We have

$$\begin{aligned}
f_+(0) = \lambda\alpha_+ - m_+(\xi') \ge \lambda\alpha_+ - |\xi'|\|m_+\|_{L^\infty(\mathbb{S}^{n-2})} \ge \lambda\big(\alpha_+ - \epsilon_0\|m_-\|_{L^\infty(\mathbb{S}^{n-2})}\big)&\\
\ge \lambda\alpha_+/2,&
\end{aligned}$$

provided

$$2\epsilon_0\|m_+\|_{L^\infty(\mathbb{S}^{n-2})} \le \alpha_+. \tag{2.3.31}$$

We apply Lemma 2.3.6 `(i)`to get

$$\begin{aligned}
\|(D_n - if_+)(D_n - ie_+)v\|^2_{L^2(\mathbb{R}_+)} + f_+(0)|D_n v_+(0) - ie_+v_+(0)|^2&\\
\ge \lambda\beta\|(D_n - ie_+)v\|^2.&
\end{aligned} \tag{2.3.32}$$

Thanks to (2.3.26) and (2.3.30), we have $\|\mathscr{L}_\lambda v\|^2_{L^2(\mathbb{R})} \gtrsim \lambda^3|v(0)|^2 + \lambda|D_n v_-(0)|^2$ and the transmission condition (2.3.6) implies thus, along with (2.3.30),

$$\begin{aligned}
\|\mathscr{L}_\lambda v\|^2_{L^2(\mathbb{R})} \gtrsim \lambda^3|v(0)|^2 + \lambda|D_n v_-(0)|^2 + \lambda|D_n v_+(0)|^2&\\
+ \lambda^2\|D_n v_-\|^2_{L^2(\mathbb{R}_-)} + \lambda^4\|v_-\|^2_{L^2(\mathbb{R}_-)}.&
\end{aligned} \tag{2.3.33}$$

Since we have here $|\xi'| \le \lambda$, we have also

$$f_+(0)|D_n v_+(0) - ie_+v_+(0)|^2 \le 2\lambda\alpha_+|D_n v_+(0)|^2 + 2\lambda^3\alpha_+|v_+(0)|^2(\alpha_+ + \|m_+\|_{L^\infty(\mathbb{S}^{n-2})})^2.$$

This implies that we have

$$\begin{aligned}
\|\mathscr{L}_\lambda v\|^2_{L^2(\mathbb{R})} \gtrsim \lambda^3|v(0)|^2 + \lambda|D_n v_-(0)|^2 + \lambda|D_n v_+(0)|^2&\\
+ \lambda^2\|D_n v_-\|^2_{L^2(\mathbb{R}_-)} + \lambda^4\|v_-\|^2_{L^2(\mathbb{R}_-)}&\\
+ \lambda\beta\|(D_n - ie_+)v_+\|^2_{L^2(\mathbb{R}_+)}.&
\end{aligned} \tag{2.3.34}$$

Applying again Lemma 2.3.6 `(i)`, we get

$$\begin{aligned}\|\mathscr{L}_\lambda v\|^2_{L^2(\mathbb{R})} \gtrsim \lambda^3|v(0)|^2 + \lambda|D_n v_-(0)|^2 + \lambda|D_n v_+(0)|^2\\ + \lambda^2\|D_n v_-\|^2_{L^2(\mathbb{R}_-)} + \lambda^4\|v_-\|^2_{L^2(\mathbb{R}_-)}\\ + \lambda\beta\|D_n v_+\|^2_{L^2(\mathbb{R}_+)} + \lambda\beta\|e_+ v_+\|^2_{L^2(\mathbb{R}_+)}. \quad (2.3.35)\end{aligned}$$

We note that $e_+ \geq \lambda(\alpha_+ - \beta x_n) \geq \lambda\alpha_+/2$, on the support of v, provided is v supported in

$$x_n \leq \beta^{-1}\alpha_+/2. \quad (2.3.36)$$

so that we obtain eventually with $\epsilon_0 \in (0, 1]$ satisfying (2.3.29), (2.3.31), $\beta \geq 1$ and $\operatorname{supp} v \subset (-\infty, \beta^{-1}\alpha_+/2]$

$$\begin{aligned}\|\mathscr{L}_\lambda v\|^2_{L^2(\mathbb{R})} \gtrsim \lambda^3|v(0)|^2 + \lambda|D_n v_-(0)|^2 + \lambda|D_n v_+(0)|^2\\ + \lambda^2\|D_n v_-\|^2_{L^2(\mathbb{R}_-)} + \lambda^4\|v_-\|^2_{L^2(\mathbb{R}_-)}\\ + \lambda\|D_n v_+\|^2_{L^2(\mathbb{R}_+)} + \lambda^3\|v_+\|^2_{L^2(\mathbb{R}_+)}, \quad (2.3.37)\end{aligned}$$

which provides the sought estimate.

[2] **We assume now that** $1 \leq \lambda \leq \epsilon_0|\xi'|$. Then we have

$$\begin{aligned}f_-(0) &= \lambda\alpha_- - m_-(\xi')\\ &\leq \epsilon_0|\xi'|\alpha_- - |\xi'|\inf_{\mathbb{S}^{n-2}} m_- = -|\xi'|\big(\inf_{\mathbb{S}^{n-2}} m_- - \epsilon_0\alpha_-\big) \leq -|\xi'|\frac{1}{2}\inf_{\mathbb{S}^{n-2}} m_-,\end{aligned}$$

provided

$$2\epsilon_0\alpha_- \leq \inf_{\mathbb{S}^{n-2}} m_-. \quad (2.3.38)$$

We have similarly

$$f_+(0) = \lambda\alpha_+ - m_+(\xi') \leq -|\xi'|\frac{1}{2}\inf_{\mathbb{S}^{n-2}} m_+$$

provided

$$2\epsilon_0\alpha_+ \leq \inf_{\mathbb{S}^{n-2}} m_+. \quad (2.3.39)$$

In that case we have thus

$$f_\pm(0) \approx -|\xi'|, \quad e_\pm(0) \approx |\xi'|,$$

and using Proposition 2.3.4, we can follow the reasoning for the previous case (switching the role of the positive half-line with the negative half-line) to get

$$\begin{aligned}\|\mathscr{L}_\lambda v\|^2_{L^2(\mathbb{R})} \gtrsim \lambda^3|v(0)|^2 + \lambda|D_n v_-(0)|^2 + \lambda|D_n v_+(0)|^2\\ + \lambda^2\|D_n v_+\|^2_{L^2(\mathbb{R}_+)} + \lambda^4\|v_+\|^2_{L^2(\mathbb{R}_+)} + \lambda\|D_n v_-\|^2_{L^2(\mathbb{R}_-)} + \lambda^3\|v_-\|^2_{L^2(\mathbb{R}_-)}, \quad (2.3.40)\end{aligned}$$

provided v is supported in

$$x_n \geq -\beta^{-1}\alpha_-/2. \quad (2.3.41)$$

[3] **We are left with the main case $\epsilon_0|\xi'| \le \lambda \le \epsilon_0^{-1} + \epsilon_0^{-1}|\xi'|$.** Since we may assume that $\lambda \ge 2/\epsilon_0$, we can assume that

$$\frac{\epsilon_0}{2} \le \frac{|\xi'|}{\lambda} \le \frac{1}{\epsilon_0}.$$

We have always the elliptic terms $e_\pm(0) \approx \lambda$, but in that case $f_\pm(0)$ cannot be elliptic and are in fact close to 0. We use the key Proposition 2.3.4 and Lemmas 2.3.5-2.3.6. We check

$$E = \|(D_n - if_+)(D_n - ie_+)v_+\|^2_{L^2(\mathbb{R}_+)} + \|(D_n - ie_-)(D_n - if_-)v_-\|^2_{L^2(\mathbb{R}_-)}.$$

From (2.3.18) in the case $f_-(0) = \lambda\alpha_- - m_-(\xi') \ge \kappa(\lambda + |\xi'|)$ we find

$$\|(D_n - ie_-)(D_n - if_-)v_-\|^2_{L^2(\mathbb{R}_-)} \gtrsim \lambda^4\|v_-\|^2_{L^2(\mathbb{R}_-)} + \lambda^3|v(0)|^2.$$

We know that it is possible to find $\kappa > 0$ such that if

$$f_-(0) = \lambda\alpha_- - m_-(\xi') < \kappa(\lambda + |\xi'|),$$

then $f_+(0) = \lambda\alpha_+ - m_+(\xi') \le -\kappa(\lambda + |\xi'|)$. Lemma 2.3.6 provides

$$\|(D_n - if_+)(D_n - ie_+)v_+\|^2_{L^2(\mathbb{R}_+)} \gtrsim \lambda\|(D_n - ie_+)v_+\|^2_{L^2(\mathbb{R}_+)} + \lambda|D_n v_+(0) - ie_+(0)v(0)|^2.$$

Since we control also $\lambda^3|v(0)|^2$, this gives control of $\lambda|D_n v_+(0)|^2$ and with the transmission condition of $\lambda|D_n v_-(0)|^2$. Lemma 2.3.6 and a support condition for v give

$$\lambda\|(D_n - ie_+)v_+\|^2_{L^2(\mathbb{R}_+)} \gtrsim \lambda^3\|v_+\|^2_{L^2(\mathbb{R}_+)},$$

completing the proof.

2.4 Comments

2.4.1 Condition (Ψ)

We may take a look at the one-dimensional estimate

$$C\|D_t u - i\lambda f(t)u\|_{L^2(\mathbb{R})} \ge \|u\|_{L^2(\mathbb{R})}, \tag{2.4.1}$$

where λ is a large positive parameter and f is a real-valued smooth function. Some simple examples show that this estimate holds for several choices of f. We set $h = 1/\lambda$ in the sequel to get a semi-classical version of our estimates.

Creation and Annihilation operators

The one-dimensional operators

$$C_+^{[k]} = hD_t + it^{2k+1}, \quad C_-^{[k]} = hD_t - it^{2k+1} \tag{2.4.2}$$

are similar to creation and annihilation operators for $k \in \mathbb{N}$. In particular, we have with $c_k > 0$, for $v \in \mathscr{S}(\mathbb{R})$,

$$\|C_+^{[k]} v\|_0 \geq c_k h^{\frac{2k+1}{2k+2}} \|v\|_0, \quad \ker C_-^{[k]} = \mathbb{C} e^{-\frac{\pi t^{2k+2}}{h(k+1)}}. \tag{2.4.3}$$

The second assertion is obvious whereas the first deserves a proof. With a linear change of coordinate $t \mapsto th^{1/(2k+2)}$, we see that $C_+^{[k]}$ is unitarily equivalent to

$$(D_t + it^{2k+1}) h^{(2k+1)/(2k+2)},$$

so it is enough to prove the estimate for $h = 1$. For $v \in \mathscr{S}(\mathbb{R})$, we know that $\dot{v} - 2\pi t^{2k+1} v = 2i\pi C_+^{[k]} v$ so that

$$v(t) = \begin{cases} 2i\pi \int_{+\infty}^{t} e^{-\frac{\pi(s^{2k+2} - t^{2k+2})}{k+1}} (C_+^{[k]} v)(s) ds & \text{for } t \geq 0, \\ 2i\pi \int_{-\infty}^{t} e^{-\frac{\pi(s^{2k+2} - t^{2k+2})}{k+1}} (C_+^{[k]} v)(s) ds & \text{for } t \leq 0, \end{cases}$$

and since for $t \geq 0$ we have

$$\begin{aligned} &\int H(t) H(s-t) e^{-\frac{\pi(s^{2k+2} - t^{2k+2})}{k+1}} ds \\ &\qquad = \frac{1}{2k+2} \int_0^{+\infty} e^{-\frac{\pi\sigma}{k+1}} (\sigma + t^{2k+2})^{-(\frac{2k+1}{2k+2})} d\sigma \\ &\qquad\qquad \leq \frac{1}{2k+2} \int_0^{+\infty} e^{-\frac{\pi\sigma}{k+1}} \sigma^{-(\frac{2k+1}{2k+2})} d\sigma = \alpha_k < +\infty, \end{aligned}$$

and also

$$\begin{aligned} &\sup_s \int H(t) H(s-t) e^{-\frac{\pi(s^{2k+2} - t^{2k+2})}{k+1}} dt \\ &= \frac{1}{2k+2} \int_0^{s^{2k+2}} e^{-\frac{\pi\sigma}{k+1}} (s^{2k+2} - \sigma)^{-(\frac{2k+1}{2k+2})} d\sigma \\ &\leq \frac{1}{2k+2} \int_0^{+\infty} e^{-\frac{\pi\sigma}{k+1}} d\sigma + \frac{1}{2k+2} \int_{\max(0, s^{2k+2} - 1)}^{s^{2k+2}} e^{-\frac{\pi\sigma}{k+1}} (s^{2k+2} - \sigma)^{-(\frac{2k+1}{2k+2})} d\sigma \\ &\leq \frac{1}{2\pi} + e^{-\frac{\pi \max(0, s^{2k+2} - 1)}{k+1}} \leq \frac{1}{2\pi} + 1, \end{aligned}$$

along with analogous estimates for $t \leq 0$, Schur's Lemma gives

$$\|v\|_0 \leq C_k \|C_+^{[k]} v\|_0,$$

which proves (2.4.3).

The operator $C_+^{[k]}$ with (dense) domain

$$D_k = \{u \in L^2(\mathbb{R}), \partial_t u \in L^2(\mathbb{R}), t^{2k+1}u \in L^2(\mathbb{R})\} = \{u \in L^2(\mathbb{R}), C_+^{[k]}u \in L^2(\mathbb{R})\}$$

is injective and has a closed image (thanks to (2.4.3)) of codimension 1: it is a Fredholm operator with index -1. The operator $C_-^{[k]}$ is the adjoint of $C_+^{[k]}$ and is onto with a one-dimensional kernel: it is a Fredholm operator with index $+1$.

Cauchy-type operators

For $k \in \mathbb{N}$, we define

$$C_0^{[k]} = hD_t + t^{2k}\sqrt{-1}. \tag{2.4.4}$$

There exists $c_k > 0$, such that for $v \in \mathscr{S}(\mathbb{R})$,

$$\|C_0^{[k]}v\|_0 \ge c_k h^{\frac{2k}{2k+1}} \|v\|_0. \tag{2.4.5}$$

As above, the linear change of variable $t \mapsto th^{1/(2k+1)}$ shows that $C_0^{[k]}$ is unitarily equivalent to $h^{2k/(2k+1)}(D_t + \sqrt{-1}t^{2k})$ so that it suffices to prove (2.4.5) for $h = 1$. Although a direct resolution of the ODE as for proving (2.4.3) would provide the answer, we shall prove a more general lemma, implying both (2.4.4) and (2.4.3).

Lemma 2.4.1. *Let $\phi \in C^0(\mathbb{R};\mathbb{R})$ such that*

$$\phi(t) > 0, s > t \Longrightarrow \phi(s) \ge 0. \tag{2.4.6}$$

Then for all $v \in W^{1,1}(\mathbb{R})$ with $\phi v \in L^1(\mathbb{R})$, we have

$$\sup_{t\in\mathbb{R}} |v(t)| \le \int_{\mathbb{R}} |\frac{dv}{dt} - \phi v| dt, \tag{2.4.7}$$

and if v is compactly supported, diameter(supp v) $\le \delta$, $v \in H^1(\mathbb{R}), \phi v \in L^2(\mathbb{R})$,

$$\|v\|_{L^2(\mathbb{R})} \le \delta \|\frac{dv}{dt} - \phi v\|_{L^2(\mathbb{R})}. \tag{2.4.8}$$

Moreover defining for $\lambda > 0$, $m(\lambda) = |\{t \in \mathbb{R}, |\phi(t)| \le \lambda^{-1}\}|$ and assuming that $\kappa(\phi) = \inf_{\lambda>0}(m(\lambda) + \lambda) < +\infty$, we have for $v \in H^1(\mathbb{R})$ with $\phi v \in L^2(\mathbb{R})$,

$$\|v\|_{L^2(\mathbb{R})} \le 2\|\frac{dv}{dt} - \phi v\|_{L^2(\mathbb{R})}\kappa(\phi). \tag{2.4.9}$$

N.B. This lemma implies the estimates (2.4.5) and (2.4.3): first of all the hypothesis (2.4.6) holds for $t^{2k+1}, \pm t^{2k}$ (violated for $-t^{2k+1}$). Moreover for $\phi = h^{-1}t^l$, we have

$$\kappa(\phi) \le |\{t \in \mathbb{R}, h^{-1}|t|^l \le h^{-\frac{1}{l+1}}\}| + h^{\frac{1}{l+1}} \le 2h^{\frac{1}{l+1}}$$

and thus

$$\|v\| \le 4h^{\frac{1}{l+1}} \|\frac{d}{idt}v + ih^{-1}t^l v\|,$$

so that $h^{\frac{l}{l+1}}\|v\| \le 4\|h\frac{d}{idt}v + it^l v\|$ and for l even the same estimate for $h\frac{d}{idt}v - it^l v$. Also the reader may have noticed that the estimates (2.4.7) and (2.4.9) hold true without any condition on the support of v; on the other hand $\kappa(0) = +\infty$ and although the estimate (2.4.8) holds for $\phi \equiv 0$, no better estimate is true in that simple case.

Proof of the Lemma. We define

$$T = \inf\{t \in \mathbb{R}, \phi(t) > 0\} \quad (T = \pm\infty \text{ if } \pm\phi \le 0).$$

The condition (2.4.6) ensures that

$$t > T \Longrightarrow \exists t' \in (T, t) \text{ with } \phi(t') > 0 \Longrightarrow \phi(t) \ge 0, \tag{2.4.10}$$

$$t < T \Longrightarrow \phi(t) \le 0. \tag{2.4.11}$$

For $v \in C_c^1(\mathbb{R})$, we have with $\dot{v} - \phi v = f$, and $t \ge T$

$$v(t) = \int_{+\infty}^{t} f(s)e^{\int_s^t \phi(\sigma)d\sigma}ds = -\int_t^{+\infty} f(s)e^{-\int_t^s \phi(\sigma)d\sigma}ds,$$

and since $\phi \ge 0$ on $[T, +\infty)$, we get

$$\text{for } t \ge T, \quad |v(t)| \le \int_t^{+\infty} |f(s)|ds,$$

and similarly for $t \le T$, $|v(t)| \le \int_{-\infty}^t |f(s)|ds$, so that (2.4.7) follows as well as its immediate consequence (2.4.8). For future reference we give another proof of (2.4.7) which uses a more flexible energy method. We calculate with $L = \frac{d}{idt} + i\phi$ and $v \in \mathscr{S}(\mathbb{R})$

$$\text{for } t'' \ge T,\ 2\operatorname{Re}\langle Lv, iH(t - t'')v\rangle = |v(t'')|^2 + 2\int_{t''}^{+\infty} |\phi(t)||v(t)|^2 dt,$$

$$\text{for } t' \le T,\ 2\operatorname{Re}\langle Lv, -iH(t' - t)v\rangle = |v(t')|^2 + 2\int_{-\infty}^{t'} |\phi(t)||v(t)|^2 dt,$$

and we get

$$\sup_{t\in\mathbb{R}} |v(t)|^2 + 2\int_{\mathbb{R}} |\phi(t)||v(t)|^2 dt \le 2\int_{\mathbb{R}} |(Lv)(t)||v(t)|dt, \tag{2.4.12}$$

proving (2.4.7) (with a constant 2), which implies also

$$\int_{\mathbb{R}} |\phi(t)||v(t)|^2 dt \le \|Lv\|_{L^2}\|v\|_{L^2}. \tag{2.4.13}$$

Now, we have also with $\lambda > 0$,

$$\begin{aligned}\int_{\mathbb{R}} |v(t)|^2 dt &\le \int_{\lambda|\phi(t)|\le 1} |v(t)|^2 dt + \int_{\lambda|\phi(t)|>1} \lambda|\phi(t)||v(t)|^2 dt \\ &\le |\{t \in \operatorname{supp} v, |\phi(t)| \le 1/\lambda\}| \sup |v(t)|^2 + \lambda\|Lv\|_{L^2}\|v\|_{L^2} \\ &\le 2\|Lv\|_{L^2}\|v\|_{L^2}\Big(|\{t \in \operatorname{supp} v, |\phi(t)| \le 1/\lambda\}| + \lambda/2\Big),\end{aligned} \tag{2.4.14}$$

which gives (2.4.9). □

The last estimate is of particular interest when the function ϕ has a polynomial behaviour, in the sense of the following lemma.

Lemma 2.4.2. *Let $k \in \mathbb{N}^*, \delta > 0$ and $C > 0$ be given. Let I be an interval of $\mathbb{R}$ and $q : I \to \mathbb{R}$ be a C^k function such that*

$$\inf_{t \in I} |\partial_t^k q| \geq \delta. \tag{2.4.15}$$

Then for all $h > 0$, the set

$$\{t \in I, |q(t)| \leq Ch^k\} \subset_{1 \leq l \leq k} \cup J_l \tag{2.4.16}$$

where J_l is an interval with length $h(\alpha_k C\delta^{-1})^{1/k}, \alpha_k = 2^{2k}k!$. As a consequence, the Lebesgue measure of $\{t \in I, |q(t)| \leq Ch^k\}$ is smaller than

$$hC^{1/k}\delta^{-1/k}4k(k!)^{1/k} \leq hC^{1/k}\delta^{-1/k}4k^2.$$

Proof. Let $k \in \mathbb{N}^*$, $h > 0$ and set $E_k(h, C, q) = \{t \in I, |q(t)| \leq Ch^k\}$. Let us first assume $k = 1$. Assume that $t, t_0 \in E_1(h, C, q)$; then the mean value theorem and (2.4.15) imply $2Ch \geq |q(t) - q(t_0)| \geq \delta|t - t_0|$ so that

$$E_1(h, C, q) \cap \{t, |t - t_0| > h2C\delta^{-1}\} = \emptyset :$$

otherwise we would have $2Ch > \delta h2C/\delta$. As a result, for any $t_0, t \in E_1(h, C, q)$, we have $|t - t_0| \leq h2C\delta^{-1}$. Either $E_1(h, C, q)$ is empty or it is not empty and then included in an interval with length $\leq h4C\delta^{-1}$.

