Lecture Notes on
Partial Differential Equations
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Contents

Chapter 1

Introduction

1.1 Examples

What is a partial differential equation? Although the question may look too general, it is certainly a natural one for the reader opening these notes with the expectation of learning things about *PDE*, the acronym of Partial Differential Equations. Loosely speaking it is a relation involving a function u of several real variables $x_1, \ldots, x_n$ with its partial derivatives

$$\frac{\partial u}{\partial x_j}, \ \frac{\partial^2 u}{\partial x_j \partial x_k}, \ \frac{\partial^3 u}{\partial x_j \partial x_k \partial x_l}, \ldots$$

Maybe a simple example would be a better starting point than a general (and vague) definition: let us consider a C^1 function u defined on $\mathbb{R}^2$ and let $c > 0$ be given. The PDE

$$\partial_t u + c \partial_x u = 0 \qquad \textbf{(Transport Equation)} \tag{1.1.1}$$

is describing a propagation phenomenon at speed c, and a solution is given by

$$u(t, x) = \omega(x - ct), \quad \omega \in C^1(\mathbb{R}). \tag{1.1.2}$$

We have indeed $\partial_t u + c \partial_x u = \omega'(x - ct)\big(-c + c\big) = 0$. Note also that if u has the dimension of a length $\mathtt{L}$ and c of a speed $\mathtt{LT}^{-1}$, $\partial_t u$ and $c \partial_x u$ have respectively the dimension $\mathtt{LT}^{-1}$, $\mathtt{LT}^{-1}\mathtt{LL}^{-1}$ i.e. (fortunately) both $\mathtt{LT}^{-1}$. At time $t = 0$, we have $u(0, x) = \omega(x)$ and at time $t = 1$, we have $u(1, x) = \omega(x - c)$ so that ω is translated (at speed c) to the right when time increases. The equation (1.1.1) is a linear PDE, namely, if u_1, u_2 are solutions, then $u_1 + u_2$ is also a solution as well as any linear combination $c_1 u_1 + c_2 u_2$ with constants c_1, c_2. Looking at (1.1.1) as an evolution equation with respect to the time variable t, we may already ask the following question: knowing u at time 0, say $u(0, x) = \omega(x)$, is it true that (1.1.2) is the unique solution? In other words, we can set the so-called **Cauchy problem,**[1]

$$\begin{cases} \partial_t u + c \partial_x u = 0, \\ u(0) = \omega, \end{cases} \tag{1.1.3}$$

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[1] Augustin L. Cauchy (1789-1857) is a French mathematician, a prominent scientific figure of the nineteenth century, who laid many foundational concepts of infinitesimal calculus; more is available on the website [17].

and ask the question of determinism: is the law of evolution (i.e. the transport equation) and the initial state of the system (that is ω) determine uniquely the solution u? We shall see that the answer is yes. Another interesting and natural question about (1.1.1) concerns the regularity of u: of course a classical solution should be differentiable, just for the equation to make sense but, somehow, this is a pity since we would like to accept as a solution $u(t,x) = |x - ct|$ and in fact all functions $\omega(x - ct)$. We shall see that Distribution theory will provide a very complete answer to this type of questions for linear equations.

Let us consider now for u of class C^1 on $\mathbb{R}^2$,

$$\partial_t u + u\partial_x u = 0. \qquad \textbf{(Burgers Equation)} \qquad (1.1.4)$$

That equation[2] is not linear, but one may look at a linear companion equation in three independent variables (t, x, y) given by $\partial_t U + y\partial_x U = 0$. It is easy to see that $U(t, x, y) = x - y(t - T)$ is a solution of the latter equation (here T is a constant). Let us now take a function $u(t, x)$ such that $x - u(t, x)(t - T) = x_0$, where x_0 is a constant, i.e.

$$u(t,x) = \frac{x - x_0}{t - T}.$$

Now, we can verify that for $t \neq T$, the function u is a solution of (1.1.4): we check

$$\partial_t u + u\partial_x u = -\frac{(x - x_0)}{(t - T)^2} + \frac{x - x_0}{t - T}\frac{1}{t - T} = 0.$$

We shall go back to this type of equation later on, but we can notice already an interesting phenomenon for this solution: assume $T > 0$, $x_0 = 0$, then the solution at $t = 0$ is $-x/T$ (perfectly smooth and decreasing) and it blows up at $t = T$. If on the contrary, we assume $T < 0$, $x_0 = 0$, the solution at $t = 0$ is is $-x/T$ (perfectly smooth and increasing), remains smooth for all times larger than T, but blows up in the past at time $t = T$.

The **Laplace equation**[3] is the second-order PDE, $\Delta u = 0$, with

$$\Delta u = \sum_{1 \leq j \leq n} \frac{\partial^2 u}{\partial x_j^2}. \qquad (1.1.5)$$

This is a linear equation and it is called second-order because it involves partial derivatives of order at most 2. The solutions of the Laplace equation are called harmonic functions. Let us determine all the harmonic polynomials in two dimensions. Denoting the variables $(x, y) \in \mathbb{R}^2$, the equation can be written as

$$(\partial_x + i\partial_y)(\partial_x - i\partial_y)u = 0.$$

Since u is assumed to be a polynomial, we can write

$$u(x,y) = \sum_{(k,l) \in \mathbb{N}^2} u_{k,l}(x + iy)^k(x - iy)^l, \quad u_{k,l} \in \mathbb{C}, \text{ all 0 but a finite number.}$$

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[2]JAN M. BURGERS (1895-1981) is a Dutch physicist.
[3]PIERRE-SIMON LAPLACE (1749-1827) is a French mathematician, see [17].