Let us assume now that $k \geq 2$. If $E_k(h, C, q) = \emptyset$, then (2.4.16) holds true. We assume that there exists $t_0 \in E_k(h, C, q)$ and we write for $t \in I$,

$$q(t) = q(t_0) + \underbrace{\int_0^1 q'(t_0 + \theta(t - t_0))d\theta(t - t_0)}_{Q(t)}. \tag{2.4.17}$$

Then if $t \in E_k(h, C, q)$, we have $2Ch^k \geq |Q(t)(t - t_0)|$. Now for a given $\omega > 0$, either $|t - t_0| \leq \omega h/2$ and $t \in [t_0 - \omega h/2, t_0 + \omega h/2]$, or $|t - t_0| > \omega h/2$ and from the previous inequality, we infer $|Q(t)| \leq \omega^{-1}4Ch^{k-1}$, i.e. we get that

$$E_k(h, C, q) \subset [t_0 - \omega h/2, t_0 + \omega h/2] \cup E_{k-1}(h, \omega^{-1}4C, Q). \tag{2.4.18}$$

But the function Q satisfies the assumptions of the lemma with $k - 1, \delta/k$ instead of k, δ: in fact for $t \in I$,

$$Q^{(k-1)}(t) = \int_0^1 q^{(k)}(t_0 + \theta(t - t_0))\theta^{k-1}d\theta$$

and if $q^{(k)}(t) \geq \delta$ on I, we get $Q^{(k-1)}(t) \geq \delta/k$. By induction on k and using (2.4.18), we get that

$$E_k(h, C, q) \subset [t_0 - \omega h/2, t_0 + \omega h/2] \cup_{1 \leq l \leq k-1} J_l, \ |J_l| \leq h(4C\omega^{-1}k\delta^{-1}\alpha_{k-1})^{1/(k-1)}. \tag{2.4.19}$$

We choose now ω so that $\omega = (4C\omega^{-1}k\delta^{-1}\alpha_{k-1})^{1/(k-1)}$ i.e. $\omega^k = 4C\delta^{-1}k\alpha_{k-1}$, that is $\omega = (C\delta^{-1}4k\alpha_{k-1})^{1/k}$, yielding the result if $\alpha_k = 4k\alpha_{k-1}$, i.e. with $\alpha_1 = 2$,

$$\alpha_k = (4k)(4(k-1))\dots(4\times 2)\alpha_1 = 4^{k-1}k!2^2 = 2^{2k}k!.$$

The proof of the lemma is complete. □

A consequence of Lemma 2.4.2 and of the estimate (2.4.14) is that for $q:\mathbb{R}\to\mathbb{R}$ satisfying (2.4.15),(2.4.6) and $h>0$,

$$\|v\|_{L^2(\mathbb{R})} \le 2\|\dot v - h^{-1}q(t)v\|_{L^2(\mathbb{R})}\Big(\frac{h^{\frac{1}{k+1}}}{2} + |\{t\in\mathbb{R}, h^{-1}|q(t)| \le h^{-\frac{1}{k+1}}\}|\Big)$$
$$|\{t\in\mathbb{R}, h^{-1}|q(t)| \le h^{-\frac{1}{k+1}}\}| = |\{t\in\mathbb{R}, |q(t)| \le h^{\frac{k}{k+1}}\}| \le 4k^2 h^{\frac{1}{k+1}}\delta^{-\frac{1}{k}},$$

so that

$$h^{\frac{k}{k+1}}\|v\|_{L^2(\mathbb{R})} \le \|h\dot v - q(t)v\|_{L^2(\mathbb{R})}(1+8k^2\delta^{-1/k}). \tag{2.4.20}$$

On the other hand (2.4.13) implies as well

$$\int h^{-1}|q(t)||v(t)|^2 dt \le \|\dot v - h^{-1}q\|_{L^2(\mathbb{R})}\|v\|_{L^2(\mathbb{R})}$$

so that we have proven the following result.

Lemma 2.4.3. *Let $q\in C^\infty(\mathbb{R};\mathbb{R})$ such that (2.4.6) and (2.4.15) (for $I=\mathbb{R}$ and some $k\in\mathbb{N}^*$) hold. Then for all $h>0$ and all $v\in C_c^\infty(\mathbb{R})$ we have*

$$h^{\frac{k}{k+1}}\|v\|^2_{L^2(\mathbb{R})} + \int |q(t)||v(t)|^2 dt$$
$$\le \|h\dot v - q(t)v\|_{L^2(\mathbb{R})}\|v\|_{L^2(\mathbb{R})}(2+8k^2\delta^{-1/k}). \tag{2.4.21}$$

Condition (Ψ)

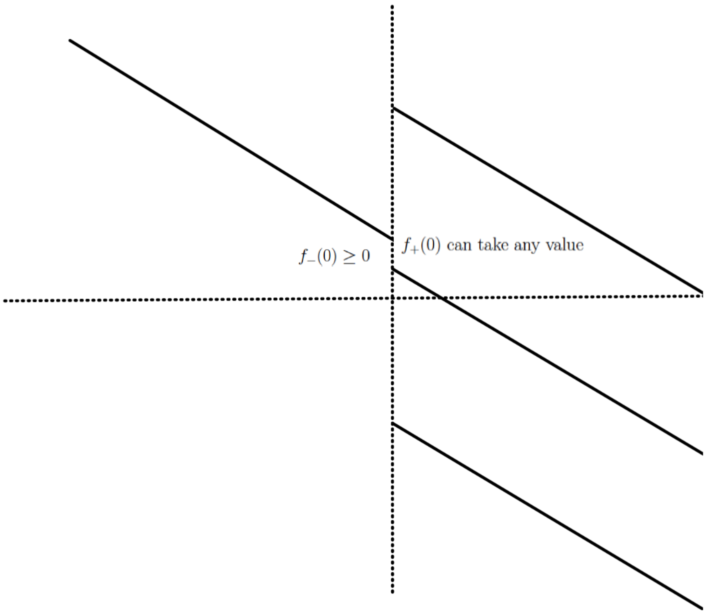
Going back to (2.4.1), we see using the previous results that the condition

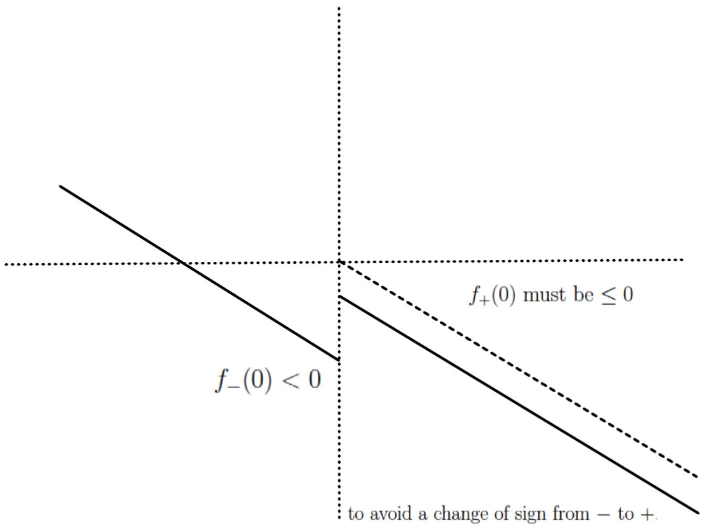
$$f(t)<0, \quad s>t \Longrightarrow f(s)\le 0, \tag{2.4.22}$$

i.e. *f does not change sign from $-$ to $+$ when t increases* is sufficient to obtain an a priori estimate of type (2.4.1). It can be proven as well that this condition is necessary (see e.g. Section 3 in [24]); Condition (2.4.22) is called condition (Ψ) for the adjoint operator $D_t + i\lambda f(t)$. When f is piecewise affine, as in our discussion, it turns out that this condition is equivalent to our main requirement expressed by (2.3.13). We have indeed

$$f_-(t) = f_-(0) - \gamma t, \quad f_+(t) = f_+(0) - \gamma t.$$

Indeed, if $f_-(0)\ge 0$, this implies that $f_-(t)\ge 0$ for $t\le 0$ and since f_+ is decreasing, no change of sign from $-$ to $+$ could occur when t increases. On the other hand, if $f_-(0) < 0$, since f_- is decreasing, no change of sign from $-$ to $+$ could occur for $t \le 0$, but we have to avoid $f_+(0) > 0$, otherwise we would have a change of sign from $-$ to $+$ when t increases for the discontinuous $t \mapsto f(t)$ near 0. So the condition (2.3.13) is exactly the expression of Condition (Ψ) for the adjoint operator D





2.4.2 Quasi-mode construction

Let us see what is happening when (2.4.22) does not hold, that is $f_-(0) < 0$ and $f_+(0) > 0$. We want to show that (2.4.1) cannot hold. We have, with a, b, γ positive

$$f(t) = H(-t)\big(f_-(0) - \gamma t\big) + H(t)\big(f_+(0) - \gamma t\big) = -H(-t)\big(a + \gamma t\big) + H(t)\big(b - \gamma t\big),$$

and we check the equation $D_t u - i\lambda f(t)u = 0$ which means

$$\dot{u} + \lambda f(t)u = 0, \quad \text{i.e.} \begin{cases} \dot{u} - \lambda(a + \gamma t) = 0, & \text{for } t \leq 0, \\ \dot{u} + \lambda(b - \gamma t) = 0, & \text{for } t \geq 0. \end{cases}$$

We get

$$\begin{cases} u = e^{\lambda(at + \gamma t^2/2)} u(0), & \text{for } t \leq 0, \\ u = e^{-\lambda(bt - \gamma t^2/2)} u(0) & \text{for } t \geq 0. \end{cases}$$

Let $\chi \in C_c^\infty(\mathbb{R}; \mathbb{R}_+)$ equal to 1 near 0 and supported where

$$\gamma t^2 \leq \min(a, b)|t|, \quad \text{i.e.} \quad |t| \leq \gamma^{-1} \min(a, b).$$

On the support of χ, we have

$$\begin{cases} |u(t)| = e^{\lambda(at + \gamma t^2/2)} |u(0)| \in [e^{-\lambda a|t|}, e^{-\frac{1}{2}\lambda a|t|}] |u(0)|, & \text{for } t \leq 0, \\ |u(t)| = e^{-\lambda(bt - \gamma t^2/2)} |u(0)| \in [e^{-\lambda b|t|}, e^{-\frac{1}{2}\lambda b|t|}] |u(0)|, & \text{for } t \geq 0. \end{cases}$$

As a result we have if $\chi = 1$ on $[-r, r]$, $u(0) = 1$,

$$\|\chi u\|_{L^2}^2 \ge |u(0)|^2 \int_{-r}^{r} (H(t)e^{-2b\lambda|t|} + H(-t)e^{-2a\lambda|t|})dt$$
$$= \frac{1-e^{-2a\lambda r}}{2a\lambda} + \frac{1-e^{-2b\lambda r}}{2b\lambda}. \quad (2.4.23)$$

We have also $\big(D_t - i\lambda f(t)\big)(\chi u) = -i\chi'(t)u(t)$ so that if χ is supported in $[-2r, 2r]$, we have

$$\begin{aligned}\|\big(D_t - i\lambda f(t)\big)(\chi u)\|_{L^2}^2 &= \int \chi'(t)^2 |u(t)|^2 dt \\ &\le \int_{r\le|t|\le 2r} \chi'(t)^2 \Big(H(-t)e^{-\lambda a|t|} + H(t)e^{-\lambda b|t|}\Big) dt \\ &\le \int_{r\le|t|\le 2r} \chi'(t)^2 \Big(H(-t)e^{-\lambda ar} + H(t)e^{-\lambda br}\Big) dt \\ &\le e^{-\lambda r \min(a,b)} \int_{r\le|t|\le 2r} \chi'(t)^2 dt. \quad (2.4.24)\end{aligned}$$

The estimates (2.4.23) and (2.4.24) make (2.4.1) impossible for $\lambda \to +\infty$: we would have

$$\begin{aligned}\frac{1-e^{-2a\lambda r}}{2a\lambda} + \frac{1-e^{-2b\lambda r}}{2b\lambda} &\le \|\chi u\|^2 \le C^2 \|(D_t - i\lambda f(t))(\chi u)\|^2 \\ &\le C^2 e^{-\lambda r \min(a,b)} \int_{r\le|t|\le 2r} \chi'(t)^2 dt,\end{aligned}$$

entailing

$$\frac{1-e^{-2a\lambda r}}{2a} + \frac{1-e^{-2b\lambda r}}{2b} \le \lambda e^{-\lambda r \min(a,b)} C^2 \int_{r\le|t|\le 2r} \chi'(t)^2 dt,$$

with a lhs with a positive limit when λ goes to $+\infty$ and a rhs with limit 0.

Since we can choose $r > 0$ as small as we like, Note that we have proven that there is no neighborhood V of 0 such that there exists $C > 0$ so that for all $u \in C_c^\infty(V)$, and all $\lambda \ge C$,

$$C\|D_t u - i\lambda f(t)u\|_{L^2} \ge \|u\|_{L^2}.$$

2.5 Open problems

2.5.1 BV elliptic matrix

The same questions can be asked for BV elliptic matrix: If we were to consider a more general framework in which the matrix $A(x)$, symmetric, positive-definite belongs to $BV(\Omega)\cap L^\infty(\Omega)$, and w is a Lipschitz continuous function on Ω the vector field Adw is in $L^\infty(\Omega)$: the second transmission condition reads in that framework $\operatorname{div}(Adw) \in L^\infty(\Omega)$. Proving a Carleman estimate in such a case is a wide open question.

2.5.2 Elliptic matrix with infinitely many jumps

However, there are simpler questions related to BV elliptic matrices: for instance take a sequence $(t_k)_{k\geq 1}$ of positive numbers strictly decreasing with limit 0. Consider the bounded real elliptic matrix

$$A(x) = B\mathbf{1}_{(-\infty,0)}(x_n) + \sum_{k\geq 1} A_k \mathbf{1}_{(t_{k+1},t_k)}(x_n) + A_0 \mathbf{1}_{(t_1,+\infty)}(x_n),$$

which has jumps on each hyperplane $\Sigma_k = \{x \in \mathbb{R}^n, x_n = t_k\}$ and at $\Sigma_0 = \{x \in \mathbb{R}^n, x_n = 0\}$. The matrix A belongs to BV with a differential

$$dA = \Big(-B\delta_0 + \sum_{k\geq 1}(A_{k-1} - A_k)\delta_0(x_n - t_k)\Big)dx_n.$$

The transmission conditions can be easily derived and the unique continuation problem is not obvious to solve: take u satisfying the transmission conditions, vanishing in some non-empty open subset of $\{x_n < 0\}$ satisfying a differential inequality

$$|\operatorname{div}(A\nabla u)| \leq C(|u| + |\nabla u|). \tag{2.5.1}$$

Using the ellipticity of B, we obtain easily that u should vanish on the whole half space $\{x_n < 0\}$. Now the main question is: does that imply that u is vanishing everywhere? Of course, to deal with these questions one should start with the present question, a priori much simpler than the previous one dealing with a general elliptic BV matrix.

2.5.3 Strong unique continuation

Staying in the framework of the present chapter with a single jump at a smooth hypersurface Σ, we may ask for a strong unique continuation property starting from a point of Σ. Assume (2.5.1) and u vanishing of infinite order at a point $x_0 \in \Sigma$, i.e.

$$\forall N \in \mathbb{N}, \quad \lim_{r\to 0_+} r^{-N} \int_{|x-x_0|\leq r} |u(x)|^p dx = 0,$$

for some $p \in [2, +\infty)$. Does that imply that u vanishes identically? Of course if the point x_0 is located outside Σ, the strong unique continuation property for Lipschitz second order real elliptic operators entails that u should vanish on one side of Σ and then by Cauchy uniqueness, we obtain the result.

If x_0 belongs to Σ, we probably need to prove a Carleman estimate with singular weights behaving like $|x-x_0|^{-\lambda}$ near the point x_0. However it is quite likely that the choice of the norm in that weight could not be isotropic and has to take into account the jump across Σ, introducing a specific singularity due to the jump. Anyhow, this problem is widely open.

Chapter 3

Conditional pseudo-convexity

Foreword. We explore here the notion of conditional pseudoconvexity of an hypersurface with respect to a differential operator. This notion was introduced in a series of papers by A. Ionescu & S. Klainerman ([14, 15]) and plays an important rôle in the proof of unique continuation properties for Lorentzian wave operators. We adopt here a phase space point of view and we provide a statement valid on a differentiable manifold not necessarily equipped with a Lorentzian structure.

3.1 Examples and counterexamples

3.1.1 The Alinhac-Baouendi counterexample

Let us consider the wave operator in 2-space dimension $\partial_t^2 - \partial_x^2 - \partial_y^2 = \Box$. There exists $V, u \in C^\infty(\mathbb{R}^3)$ with

$$\operatorname{supp} u = \{y \geq 0\}, \quad \Box u + Vu = 0. \tag{3.1.1}$$

This result and some generalizations were proven by S. Alinhac and S. Baouendi [2]. Note that this operator is with constant coefficients, so that the characteristics are straight lines and the tangential ones are included in the boundary $y = 0$. This problem is easily proven to be ill-posed since it is non hyperbolic with respect to the *timelike* hypersurface $y = 0$.

The construction of this counterexample is a highly non-trivial task and this result appears as the most significant counterexample to Cauchy uniqueness. We note in particular that this constant coefficient operator (also of real principal type) is locally solvable, which is not the case of P. Cohen's vector fields counterexamples (see e.g. Theorem 8.9.2) in [8]: typically the operator in two dimensions, $\partial_t + ib(t,x)\partial_x$ fails to satisfy Cauchy uniqueness with respect to $t = 0$ if $t \mapsto b(t,x)$ is highly oscillatory around 0; here also the construction is pretty involved but since the Nirenberg-Treves condition (P) is violated for this vector field, it is not locally solvable. So the non-uniqueness property in that case is somehow less interesting than for an operator having plenty of local solutions. The article [1] contains much more information on non-uniqueness results.

3.1.2 Hörmander-Tataru-Robbiano-Zuily's uniqueness result

A recurrent question about the counterexample (3.1.1) was for long time if such a phenomenon could hold if V does not depend on the time variable. A negative answer was given by D. Tataru's [35], L. Hörmander's [11], L. Robbiano & C. Zuily in [30] who proved uniqueness for $\square + V(t,x,y)$ with respect to $\{y=0\}$ when V is a smooth function depending analytically of the variable t. Several geometric statements are given in that series of articles which go much beyond this example.

3.2 Background

3.2.1 Cauchy uniqueness

Let $P = \sum_{|\alpha|\le m} a_\alpha(x) D_x^\alpha$ be a differential operator[1] and Σ be an oriented hypersurface of an open subset Ω of $\mathbb{R}^n$; we have in particular the partition of Ω

$$\Omega = \Sigma_- \cup \Sigma \cup \Sigma_+, \quad \Sigma_\pm \text{ open}, \quad \Sigma_\pm \cup \Sigma = \overline{\Sigma_\pm}.$$

We are interested in the uniqueness for the Cauchy problem for P across Σ in the following sense. We shall say that P has the stable Cauchy uniqueness property across the oriented Σ if the conditions

$$|(Pu)(x)| \le \sum_{|\alpha|\le m-1} |V_\alpha(x) D_x^\alpha u(x)|, \quad \text{on } \Omega \text{ for some } V_\alpha \in L^\infty_{loc}(\Omega), \tag{3.2.1}$$

$$u_{|\Sigma_-} = 0, \tag{3.2.2}$$

imply $u = 0$ in a neighborhood of Σ.

3.2.2 Pseudo-convexity

The principal symbol of P is defined on $\Omega \times \mathbb{R}^n$ by

$$p(x,\xi) = \sum_{|\alpha|=m} a_\alpha(x)\xi^\alpha.$$

We shall always assume that, if $p(x,\xi)=0$, there exists a neighborhood V of (x,ξ) in the sphere bundle, such that, on V, the following inequality holds:

$$\{\operatorname{Re} p, \operatorname{Im} p\} \ge -C|p|. \tag{3.2.3}$$

Note that this condition is trivially satisfied when the coefficients of P are real-valued (the case which interests us here anyway) and moreover that, since the polynomial $\{\operatorname{Re} p, \operatorname{Im} p\}$ (in the ξ variable) has odd degree $2m-1$, this inequality is equivalent to

$$|\{\operatorname{Re} p, \operatorname{Im} p\}(x,\xi)| \le C|p(x,\xi)||\xi|^{m-1}, \tag{3.2.4}$$

[1] We have used the standard notation $D_{x_j} = \frac{1}{i}\frac{\partial}{\partial x_j}$, $\quad D_x^\alpha = D_{x_1}^{\alpha_1}\dots D_{x_n}^{\alpha_n}$.

a property called principal normality. The hypersurface Σ is said to be strongly pseudoconvex at $x_0 \in \Sigma$ if, whenever φ is a defining function of the oriented Σ, $(\Sigma = \{x \in \Omega, \varphi(x) = 0\}, d\varphi \neq 0$ at Σ, $\Sigma_+ = \{x \in \Omega, \varphi(x) > 0\})$,

$$p(x_0, \overbrace{\xi_0 - i\tau_0 d\varphi(x_0)}^{\zeta_0}) = \frac{\partial p}{\partial \xi}(x_0, \zeta_0) \cdot d\varphi(x_0) = 0, \tag{3.2.5}$$

for some $\mathbb{R}^n \times \mathbb{R}_+ \ni (\xi_0, \tau_0) \neq (0,0)$ imply with $\zeta = \xi_0 - i\tau d\varphi(x_0)$ that

$$\lim_{\tau \to (\tau_0)_+} \frac{1}{\tau} \underbrace{\operatorname{Im}\left(\overline{\frac{\partial p}{\partial \xi}(x_0, \zeta)} \cdot \frac{\partial p}{\partial x}(x_0, \zeta)\right)}_{q} - \varphi''(x_0)\overline{\frac{\partial p}{\partial \xi}(x_0, \zeta_0)} \cdot \frac{\partial p}{\partial \xi}(x_0, \zeta_0) > 0. \tag{3.2.6}$$

Note that if $\tau_0 > 0$, the limit above is pointless and if $\tau_0 = 0$, the function $q(\tau)$ is vanishing at $\tau = 0$, thanks to (3.2.4) and the limit is simply $q'(0)$. In other words, one can rewrite (3.2.6) when $\tau_0 = 0$ as

$$\operatorname{Re}\{\bar{p}, \{p, \varphi\}\}(x_0, \xi_0) < 0. \tag{3.2.7}$$

Note also that this notion does not depend on the choice of the function φ with a non-vanishing gradient defining Σ: in the first place, the conditions (3.2.5) use only the conormal vector $N_0 = d\varphi(x_0)$ of Σ at x_0 and changing φ into $a\varphi$ with a positive function a, will give a term

$$(a\varphi'' + 2\nabla a \nabla \varphi + a''\varphi)\overline{\frac{\partial p}{\partial \xi}(x_0, \zeta_0)} \cdot \frac{\partial p}{\partial \xi}(x_0, \zeta_0)$$

and since $\varphi(x_0) = 0$ and $\nabla \varphi \frac{\partial p}{\partial \xi}(x_0, \zeta_0) = 0$, the second term in (3.2.6) is only multiplied by $a(x_0)$. Moreover $\zeta = \xi_0 - i\tau a(x_0) d\varphi(x_0)$ so that τ is also multiplied by $a(x_0)$, thus as well as the first term. The positivity of (3.2.6) is left unchanged.

3.2.3 Examples

Simple roots

Let us assume that the hypersurface Σ is non-characteristic for the operator P at x_0: with $N_0 = d\varphi(x_0)$, it means that $p(x_0, N_0) \neq 0$, where p is the principal symbol of P. Choosing the coordinate system such that the hypersurface Σ is the hyperplane $x_n = 0$, we get that

$$p(x', x_n; \xi', \xi_n) \qquad \text{(homogeneous polynomial of degree } m \text{ in } (\xi', \xi_n) \in \mathbb{R}^{n-1} \times \mathbb{R})$$

is a polynomial of degree m in the variable ξ_n. Now if the roots of $\zeta_n \mapsto p(0, 0; \xi', \zeta_n)$ are simple, i.e. if for $\mathbb{R}^{n-1} \ni \xi' \neq 0$,

$$p(0, 0; \xi', \zeta_n) = 0 \Longrightarrow \frac{\partial p}{\partial \xi_n}(0, 0; \xi', \zeta_n) \neq 0 \tag{3.2.8}$$

the pseudo-convexity hypothesis is satisfied since the situation (3.2.5) does not occur. This is the hypothesis used by A. Calderón (say for operators with real coefficients).

Second order real operators

If P is a second order operator with real coefficients in the principal part and Σ is non-characteristic with respect to P, the pseudo-convexity hypothesis means that for $\mathbb{R}^n \ni \xi \neq 0$

$$p(x_0, \xi) = \{p, \varphi\}(x_0, \xi) = 0 \Longrightarrow \{p, \{p, \varphi\}\}(x_0, \xi) < 0. \tag{3.2.9}$$

In fact non-real roots cannot be double since they occur in conjugate pair. In other words, Σ is "above" the tangential characteristics, a sort of convexity assumption. The integral curves of H_p in the phase space are the bicharacteristic curves and the characteristic curves are simply their first projection. The bicharacteristics are defined by

$$\dot{x}(t) = \frac{\partial p}{\partial \xi}(x(t), \xi(t)), \quad \dot{\xi}(t) = -\frac{\partial p}{\partial x}(x(t), \xi(t))$$

so that, calculating

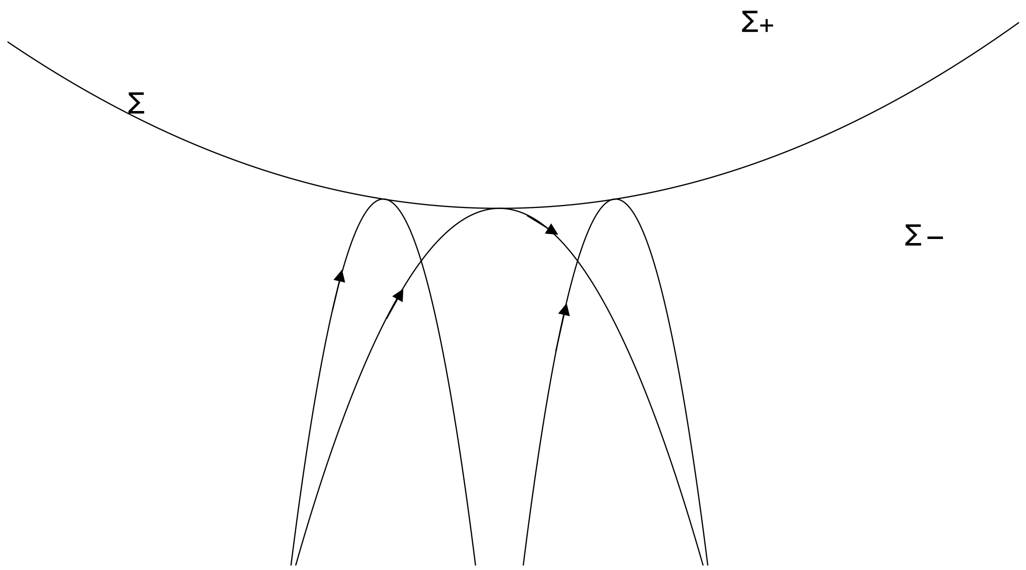


Figure 3.1: Pseudoconvex Σ with respect to the characteristic curves of P

$$\frac{d}{dt}\big(\varphi(x(t))\big) = H_p(\varphi)(x(t), \xi(t)), \quad \frac{d^2}{dt^2}\big(\varphi(x(t))\big) = H_p^2(\varphi)(x(t), \xi(t))$$

and with $\varphi(x_0) = H_p(\varphi)(x_0, \xi_0) = 0$, the pseudo-convexity condition is indeed $H_p^2(\varphi)(x_0, \xi_0) < 0$.

Constant coefficients

When P has constant coefficients and Σ non-characteristic with respect to P is given by the equation $x_n = f(x')$ with $f(0) = 0, f'(0) = 0$, the pseudoconvexity condition is

$$p(\xi', \xi_n - i\tau) = \frac{\partial p}{\partial \xi_n}(\xi', \xi_n - i\tau) = 0 \Longrightarrow f''(0)\overline{\frac{\partial p}{\partial \xi'}(\xi', \xi_n - i\tau)}\frac{\partial p}{\partial \xi'}(\xi', \xi_n - i\tau) > 0$$

and for principal type operators, this follows from the convexity of f, i.e. of Σ_+.

3.2.4 Uniqueness under pseudo-convexity

Theorem 3.2.1 (Calderón, Hörmander). *Let P be a principally normal differential operator with C^∞ coefficients (resp. an operator with Lipschitz-continuous real coefficients in the principal part) and Σ a strongly pseudo-convex C^2 hypersurface at x_0. Then there exists a neighborhood ω of x_0 such that P has the stable uniqueness for the Cauchy problem on ω with respect to the oriented Σ.*

This theorem was proven by Calderón for operators with real coefficients and simple roots, using a pseudodifferential factorization; as a matter of fact, the paper [4] was the starting point of microlocal methods in local analysis of PDE and it is somewhat paradoxical that L. Hörmander, who became one of the main architects of pseudodifferential operator theory, found a generalization of Calderón's result via a local method of proof, introducing the notion of pseudo-convexity.

The regularity issues are important, in particular for applications to quasi-linear equations. The most general notion (3.2.4) of principal normality given in Definition 28.2.4 of [10] (see also [20]) is useful only for non-real operators but seems to require the strength of Fefferman-Phong inequality, a method greedy with derivatives (the more restrictive notion of principal normality used in Definition 8.5.1 of [8] was using only C^2 regularity). However C^1 (and even Lipschitz continuity) is enough is the real case as well as in the elliptic case. Andrzej Pliś has shown in 1963 ([28]) that Hölder continuity (any index < 1) of the coefficients is not enough to get unique continuation for real second-order elliptic operators, a result precised later by K. Miller ([26]) and N. Filonov [7] with counterexamples in divergence form. We know thus that for real second-order operator, Lipschitz continuity is enough to get unique continuation under a pseudo-convexity hypothesis via a Carleman estimate, whereas Hölder continuity alone could ruin unique continuation. However for elliptic operators, coefficients jumping on a smooth hypersurface can be handled, and some Carleman estimate can be proven in that case (see [18] and the references therein).

For operators with smooth complex coefficients, principal normality plays an important rôle and can be seen as a strengthening of Nirenberg-Treves condition (P). In fact, a Carleman estimate will imply local solvability which is characterized by condition (P) for differential operators of principal type. The first counterexample to Cauchy uniqueness was found by P. Cohen (see e.g [31] and the references therein) and is a complex vector field violating condition (P) (and thus principal normality); in particular that vector field is not locally solvable[2]. On the contrary, the counterexample by Alinhac & Baouendi [2] is the wave-operator in two dimensions with respect to a timelike hypersurface: there exist smooth u, V in $\mathbb{R}^3$ such that

$$(\partial_t^2 - \partial_{x_1}^2 - \partial_{x_2}^2)u + V(t, x_1, x_2)u = 0, \quad \operatorname{supp} u = \{x_1 \geq 0\}.$$

Note that V is complex-valued and that feature is important in the construction (no counterexample is known for V real-valued for second-order operators). Of course pseudo-convexity is violated since the characteristics are straight lines and the tangent ones stay in the "initial" hypersurface $\{x_1 = 0\}$. On the other hand the paper [30] (see also the references therein) implies Cauchy uniqueness for the operator & hypersurface above as soon as V is analytic with respect to t.

3.3 Conditional pseudo-convexity

3.3.1 The result

Definition 3.3.1. Let Ω be an open subset of $\mathbb{R}^n$, P_2 be a second order differential operator with real Lipschitz-continuous coefficients and principal symbol p_2, Σ be a smooth hypersurface noncharacteristic with respect to P_2 and $x_0 \in \Sigma$. Let P_1 be a first-order differential operator with continuous coefficients and principal symbol p_1. We shall say that Σ is strongly pseudo-convex with respect to P_2 conditionally with respect to P_1 at x_0 if for all $\xi \in \mathbb{R}^n, \xi \neq 0$,

$$p_2(x_0, \xi) = \{p_2, \varphi\}(x_0, \xi) = p_1(x_0, \xi) = 0 \Longrightarrow H_{p_2}^2(\varphi)(x_0, \xi) < 0,$$

where $\Sigma_+ = \{x \in \Omega, \varphi(x) > 0\}$.

Theorem 3.3.2. *Let $\Omega, P_2, P_1, \Sigma, \varphi, x_0$ as in Definition 3.3.1. Then there exists a neighborhood V of x_0 and a neighborhood $\mathcal{V}$ of φ in $C^2(V)$ such that for any $\psi \in \mathcal{V}$, there exists a constant $C > 0$ such that for all $v \in C_c^\infty(V)$, all $\tau \geq C$,*

$$C\|e^{-\tau\psi}P_2 v\|_{L^2} + C\tau^{1/2}\|e^{-\tau\psi}P_1 v\|_{L^2} \geq \tau^{3/2}\|e^{-\tau\psi}v\|_{L^2} + \tau^{1/2}\|e^{-\tau\psi}\nabla v\|_{L^2}.$$

Corollary 3.3.3. *Let $\Omega, P_2, P_1, \Sigma, \varphi, x_0$ as in Definition 3.3.1. Then there exists a neighborhood W of x_0 such that if on W*

$$|(P_2 u)(x)| \leq |V_0(x)u(x)| + |V_1(x)\nabla u(x)|, \quad |(P_1 u)(x)| \leq |V_0(x)u(x)|$$

with $V_j \in L^\infty_{loc}(\Omega)$ and if $\operatorname{supp} u \subset \Sigma_+$, then $u = 0$ on W.

[2]Paul Cohen's achievement in finding a smooth vector field without Cauchy uniqueness with respect to a non-characteristic hypersurface remains a landmark in the history of mathematics and we leave to the reader the philosophical question about the relevance of uniqueness for operators without (much) solutions.

3.3.2 A more general result

Definition 3.3.4. Let Ω be an open subset of $\mathbb{R}^n$, P_m be a differential operator of order m with C^∞ coefficients, principally normal (i.e. satisfying (3.2.4)) with principal symbol p_m, Σ be a smooth hypersurface and $x_0 \in \Sigma$. Let P_{m-1} be a differential operator with C^∞ coefficients, of order $m-1$ with principal symbol p_{m-1}.

We shall say that Σ is strongly pseudo-convex with respect to P_m conditionally with respect to P_{m-1} at x_0 if for all $\xi \in \mathbb{R}^n, \xi \neq 0$ such that $p_{m-1}(x_0, \xi) = 0$, (3.2.5) implies (3.2.6) where $\Sigma_+ = \{x \in \Omega, \varphi(x) > 0\}$.

Theorem 3.3.5. *Let $\Omega, P_m, P_{m-1}, \Sigma, \varphi, x_0$ as in Definition 3.3.4. Then there exists a neighborhood V of x_0 and a neighborhood $\mathcal{V}$ of φ in $C^2(V)$ such that for any $\psi \in \mathcal{V}$, there exists a constant $C > 0$ such that for all $v \in C_c^\infty(V)$, all $\tau \geq C$,*

$$C\|e^{-\tau\psi}P_m v\|_{L^2} + C\tau^{1/2}\|e^{-\tau\psi}P_{m-1}v\|_{L^2} \geq \sum_{0\leq j\leq m-1} \tau^{m-j-\frac{1}{2}}\|e^{-\tau\psi}\nabla^j v\|_{L^2}.$$

Corollary 3.3.6. *Let $\Omega, P_m, P_1, \Sigma, \varphi, x_0$ as in Definition 3.3.4. Then there exists a neighborhood W of x_0 such that if on W*

$$|(P_m u)(x)| \leq \sum_{0\leq j\leq m-1} |V_j(x)\nabla^j u(x)|, \quad |(P_{m-1}u)(x)| \leq \sum_{0\leq j\leq m-2} |V_j(x)\nabla^j u(x)|,$$

with $V_j \in L^\infty_{loc}(\Omega)$ and if $\operatorname{supp} u \subset \Sigma_+$*, then $u = 0$ on W.*

3.4 Proofs

3.4.1 Proof of theorem 3.3.5

The most general result deals with operators with C^∞ coefficients and can be proven using Fefferman-Phong inequality, following the standard argument using pseudodifferential calculus with large parameter. We assume that Σ is strongly pseudo-convex with respect to P_m conditionally with respect to P_{m-1} at $x_0 \in \Sigma$. We introduce the weight function

$$\psi(x) = \varphi(x) - \frac{\mu}{2}\varphi(x)^2 + \frac{1}{2\mu^2}|x - x_0|^2$$

where φ is a defining function for the oriented Σ and μ is a positive parameter. We note that there exists a neighborhood Ω_μ of x_0 in Ω such that the level surface $\{x \in \Omega_\mu, x \neq x_0, \psi(x) = 0\}$ is included in $\Sigma_- \cap \Omega_\mu = \{x \in \Omega_\mu, \varphi(x) < 0\}$ and that for $b > 0$ small enough, there exists $a > 0$ such that

$$\{x, b \leq |x - x_0| \leq 2b\} \cap \Sigma_+ \subset \{x \in \Omega_\mu, \psi(x) \geq a\}.$$

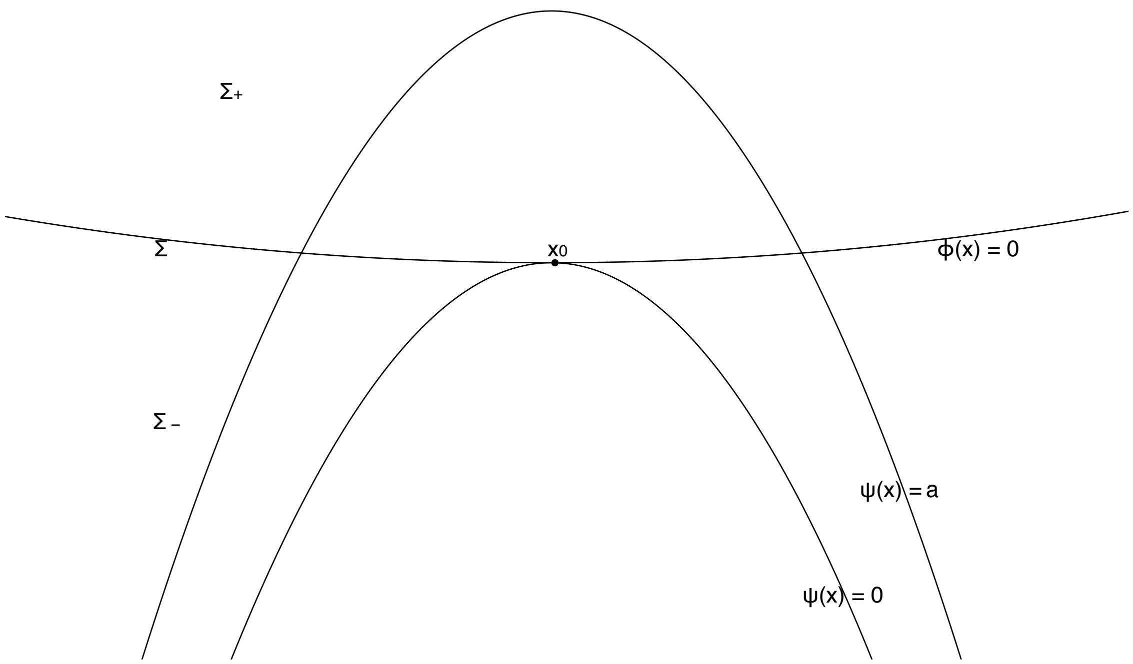


Figure 3.2: Convexification: level surfaces of the weight ψ and Σ $(a > 0)$.

We consider the symbol

$$a(x,\xi,\tau) = p_m(x,\xi - i\tau d\psi(x))$$

and assuming as we may that a is defined globally on $\mathbb{R}^n_x \times \mathbb{R}^n_\xi \times [1,+\infty)_\tau$, we note that we can as well assume that $a \in \mathcal{S}^m$, i.e.

$$|(\partial_x^\alpha \partial_\xi^\beta a)(x,\xi,\tau)| \leq C_{\alpha\beta}(1+|\xi|+\tau)^{m-|\beta|}. \tag{3.4.1}$$

Now, considering

$$e^{-\tau\psi} P_m e^{\tau\psi} = \sum_{|\alpha|=m} a_\alpha(x)(D_x - i\tau d\varphi(x))^\alpha,$$

we see that the Weyl symbol of $e^{-\tau\psi}P_m e^{\tau\psi}$ is equal to $p_m(x,\xi - i\tau d\psi(x))$ modulo $\mathcal{S}^{m-1}$. Similarly the Weyl symbol of $\tau^{1/2}e^{-\tau\psi}P_{m-1}e^{\tau\psi}$ is equal to

$$b(x,\xi,\tau) = \underbrace{\tau^{1/2}p_{m-1}(x,\xi - i\tau d\psi(x))}_{\in \mathcal{S}^{m-1/2}} \text{ modulo } \mathcal{S}^{m-3/2}.$$

We need to calculate ($a\sharp b$ is the composition of symbols, corresponding to the composition of operators)

$$\bar{a}\sharp a + \mu\bar{b}\sharp b \equiv |a|^2 + \frac{1}{2i}\{\bar{a}, a\} + \mu|b|^2 = c \mod \mathcal{S}^{2m-2}.$$

We have

$$c_{2m-1}(x,\xi,\tau) = |p_m(x, \underbrace{\xi - i\tau d\psi(x)}_{\zeta})|^2 + \mu\tau|p_{m-1}(x,\zeta)|^2$$
$$+ \operatorname{Im}\left(\overline{\frac{\partial p_m}{\partial \xi}(x,\zeta)} \cdot \frac{\partial p_m}{\partial x}(x,\zeta)\right) - \tau\psi''(x)\overline{\frac{\partial p_m}{\partial \xi}(x,\zeta)}\frac{\partial p_m}{\partial \xi}(x,\zeta),$$

so that using

$$\psi'' = \varphi'' - \mu\varphi'^2 - \mu\varphi\varphi'' + \mu^{-2},$$

we get

$$c_{2m-1} = |p_m(x,\zeta)|^2 + \mu\tau|p_{m-1}(x,\zeta)|^2 + \mu\tau|\{p_m,\varphi\}(x,\zeta)|^2$$
$$+ \operatorname{Im}\left(\overline{\frac{\partial p_m}{\partial \xi}(x,\zeta)} \cdot \frac{\partial p_m}{\partial x}(x,\zeta)\right) - \tau\varphi''(1-\mu\varphi)\overline{\frac{\partial p_m}{\partial \xi}(x,\zeta)}\frac{\partial p_m}{\partial \xi}(x,\zeta)$$
$$- \mu^{-2}\tau|\frac{\partial p_m}{\partial \xi}(x,\zeta)|^2. \quad (3.4.2)$$

Lemma 3.4.1. *There exists $\mu \ge 1$ such that for all (x,ξ,τ) with $|x-x_0| \le \mu^{-2}, \xi \in \mathbb{R}^n, \tau \ge \mu^3$,*

$$c_{2m-1}(x,\xi,\tau) \ge \mu^{-1}\tau(|\xi|^2+\tau^2)^{m-1},$$

with c_{2m-1} defined in (3.4.2).

Proof. Reductio ad absurdum: otherwise for all $k \ge 1$, we would find x_k, ξ_k, τ_k with $|x_k - x_0| \le k^{-2}, \xi_k \in \mathbb{R}^n, \tau_k \ge k^3$ so that

$$c_{2m-1}(x_k,\xi_k,\tau_k) < k^{-1}\tau_k(|\xi_k|^2+\tau_k^2)^{m-1}. \quad (3.4.3)$$

We note first that

$$\psi'(x_k) = \varphi'(x_k) - k\varphi(x_k)\varphi'(x_k) + k^{-2}(x_k - x_0),$$

and since $\varphi(x_0) = 0$, we have $\varphi(x_k) = O(k^{-2})$, we get that $\lim_k \psi'(x_k) = \varphi'(x_0) = N_0$ and we may assume that $|N_0| = 1$. On the other hand, we may assume by compactness (extracting a subsequence) that $\lim_k \frac{(\xi_k,\tau_k)}{(|\xi_k|^2+\tau_k^2)^{1/2}} = (\Xi_0, \sigma_0) \in \mathbb{S}^n$, so that with

$$\zeta_k = \xi_k - i\tau_k\psi'(x_k), \quad \lim_k\big(Z_k = \frac{\zeta_k}{|\zeta_k|}\big) = Z_0 = \Xi_0 - i\sigma_0 N_0, \quad \sigma_0 \ge 0, |\Xi_0|^2 + \sigma_0^2 = 1.$$

Multiplying the inequality (3.4.3) by $|\zeta_k|^{-2m}$, we obtain

$$|p_m(x_k, Z_k)|^2 + O(k|\zeta_k|^{-1}) \le O(k^{-1}|\zeta_k|^{-1}),$$

and since $|\zeta_k| \ge \tau_k \ge k^3$, this gives

$$p_m(x_0, Z_0) = 0. \quad (3.4.4)$$

Multiplying the inequality (3.4.3) by $k^{-1}\tau_k^{-1}|\zeta_k|^{2-2m}$, we obtain

$$\begin{aligned}|p_m(x_k,Z_k)|^2 k^{-1}\tau_k^{-1}|\zeta_k|^2 + |p_{m-1}(x_k,Z_k)|^2 + |\left\{p_m,\varphi\right\}(x_k,Z_k)|^2 \\ + k^{-1}\tau_k^{-1}|\zeta_k| q_{2m-1}(x_k,Z_k) \\ - k^{-1}\varphi''(x_k)\big(1-\underbrace{k\varphi(x_k)}_{O(k^{-1})}\big)\overline{\frac{\partial p_m}{\partial \xi}(x_k,Z_k)}\frac{\partial p_m}{\partial \xi}(x_k,Z_k) - k^{-3}|\frac{\partial p_m}{\partial \xi}(x_k,Z_k)|^2 \\ < k^{-2},\end{aligned}$$

with $q_{2m-1}(x,\zeta)=\operatorname{Im}\left(\overline{\frac{\partial p_m}{\partial \xi}(x,\zeta)}\cdot\frac{\partial p_m}{\partial x}(x,\zeta)\right)$ so that, with

$$Z_k=\Xi_k-i\sigma_k N_k,$$

$(\lim_k N_k=N_0, \lim_k Z_k=Z_0=\Xi_0-i\sigma_0 N_0)$

$$\begin{aligned}|p_m(x_k,Z_k)|^2 k^{-1}\tau_k^{-1}|\zeta_k|^2 + |p_{m-1}(x_k,Z_k)|^2 + |\left\{p_m,\varphi\right\}(x_k,Z_k)|^2 \\ + k^{-1}\sigma_k^{-1}q_{2m-1}(x_k,Z_k)\le O(k^{-1}).\end{aligned}$$

If $\sigma_0>0$, we get

$$p_{m-1}(x_0,Z_0)=0,\quad \left\{p_m,\varphi\right\}(x_0,Z_0)=0. \tag{3.4.5}$$

If $\sigma_0=0$, then $|\Xi_0|=1$ we know by the principal normality that near (x_0,Ξ_0) in $\mathbb{R}^n\times\mathbb{S}^{n-1}$,

$$q_{2m-1}(x,\Xi)\ge -C_0|p_m(x,\Xi)|$$

and thus

$$q_{2m-1}(x_k,Z_k)\ge -C_0|p_m(x_k,\Xi_k)|+O(\sigma_k)\ge -C_0|p_m(x_k,Z_k)|+O(\sigma_k),$$

implying

$$\begin{aligned}|p_m(x_k,Z_k)|^2 k^{-1}\sigma_k^{-1}|\zeta_k| + |p_{m-1}(x_k,Z_k)|^2 + |\left\{p_m,\varphi\right\}(x_k,Z_k)|^2 \\ - C_0 k^{-1}\sigma_k^{-1}|p_m(x_k,Z_k)|\le O(k^{-1}),\end{aligned}$$

so that

$$\begin{aligned}|p_m(x_k,Z_k)|^2 k^{-1}\sigma_k^{-1}|\zeta_k| + |p_{m-1}(x_k,Z_k)|^2 + |\left\{p_m,\varphi\right\}(x_k,Z_k)|^2 \\ -\frac{1}{2}k^{-1}\sigma_k^{-1}|p_m(x_k,Z_k)|^2|\zeta_k| - \frac{1}{2}C_0^2\underbrace{k^{-1}\sigma_k^{-1}|\zeta_k|^{-1}}_{=k^{-1}\tau_k^{-1}}\le O(k^{-1}),\end{aligned}$$

implying as well (3.4.5). From the hypothesis, we obtain

$$p_m(x_0,\underbrace{\Xi_0-i\sigma_0 N_0}_{Z_0})=\left\{p_m,\varphi\right\}(x_0,Z_0)=p_{m-1}(x_0,Z_0)=0\Longrightarrow (3.2.6)>0.$$

Multiplying the inequality (3.4.3) by $\tau_k^{-1}|\zeta_k|^{2-2m}$, we obtain

$$|p_m(x_k,Z_k)|^2\tau_k^{-1}|\zeta_k|^2+|p_{m-1}(x_k,Z_k)|^2+|\{p_m,\varphi\}(x_k,Z_k)|^2$$
$$+\sigma_k^{-1}q_{2m-1}(x_k,Z_k)-\varphi''(x_k)\overline{\frac{\partial p_m}{\partial\xi}(x_k,Z_k)}\frac{\partial p_m}{\partial\xi}(x_k,Z_k)\le O(k^{-1}),$$

which is impossible if $\sigma_0>0$ since the limit

$$\sigma_0^{-1}q_{2m-1}(x_0,Z_0)-\varphi''(x_0)\overline{\frac{\partial p_m}{\partial\xi}(x_0,Z_0)}\frac{\partial p_m}{\partial\xi}(x_0,Z_0)>0.$$

If $\sigma_0=0$, we get

$$|p_m(x_k,Z_k)|^2\tau_k^{-1}|\zeta_k|^2+|p_{m-1}(x_k,Z_k)|^2+|\{p_m,\varphi\}(x_k,Z_k)|^2$$
$$+\sigma_k^{-1}q_{2m-1}(x_k,Z_k)-\varphi''(x_k)\overline{\frac{\partial p_m}{\partial\xi}(x_k,Z_k)}\frac{\partial p_m}{\partial\xi}(x_k,Z_k)\le O(k^{-1}),$$

and we know also

$$q_{2m-1}(x_k,Z_k)=q_{2m-1}(x_k,\Xi_k)+\alpha_k\sigma_k+O(\sigma_k^2)\ge\alpha_k\sigma_k-C_0|p_m(x_k,\Xi_k)|+O(\sigma_k^2)$$

with

$$\lim_k\alpha_k-\varphi''(x_0)\overline{\frac{\partial p_m}{\partial\xi}(x_0,Z_0)}\frac{\partial p_m}{\partial\xi}(x_0,Z_0)>0. \tag{3.4.6}$$

To handle the next term, we note that

$$C_0\sigma_k^{-1}|p_m(x_k,\Xi_k)|\le\frac{1}{2}\sigma_k^{-1}|\zeta_k||p_m(x_k,\Xi_k)|^2+\frac{C_0^2}{2}\underbrace{\sigma_k^{-1}|\zeta_k|^{-1}}_{=\tau_k^{-1}},$$

and we obtain

$$\frac{1}{2}|p_m(x_k,Z_k)|^2\tau_k^{-1}|\zeta_k|^2+|p_{m-1}(x_k,Z_k)|^2+|\{p_m,\varphi\}(x_k,Z_k)|^2$$
$$+\alpha_k+O(\sigma_k)+O(\tau_k^{-1})-\varphi''(x_k)\overline{\frac{\partial p_m}{\partial\xi}(x_k,Z_k)}\frac{\partial p_m}{\partial\xi}(x_k,Z_k)\le O(k^{-1}),$$

which is impossible from (3.4.6) □

Lemma 3.4.2. *The operator c_{2m-1}^w with Weyl symbol $c(x,\xi,\tau)$ is such that*

$$c_{2m-1}^w-\frac{\tau}{\mu}(|D_x|^2+\tau^2)^{m-1}\ge-C(|D_x|^2+\tau^2)^{m-1}$$

when acting on functions supported near x_0 and μ is large enough.

Proof. A simple consequence of the Fefferman-Phong inequality. □

Theorem 3.3.5 is then an immediate consequence of the last lemma.

3.4.2 Less generality

The proof above may seem to have two different downsides.

First of all, who cares about complex coefficients and why do you make your life difficult with this principal normality business? Well, clearly complex coefficients are not useful for many applications but Paul Cohen's counterexample to Cauchy uniqueness as well as Hans Lewy's counterexample to local solvability where both vector fields with smooth complex-valued coefficients and it sounds worth while to understand the geometric defects explaining these pathologies.

Next, using the Fefferman-Phong inequality looks like an unnecessary refinement. This is probably true and anyhow for non-characteristic second-order operators with real coefficients, elementary proofs are already available (see e.g. [8], Theorems 8.3.1, 8.4.1); Lipschitz continuity should be sufficient.

3.4.3 Lorentzian geometry setting

We refer the reader to the appendix 4.5 for a reminder on Lorentzian geometry. The principal symbol of the wave operator is

$$p(x,\xi) = \langle \underbrace{g(x)^{-1}\xi}_{=X}, \xi\rangle = \big(X, X\big)_g, \tag{3.4.7}$$

and for a function ϕ of the variable x, we have

$$H_p(\phi) = \frac{\partial p}{\partial \xi} \cdot d\phi(x) = 2g(x)^{-1}\xi \cdot d\phi(x) = 2\big(g(x)^{-1}\xi, \nabla\phi\big)_g = 2\big(X, \nabla\phi\big)_g. \tag{3.4.8}$$

Moreover, we have

$$\begin{aligned}
H_p^2(\phi) &= 2g^{-1}(x)\xi \cdot \frac{\partial}{\partial x}\Big(2\big(g(x)^{-1}\xi, \nabla\phi\big)_g\Big) \\
&\qquad\qquad - \frac{\partial}{\partial x}\big(\langle g(x)^{-1}\xi, \xi\rangle\big) \cdot \frac{\partial}{\partial \xi}\Big(2g(x)^{-1}\xi \cdot d\phi(x)\Big) \\
&= 4D_X\Big(\big(g(x)^{-1}\xi, \nabla\phi\big)_g\Big) - 2\nabla\phi\big(\langle g(x)^{-1}\xi, \xi\rangle\big) \\
&= 4\big(D_X\nabla\phi, X\big)_g + 4\big(\nabla\phi, D_X g(x)^{-1}\xi\big)_g - 2D_{\nabla\phi}\big(\langle g(x)^{-1}\xi, \xi\rangle\big) \\
&= 4\big(D_X\nabla\phi, X\big)_g = 4(\nabla^2\phi)(X, X)
\end{aligned}$$

since $D_Y(g) = 0$. The pseudo-convexity hypothesis in this Lorentzian setting is thus

$$\forall X \neq 0, \quad \big(X, X\big)_g = \big(X, \nabla\rho\big)_g = 0 \Longrightarrow (\nabla^2\rho)(X, X) < 0. \tag{3.4.9}$$

Let's perform a coordinate-dependent calculation:

$$\begin{aligned}
H_p^2\phi &= \Big\{g^{-1}\xi^2, 2g^{jk}\xi_k\frac{\partial\phi}{\partial x^j}\Big\} \\
&= 4g^{-1}\xi \cdot \frac{\partial(g^{jk}\partial_j\phi)}{\partial x}\xi_k - 2\frac{\partial(g^{-1}\xi^2)}{\partial x} \cdot \frac{\partial(g^{jk}\partial_j\phi\xi_k)}{\partial \xi} \\
&= 4g^{lm}\xi_m\partial_l(g^{jk}\partial_j\phi)\xi_k - 2\partial_l(g^{pq})\xi_p\xi_q g^{jl}\partial_j\phi
\end{aligned}$$

$$
\begin{aligned}
&= \xi_k \xi_m \Big(4 g^{lm} \partial_l (g^{jk}) \partial_j \phi + 4 g^{lm} g^{jk} \partial_l \partial_j \phi - 2 \partial_l (g^{km}) g^{jl} \partial_j \phi \Big) \\
&= 4 g_{kp} X^p g_{mq} X^q \Big(g^{lm} \partial_l (g^{jk}) \partial_j \phi + g^{lm} g^{jk} \partial_l \partial_j \phi - \frac{1}{2} \partial_l (g^{km}) g^{jl} \partial_j \phi \Big) \\
&= 4 X^p X^q \Big(g_{kp} g_{mq} g^{lm} \partial_l (g^{jk}) \partial_j \phi + g_{kp} g_{mq} g^{lm} g^{jk} \partial_l \partial_j \phi - \frac{1}{2} g_{kp} g_{mq} \partial_l (g^{km}) g^{jl} \partial_j \phi \Big) \\
&= 4 X^p X^q \Big(g_{kp} \partial_q (g^{jk}) \partial_j \phi + g_{kp} g^{jk} \partial_q \partial_j \phi - \frac{1}{2} g_{kp} g_{mq} \partial_l (g^{km}) g^{jl} \partial_j \phi \Big) \\
&= 4 X^p X^q \Big(\partial_q \partial_p \phi + \big(g_{kp} \partial_q (g^{jk}) - \frac{1}{2} g_{kp} g_{mq} \partial_l (g^{km}) g^{jl} \big) \partial_j \phi \Big).
\end{aligned}
$$

We have in factor of $\partial_j \phi$

$$
\begin{aligned}
g_{kp} \partial_q (g^{jk}) - \frac{1}{2} g_{kp} g_{mq} \partial_l (g^{km}) g^{jl} &= -\partial_q (g_{kp}) g^{jk} + \frac{1}{2} g_{kp} \partial_l (g_{mq}) g^{km} g^{jl} \\
&= -\partial_q (g_{kp}) g^{jk} + \frac{1}{2} \partial_l (g_{pq}) g^{jl},
\end{aligned}
$$

so that

$$
\begin{aligned}
H_p^2(\phi) &= 4 X^p X^q \Big(\partial_q \partial_p \phi - \partial_j \phi \big[\partial_q (g_{kp}) g^{jk} - \frac{1}{2} \partial_k (g_{pq}) g^{jk} \big] \Big) \\
&= 4 X^p X^q \Big(\partial_q \partial_p \phi - (\partial_j \phi) g^{jk} \big[\frac{1}{2} \partial_p (g_{kq}) + \frac{1}{2} \partial_q (g_{kp}) - \frac{1}{2} \partial_k (g_{pq}) \big] \Big) \\
&= 4 X^p X^q \Big(\partial_q \partial_p \phi - (\partial_j \phi) \Gamma^j_{pq} \Big) = 4 (\nabla^2 \phi)(X, X), \quad \text{qed.}
\end{aligned}
$$

Chapter 4

Appendix

4.1 Fourier transformation

4.1.1 Fourier Transform of tempered distributions

The Fourier transformation on $\mathscr{S}(\mathbb{R}^n)$

Definition 4.1.1. Let $n \geq 1$ be an integer. The Schwartz space $\mathscr{S}(\mathbb{R}^n)$ is defined as the vector space of C^∞ functions u from $\mathbb{R}^n$ to $\mathbb{C}$ such that, for all multi-indices. $\alpha, \beta \in \mathbb{N}^n$,

$$\sup_{x \in \mathbb{R}^n} |x^\alpha \partial_x^\beta u(x)| < +\infty.$$

Here we have used the multi-index notation: for $\alpha = (\alpha_1, \dots, \alpha_n) \in \mathbb{N}^n$ we define

$$x^\alpha = x_1^{\alpha_1} \dots x_n^{\alpha_n}, \quad \partial_x^\alpha = \partial_{x_1}^{\alpha_1} \dots \partial_{x_n}^{\alpha_n}, \quad |\alpha| = \sum_{1 \leq j \leq n} \alpha_j. \tag{4.1.1}$$

A simple example of such a function is $e^{-|x|^2}$, ($|x|$ is the Euclidean norm of x) and more generally, if A is a symmetric positive definite $n \times n$ matrix, the function

$$v_A(x) = e^{-\pi \langle Ax, x \rangle} \tag{4.1.2}$$

belongs to the Schwartz class. The space $\mathscr{S}(\mathbb{R}^n)$ is a Fréchet space equipped with the countable family of semi-norms $(p_k)_{k \in \mathbb{N}}$

$$p_k(u) = \sup_{\substack{x \in \mathbb{R}^n \\ |\alpha|, |\beta| \leq k}} |x^\alpha \partial_x^\beta u(x)|. \tag{4.1.3}$$

Definition 4.1.2. For $u \in \mathscr{S}(\mathbb{R}^n)$, we define its Fourier transform $\hat{u}$ as

$$\hat{u}(\xi) = \int_{\mathbb{R}^n} e^{-2i\pi x \cdot \xi} u(x) dx. \tag{4.1.4}$$

Lemma 4.1.3. *The Fourier transform sends continuously $\mathscr{S}(\mathbb{R}^n)$ into itself.*

Proof. Just notice that

$$\xi^\alpha \partial_\xi^\beta \hat{u}(\xi) = \int e^{-2i\pi x \xi} \partial_x^\alpha (x^\beta u)(x) dx (2i\pi)^{|\beta|-|\alpha|} (-1)^{|\beta|},$$

and since $\sup_{x\in\mathbb{R}^n} (1+|x|)^{n+1} |\partial_x^\alpha (x^\beta u)(x)| < +\infty$, we get the result. □

Lemma 4.1.4. *For a symmetric positive definite $n \times n$ matrix A, we have*

$$\widehat{v_A}(\xi) = (\det A)^{-1/2} e^{-\pi \langle A^{-1}\xi, \xi\rangle}, \tag{4.1.5}$$

where v_A is given by (4.1.2).

Proof. In fact, diagonalizing the symmetric matrix A, it is enough to prove a one-dimensional version, i.e. to check

$$\int e^{-2i\pi x \xi} e^{-\pi x^2} dx = \int e^{-\pi (x+i\xi)^2} dx e^{-\pi \xi^2} = e^{-\pi \xi^2},$$

where the second equality is obtained by taking the ξ-derivative of $\int e^{-\pi(x+i\xi)^2} dx$: we have indeed

$$\frac{d}{d\xi}\Big(\int e^{-\pi(x+i\xi)^2} dx\Big) = \int e^{-\pi(x+i\xi)^2} (-2i\pi)(x+i\xi) dx$$
$$= -i \int \frac{d}{dx}\big(e^{-\pi(x+i\xi)^2}\big) dx = 0.$$

For $a > 0$, we obtain $\int_{\mathbb{R}} e^{-2i\pi x\xi} e^{-\pi a x^2} dx = a^{-1/2} e^{-\pi a^{-1} \xi^2}$, which is the sought result in one dimension. If $n \ge 2$, and A is a positive definite symmetric matrix, there exists an orthogonal $n \times n$ matrix P (i.e. ${}^tPP = \mathrm{Id}$) such that

$$D = {}^tPAP, \quad D = \operatorname{diag}(\lambda_1, \dots, \lambda_n), \text{ all } \lambda_j > 0.$$

As a consequence, we have, since $|\det P| = 1$,

$$\begin{aligned}
\int_{\mathbb{R}^n} e^{-2i\pi x\cdot\xi} e^{-\pi\langle Ax, x\rangle} dx &= \int_{\mathbb{R}^n} e^{-2i\pi (Py)\cdot\xi} e^{-\pi \langle APy, Py\rangle} dy \\
&= \int_{\mathbb{R}^n} e^{-2i\pi y\cdot({}^tP\xi)} e^{-\pi\langle Dy, y\rangle} dy \\
(\text{with } \eta = {}^tP\xi) \quad &= \prod_{1\le j\le n} \int_{\mathbb{R}} e^{-2i\pi y_j \eta_j} e^{-\pi \lambda_j y_j^2} dy_j = \prod_{1\le j \le n} \lambda_j^{-1/2} e^{-\pi \lambda_j^{-1} \eta_j^2} \\
&= (\det A)^{-1/2} e^{-\pi \langle D^{-1}\eta, \eta\rangle} = (\det A)^{-1/2} e^{-\pi \langle {}^tPA^{-1}P\ {}^tP\xi, {}^tP\xi\rangle} \\
&= (\det A)^{-1/2} e^{-\pi\langle A^{-1}\xi, \xi\rangle}.
\end{aligned}$$

□

Proposition 4.1.5. *The Fourier transformation is an isomorphism of the Schwartz class and for $u \in \mathscr{S}(\mathbb{R}^n)$, we have*

$$u(x) = \int e^{2i\pi x\xi} \hat{u}(\xi) d\xi. \tag{4.1.6}$$

Proof. Using (4.1.5) we calculate for $u \in \mathscr{S}(\mathbb{R}^n)$ and $\epsilon > 0$, dealing with absolutely converging integrals,

$$\begin{aligned} u_\epsilon(x) &= \int e^{2i\pi x\xi}\hat{u}(\xi)e^{-\pi\epsilon^2|\xi|^2}d\xi \\ &= \iint e^{2i\pi x\xi}e^{-\pi\epsilon^2|\xi|^2}u(y)e^{-2i\pi y\xi}dyd\xi \\ &= \int u(y)e^{-\pi\epsilon^{-2}|x-y|^2}\epsilon^{-n}dy \\ &= \int \underbrace{\big(u(x+\epsilon y)-u(x)\big)}_{\text{with absolute value}\le\epsilon|y|\|u'\|_{L^\infty}} e^{-\pi|y|^2}dy + u(x). \end{aligned}$$

Taking the limit when ϵ goes to zero, we get the Fourier inversion formula

$$u(x) = \int e^{2i\pi x\xi}\hat{u}(\xi)d\xi. \tag{4.1.7}$$

We have also proven for $u \in \mathscr{S}(\mathbb{R}^n)$ and $\check{u}(x) = u(-x)$

$$u = \check{\hat{\hat{u}}}. \tag{4.1.8}$$

Since $u \mapsto \hat{u}$ and $u \mapsto \check{u}$ are continuous homomorphisms of $\mathscr{S}(\mathbb{R}^n)$, this completes the proof of the proposition. □

Proposition 4.1.6. *Using the notation*

$$D_{x_j} = \frac{1}{2i\pi}\frac{\partial}{\partial x_j}, \quad D_x^\alpha = \prod_{j=1}^n D_{x_j}^{\alpha_j} \quad \textit{with } \alpha = (\alpha_1, \dots, \alpha_n) \in \mathbb{N}^n, \tag{4.1.9}$$

we have, for $u \in \mathscr{S}(\mathbb{R}^n)$

$$\widehat{D_x^\alpha u}(\xi) = \xi^\alpha \hat{u}(\xi), \qquad (D_\xi^\alpha \hat{u})(\xi) = (-1)^{|\alpha|}\widehat{x^\alpha u(x)}(\xi) \tag{4.1.10}$$

Proof. We have for $u \in \mathscr{S}(\mathbb{R}^n)$, $\hat{u}(\xi) = \int e^{-2i\pi x\cdot\xi}u(x)dx$ and thus

$$(D_\xi^\alpha \hat{u})(\xi) = (-1)^{|\alpha|}\int e^{-2i\pi x\cdot\xi}x^\alpha u(x)dx,$$

$$\xi^\alpha\hat{u}(\xi) = \int (-2i\pi)^{-|\alpha|}\partial_x^\alpha\big(e^{-2i\pi x\cdot\xi}\big)u(x)dx = \int e^{-2i\pi x\cdot\xi}(2i\pi)^{-|\alpha|}(\partial_x^\alpha u)(x)dx,$$

proving both formulas. □

N.B. The normalization factor $\frac{1}{2i\pi}$ leads to a simplification in Formula (4.1.10), but the most important aspect of these formulas is certainly that the Fourier transformation exchanges the operation of derivation with the operation of multiplication. For instance with

$$P(D) = \sum_{|\alpha|\le m} a_\alpha D_x^\alpha,$$

we have for $u \in \mathscr{S}(\mathbb{R}^n)$, $\widehat{Pu}(\xi) = \sum_{|\alpha|\le m} a_\alpha \xi^\alpha \hat{u}(\xi) = P(\xi)\hat{u}(\xi)$, and thus

$$(Pu)(x) = \int_{\mathbb{R}^n} e^{2i\pi x\cdot\xi} P(\xi)\hat{u}(\xi) d\xi. \tag{4.1.11}$$

Proposition 4.1.7. *Let ϕ, ψ be functions in $\mathscr{S}(\mathbb{R}^n)$. Then the convolution $\phi * \psi$ belongs to the Schwartz space and the mapping*

$$\mathscr{S}(\mathbb{R}^n) \times \mathscr{S}(\mathbb{R}^n) \ni (\phi, \psi) \mapsto \phi * \psi \in \mathscr{S}(\mathbb{R}^n)$$

is continuous. Moreover we have

$$\widehat{\phi * \psi} = \hat{\phi}\hat{\psi}. \tag{4.1.12}$$

Proof. The mapping $(x, y) \mapsto F(x, y) = \phi(x - y)\psi(y)$ belongs to $\mathscr{S}(\mathbb{R}^{2n})$ since x, y derivatives of the smooth function F are linear combinations of products $(\partial^\alpha\phi)(x - y)(\partial^\beta\psi)(y)$ and moreover

$$\begin{aligned}(1 + |x| + |y|)^N |(\partial^\alpha\phi)(x - y)(\partial^\beta\psi)(y)| \\ \le (1 + |x - y|)^N |(\partial^\alpha\phi)(x - y)| (1 + 2|y|)^N |(\partial^\beta\psi)(y)| \\ \le p(\phi)q(\psi),\end{aligned}$$

where p, q are semi-norms on $\mathscr{S}(\mathbb{R}^n)$. This proves that the bilinear mapping $(\phi, \psi) \mapsto F(\phi, \psi)$ is continuous from $\mathscr{S}(\mathbb{R}^n) \times \mathscr{S}(\mathbb{R}^n)$ into $\mathscr{S}(\mathbb{R}^{2n})$. We have now directly $\partial_x^\alpha(\phi * \psi) = (\partial_x^\alpha\phi) * \psi$ and

$$\begin{aligned}(1 + |x|)^N |\partial_x^\alpha(\phi * \psi)| &\le \int |F(\partial^\alpha\phi, \psi)(x, y)|(1 + |x|)^N dy \\ &\le \int \underbrace{|F(\partial^\alpha\phi, \psi)(x, y)|(1 + |x|)^N (1 + |y|)^{n+1}}_{\le p(F)} (1 + |y|)^{-n-1} dy,\end{aligned}$$

where p is a semi-norm of F (thus bounded by a product of semi-norms of ϕ and ψ), proving the continuity property. Also we obtain from Fubini's Theorem

$$(\widehat{\phi * \psi})(\xi) = \iint e^{-2i\pi(x-y)\cdot\xi} e^{-2i\pi y\cdot\xi} \phi(x - y)\psi(y) dy dx = \hat{\phi}(\xi)\hat{\psi}(\xi),$$

completing the proof of the proposition. □

The Fourier transformation on $\mathscr{S}'(\mathbb{R}^n)$

Definition 4.1.8. Let n be an integer ≥ 1. We define the space $\mathscr{S}'(\mathbb{R}^n)$ as the topological dual of the Fréchet space $\mathscr{S}(\mathbb{R}^n)$: this space is called the space of *tempered distributions* on $\mathbb{R}^n$.

We note that the mapping

$$\mathscr{S}(\mathbb{R}^n) \ni \phi \mapsto \frac{\partial \phi}{\partial x_j} \in \mathscr{S}(\mathbb{R}^n),$$

is continuous since for all $k \in \mathbb{N}$, $p_k(\partial\phi/\partial x_j) \le p_{k+1}(\phi)$, where the semi-norms p_k are defined in (4.1.3). This property allows us to define by duality the derivative of a tempered distribution.

Definition 4.1.9. Let $u \in \mathscr{S}'(\mathbb{R}^n)$. We define $\partial u/\partial x_j$ as an element of $\mathscr{S}'(\mathbb{R}^n)$ by

$$\langle \frac{\partial u}{\partial x_j}, \phi \rangle_{\mathscr{S}', \mathscr{S}} = -\langle u, \frac{\partial \phi}{\partial x_j} \rangle_{\mathscr{S}', \mathscr{S}}. \tag{4.1.13}$$

The mapping $u \mapsto \partial u/\partial x_j$ is a well-defined endomorphism of $\mathscr{S}'(\mathbb{R}^n)$ since the estimates

$$\forall \phi \in \mathscr{S}(\mathbb{R}^n), \quad |\langle \frac{\partial u}{\partial x_j}, \phi \rangle| \le C_u p_{k_u}(\frac{\partial \phi}{\partial x_j}) \le C_u p_{k_u+1}(\phi),$$

ensure the continuity on $\mathscr{S}(\mathbb{R}^n)$ of the linear form $\partial u/\partial x_j$.

Definition 4.1.10. Let $u \in \mathscr{S}'(\mathbb{R}^n)$ and let P be a polynomial in n variables with complex coefficients. We define the product Pu as an element of $\mathscr{S}'(\mathbb{R}^n)$ by

$$\langle Pu, \phi \rangle_{\mathscr{S}', \mathscr{S}} = \langle u, P\phi \rangle_{\mathscr{S}', \mathscr{S}}. \tag{4.1.14}$$

The mapping $u \mapsto Pu$ is a well-defined endomorphism of $\mathscr{S}'(\mathbb{R}^n)$ since the estimates

$$\forall \phi \in \mathscr{S}(\mathbb{R}^n), \quad |\langle Pu, \phi \rangle| \le C_u p_{k_u}(P\phi) \le C_u p_{k_u+D}(\phi),$$

where D is the degree of P, ensure the continuity on $\mathscr{S}(\mathbb{R}^n)$ of the linear form Pu.

Lemma 4.1.11. *Let Ω be an open subset of $\mathbb{R}^n$, $f \in L^1_{loc}(\Omega)$ such that, for all $\varphi \in C_c^\infty(\Omega)$, $\int f(x)\varphi(x)dx = 0$. Then we have $f = 0$.*

Proof. Let K be a compact subset of Ω and let $\chi \in C_c^\infty(\Omega)$ equal to 1 on a neighborhood of K. With $\rho \in C_c^\infty$ with integral 1, we get that

$$\lim_{\epsilon \to 0_+} \rho_\epsilon * (\chi f) = \chi f \quad \text{in } L^1(\mathbb{R}^n).$$

We have $\big(\rho_\epsilon * (\chi f)\big)(x) = \displaystyle\int f(y) \underbrace{\chi(y)\rho\big((x-y)\epsilon^{-1}\big)\epsilon^{-n}}_{=\varphi_x(y)} dy$, with $\operatorname{supp}\varphi_x \subset \operatorname{supp}\chi$, $\varphi_x \in C_c^\infty(\Omega)$, and from the assumption of the lemma, we obtain $\big(\rho_\epsilon * (\chi f)\big)(x) = 0$ for all x, implying $\chi f = 0$ from the convergence result and thus $f = 0$, a.e. on K; the conclusion of the lemma follows since Ω is a countable union of compact sets. □

Definition 4.1.12 (support of a distribution)**.** For $u \in \mathscr{S}'(\mathbb{R}^n)$, we define the support of u and we note $\operatorname{supp} u$ the closed subset of $\mathbb{R}^n$ defined by

$$(\operatorname{supp} u)^c = \{x \in \mathbb{R}^n, \exists V \text{ open} \in \mathscr{V}_x, \quad u_{|V} = 0\}, \tag{4.1.15}$$

where $\mathscr{V}_x$ stands for the set of neighborhoods of x and $u_{|V} = 0$ means that for all $\phi \in C_c^\infty(V)$, $\langle u, \phi \rangle = 0$.

Proposition 4.1.13.
(1) *We have $\mathscr{S}'(\mathbb{R}^n) \supset \cup_{1 \le p \le +\infty} L^p(\mathbb{R}^n)$, with a continuous injection of each $L^p(\mathbb{R}^n)$ into $\mathscr{S}'(\mathbb{R}^n)$. As a consequence $\mathscr{S}'(\mathbb{R}^n)$ contains as well all the derivatives in the sense (4.1.13) of all the functions in some $L^p(\mathbb{R}^n)$.*
(2) *For $u \in C^1(\mathbb{R}^n)$ such that*

$$\big(|u(x)| + |du(x)|\big)(1 + |x|)^{-N} \in L^1(\mathbb{R}^n), \tag{4.1.16}$$

for some non-negative N, the derivative in the sense (4.1.13) coincides with the ordinary derivative.

Proof. (1) For $u \in L^p(\mathbb{R}^n)$ and $\phi \in \mathscr{S}(\mathbb{R}^n)$, we can define

$$\langle u, \phi \rangle_{\mathscr{S}', \mathscr{S}} = \int_{\mathbb{R}^n} u(x)\phi(x)dx, \tag{4.1.17}$$

which is a continuous linear form on $\mathscr{S}(\mathbb{R}^n)$:

$$|\langle u, \phi \rangle_{\mathscr{S}', \mathscr{S}}| \le \|u\|_{L^p(\mathbb{R}^n)} \|\phi\|_{L^{p'}(\mathbb{R}^n)},$$

$$\|\phi\|_{L^{p'}(\mathbb{R}^n)} \le \sup_{x \in \mathbb{R}^n} \big((1 + |x|)^{\frac{n+1}{p'}} |\phi(x)|\big) C_{n,p} \le C_{n,p} p_k(\phi), \text{ for } k \ge k_{n,p} = \frac{n+1}{p'},$$

with p_k given by (4.1.3) (when $p = 1$, we can take $k = 0$). We indeed have a continuous injection of $L^p(\mathbb{R}^n)$ into $\mathscr{S}'(\mathbb{R}^n)$: in the first place the mapping described above is well-defined and continuous from the estimate

$$|\langle u, \phi \rangle| \le \|u\|_{L^p} C_{n,p} p_{k_{n,p}}(\phi).$$

Moreover, this mapping is linear and injective from Lemma 4.1.11.

(2) We have for $\phi \in \mathscr{S}(\mathbb{R}^n)$, $\chi_0 \in C_c^\infty(\mathbb{R}^n)$, $\chi_0 = 1$ near the origin,

$$A = \langle \frac{\partial u}{\partial x_j}, \phi \rangle_{\mathscr{S}', \mathscr{S}} = -\langle u, \frac{\partial \phi}{\partial x_j} \rangle_{\mathscr{S}', \mathscr{S}} = -\int_{\mathbb{R}^n} u(x) \frac{\partial \phi}{\partial x_j}(x) dx,$$

so that, using Lebesgue's dominated convergence theorem, we find

$$A = -\lim_{\epsilon \to 0_+} \int_{\mathbb{R}^n} u(x) \frac{\partial \phi}{\partial x_j}(x) \chi_0(\epsilon x) dx.$$

Performing an integration by parts on C^1 functions with compact support, we get

$$A = \lim_{\epsilon \to 0_+} \Big\{ \int_{\mathbb{R}^n} (\partial_j u)(x) \phi(x) \chi_0(\epsilon x) dx + \epsilon \int_{\mathbb{R}^n} u(x) \phi(x) (\partial_j \chi_0)(\epsilon x) dx \Big\},$$

with $\partial_j u$ standing for the ordinary derivative. We have also

$$\int_{\mathbb{R}^n} |u(x)\phi(x)(\partial_j\chi_0)(\epsilon x)|dx \le \|\partial_j\chi_0)\|_{L^\infty(\mathbb{R}^n)} \int |u(x)|(1+|x|)^{-N}dx\ p_N(\phi) < +\infty,$$

so that

$$\langle\frac{\partial u}{\partial x_j}, \phi\rangle_{\mathscr{S}',\mathscr{S}} = \lim_{\epsilon\to 0_+}\int_{\mathbb{R}^n}(\partial_j u)(x)\phi(x)\chi_0(\epsilon x)dx.$$

Since the lhs is a continuous linear form on $\mathscr{S}(\mathbb{R}^n)$ so is the rhs. On the other hand for $\phi \in C_c^\infty(\mathbb{R}^n)$, the rhs is $\int_{\mathbb{R}^n}(\partial_j u)(x)\phi(x)dx$. Since $C_c^\infty(\mathbb{R}^n)$ is dense in $\mathscr{S}(\mathbb{R}^n)$ (Exercise), we find that

$$\langle\frac{\partial u}{\partial x_j}, \phi\rangle_{\mathscr{S}',\mathscr{S}} = \int_{\mathbb{R}^n}(\partial_j u)(x)\phi(x)dx,$$

since the mapping $\phi \mapsto \int_{\mathbb{R}^n}(\partial_j u)(x)\phi(x)dx$ belongs to $\mathscr{S}'(\mathbb{R}^n)$, thanks to the assumption on du in (4.1.16). This proves that $\frac{\partial u}{\partial x_j} = \partial_j u$. □

The Fourier transformation can be extended to $\mathscr{S}'(\mathbb{R}^n)$. We start with noticing that for T, ϕ in the Schwartz class we have, using Fubini Theorem,

$$\int \hat{T}(\xi)\phi(\xi)d\xi = \iint T(x)\phi(\xi)e^{-2i\pi x\cdot\xi}dxd\xi = \int T(x)\hat{\phi}(x)dx,$$

and we can use the latter formula as a definition.

Definition 4.1.14. Let T be a tempered distribution ; the Fourier transform $\hat{T}$ of T is the tempered distribution defined by the formula

$$\langle\hat{T}, \varphi\rangle_{\mathscr{S}',\mathscr{S}} = \langle T, \hat{\varphi}\rangle_{\mathscr{S}',\mathscr{S}}. \tag{4.1.18}$$

The linear form $\hat{T}$ is obviously a tempered distribution since the Fourier transformation is continuous on $\mathscr{S}$. Thanks to Lemma 4.1.11, if $T \in \mathscr{S}$, the present definition of $\hat{T}$ and (4.1.4) coincide.

This definition gives that, with δ_0 standing as the Dirac mass at 0, $\langle\delta_0, \phi\rangle_{\mathscr{S}',\mathscr{S}} = \phi(0)$ (obviously a tempered distribution), we have

$$\widehat{\delta_0} = 1, \tag{4.1.19}$$

since $\langle\widehat{\delta_0}, \varphi\rangle = \langle\delta_0, \widehat{\varphi}\rangle = \widehat{\varphi}(0) = \int\varphi(x)dx = \langle 1, \varphi\rangle$.

Theorem 4.1.15. *The Fourier transformation is an isomorphism of $\mathscr{S}'(\mathbb{R}^n)$. Let T be a tempered distribution. Then we have*[1]

$$T = \check{\hat{\hat{T}}}, \quad \check{\hat{T}} = \hat{\check{T}}. \tag{4.1.20}$$

With obvious notations, we have the following extensions of (4.1.10),

$$\widehat{D_x^\alpha T}(\xi) = \xi^\alpha\hat{T}(\xi), \qquad (D_\xi^\alpha\hat{T})(\xi) = (-1)^{|\alpha|}\widehat{x^\alpha T(x)}(\xi). \tag{4.1.21}$$

[1] We define $\check{T}$ as the distribution given by $\langle\check{T}, \varphi\rangle = \langle T, \check{\varphi}\rangle$ and if $T \in \mathscr{S}'$, $\check{T}$ is also a tempered distribution since $\varphi \mapsto \check{\varphi}$ is an involutive isomorphism of $\mathscr{S}$.

Proof. We have for $T \in \mathscr{S}'$

$$\langle \check{\hat{\hat{T}}}, \varphi \rangle_{\mathscr{S}',\mathscr{S}} = \langle \hat{\hat{T}}, \check{\varphi} \rangle_{\mathscr{S}',\mathscr{S}} = \langle \hat{T}, \hat{\check{\varphi}} \rangle_{\mathscr{S}',\mathscr{S}} = \langle T, \hat{\hat{\check{\varphi}}} \rangle_{\mathscr{S}',\mathscr{S}} = \langle T, \varphi \rangle_{\mathscr{S}',\mathscr{S}},$$

where the last equality is due to the fact that $\varphi \mapsto \check{\varphi}$ commutes[2] with the Fourier transform and (4.1.7) means

$$\check{\hat{\hat{\varphi}}} = \varphi,$$

a formula also proven true on $\mathscr{S}'$ by the previous line of equality. Formula (4.1.10) is true as well for $T \in \mathscr{S}'$ since, with $\varphi \in \mathscr{S}$ and $\varphi_\alpha(\xi) = \xi^\alpha \varphi(\xi)$, we have

$$\langle \widehat{D^\alpha T}, \varphi \rangle_{\mathscr{S}',\mathscr{S}} = \langle T, (-1)^{|\alpha|} D^\alpha \hat{\varphi} \rangle_{\mathscr{S}',\mathscr{S}} = \langle T, \widehat{\varphi_\alpha} \rangle_{\mathscr{S}',\mathscr{S}} = \langle \hat{T}, \varphi_\alpha \rangle_{\mathscr{S}',\mathscr{S}},$$

and the other part is proven the same way. □

4.1.2 The Fourier transformation on $L^1(\mathbb{R}^n)$ and $L^2(\mathbb{R}^n)$

Theorem 4.1.16. *The Fourier transformation is linear continuous from $L^1(\mathbb{R}^n)$ into $L^\infty(\mathbb{R}^n)$ and for $u \in L^1(\mathbb{R}^n)$, we have*

$$\hat{u}(\xi) = \int e^{-2i\pi x \cdot \xi} u(x) dx, \quad \|\hat{u}\|_{L^\infty(\mathbb{R}^n)} \le \|u\|_{L^1(\mathbb{R}^n)}. \tag{4.1.22}$$

Proof. Formula (4.1.4) can be used to define directly the Fourier transform of a function in $L^1(\mathbb{R}^n)$ and this gives an $L^\infty(\mathbb{R}^n)$ function which coincides with the Fourier transform: for a test function $\varphi \in \mathscr{S}(\mathbb{R}^n)$, and $u \in L^1(\mathbb{R}^n)$, we have by the definition (4.1.18) above and Fubini theorem

$$\langle \hat{u}, \varphi \rangle_{\mathscr{S}',\mathscr{S}} = \int u(x) \hat{\varphi}(x) dx = \iint u(x) \varphi(\xi) e^{-2i\pi x \cdot \xi} dx d\xi = \int \widetilde{u}(\xi) \varphi(\xi) d\xi$$

with $\widetilde{u}(\xi) = \int e^{-2i\pi x \cdot \xi} u(x) dx$ which is thus the Fourier transform of u. □

Theorem 4.1.17 (Plancherel formula)**.**
The Fourier transformation can be extended to a unitary operator of $L^2(\mathbb{R}^n)$, i.e. there exists a unique bounded linear operator $F : L^2(\mathbb{R}^n) \longrightarrow L^2(\mathbb{R}^n)$, such that for $u \in \mathscr{S}(\mathbb{R}^n)$, $Fu = \hat{u}$ and we have $F^*F = FF^* = \mathrm{Id}_{L^2(\mathbb{R}^n)}$. Moreover

$$F^* = CF = FC, \quad F^2 C = \mathrm{Id}_{L^2(\mathbb{R}^n)}, \tag{4.1.23}$$

where C is the involutive isomorphism of $L^2(\mathbb{R}^n)$ defined by $(Cu)(x) = u(-x)$. This gives the Plancherel formula: for $u, v \in L^2(\mathbb{R}^n)$,

$$\int_{\mathbb{R}^n} \hat{u}(\xi) \overline{\hat{v}(\xi)} d\xi = \int u(x) \overline{v(x)} dx. \tag{4.1.24}$$

[2] If $\varphi \in \mathscr{S}$, we have $\widehat{\check{\varphi}}(\xi) = \int e^{-2i\pi x \cdot \xi} \varphi(-x) dx = \int e^{2i\pi x \cdot \xi} \varphi(x) dx = \hat{\varphi}(-\xi) = \check{\hat{\varphi}}(\xi)$.

Proof. For test functions $\varphi, \psi \in \mathscr{S}(\mathbb{R}^n)$, using Fubini theorem and (4.1.7), we get[3]

$$(\hat{\psi}, \hat{\varphi})_{L^2(\mathbb{R}^n)} = \int \hat{\psi}(\xi)\overline{\hat{\varphi}(\xi)}d\xi = \iint \hat{\psi}(\xi)e^{2i\pi x\cdot\xi}\overline{\varphi(x)}dxd\xi = (\psi, \varphi)_{L^2(\mathbb{R}^n)}.$$

Next, the density of $\mathscr{S}$ in L^2 shows that there is a unique continuous extension F of the Fourier transform to L^2 and that extension is an isometric operator (i.e. satisfying for all $u \in L^2(\mathbb{R}^n), \|Fu\|_{L^2} = \|u\|_{L^2}$, i.e. $F^*F = \mathrm{Id}_{L^2}$). We note that the operator C defined by $Cu = \check{u}$ is an involutive isomorphism of $L^2(\mathbb{R}^n)$ and that for $u \in \mathscr{S}(\mathbb{R}^n)$,

$$CF^2u = u = FCFu = F^2Cu.$$

By the density of $\mathscr{S}(\mathbb{R}^n)$ in $L^2(\mathbb{R}^n)$, the bounded operators

$$CF^2,\ \mathrm{Id}_{L^2(\mathbb{R}^n)},\ FCF,\ F^2C,$$

are all equal. On the other hand for $u, \varphi \in \mathscr{S}(\mathbb{R}^n)$, we have

$$(F^*u, \varphi)_{L^2} = (u, F\varphi)_{L^2} = \int u(x)\overline{\hat{\varphi}(x)}dx$$
$$= \iint u(x)\bar{\varphi}(\xi)e^{2i\pi x\cdot\xi}dxd\xi = (CFu, \varphi)_{L^2},$$

so that $F^*u = CFu$ for all $u \in \mathscr{S}$ and by continuity $F^* = CF$ as bounded operators on $L^2(\mathbb{R}^n)$, thus $FF^* = FCF = \mathrm{Id}$. The proof is complete. □

4.1.3 Some standard examples of Fourier transform

Let us consider the Heaviside function defined on $\mathbb{R}$ by $H(x) = 1$ for $x > 0$, $H(x) = 0$ for $x \le 0$; as a bounded measurable function, it is a tempered distribution, so that we can compute its Fourier transform. With the notation of this section, we have, with δ_0 the Dirac mass at 0, $\check{H}(x) = H(-x)$,

$$\widehat{H} + \widehat{\check{H}} = \hat{1} = \delta_0, \quad \widehat{H} - \widehat{\check{H}} = \widehat{\mathrm{sign}}, \qquad \frac{1}{i\pi} = \frac{1}{2i\pi}2\widehat{\delta_0}(\xi) = \widehat{D\,\mathrm{sign}}(\xi) = \xi\widehat{\mathrm{sign}}\xi.$$

We note that $\mathbb{R} \mapsto \ln|x|$ belongs to $\mathscr{S}'(\mathbb{R})$ and[4] we define the so-called principal value of $1/x$ on $\mathbb{R}$ by

$$\mathrm{pv}\big(\frac{1}{x}\big) = \frac{d}{dx}(\ln|x|), \tag{4.1.25}$$

$$\text{so that, } \langle \mathrm{pv}\,\frac{1}{x}, \phi\rangle = -\int \phi'(x)\ln|x|dx = -\lim_{\epsilon\to 0_+}\int_{|x|\ge\epsilon}\phi'(x)\ln|x|dx$$
$$= \lim_{\epsilon\to 0_+}\Big(\int_{|x|\ge\epsilon}\phi(x)\frac{1}{x}dx + \underbrace{\big(\phi(\epsilon) - \phi(-\epsilon)\big)\ln\epsilon}_{\to 0}\Big)$$
$$= \lim_{\epsilon\to 0_+}\int_{|x|\ge\epsilon}\phi(x)\frac{1}{x}dx. \tag{4.1.26}$$

[3]We have to pay attention to the fact that the scalar product $(u, v)_{L^2}$ in the complex Hilbert space $L^2(\mathbb{R}^n)$ is linear with respect to u and antilinear with respect to v: for $\lambda, \mu \in \mathbb{C}, (\lambda u, \mu v)_{L^2} = \lambda\bar{\mu}(u, v)_{L^2}$.

[4]For $\phi \in \mathscr{S}(\mathbb{R})$, we have $\langle \ln|x|, \phi(x)\rangle_{\mathscr{S}'(\mathbb{R}),\mathscr{S}(\mathbb{R})} = \int_{\mathbb{R}}\phi(x)\ln|x|dx$.

This entails $\xi\bigl(\widehat{\operatorname{sign}}\xi - \frac{1}{i\pi}pv(1/\xi)\bigr) = 0$ and we get

$$\widehat{\operatorname{sign}}\xi - \frac{1}{i\pi}pv(1/\xi) = c\delta_0,$$

with $c = 0$ since the lhs is odd[5]. We obtain

$$\widehat{\operatorname{sign}}(\xi) = \frac{1}{i\pi}pv\frac{1}{\xi}, \tag{4.1.27}$$

$$\widehat{pv(\frac{1}{\pi x})} = -i \operatorname{sign}\xi, \tag{4.1.28}$$

$$\hat{H} = \frac{\delta_0}{2} + \frac{1}{2i\pi}pv(\frac{1}{\xi}) = \frac{1}{(x-i0)}\frac{1}{2i\pi} \quad . \tag{4.1.29}$$

Let us consider now for $0 < \alpha < n$ the $L^1_{\text{loc}}(\mathbb{R}^n)$ function $u_\alpha(x) = |x|^{\alpha-n}$ ($|x|$ is the Euclidean norm of x); since u_α is also bounded for $|x| \geq 1$, it is a tempered distribution. Let us calculate its Fourier transform v_α. Since u_α is homogeneous of degree $\alpha - n$, we get that v_α is a homogeneous distribution of degree $-\alpha$. On the other hand, if $S \in \mathrm{O}(\mathbb{R}^n)$ (the orthogonal group), we have in the distribution sense[6] since u_α is a radial function, i.e. such that

$$v_\alpha(S\xi) = v_\alpha(\xi). \tag{4.1.30}$$

The distribution $|\xi|^\alpha v_\alpha(\xi)$ is homogeneous of degree 0 on $\mathbb{R}^n\backslash\{0\}$ and is also "radial", i.e. satisfies (4.1.30). Moreover on $\mathbb{R}^n\backslash\{0\}$, the distribution v_α is a C^1 function which coincides with[7]

$$\int e^{-2i\pi x\cdot\xi}\chi_0(x)|x|^{\alpha-n}dx + |\xi|^{-2N}\int e^{-2i\pi x\cdot\xi}|D_x|^{2N}\bigl(\chi_1(x)|x|^{\alpha-n}\bigr)dx,$$

where $\chi_0 \in C^\infty_c(\mathbb{R}^n)$ is 1 near 0 and $\chi_1 = 1-\chi_0$, $N \in \mathbb{N}, \alpha + 1 < 2N$. As a result $|\xi|^\alpha v_\alpha(\xi) = c_\alpha$ on $\mathbb{R}^n\backslash\{0\}$ and the distribution on $\mathbb{R}^n$ (note that $\alpha < n$)

$$T = v_\alpha(\xi) - c_\alpha|\xi|^{-\alpha}$$

is supported in $\{0\}$ and homogeneous (on $\mathbb{R}^n$) with degree $-\alpha$. The condition $0 < \alpha < n$ gives $v_\alpha = c_\alpha|\xi|^{-\alpha}$. To find c_α, we compute

$$\int_{\mathbb{R}^n}|x|^{\alpha-n}e^{-\pi x^2}dx = \langle u_\alpha, e^{-\pi x^2}\rangle = c_\alpha\int_{\mathbb{R}^n}|\xi|^{-\alpha}e^{-\pi\xi^2}d\xi$$

---

[5] A distribution T on $\mathbb{R}^n$ is said to be odd (resp. even) when $\check{T} = -T$ (resp. T).

[6] For $M \in Gl(n,\mathbb{R})$, $T \in \mathscr{S}'(\mathbb{R}^n)$, we define $\langle T(Mx), \phi(x)\rangle = \langle T(y), \phi(M^{-1}y)\rangle|\det M|^{-1}$.

[7] We have $\widehat{u_\alpha} = \widehat{\chi_0 u_\alpha} + \widehat{\chi_1 u_\alpha}$ and for ϕ supported in $\mathbb{R}^n\backslash\{0\}$ we get,

$$\langle\widehat{\chi_1 u_\alpha}, \phi\rangle = \langle\widehat{\chi_1 u_\alpha}|\xi|^{2N}, \phi(\xi)|\xi|^{-2N}\rangle = \langle\widehat{|D_x|^{2N}\chi_1 u_\alpha}, \phi(\xi)|\xi|^{-2N}\rangle.$$

which yields

$$2^{-1}\Gamma(\frac{\alpha}{2})\pi^{-\frac{\alpha}{2}} = \int_0^{+\infty} r^{\alpha-1}e^{-\pi r^2}dr = c_\alpha \int_0^{+\infty} r^{n-\alpha-1}e^{-\pi r^2}dr$$
$$= c_\alpha 2^{-1}\Gamma(\frac{n-\alpha}{2})\pi^{-(\frac{n-\alpha}{2})}.$$

We have proven the following lemma.

Lemma 4.1.18. *Let $n \in \mathbb{N}^*$ and $\alpha \in (0, n)$. The function $u_\alpha(x) = |x|^{\alpha-n}$ is $L^1_{loc}(\mathbb{R}^n)$ and also a temperate distribution on $\mathbb{R}^n$. Its Fourier transform v_α is also $L^1_{loc}(\mathbb{R}^n)$ and given by*

$$v_\alpha(\xi) = |\xi|^{-\alpha}\pi^{\frac{n}{2}-\alpha}\frac{\Gamma(\frac{\alpha}{2})}{\Gamma(\frac{n-\alpha}{2})}.$$

Fourier transform of Gaussian functions

Proposition 4.1.19. *Let A be a symmetric nonsingular $n \times n$ matrix with complex entries such that $\operatorname{Re} A \geq 0$. We define the Gaussian function v_A on $\mathbb{R}^n$ by $v_A(x) = e^{-\pi\langle Ax,x\rangle}$. The Fourier transform of v_A is*

$$\widehat{v_A}(\xi) = (\det A)^{-1/2}e^{-\pi\langle A^{-1}\xi,\xi\rangle}, \tag{4.1.31}$$

where $(\det A)^{-1/2}$ is defined according above. In particular, when $A = -iB$ with a symmetric real nonsingular matrix B, we get

$$\operatorname{Fourier}(e^{i\pi\langle Bx,x\rangle})(\xi) = \widehat{v_{-iB}}(\xi) = |\det B|^{-1/2}e^{i\frac{\pi}{4}\operatorname{sign} B}e^{-i\pi\langle B^{-1}\xi,\xi\rangle}. \tag{4.1.32}$$

Proof. Let us define Υ_+^* as the set of symmetric $n \times n$ complex matrices with a positive definite real part (naturally these matrices are nonsingular since $Ax = 0$ for $x \in \mathbb{C}^n$ implies $0 = \operatorname{Re}\langle Ax, \bar{x}\rangle = \langle(\operatorname{Re} A)x, \bar{x}\rangle$, so that $\Upsilon_+^* \subset \Upsilon_+$).

Let us assume first that $A \in \Upsilon_+^*$; then the function v_A is in the Schwartz class (and so is its Fourier transform). The set Υ_+^* is an open convex subset of $\mathbb{C}^{n(n+1)/2}$ and the function $\Upsilon_+^* \ni A \mapsto \widehat{v_A}(\xi)$ is holomorphic and given on $\Upsilon_+^* \cap \mathbb{R}^{n(n+1)/2}$ by (4.1.31). On the other hand the function

$$\Upsilon_+^* \ni A \mapsto e^{-\frac{1}{2}\operatorname{trace}\log A}e^{-\pi\langle A^{-1}\xi,\xi\rangle},$$

is also holomorphic and coincides with previous one on $\mathbb{R}^{n(n+1)/2}$. By analytic continuation this proves (4.1.31) for $A \in \Upsilon_+^*$.

If $A \in \Upsilon_+$ and $\varphi \in \mathscr{S}(\mathbb{R}^n)$, we have $\langle\widehat{v_A}, \varphi\rangle_{\mathscr{S}',\mathscr{S}} = \int v_A(x)\hat{\varphi}(x)dx$ so that $\Upsilon_+ \ni A \mapsto \langle\widehat{v_A}, \varphi\rangle$ is continuous and thus (note that the mapping $A \mapsto A^{-1}$ is an homeomorphism of Υ_+), using the previous result on Υ_+^*,

$$\langle\widehat{v_A}, \varphi\rangle = \lim_{\epsilon\to 0_+}\langle\widehat{v_{A+\epsilon I}}, \varphi\rangle = \lim_{\epsilon\to 0_+}\int e^{-\frac{1}{2}\operatorname{trace}\log(A+\epsilon I)}e^{-\pi\langle(A+\epsilon I)^{-1}\xi,\xi\rangle}\varphi(\xi)d\xi,$$

and by continuity of log on Υ_+ and dominated convergence,

$$\langle\widehat{v_A}, \varphi\rangle = \int e^{-\frac{1}{2}\operatorname{trace}\log A}e^{-\pi\langle A^{-1}\xi,\xi\rangle}\varphi(\xi)d\xi,$$

which is the sought result. □

4.1.4 Multipliers of $\mathscr{S}'(\mathbb{R}^n)$

Definition 4.1.20. The space $\mathscr{O}_M(\mathbb{R}^n)$ of multipliers of $\mathscr{S}(\mathbb{R}^n)$ is the subspace of the functions $f \in C^\infty(\mathbb{R}^n)$ such that,

$$\forall \alpha \in \mathbb{N}^n, \exists C_\alpha > 0, \exists N_\alpha \in \mathbb{N}, \quad \forall x \in \mathbb{R}^n, \quad |(\partial_x^\alpha f)(x)| \le C_\alpha (1+|x|)^{N_\alpha}. \tag{4.1.33}$$

It is easy to check that, for $f \in \mathscr{O}_M(\mathbb{R}^n)$, the operator $u \mapsto fu$ is continuous from $\mathscr{S}(\mathbb{R}^n)$ into itself, and by transposition from $\mathscr{S}'(\mathbb{R}^n)$ into itself: we define for $T \in \mathscr{S}'(\mathbb{R}^n)$, $f \in \mathscr{O}_M(\mathbb{R}^n)$,

$$\langle fT, \varphi\rangle_{\mathscr{S}',\mathscr{S}} = \langle T, f\varphi\rangle_{\mathscr{S}',\mathscr{S}},$$

and if p is a semi-norm of $\mathscr{S}$, the continuity on $\mathscr{S}$ of the multiplication by f implies that there exists a semi-norm q on $\mathscr{S}$ such that for all $\varphi \in \mathscr{S}$, $p(f\varphi) \le q(\varphi)$. A typical example of a function in $\mathscr{O}_M(\mathbb{R}^n)$ is $e^{iP(x)}$ where P is a real-valued polynomial: in fact the derivatives of $e^{iP(x)}$ are of type $Q(x)e^{iP(x)}$ where Q is a polynomial so that (4.1.33) holds.

Definition 4.1.21. Let T, S be tempered distributions on $\mathbb{R}^n$ such that $\hat{T}$ belongs to $\mathscr{O}_M(\mathbb{R}^n)$. We define the convolution $T * S$ by

$$\widehat{T * S} = \hat{T}\hat{S}. \tag{4.1.34}$$

Note that this definition makes sense since $\hat{T}$ is a multiplier so that $\hat{T}\hat{S}$ is indeed a tempered distribution whose inverse Fourier transform is meaningful. We have

$$\langle T * S, \phi\rangle_{\mathscr{S}'(\mathbb{R}^n),\mathscr{S}(\mathbb{R}^n)} = \langle \widehat{T * S}, \hat{\check{\phi}}\rangle_{\mathscr{S}'(\mathbb{R}^n),\mathscr{S}(\mathbb{R}^n)} = \langle \hat{S}, \hat{T}\hat{\check{\phi}}\rangle_{\mathscr{S}'(\mathbb{R}^n),\mathscr{S}(\mathbb{R}^n)}.$$

Proposition 4.1.22. *Let T be a distribution on $\mathbb{R}^n$ such that T is compactly supported. Then $\hat{T}$ is a multiplier which can be extended to an entire function on $\mathbb{C}^n$ such that if* $\operatorname{supp} T \subset \bar{B}(0, R_0)$,

$$\exists C_0, N_0 \ge 0, \forall \zeta \in \mathbb{C}^n, \quad |\hat{T}(\zeta)| \le C_0(1+|\zeta|)^{N_0} e^{2\pi R_0 |\operatorname{Im}\zeta|}. \tag{4.1.35}$$

In particular, for $S \in \mathscr{S}'(\mathbb{R}^n)$, we may define according to (4.1.34) the convolution $T * S$.

Proof. Let us first check the case $R_0 = 0$: then the distribution T is supported at $\{0\}$ and is a linear combination of derivatives of the Dirac mass at 0. Formulas (4.1.19), (4.1.21) imply that $\hat{T}$ is a polynomial, so that the conclusions of Proposition 4.1.22 hold in that case.

Let us assume that $R_0 > 0$ and let us consider a function χ is equal to 1 in neighborhood of $\operatorname{supp} T$ (this implies $\chi T = T$) and

$$\langle \widehat{T}, \phi\rangle_{\mathscr{S}',\mathscr{S}} = \langle \widehat{\chi T}, \phi\rangle_{\mathscr{S}',\mathscr{S}} = \langle T, \chi\hat{\phi}\rangle_{\mathscr{S}',\mathscr{S}}. \tag{4.1.36}$$

On the other hand, defining for $\zeta \in \mathbb{C}^n$ (with $x \cdot \zeta = \sum x_j \zeta_j$ for $x \in \mathbb{R}^n$),

$$F(\zeta) = \langle T(x), \chi(x) e^{-2i\pi x\cdot\zeta}\rangle_{\mathscr{S}',\mathscr{S}}, \tag{4.1.37}$$

we see that F is an entire function (i.e. holomorphic on $\mathbb{C}^n$): calculating

$$\begin{aligned} F(\zeta+h) - F(\zeta) &= \langle T(x), \chi(x) e^{-2i\pi x\cdot\zeta}(e^{-2i\pi x\cdot h} - 1)\rangle \\ &= \langle T(x), \chi(x) e^{-2i\pi x\cdot\zeta}(-2i\pi x\cdot h)\rangle \\ &\quad + \langle T(x), \chi(x) e^{-2i\pi x\cdot\zeta} \int_0^1 (1-\theta) e^{-2i\theta\pi x\cdot h} d\theta (-2i\pi x \cdot h)^2\rangle, \end{aligned}$$

and applying to the last term the continuity properties of the linear form T, we obtain that the complex differential of F is

$$\sum_{1\le j\le n} \langle T(x), \chi(x) e^{-2i\pi x\cdot\zeta}(-2i\pi x_j)\rangle d\zeta_j.$$

Moreover the derivatives of (4.1.37) are

$$F^{(k)}(\zeta) = \langle T(x), \chi(x) e^{-2i\pi x\cdot\zeta}(-2i\pi x)^k\rangle_{\mathscr{S}',\mathscr{S}}. \tag{4.1.38}$$

To evaluate the semi-norms of $x \mapsto \chi(x) e^{-2i\pi x\cdot\zeta}(-2i\pi x)^k$ in the Schwartz space, we have to deal with a finite sum of products of type

$$\left|x^\gamma (\partial^\alpha \chi)(x) e^{-2i\pi x\cdot\zeta}(-2i\pi\zeta)^\beta\right| \le (1+|\zeta|)^{|\beta|} \sup_{x\in\mathbb{R}^n} |x^\gamma (\partial^\alpha\chi)(x) e^{2\pi|x||\operatorname{Im}\zeta|}|.$$

We may now choose a function χ_0 equal to 1 on $B(0,1)$, supported in $B(0, \frac{R_0+2\epsilon}{R_0+\epsilon})$ such that $\|\partial^\beta \chi_0\|_{L^\infty} \le c(\beta)\epsilon^{-|\beta|}$ with $\epsilon = \frac{R_0}{1+|\zeta|}$. We find with

$$\chi(x) = \chi_0(x/(R_0+\epsilon)) \quad \text{(which is 1 on a neighborhood of } B(0,R_0)),$$

$$\begin{aligned} \sup_{x\in\mathbb{R}^n} |x^\gamma (\partial^\alpha \chi)(x) e^{2\pi|x||\operatorname{Im}\zeta|}| &\le (R_0+2\epsilon)^{|\gamma|} \sup_{y\in\mathbb{R}^n} |(\partial^\alpha\chi_0)(y) e^{2\pi(R_0+2\epsilon)|\operatorname{Im}\zeta|}| \\ &\le (R_0+2\epsilon)^{|\gamma|} e^{2\pi(R_0+2\epsilon)|\operatorname{Im}\zeta|} c(\alpha)\epsilon^{-|\alpha|} \\ &= (R_0 + 2\frac{R_0}{1+|\zeta|})^{|\gamma|} e^{2\pi(R_0+2\frac{R_0}{1+|\zeta|})|\operatorname{Im}\zeta|} c(\alpha) (\frac{1+|\zeta|}{R_0})^{|\alpha|} \\ &\le (3R_0)^{|\gamma|} e^{2\pi R_0|\operatorname{Im}\zeta|} e^{4\pi R_0} c(\alpha) R_0^{-|\alpha|}(1+|\zeta|)^{|\alpha|} \end{aligned}$$

yielding

$$|F^{(k)}(\zeta)| \le e^{2\pi R_0|\operatorname{Im}\zeta|} C_k (1+|\zeta|)^{N_k},$$

which implies that $\mathbb{R}^n \ni \xi \mapsto F(\xi)$ is indeed a multiplier. We have also

$$\langle T, \chi\hat{\phi}\rangle_{\mathscr{S}',\mathscr{S}} = \langle T(x), \chi(x) \int_{\mathbb{R}^n} \phi(\xi) e^{-2i\pi x\xi} d\xi\rangle_{\mathscr{S}',\mathscr{S}}.$$

Since the function F is entire we have for $\phi \in C_c^\infty(\mathbb{R}^n)$, using (4.1.38) and Fubini Theorem on $\ell^1(\mathbb{N}) \times L^1(\mathbb{R}^n)$,

$$\int_{\mathbb{R}^n} F(\xi)\phi(\xi)d\xi = \sum_{k\geq 0}\langle T(x), \chi(x)(-2i\pi x)^k\rangle \int_{\operatorname{supp}\phi} \frac{\xi^k}{k!}\phi(\xi)d\xi. \tag{4.1.39}$$

On the other hand, since $\hat{\phi}$ is also entire (from the discussion on F or directly from the integral formula for the Fourier transform of $\phi \in C_c^\infty(\mathbb{R}^n)$), we have

$$\begin{aligned}\langle T, \chi\hat{\phi}\rangle &= \langle T(x), \chi(x)\sum_{k\geq 0}(\hat{\phi})^{(k)}(0)x^k/k!\rangle \\ &= \underbrace{\langle T(x), \chi(x)\lim_{N\to+\infty}\sum_{0\leq k\leq N}(\hat{\phi})^{(k)}(0)x^k/k!\rangle}_{\text{convergence in } C_c^\infty(\mathbb{R}^n)} \\ &= \lim_{N\to+\infty}\sum_{0\leq k\leq N}\langle T(x), \chi(x)x^k/k!\rangle \int_{\mathbb{R}^n}\phi(\xi)(-2i\pi\xi)^k d\xi.\end{aligned}$$

Thanks to (4.1.39), that quantity is equal to $\int_{\mathbb{R}^n} F(\xi)\phi(\xi)d\xi$. As a result, the tempered distributions $\hat{T}$ and F coincide on $C_c^\infty(\mathbb{R}^n)$, which is dense in $\mathscr{S}(\mathbb{R}^n)$ and so $\hat{T} = F$, concluding the proof. □

4.2 Gårding's inequality

4.2.1 The Wick calculus of pseudodifferential operators

Wick quantization

We recall here some facts on the so-called Wick quantization, as used in [21], [22], [23].

Definition 4.2.1. Let $Y = (y, \eta)$ be a point in $\mathbb{R}^n \times \mathbb{R}^n$. The operator Σ_Y is defined as $\left[2^n e^{-2\pi|\cdot - Y|^2}\right]^w$. Let a be in $L^\infty(\mathbb{R}^{2n})$. The Wick quantization of a is defined as

$$a^{\mathrm{Wick}} = \int_{\mathbb{R}^{2n}} a(Y)\Sigma_Y dY. \tag{4.2.1}$$

Remark 4.2.2. The operator Σ_Y is a rank-one orthogonal projection: we have

$$\Sigma_Y u = (Wu)(Y)\tau_Y\varphi_0 \quad \text{with } (Wu)(Y) = \langle u, \tau_Y\varphi_0\rangle_{L^2(\mathbb{R}^n)}, \tag{4.2.2}$$

$$\text{where } \varphi_0(x) = 2^{n/4}e^{-\pi|x|^2} \text{ and } (\tau_{y,\eta}\varphi_0)(x) = \varphi_0(x-y)e^{2i\pi\langle x-\frac{y}{2},\eta\rangle}. \tag{4.2.3}$$

In fact we get from the definition of Σ_Y that, for $u \in \mathscr{S}(\mathbb{R}^n)$

$$\begin{aligned}(\Sigma_{y,\eta}u)(x) &= \iint u(z)e^{2i\pi(x-z)\cdot\xi}2^n e^{-2\pi|\frac{x+z}{2}-y|^2}e^{-2\pi|\xi-\eta|^2}dzd\xi \\ &= \int u(z)e^{2i\pi(x-z)\cdot\eta}2^{n/2}e^{-2\pi|\frac{x+z}{2}-y|^2}e^{-\frac{\pi}{2}|x-z|^2}dz \\ &= \int u(z)e^{-2i\pi(z-\frac{y}{2})\cdot\eta}2^{n/4}e^{-\pi|z-y|^2}dz\ 2^{n/4}e^{-\pi|x-y|^2}e^{2i\pi(x-\frac{y}{2})\cdot\eta} \\ &= \langle u, \tau_{y,\eta}\varphi_0\rangle\tau_{y,\eta}\varphi_0.\end{aligned}$$

Proposition 4.2.3.

(1) *Let a be in $L^\infty(\mathbb{R}^{2n})$. Then $a^{Wick} = W^* a^\mu W$ and $1^{Wick} = Id_{L^2(\mathbb{R}^n)}$ where W is the isometric mapping from $L^2(\mathbb{R}^n)$ to $L^2(\mathbb{R}^{2n})$ given above, and a^μ the operator of multiplication by a in $L^2(\mathbb{R}^{2n})$. The operator $\pi_{\mathcal{H}} = WW^*$ is the orthogonal projection on a closed proper subspace $\mathcal{H}$ of $L^2(\mathbb{R}^{2n})$ and has the kernel*

$$\Pi(X,Y) = e^{-\frac{\pi}{2}|X-Y|^2}e^{-i\pi[X,Y]}, \tag{4.2.4}$$

where $[,]$ is the symplectic form. Moreover, we have

$$\|a^{\mathrm{Wick}}\|_{\mathcal{L}(L^2(\mathbb{R}^n))} \le \|a\|_{L^\infty(\mathbb{R}^{2n})}, \tag{4.2.5}$$

$$a(X) \ge 0 \text{ for all } X \text{ implies } a^{Wick} \ge 0. \tag{4.2.6}$$

(2) *Let m be a real number, and $p \in S(\Lambda^m, \Lambda^{-1}\Gamma)$, where Γ is the Euclidean norm on $\mathbb{R}^{2n}$. Then $p^{Wick} = p^w + r(p)^w$, with $r(p) \in S(\Lambda^{m-1}, \Lambda^{-1}\Gamma)$ so that the mapping $p \mapsto r(p)$ is continuous. More precisely, one has*

$$r(p)(X) = \int_0^1\int_{\mathbb{R}^{2n}}(1-\theta)p''(X+\theta Y)Y^2e^{-2\pi\Gamma(Y)}2^n dY d\theta.$$

Note that $r(p) = 0$ if p is affine and $r(p) = \frac{1}{8\pi}\operatorname{trace} p''$ if p is a polynomial with degree ≤ 2.

(3) *For $a \in L^\infty(\mathbb{R}^{2n})$, the Weyl symbol of a^{Wick} is*

$$a * 2^n \exp -2\pi\Gamma, \text{ which belongs to } S(1,\Gamma) \text{ with } k^{th}\text{-seminorm } c(k)\|a\|_{L^\infty}. \tag{4.2.7}$$

(4) *Let $\mathbb{R} \ni t \mapsto a(t,X) \in \mathbb{R}$ such that, for $t \le s$, $a(t,X) \le a(s,X)$. Then, for $u \in C_c^1\big(\mathbb{R}_t, L^2(\mathbb{R}^n)\big)$, assuming $a(t,\cdot) \in L^\infty(\mathbb{R}^{2n})$,*

$$\int_{\mathbb{R}} \mathrm{Re}\langle D_t u(t), ia(t)^{Wick}u(t)\rangle_{L^2(\mathbb{R}^n)}dt \ge 0. \tag{4.2.8}$$

(5) *With the operator Σ_Y given in Definition 4.2.1, we have the estimate*

$$\|\Sigma_Y\Sigma_Z\|_{\mathcal{L}(L^2(\mathbb{R}^n))} \le 2^n e^{-\frac{\pi}{2}\Gamma(Y-Z)}. \tag{4.2.9}$$

(6) *More precisely, the Weyl symbol of* $\Sigma_Y\Sigma_Z$ *is, as a function of the variable* $X \in \mathbb{R}^{2n}$*, setting* $\Gamma(T) = |T|^2$

$$e^{-\frac{\pi}{2}|Y-Z|^2} e^{-2i\pi[X-Y,X-Z]} 2^n e^{-2\pi|X-\frac{Y+Z}{2}|^2}. \tag{4.2.10}$$

Remark 4.2.4. Part of this proposition is well summarized by the following diagram:

$$\begin{array}{ccc} L^2(\mathbb{R}^{2n}) & \xrightarrow[\text{(multiplication by } a)]{a} & L^2(\mathbb{R}^{2n}) \\ W\big\uparrow & & \big\downarrow W^* \\ L^2(\mathbb{R}^n) & \xrightarrow[a^{\text{Wick}}]{} & L^2(\mathbb{R}^n) \end{array}$$

Proof. For $u, v \in \mathscr{S}(\mathbb{R}^n)$, we have

$$\langle a^{\text{Wick}} u, v\rangle = \int_{\mathbb{R}^{2n}} a(Y)\langle \Sigma_Y u, v\rangle_{L^2(\mathbb{R}^n)} dY = \int_{\mathbb{R}^{2n}} a(Y)(Wu)(Y)\overline{(Wv)(Y)} dY,$$

which gives

$$a^{\text{Wick}} = W^* a^{\mu} W. \tag{4.2.11}$$

Also we have from (4.2.1) that $1^{\text{Wick}} = \text{Id}$, since

$$1^{\text{Wick}} = \int_{\mathbb{R}^{2n}} \Sigma_Y dY \quad \text{has Weyl symbol} \int_{\mathbb{R}^{2n}} 2^n e^{-2\pi|X-Y|^2} dY = 1.$$

This implies that

$$W^*W = \text{Id},$$

i.e. W is isometric from $L^2(\mathbb{R}^n)$ into $L^2(\mathbb{R}^{2n})$. The operator WW^* is bounded selfadjoint and is a projection since $WW^*WW^* = WW^*$. Defining $\mathcal{H}$ as $\operatorname{ran} W$, we get that WW^* is the orthogonal projection onto $\mathcal{H}$, since the range of WW^* is included in the range of W, and for $\Phi \in \mathcal{H}$, we have

$$\Phi = Wu = WW^*Wu \in \operatorname{ran}(WW^*).$$

Moreover $\operatorname{ran} W$ is closed since W is isometric, that latter property implying also, using (4.2.11), the property (4.2.5), whereas (4.2.6) follows from (4.2.1) and $\Sigma_Y \ge 0$ as an orthogonal projection. The kernel of the operator WW^* is, from (4.2.2), (4.2.3), with $X = (x, \xi), Y = (y, \eta)$,

$$\begin{aligned} \Pi(X,Y) &= \langle \tau_Y \varphi_0, \tau_X \varphi_0\rangle_{L^2(\mathbb{R}^n)} \\ &= 2^{n/2} \int_{\mathbb{R}^n} e^{-\pi|t-x|^2} e^{-\pi|t-y|^2} e^{2i\pi(t-\frac{y}{2})\cdot\eta} e^{-2i\pi(t-\frac{x}{2})\cdot\xi} dt \\ &= e^{-\frac{\pi}{2}|x-y|^2} 2^{n/2} \int_{\mathbb{R}^n} e^{-\frac{\pi}{2}|2t-x-y|^2} e^{2i\pi t\cdot(\eta-\xi)} dt e^{i\pi(x\cdot\xi-y\cdot\eta)} \\ &= e^{-\frac{\pi}{2}|x-y|^2} 2^{n/2} \int_{\mathbb{R}^n} e^{-2\pi|t|^2} e^{2i\pi(t+\frac{x+y}{2})\cdot(\eta-\xi)} dt e^{i\pi(x\cdot\xi-y\cdot\eta)} \\ &= e^{-\frac{\pi}{2}|x-y|^2} e^{-\frac{\pi}{2}|\xi-\eta|^2} e^{i\pi(x+y)\cdot(\eta-\xi)} e^{i\pi(x\cdot\xi-y\cdot\eta)} \\ &= e^{-\frac{\pi}{2}|x-y|^2} e^{-\frac{\pi}{2}|\xi-\eta|^2} e^{i\pi(x\eta-y\xi)} = e^{-\frac{\pi}{2}|X-Y|^2} e^{-i\pi[X,Y]}, \end{aligned}$$

which is (4.2.4). Postponing the proof of $\mathcal{H} \neq L^2(\mathbb{R}^{2n})$ until after the proof of (2), we have proven (1). To obtain (2), we note that (4.2.1) gives directly that

$$a^{\text{Wick}} = (a * 2^n \exp -2\pi\Gamma)^w$$

and the second order Taylor expansion gives (2) while (3) is obvious from the convolution formula. Note also that $u \in \mathscr{S}(\mathbb{R}^n)$ implies $Wu \in \mathscr{S}(\mathbb{R}^{2n})$ since $e^{-i\pi y\cdot\eta}(Wu)(y,\eta)$ is the partial Fourier transform with respect to x of $\mathbb{R}^n \times \mathbb{R}^n \ni (x,y) \mapsto u(x)2^{n/4}e^{-\pi|x-y|^2}$: this gives also another proof of W isometric since

$$\iint |u(x)|^2 2^{n/2} e^{-2\pi|x-y|^2} dxdy = \|u\|^2_{L^2(\mathbb{R}^n)}.$$

We calculate now, for $u \in \mathscr{S}(\mathbb{R}^n)$ with L^2 norm 1, using the already proven (2) on the Wick quantization of linear forms,

$$\begin{aligned}2\operatorname{Re}\langle \pi_{\mathcal{H}}\xi_1 Wu, ix_1 Wu\rangle_{L^2(\mathbb{R}^{2n})} &= 2\operatorname{Re}\langle W^*\xi_1 Wu, iW^*x_1 Wu\rangle_{L^2(\mathbb{R}^n)}\\ &= 2\operatorname{Re}\langle \xi_1^{\text{Wick}}u, ix_1^{\text{Wick}}u\rangle_{L^2(\mathbb{R}^n)} = 2\operatorname{Re}\langle D_1 u, ix_1 u\rangle_{L^2(\mathbb{R}^n)} = 1/2\pi.\end{aligned}$$

If $\mathcal{H}$ were the whole $L^2(\mathbb{R}^{2n})$, the projection $\pi_{\mathcal{H}}$ would be the identity and we would have

$$0 = 2\operatorname{Re}\langle \xi_1 Wu, ix_1 Wu\rangle_{L^2(\mathbb{R}^{2n})} = 2\operatorname{Re}\langle \pi_{\mathcal{H}}\xi_1 Wu, ix_1 Wu\rangle_{L^2(\mathbb{R}^{2n})} = 1/2\pi.$$

Let us prove (4). We have from the Lebesgue dominated convergence theorem,

$$\begin{aligned}\alpha &= \int_{\mathbb{R}} \operatorname{Re}\langle D_t u(t), ia(t)^{\text{Wick}}u(t)\rangle_{L^2(\mathbb{R}^n)}dt\\ &= -\lim_{h\to 0_+}\int_{\mathbb{R}} \frac{1}{2\pi h}\operatorname{Re}\langle u(t+h)-u(t), a(t)^{\text{Wick}}u(t+h)\rangle_{L^2(\mathbb{R}^n)}dt\\ &= \lim_{h\to 0_+}\frac{1}{2\pi h}\Big(-\int_{\mathbb{R}}\operatorname{Re}\langle u(t), a(t-h)^{\text{Wick}}u(t)\rangle_{L^2(\mathbb{R}^n)}dt\\ &\qquad\qquad + \int_{\mathbb{R}}\operatorname{Re}\langle u(t), a(t)^{\text{Wick}}u(t+h)\rangle_{L^2(\mathbb{R}^n)}dt\Big)\end{aligned}$$

$$\begin{aligned}= \lim_{h\to 0_+}\Big\{&\underbrace{\frac{1}{2\pi h}\int_{\mathbb{R}}\operatorname{Re}\langle\big(a(t)-a(t-h)\big)^{\text{Wick}}u(t), u(t)\rangle_{L^2(\mathbb{R}^n)}dt}_{=\beta(h)}\\ &+\underbrace{\int_{\mathbb{R}}\operatorname{Re}\langle\frac{-1}{2\pi hi}\big(u(t+h)-u(t)\big), ia(t)^{\text{Wick}}u(t)\rangle_{L^2(\mathbb{R}^n)}dt}_{\text{with limit } -\alpha}\Big\}.\end{aligned}$$

The previous calculation shows that $\beta(h)$ has a limit when $h \to 0_+$ and $2\alpha = \lim_{h\to 0_+}\beta(h)$. Since the function $a(t)-a(t-h)$ is non-negative, the already proven

(4.2.6) implies that the operator $(a(t)-a(t-h))^{\mathrm{Wick}}$ is also non-negative, implying $\beta(h)\geq 0$ which gives $\alpha\geq 0$, i.e. (4.2.8)[8]. Since for the Weyl quantization, $\|a^w\|_{\mathcal{L}(L^2(\mathbb{R}^n))}\leq 2^n\|a\|_{L^1(\mathbb{R}^{2n})}$, we get the result (4.2.9) from (4.2.10). Let us finally prove the latter formula. From the composition formula (4.3.5), we obtain that the Weyl symbol ω of $\Sigma_Y\Sigma_Z$ is

$$\begin{aligned}
\omega(X) &= 2^{2n}\iint e^{-4i\pi[X-X_1,X-X_2]}2^{2n}e^{-2\pi|X_1-Y|^2}e^{-2\pi|X_2-Z|^2}dX_1dX_2\\
&= 2^{4n}\iint e^{-4i\pi[X-Y,X-X_2]}e^{-2i\pi\langle X_1,2\sigma(X-X_2)\rangle}e^{-2\pi|X_1|^2}e^{-2\pi|X_2-Z|^2}dX_1dX_2\\
&= 2^{3n}\int e^{-4i\pi[X-Y,X-X_2]}e^{-2\pi|X-X_2|^2}e^{-2\pi|X_2-Z|^2}dX_2\\
&= 2^{3n}e^{-\pi|X-Z|^2}\int e^{-4i\pi[X-Y,X-X_2]}e^{-\pi|X+Z-2X_2|^2}dX_2\\
&= 2^{3n}e^{-\pi|X-Z|^2}e^{-2i\pi[X-Y,X-Z]}\int e^{-4i\pi[X-Y,-X_2]}e^{-4\pi|X_2|^2}dX_2\\
&= 2^{n}e^{-\pi|X-Z|^2}e^{-2i\pi[X-Y,X-Z]}e^{-\pi|X-Y|^2}\\
&= 2^{n}e^{-2i\pi[X-Y,X-Z]}e^{-2\pi|X-\frac{Y+Z}{2}|^2}e^{-\frac{\pi}{2}|Y-Z|^2}.
\end{aligned}$$

□

Fock-Bargmann spaces

There are also several links with the so-called Fock-Bargmann spaces (the space $\mathcal{H}$ above), that we can summarize with the following definitions and properties.

Proposition 4.2.5. *With $\mathcal{H}$ defined in Proposition 4.2.3 we have*

$$\mathcal{H}=\{\Phi\in L^2(\mathbb{R}^{2n}_{y,\eta}),\quad \Phi=f(z)\exp-\frac{\pi}{2}|z|^2,\ z=\eta+iy\ ,\ f\ \text{entire}\},\tag{4.2.12}$$

i.e. $\mathcal{H}=\mathrm{ran}W=L^2(\mathbb{R}^{2n})\cap\ker(\bar\partial+\frac{\pi}{2}z)$.

Proof. For $v\in L^2(\mathbb{R}^n)$, we have, with the notation $z^2=\sum_{1\leq j\leq n}z_j^2$ for $z\in\mathbb{C}^n$,

$$\begin{aligned}
(Wv)(y,\eta) &= \int_{\mathbb{R}^n}v(x)2^{n/4}e^{-\pi(x-y)^2}e^{-2i\pi(x-\frac{y}{2})\eta}dx\\
&= \int_{\mathbb{R}^n}v(x)2^{n/4}e^{-\pi(x-y+i\eta)^2}dxe^{-\frac{\pi}{2}(y^2+\eta^2)}e^{-\frac{\pi}{2}(\eta+iy)^2}
\end{aligned}\tag{4.2.13}$$

and we see that $Wv\in L^2(\mathbb{R}^{2n})\cap\ker(\bar\partial+\frac{\pi}{2}z)$. Conversely, if $\Phi\in L^2(\mathbb{R}^{2n})\cap\ker(\bar\partial+\frac{\pi}{2}z)$,

[8] Note that (4.2.8) is simply a way of writing that $\frac{d}{dt}\left(a(t)^{\mathrm{Wick}}\right)\geq 0$, which is a consequence of (4.2.6) and of the non-decreasing assumption made on $t\mapsto a(t,X)$.

we have $\Phi(x,\xi)=e^{-\frac{\pi}{2}(x^2+\xi^2)}f(\xi+ix)$ with $\Phi\in L^2(\mathbb{R}^{2n})$ and f entire. This gives

$$\begin{aligned}
(WW^*\Phi)(x,\xi) &= \iint e^{-\frac{\pi}{2}\left((\xi-\eta)^2+(x-y)^2+2i\xi y-2i\eta x\right)}\Phi(y,\eta)dyd\eta\\
&= e^{-\frac{\pi}{2}(\xi^2+x^2)}\iint e^{-\frac{\pi}{2}(\eta^2-2\xi\eta+y^2-2xy+2i\xi y-2i\eta x)}\Phi(y,\eta)dyd\eta\\
&= e^{-\frac{\pi}{2}(\xi^2+x^2)}\iint e^{-\frac{\pi}{2}\left(\eta^2+y^2+2iy(\xi+ix)-2\eta(\xi+ix)\right)}\Phi(y,\eta)dyd\eta\\
&= e^{-\frac{\pi}{2}(\xi^2+x^2)}\iint e^{-\pi(y^2+\eta^2)}e^{\pi(\eta-iy)(\xi+ix)}f(\eta+iy)dyd\eta\\
&= e^{-\frac{\pi}{2}|z|^2}\iint e^{-\pi|\zeta|^2}e^{\pi\bar\zeta z}f(\zeta)dyd\eta \quad (\zeta=\eta+iy,\ z=\xi+ix)\\
&= e^{-\frac{\pi}{2}|z|^2}\iint f(\zeta)\prod_{1\le j\le n}\frac{1}{\pi(z_j-\zeta_j)}\frac{\partial}{\partial\bar\zeta_j}\left(e^{-\pi|\zeta|^2}e^{\pi\bar\zeta z}\right)dyd\eta\\
&= e^{-\frac{\pi}{2}|z|^2}\langle f(\zeta)\prod_{1\le j\le n}\frac{\partial}{\partial\bar\zeta_j}\left(\frac{1}{\pi(\zeta_j-z_j)}\right), e^{-\pi|\zeta|^2}e^{\pi\bar\zeta z}\rangle_{\mathscr{S}'(\mathbb{R}^{2n}),\mathscr{S}(\mathbb{R}^{2n})}\\
&= e^{-\frac{\pi}{2}|z|^2}f(z),
\end{aligned}$$

since f is entire. This implies $WW^*\Phi=\Phi$ and $\Phi\in\operatorname{ran}W$, completing the proof of the proposition. □

Proposition 4.2.6. *Defining*

$$\mathscr{H}=\ker(\bar\partial+\frac{\pi}{2}z)\cap\mathscr{S}'(\mathbb{R}^{2n}), \tag{4.2.14}$$

the operator W given by (4.2.2) can be extended as a continuous mapping from $\mathscr{S}'(\mathbb{R}^n)$ onto $\mathscr{H}$ (the $L^2(\mathbb{R}^n)$ dot-product is replaced by a bracket of (anti)duality). The operator $\widetilde\Pi$ with kernel Π given by (4.2.4) defines a continuous mapping from $\mathscr{S}(\mathbb{R}^{2n})$ into itself and can be extended as a continuous mapping from $\mathscr{S}'(\mathbb{R}^{2n})$ onto $\mathscr{H}$. It verifies

$$\widetilde\Pi^2=\widetilde\Pi,\quad \widetilde\Pi_{|\mathscr{H}}=\mathrm{Id}_{\mathscr{H}}. \tag{4.2.15}$$

Proof. As above we use that $e^{-i\pi y\eta}(Wv)(y,\eta)$ is the partial Fourier transform w.r.t. x of the tempered distribution on $\mathbb{R}^{2n}_{x,y}$

$$v(x)2^{n/4}e^{-\pi(x-y)^2}.$$

Since $e^{\pm i\pi y\eta}$ are in the space $\mathscr{O}_M(\mathbb{R}^{2n})$ of multipliers of $\mathscr{S}(\mathbb{R}^{2n})$, that transformation is continuous and injective from $\mathscr{S}'(\mathbb{R}^n)$ into $\mathscr{S}'(\mathbb{R}^{2n})$. Replacing in (4.2.13) the integrals by brackets of duality, we see that $W(\mathscr{S}'(\mathbb{R}^n))\subset\mathscr{H}$. Conversely, if $\Phi\in\mathscr{H}$, the same calculations as above give (4.2.15) and (4.2.14). □

4.2.2 The Gårding inequality with gain of one derivative

We want to prove in this section that a non-negative symbol of order 1, related to an admissible metric, is quantized by an operator which is semi-bounded from below.

To be of order 1 for a symbol a means that $a \in S(\lambda_g, g)$. The main point in this generalization is that the non-negativity for the operator as a consequence of the non-negativity of its symbol holds true as well for any admissible metric g.

However, we want also to deal with systems, including infinite-dimensional systems and prove our inequality in that framework. So far we have dealt only with scalar-valued (complex-valued) symbols ; let us first consider a symbol a defined on $\mathbb{R}^{2n}$ but valued in the algebra of $N \times N$ matrices. It means simply that $a = (a_{jk})_{1\le j,k\le N}$ where each a_{jk} belongs to $S(m, g)$ for some g-admissible weight m. Although many results can be extended without much change to this "matrix-valued" case, it is very important to keep in mind that $\mathscr{B}(\mathbb{H})$ is not commutative as soon as $\dim \mathbb{H} > 1$ and that the composition formula and the Poisson bracket should be given the proper definition, taking into account the position of the various terms. Anyhow, we shall skip checking all the details of that calculus for systems of pseudodifferential operators and take advantage of the very simple proof using the Wick calculus to extend the result to that case.

Theorem 4.2.7. *Let g be an admissible metric on $\mathbb{R}^{2n}$, $\mathbb{H}$ be a Hilbert space, a be a symbol in $S(\lambda_g, g)$ valued in the non-negative symmetric bounded operators on $\mathbb{H}$. Then the operator a^w is semi-bounded from below, and more precisely, there exists $l \in \mathbb{N}$ and C depending only on n such that*

$$\forall u \in \mathscr{S}(\mathbb{R}^n; \mathbb{H}), \quad \langle a^w u, u\rangle + C\|a\|^{(l)}_{S(\lambda_g,g)}\|u\|^2_{L^2(\mathbb{R}^n;\mathbb{H})} \ge 0. \tag{4.2.16}$$

Under the same hypothesis, the same result is true with

$$\langle a^w u, u\rangle \text{ replaced by } \operatorname{Re}\langle a(x,D)u, u\rangle.$$

Proof. We can find a family $(\varphi_Y)_{Y\in\mathbb{R}^{2n}}$ of functions uniformly in $S(1, g)$ supported in $U_{Y,r}$, nonnegative, such that $\int \varphi_Y |g_Y|^{1/2} dY = 1$. With $(\psi_Y)_{Y\in\mathbb{R}^{2n}}$ uniformly in $S(1, g)$ and real-valued, supported in $U_{Y,2r}$, equal to 1 on $U_{Y,r}$, we have

$$\psi_Y \sharp \varphi_Y a \sharp \psi_Y = \varphi_Y a + r_Y, \tag{4.2.17}$$

and we get that $(r_Y)_{Y\in\mathbb{R}^{2n}}$ is a uniformly confined family of symbols, so that

$$a^w \equiv \int_{\mathbb{R}^{2n}} \psi_Y^w (\varphi_Y a)^w \psi_Y^w |g_Y|^{1/2} dY, \qquad \operatorname{mod} \mathscr{L}(L^2(\mathbb{R}^n)). \tag{4.2.18}$$

The symbol $\varphi_Y a$ belongs uniformly to $S(\lambda_g(Y), g_Y) \subset S(\lambda_g(Y), \lambda_g(Y)^{-1} g_Y^\natural)$, and $g_Y^\natural = (g_Y^\natural)^\sigma$. Using a linear symplectic mapping and Segal's formula, we get that $(\varphi_Y a)^w$ is unitary equivalent to some α^w with $0 \le \alpha \in S(\mu, \mu^{-1}|dX|^2)$ with seminorms bounded above independently of Y and $\mu = \lambda_g(Y)$. Proposition 4.2.3(1)(2) imply that $\alpha^w + C \ge 0$, where C is a seminorm of α and thus of a, so that $(a\varphi_Y)^w + C \ge 0$. Plugging this in (4.2.18), we get the result since

$$\int \psi_Y^w \psi_Y^w |g_Y|^{1/2} dY \in \mathscr{L}(L^2(\mathbb{R}^n)), \tag{4.2.19}$$

thanks to Cotlar's lemma. □

Remark 4.2.8. The reader may think that we did not pay much attention to the fact that the symbol was valued in $\mathscr{B}(\mathbb{H})$; in fact, since the ψ_Y, χ_Y are scalar-valued, the formulas (4.2.17), (4.2.18) hold without change (except that $L^2(\mathbb{R}^n)$ becomes $L^2(\mathbb{R}^n; \mathbb{H})$) and it is a simple matter to check that the $\mathscr{B}(\mathbb{H})$-valued version of Proposition 4.2.3 holds true, with the non-negativity condition $a(X) \geq 0$ meaning $a(X)$ nonnegative symmetric bounded operator in $\mathbb{H}$.

4.3 Weyl quantization

A much more detailed account is given in the book [24] (see in particular Section 2.1.3). We are given a function a defined on the phase space $\mathbb{R}^n \times \mathbb{R}^n$ (a is a "Hamiltonian") and we wish to associate to this function an operator. For instance, we may introduce the one-parameter formulas, op_t for $t \in \mathbb{R}$,

$$(\operatorname{op}_t(a))u(x) = \iint e^{2i\pi(x-y)\cdot\xi} a\big((1-t)x + ty, \xi\big)u(y)dyd\xi. \tag{4.3.1}$$

When $t = 0$, we recognize the standard quantization, quantizing $a(x)\xi_j$ in $a(x)D_{x_j}$. However, one may wish to multiply first and take the derivatives afterwards: this is what the choice $t = 1$ does, quantizing $a(x)\xi_j$ in $D_{x_j}a(x)$. The more symmetrical choice $t = 1/2$ was done by Hermann Weyl: we have

$$(\operatorname{op}_{1/2}(a))u(x) = \iint e^{2i\pi(x-y)\cdot\xi} a\big(\frac{x+y}{2}, \xi\big)u(y)dyd\xi, \tag{4.3.2}$$

and thus

$$\operatorname{op}_{1/2}(a(x)\xi_j) = \frac{1}{2}\left(a(x)D_{x_j} + D_{x_j}a(x)\right).$$

This quantization is widely used in quantum mechanics, because a real-valued Hamiltonian gets quantized by a (formally) selfadjoint operator.[9] The reader may be embarrassed by the fact that we did not bother about the convergence of the integrals above. Before providing a definition, we may assume that $a \in \mathscr{S}(\mathbb{R}^{2n}), u, v \in \mathscr{S}(\mathbb{R}^n), t \in \mathbb{R}$ and compute

$$\begin{aligned}
\langle(\operatorname{op}_t a)u, v\rangle &= \iiint a\big((1-t)x + ty, \xi\big)e^{2i\pi(x-y)\cdot\xi}u(y)\bar{v}(x)dyd\xi dx \\
&= \iiint a(z,\xi)e^{-2i\pi s\cdot\xi}u(z + (1-t)s)\bar{v}(z - ts)dzd\xi ds \\
&= \iiint a(x,\xi)e^{-2i\pi z\cdot\xi}u(x + (1-t)z)\bar{v}(x - tz)dxd\xi dz,
\end{aligned}$$

so that with

$$\Omega_{u,v}(t)(x,\xi) = \int e^{-2i\pi z\cdot\xi}u(x + (1-t)z)\bar{v}(x - tz)dz, \tag{4.3.3}$$

[9] The most important property of that quantization remains its symplectic invariance.

which is easily seen[10] to be in $\mathscr{S}(\mathbb{R}^{2n})$ when $u, v \in \mathscr{S}(\mathbb{R}^n)$, we can give the following definition.

Definition 4.3.1. Let $a \in \mathscr{S}'(\mathbb{R}^{2n})$ be a tempered distribution and $t \in \mathbb{R}$. We define the operator $\mathrm{op}_t a : \mathscr{S}(\mathbb{R}^n) \longrightarrow \mathscr{S}^*(\mathbb{R}^n)$ by the formula

$$\langle (\mathrm{op}_t a)u, v\rangle_{\mathscr{S}^*(\mathbb{R}^n),\mathscr{S}(\mathbb{R}^n)} = \prec a, \Omega_{u,v}(t) \succ_{\mathscr{S}'(\mathbb{R}^{2n}),\mathscr{S}(\mathbb{R}^{2n})},$$

where $\mathscr{S}^*(\mathbb{R}^n)$ is the antidual of $\mathscr{S}(\mathbb{R}^n)$ (continuous antilinear forms).

Proposition 4.3.2. *Let $a \in \mathscr{S}'(\mathbb{R}^{2n})$ be a tempered distribution and $t \in \mathbb{R}$. We have*

$$op_t(a) = op_0(J^t a) = (J^t a)(x, D),$$

with $J^t = e^{2i\pi t D_x \cdot D_\xi}$.

Proof. Let $u, v \in \mathscr{S}(\mathbb{R}^n)$. With the $\mathscr{S}(\mathbb{R}^{2n})$ function $\Omega_{u,v}(t)$ given above, we have for $t \neq 0$,

$$\begin{aligned}\big(J^t\Omega_{u,v}(0)\big)(x,\xi) &= |t|^{-n}\iint e^{-2i\pi t^{-1}(x-y)\cdot(\xi-\eta)}\Omega_{u,v}(0)(y,\eta)dyd\eta \\ &= |t|^{-n}\iint e^{-2i\pi t^{-1}(x-y)\cdot(\xi-\eta)}\hat u(\eta)\bar v(y)e^{2i\pi y\cdot\eta}dyd\eta \\ &= \iint e^{-2i\pi z\cdot(\xi-\eta)}\hat u(\eta)\bar v(x-tz)e^{2i\pi(x-tz)\cdot\eta}dzd\eta \\ &= \int e^{-2i\pi z\cdot\xi}u(x+(1-t)z)\bar v(x-tz)dz = \Omega_{u,v}(t)(x,\xi), \qquad (4.3.4)\end{aligned}$$

so that

$$\begin{aligned}\langle (\mathrm{op}_t a)u, v\rangle_{\mathscr{S}^*(\mathbb{R}^n),\mathscr{S}(\mathbb{R}^n)} &= \prec a, \Omega_{u,v}(t) \succ_{\mathscr{S}'(\mathbb{R}^{2n}),\mathscr{S}(\mathbb{R}^{2n})} && \text{(definition 4.3.1)}\\ &= \prec a, J^t\Omega_{u,v}(0) \succ_{\mathscr{S}'(\mathbb{R}^{2n}),\mathscr{S}(\mathbb{R}^{2n})} && \text{(proerty (4.3.4))}\\ &= \prec J^t a, \Omega_{u,v}(0) \succ_{\mathscr{S}'(\mathbb{R}^{2n}),\mathscr{S}(\mathbb{R}^{2n})} && \text{(easy identity for } J^t)\\ &= \langle (J^t a)(x,D)u, v\rangle_{\mathscr{S}^*(\mathbb{R}^n),\mathscr{S}(\mathbb{R}^n)}\end{aligned}$$

completing the proof. □

Remark 4.3.3. We get in particular that

$$a(x,D)^* = \mathrm{op}_1(\bar a) = (J\bar a)(x,D),$$

a formula which in fact motivates the study of the group J^t. On the other hand, using the Weyl quantization simplifies somewhat the matter of taking adjoints since we have,

$$\big(\mathrm{op}_{1/2}(a)\big)^* = \big(\mathrm{op}_0(J^{1/2}a)\big)^* = \mathrm{op}_0(J(\overline{J^{1/2}a})) = \mathrm{op}_0(J^{1/2}\bar a) = \mathrm{op}_{1/2}(\bar a)$$

and in particular **if a is real-valued, $\mathrm{op}_{1/2}(a)$ is formally selfadjoint.**

[10] In fact the linear mapping $\mathbb{R}^n \times \mathbb{R}^n \ni (x,z) \mapsto (x-tz, x+(1-t)z)$ has determinant 1 and $\Omega_{u,v}(t)$ appears as the partial Fourier transform of the function $\mathbb{R}^n \times \mathbb{R}^n \ni (x,z) \mapsto \bar v(x-tz)u(x+(1-t)z)$, which is in the Schwartz class.

Composition formula

It is easy to see that the operator a^w is continuous from $\mathscr{S}(\mathbb{R}^n)$ into itself whenever $a \in C_b^\infty(\mathbb{R}^{2n})$ and thus in particular when $a \in \mathscr{S}(\mathbb{R}^{2n})$. For $a, b \in \mathscr{S}(\mathbb{R}^{2n})$, we obtain

$$a^w b^w = \iint_{\mathbb{R}^{2n}\times\mathbb{R}^{2n}} a(Y)b(Z)2^{2n}\sigma_Y\sigma_Z dY dZ.$$

We get $a^w b^w = (a\sharp b)^w$ with

$$(a\sharp b)(X) = 2^{2n}\iint_{\mathbb{R}^{2n}\times\mathbb{R}^{2n}} e^{-4i\pi[X-Y,X-Z]}a(Y)b(Z)dY dZ. \tag{4.3.5}$$

We can compare this with the classical composition formula, a namely $\operatorname{op}(a)\operatorname{op}(b) = \operatorname{op}(a\diamond b)$ with

$$(a\diamond b)(x,\xi) = \iint_{\mathbb{R}^n\times\mathbb{R}^n} e^{-2i\pi y\cdot\eta}a(x,\xi+\eta)b(y+x,\xi)dyd\eta.$$

Another method to perform that calculation would be to use the kernels of the operators a^w, b^w. For future reference, we note that the distribution kernel k_a of the operator $a(x,D)$ (for $a\in\mathscr{S}'(\mathbb{R}^{2n})$) is

$$k_a(x,y) = \int e^{2i\pi(x-y)\cdot\xi}a(x,\xi)d\xi = \widehat{a}^2(x,y-x) \tag{4.3.6}$$

so that $\widehat{a}^2(x,y) = k_a(x,y+x)$ and in the distribution sense

$$a(x,\xi) = \int e^{2i\pi y\cdot\xi}k_a(x,y+x)dy. \tag{4.3.7}$$

The distribution kernel κ_a of the operator a^w (for $a\in\mathscr{S}'(\mathbb{R}^{2n})$) is (in the distribution sense)

$$\kappa_a(x,y) = \int e^{2i\pi(x-y)\cdot\xi}a(\frac{x+y}{2},\xi)d\xi \tag{4.3.8}$$

so that $\kappa_a(x-\frac{t}{2},x+\frac{t}{2}) = \int e^{-2i\pi t\cdot\xi}a(x,\xi)d\xi = \widehat{a}^2(x,t)$ and thus

$$a(x,\xi) = \int \kappa_a(x-\frac{t}{2},x+\frac{t}{2})e^{2i\pi t\xi}dt. \tag{4.3.9}$$

Remark 4.3.4. For $a_j \in S^{m_j}_{1,0}, j=1,2$ we have

$$a_1\sharp a_2 = a_1a_2 + \frac{1}{4i\pi}\{a_1,a_2\} \mod S^{m_1+m_2-2}_{1,0}, \tag{4.3.10}$$

$$a_1\sharp a_2 + a_2\sharp a_1 = 2a_1a_2 \mod S^{m_1+m_2-2}_{1,0}, \tag{4.3.11}$$

$$a_1\sharp a_2 - a_2\sharp a_1 = \frac{1}{2i\pi}\{a_1,a_2\} \mod S^{m_1+m_2-3}_{1,0}. \tag{4.3.12}$$

4.4 Fefferman-Phong inequality

The Fefferman-Phong inequality would deserve a full lecture and is certainly too difficult to be thoroughly treated in a simple appendix. We refer the reader to the detailed treatment given in Section 2.5.3 of [24] or to Theorem 18.6.8 in [13]. For the reader interested solely in Carleman estimates, it should be noted that Fefferman-Phong's inequality was used only to tackle the class of principally normal operators with complex coefficients.

4.5 Riemannian-Lorentzian geometry glossary

4.5.1 Differential geometry

We assume that the reader is familiar with the notion of differentiable manifold, exterior differentiation and tensors. Let X be a vector field and ω be a p-form. We define the Lie derivative $\mathscr{L}_X(\omega)$ as

$$\mathscr{L}_X(\omega) = d(\omega\rfloor X) + d\omega\rfloor X, \tag{4.5.1}$$

where $\rfloor$ stands for the interior product: for a p-form ω and vector fields $X, Y_2, \dots, Y_p$,

$$\langle \omega\rfloor X, Y_2 \wedge \dots \wedge Y_p\rangle = \langle \omega, X \wedge Y_2 \wedge \dots \wedge Y_p\rangle$$

In particular if f is a function, we have $\mathscr{L}_X(f) = \langle df, X\rangle = Xf$. The Lie derivative preserves tensor type and acts as a derivation on tensor products:

$$\mathscr{L}_X(S \otimes T) = \mathscr{L}_X(S) \otimes T + S \otimes \mathscr{L}_X(T). \tag{4.5.2}$$

For X, Y vector fields, we have

$$\mathscr{L}_X(Y) = [X, Y]. \tag{4.5.3}$$

Indeed for a function f, using that the Lie derivative obeys Leibniz' rule with respect to contraction, we get

$$\langle df, \mathscr{L}_X(Y)\rangle = \mathscr{L}_X(\langle df, Y\rangle) - \langle \mathscr{L}_X(df), Y\rangle = XYf - \langle d(df\rfloor X), Y\rangle = XYf - YXf.$$

On the other hand, the Lie derivative commutes with the exterior differentiation: for ω a p-form, we have

$$\mathscr{L}_X(d\omega) = d(\mathscr{L}_X\omega). \tag{4.5.4}$$

This follows from (4.5.1): $d(\mathscr{L}_X\omega) = d(d\omega\rfloor X) = \mathscr{L}_X(d\omega)$.

4.5.2 Riemaniann-Lorentzian geometry

Let $(\mathcal{M}, g)$ be a Riemannian (resp. Lorentzian) manifold, i.e. a differentiable manifold equipped with a section of the bundle of $(0, 2)$ tensors which is positive definite (resp. non degenerate with index 1). In a coordinate chart W it means that we are given a smooth mapping $W \ni x \mapsto g(x)$ which is a symmetric $n \times n$ matrix, positive definite in the Riemaniann case, with signature $(n - 1, 1)$ in the Lorentzian case.

The Laplace-Beltrami operator Δ_g in the Riemannian case, the wave operator $\square_g$ in the Lorentzian case

are given in a coordinate chart by

$$|g|^{-1/2} \sum_{1\le j,k\le n} \frac{\partial}{\partial x^j} |g|^{1/2} g^{jk}(x) \frac{\partial}{\partial x^k}, \quad |g| = |\det g|, \quad (g^{jk}(x)) = g(x)^{-1}. \tag{4.5.5}$$

We note the formal selfadjointness of that operator: for u, v smooth compactly supported in a coordinate chart W, we have, using Einstein convention,

$$\begin{aligned}\langle \square_g u, v\rangle_{L^2(\mathcal{M})} &= \int_{\mathbb{R}^n} |g(x)|^{-1/2} \frac{\partial}{\partial x^j}\Big(|g|^{1/2} g^{jk}(x) \frac{\partial u}{\partial x^k}\Big) \bar v(x) |g(x)|^{1/2} dx \\ &= -\int_{\mathbb{R}^n} |g|^{1/2} g^{jk}(x) \frac{\partial u}{\partial x^k}\frac{\partial \bar v}{\partial x^j} dx \\ &= \int_{\mathbb{R}^n} |g(x)|^{-1/2} \frac{\partial}{\partial x^k}\Big(|g|^{1/2} g^{jk}(x) \frac{\partial \bar v}{\partial x^j}\Big) u(x) |g(x)|^{1/2} dx \\ &= \langle u, \square_g v\rangle_{L^2(\mathcal{M})}.\end{aligned}$$

The Levi-Civita connection D

acts linearly on tensors, is a derivation with respect to contraction, preserves the metric $Dg = 0$, and is torsion-free: for X, Y vector fields

$$D_X(Y) - D_Y(X) = [X, Y]. \tag{4.5.6}$$

For X, Y, Z vector fields, we have

$$\begin{aligned}&X(g(Y,Z)) + Y(g(Z,X)) - Z(g(X,Y)) \\ &= g(D_X(Y), Z) + g(Y, D_X(Z)) + g(D_Y(Z), X) + g(Z, D_Y(X)) \\ &\qquad\qquad - g(D_Z(X), Y) - g(X, D_Z(Y)) \\ &= \big(D_X(Y) + D_Y(X), Z\big)_g + \big([X,Z], Y\big)_g + \big([Y,Z], X\big)_g \\ &= \big(2D_X(Y) - [X,Y], Z\big)_g + \big([X,Z], Y\big)_g + \big([Y,Z], X\big)_g\end{aligned}$$

so that

$$\begin{aligned}\big(2D_X(Y), Z\big)_g = X(g(Y,Z)) &+ Y(g(Z,X)) - Z(g(X,Y)) \\ &- \big([X,Z], Y\big)_g - \big([Y,Z], X\big)_g + \big([X,Y], Z\big)_g,\end{aligned}$$

proving the determination of the Levi-Civita connection by the previous axioms. We may define the *Christoffel symbols*

$$D_{e_j}(e_k) = \Gamma^l_{jk} e_l, \qquad \text{(note that (4.5.6) implies } \Gamma^l_{jk} = \Gamma^l_{kj}).$$

Since we have from the previous formula

$$\frac{1}{2}\Big(\partial_{x_j}(g_{km}) + \partial_{x_k}(g_{jm}) - \partial_{x_m}(g_{jk})\Big) = (D_{e_j}(e_k), e_m)_g = \Gamma^l_{jk} g_{lm}$$

we obtain

$$\frac{g^{mp}}{2}\Big(\partial_{x_j}(g_{km})+\partial_{x_k}(g_{jm})-\partial_{x_m}(g_{jk})\Big)=\Gamma^l_{jk}g_{lm}g^{mp}=\Gamma^l_{jk}\delta_{l,p}$$

so that

$$\Gamma^l_{jk}=\frac{g^{lm}}{2}\Big(\partial_{x_j}(g_{mk})+\partial_{x_k}(g_{mj})-\partial_{x_m}(g_{jk})\Big). \tag{4.5.7}$$

The gradient, the Hessian.

Let f be a smooth function we define the vector field ∇f by the identity satisfied for any vector field X,

$$\big(\nabla f, X\big)_g=\langle df, X\rangle = Xf, \quad \text{so that } \nabla f = g^{-1}df.$$

The Hessian of f is $\nabla^2 f$ which is a $(0,2)$ tensor: for X, Y vector fields, we have

$$D_X\big(\nabla f, Y\big)_g=\big(D_X\nabla f, Y\big)_g+\big(\nabla f, D_XY\big)_g,$$

so that $\big(D_X\nabla f, Y\big)_g = XYf - D_X(Y)f$. This gives as well

$$\big(D_Y\nabla f, X\big)_g = YXf - D_Y(X)f.$$

and we note that

$$XYf-D_X(Y)f=YXf-D_Y(X)f,$$

since the Levi-Civita connection is torsion free (see (4.5.6)): $[X,Y]=D_X(Y)-D_Y(X)$. As a result

$$\nabla^2 f(X,Y)=\frac{1}{2}\big(XY+YX-D_X(Y)-D_Y(X)\big)f = XYf-D_X(Y)f. \tag{4.5.8}$$

In coordinate, we get

$$\nabla^2 f(e_j,e_k)=\frac{\partial^2 f}{\partial x^j\partial x^k}-\Gamma^l_{jk}\frac{\partial f}{\partial x^l}. \tag{4.5.9}$$

We have also indeed a (0,2) symmetric tensor since

$$\nabla^2 f(a_j\frac{\partial}{\partial x^j}, a_k\frac{\partial}{\partial x^k}) = a_ja_k\frac{\partial^2 f}{\partial x^j\partial x^k}+a_j\frac{\partial a_k}{\partial x_j}\frac{\partial f}{\partial x_k}-D_{a_je_j}(a_ke_k)f$$

$$=a_ja_k\frac{\partial^2 f}{\partial x^j\partial x^k}+a_j\frac{\partial a_k}{\partial x_j}\frac{\partial f}{\partial x_k}-a_j\Gamma^l_{jk}a_k\frac{\partial f}{\partial x^l}-a_j\frac{\partial a_k}{\partial x^j}\frac{\partial f}{\partial x_k},$$

which coincides with (4.5.9).

Bibliography

[1] S. Alinhac, *Non-unicité du problème de Cauchy*, Ann. of Math. (2) **117** (1983), no. 1, 77–108. MR 683803 (85g:35011)

[2] S. Alinhac and M. S. Baouendi, *A nonuniqueness result for operators of principal type*, Math. Z. **220** (1995), no. 4, 561–568. MR 1363855 (96j:35003)

[3] Richard Beals and Charles Fefferman, *On local solvability of linear partial differential equations*, Ann. of Math. (2) **97** (1973), 482–498. MR 0352746 (50 #5233)

[4] A.-P. Calderón, *Uniqueness in the Cauchy problem for partial differential equations.*, Amer. J. Math. **80** (1958), 16–36. MR 0104925 (21 #3675)

[5] T. Carleman, *Sur un problème d'unicité pur les systèmes d'équations aux dérivées partielles à deux variables indépendantes*, Ark. Mat., Astr. Fys. **26** (1939), no. 17, 9. MR 0000334 (1,55f)

[6] A. Doubova, A. Osses, and J.-P. Puel, *Exact controllability to trajectories for semilinear heat equations with discontinuous diffusion coefficients*, ESAIM Control Optim. Calc. Var. **8** (2002), 621–661, A tribute to J. L. Lions.

[7] N. Filonov, *Second-order elliptic equation of divergence form having a compactly supported solution*, J. Math. Sci. (New York) **106** (2001), no. 3, 3078–3086, Function theory and phase transitions.

[8] Lars Hörmander, *Linear partial differential operators*, Springer Verlag, Berlin, 1976. MR 0404822 (53 #8622)

[9] ______, *The analysis of linear partial differential operators. IV*, Grundlehren der Mathematischen Wissenschaften [Fundamental Principles of Mathematical Sciences], vol. 275, Springer-Verlag, Berlin, 1985, Fourier integral operators. MR 781537 (87d:35002b)

[10] ______, *The analysis of linear partial differential operators. IV*, Grundlehren der Mathematischen Wissenschaften [Fundamental Principles of Mathematical Sciences], vol. 275, Springer-Verlag, Berlin, 1994, Fourier integral operators, Corrected reprint of the 1985 original. MR 1481433 (98f:35002)

[11] ———, *On the uniqueness of the Cauchy problem under partial analyticity assumptions*, Geometrical optics and related topics (Cortona, 1996), Progr. Nonlinear Differential Equations Appl., vol. 32, Birkhäuser Boston, Boston, MA, 1997, pp. 179–219. MR 2033496

[12] ———, *The analysis of linear partial differential operators. I*, Classics in Mathematics, Springer-Verlag, Berlin, 2003, Distribution theory and Fourier analysis, Reprint of the second (1990) edition [Springer, Berlin; MR1065993 (91m:35001a)]. MR 1996773

[13] ———, *The analysis of linear partial differential operators. III*, Classics in Mathematics, Springer, Berlin, 2007, Pseudo-differential operators, Reprint of the 1994 edition. MR 2304165 (2007k:35006)

[14] Alexandru D. Ionescu and Sergiu Klainerman, *On the uniqueness of smooth, stationary black holes in vacuum*, Invent. Math. **175** (2009), no. 1, 35–102. MR 2461426 (2009j:83053)

[15] ———, *Uniqueness results for ill-posed characteristic problems in curved space-times*, Comm. Math. Phys. **285** (2009), no. 3, 873–900. MR 2470908 (2010j:83075)

[16] V. Ja. Ivriĭ and V. M. Petkov, *Necessary conditions for the correctness of the Cauchy problem for non-strictly hyperbolic equations*, Uspehi Mat. Nauk **29** (1974), no. 5(179), 3–70, Collection of articles dedicated to the memory of Ivan Georgievič Petrovskiĭ (1901–1973), III. MR 0427843 (55 #873)

[17] Peter D. Lax, *Asymptotic solutions of oscillatory initial value problems*, Duke Math. J. **24** (1957), 627–646. MR 0097628 (20 #4096)

[18] Jérôme Le Rousseau and Nicolas Lerner, *Carleman estimates for anisotropic elliptic operators with jumps at an interface*, Anal. PDE **6** (2013), no. 7, 1601–1648. MR 3148062

[19] Jérôme Le Rousseau and Luc Robbiano, *Local and global Carleman estimates for parabolic operators with coefficients with jumps at interfaces*, Invent. Math. **183** (2011), no. 2, 245–336. MR 2772083 (2012k:35208)

[20] Nicolas Lerner, *Unicité de Cauchy pour des opérateurs différentiels faiblement principalement normaux*, J. Math. Pures Appl. (9) **64** (1985), no. 1, 1–11. MR 802381 (86j:35006)

[21] ———, *Energy methods via coherent states and advanced pseudo-differential calculus*, Multidimensional complex analysis and partial differential equations (São Carlos, 1995), Contemp. Math., vol. 205, Amer. Math. Soc., Providence, RI, 1997, pp. 177–201. MR MR1447224 (98d:35244)

[22] ______, *Perturbation and energy estimates*, Ann. Sci. École Norm. Sup. (4) **31** (1998), no. 6, 843–886. MR MR1664214 (2000b:35289)

[23] ______, *When is a pseudo-differential equation solvable?*, Ann. Inst. Fourier (Grenoble) **50** (2000), no. 2, 443–460. MR MR1775357 (2001e:35188)

[24] ______, *Metrics on the phase space and nonselfadjoint pseudodifferential operators*, Pseudo-Differential Operators. Theory and Applications, vol. 3, Birkhäuser Verlag, Basel, 2010. MR 2599384 (2011b:35002)

[25] G. Métivier, *Counterexamples to Hölmgren's uniqueness for analytic nonlinear Cauchy problems*, Invent. Math. **112** (1993), no. 1, 217–222. MR 1207483 (94b:35010)

[26] K. Miller, *Nonunique continuation for uniformly parabolic and elliptic equations in self-adjoint divergence form with Hölder continuous coefficients*, Arch. Rational Mech. Anal. **54** (1974), 105–117.

[27] Sigeru Mizohata, *Some remarks on the Cauchy problem*, J. Math. Kyoto Univ. **1** (1961/1962), 109–127. MR 0170112 (30 #353)

[28] A. Pliś, *A smooth linear elliptic differential equation without any solution in a sphere*, Comm. Pure Appl. Math. **14** (1961), 599–617. MR 0136846 (25 #307)

[29] ______, *On non-uniqueness in Cauchy problem for an elliptic second order differential equation*, Bull. Acad. Polon. Sci. Sér. Sci. Math. Astronom. Phys. **11** (1963), 95–100.

[30] Luc Robbiano and Claude Zuily, *Uniqueness in the Cauchy problem for operators with partially holomorphic coefficients*, Invent. Math. **131** (1998), no. 3, 493–539. MR 1614547 (99e:35004)

[31] Jean-Pierre Rosay, *CR functions vanishing on open sets. (Almost) complex structures and Cohen's example*, Indag. Math. (N.S.) **9** (1998), no. 2, 289–303. MR 1691491 (2000d:32053)

[32] Xavier Saint Raymond, *L'unicité pour les problèmes de Cauchy linéaires du premier ordre*, Enseign. Math. (2) **32** (1986), no. 1-2, 1–55. MR 850550 (87m:35041)

[33] ______, *Remarks on Gårding inequalities for differential operators*, Ann. Sc. Norm. Super. Pisa Cl. Sci. (5) **1** (2002), no. 1, 169–185. MR 1994806 (2004f:35200)

[34] Monty Strauss and François Trèves, *First-order linear PDEs and uniqueness in the Cauchy problem*, J. Differential Equations **15** (1974), 195–209. MR 0330739 (48 #9076)

[35] Daniel Tataru, *Unique continuation for solutions to partial differential equations; between Hörmander's theorem and Holmgren's theorem*, Comm. Partial Differential Equations **20** (1995), no. 5-6, 855–884.

[36] François Trèves, *Basic linear partial differential equations*, Academic Press [A subsidiary of Harcourt Brace Jovanovich, Publishers], New York-London, 1975, Pure and Applied Mathematics, Vol. 62. MR 0447753 (56 #6063)

[37] Claude Zuily, *Uniqueness and nonuniqueness in the Cauchy problem*, Progress in Mathematics, vol. 33, Birkhäuser Boston Inc., Boston, MA, 1983. MR 701544 (85d:35009)