Now we note that $(\partial_x + i\partial_y)(x+iy)^l = l(x+iy)^{l-1}(1+i^2) = 0$ and $(\partial_x - i\partial_y)(x-iy)^l = 0$. As a result u is a harmonic polynomial if $u_{k,l} = 0$ when $kl \neq 0$. Conversely, noting that $(\partial_x + i\partial_y)(x-iy)^l = l(x-iy)^{l-1}2$ and $(\partial_x - i\partial_y)(x+iy)^k = k(x+iy)^{k-1}2$, we have (the finite sum)

$$\Delta u = \sum_{(k,l)\in(\mathbb{N}^*)^2} u_{k,l} 4kl(x+iy)^{k-1}(x-iy)^{l-1}$$

and thus for $kl \neq 0$, $u_{k,l} = 0$, from the following remark: *If the polynomial $P = \sum_{p,q\in\mathbb{N}} a_{p,q} z^p \bar{z}^q$ vanishes identically for $z \in \mathbb{C}$, then all $a_{p,q} = 0$.* To prove this remark, we shall note with $z = x + iy$,

$$\frac{\partial}{\partial\bar{z}} = \frac{1}{2}\Big(\frac{\partial}{\partial x} + i\frac{\partial}{\partial y}\Big), \quad \frac{\partial}{\partial z} = \frac{1}{2}\Big(\frac{\partial}{\partial x} - i\frac{\partial}{\partial y}\Big), \text{ so that } \frac{\partial}{\partial\bar{z}}\bar{z} = 1, \frac{\partial}{\partial\bar{z}}z = 0, \frac{\partial}{\partial z}\bar{z} = 0, \frac{\partial}{\partial z}z = 1,$$

$$0 = \frac{1}{p!q!}\Big(\frac{\partial^p}{\partial\bar{z}^p}\frac{\partial^q}{\partial z^q}P\Big)(0) = a_{p,q}.$$

Finally the harmonic polynomials in two dimensions are

$$u(x,y) = f(x+iy) + g(x-iy), \quad f,g \text{ polynomials in } \mathbb{C}[X]. \tag{1.1.6}$$

Requiring moreover that they should be real-valued leads to, using the standard notation $x + iy = re^{i\theta}$, $r \geq 0, \theta \in \mathbb{R}$,

$$u(x,y) = \sum_{k\in\mathbb{N}} \operatorname{Re}\big((a_k - ib_k)(x+iy)^k\big) = \sum_{k\in\mathbb{N}} r^k \operatorname{Re}\big((a_k - ib_k)e^{i\theta k}\big)$$

$$= \sum_{k\in\mathbb{N}} \big(a_k \cos(k\theta) + b_k \sin(k\theta)\big)r^k, \quad a_k, b_k \in \mathbb{R} \text{ all 0 but a finite number.}$$

We see also that for a sequence $(c_k)_{k\in\mathbb{Z}} \in \ell^1$,

$$v(x,y) = c_0 + \sum_{k\in\mathbb{N}^*} (c_k z^k + c_{-k}\bar{z}^k) \tag{1.1.7}$$

is a harmonic function in the unit disk $D_1 = \{z \in \mathbb{C}, |z| < 1\}$ such that

$$v_{|\partial D_1}(e^{i\theta}) = \sum_{k\in\mathbb{Z}} c_k e^{ik\theta}.$$

As a consequence the function (1.1.7) is solving the **Dirichlet problem**[4] for the Laplace operator in the unit disk D_1 with

$$\begin{cases} \Delta v = 0 & \text{on } D_1, \\ v = \nu & \text{on } \partial D_1, \end{cases} \tag{1.1.8}$$

where ν is given by its Fourier series expansion $\nu(e^{i\theta}) = \sum_{k\in\mathbb{Z}} c_k e^{ik\theta}$. The **boundary condition** $v = \nu$ on ∂D_1 is called a Dirichlet boundary condition. The Laplace

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[4] JOHANN P. DIRICHLET (1805-1859) is a German mathematician, see [17].

equation is a "stationary" equation, i.e. there is no time variable and that boundary condition should not be confused with an initial condition occurring for the Cauchy problem (1.1.3).

The **eikonal equation** is a non-linear equation

$$|\nabla \phi| = 1, \quad \text{i.e.} \quad \sum_{1 \le j \le n} |\partial_{x_j} \phi|^2 = 1. \tag{1.1.9}$$

Note that for $\xi \in \mathbb{R}^n$ with Euclidean norm equal to 1, $\phi(x) = \xi \cdot x$ is a solution of (1.1.9). The notation $\nabla \phi$ (`nabla` ϕ) stands for the vector

$$\nabla \phi = \left(\frac{\partial \phi}{\partial x_1}, \dots, \frac{\partial \phi}{\partial x_n} \right). \tag{1.1.10}$$

We shall study as well the **Hamilton-Jacobi equation**[5]

$$\partial_t u + H(x, \nabla u) = 0, \tag{1.1.11}$$

which is a non-linear evolution equation.

The **Helmholtz**[6] **equation** $-\Delta u = \lambda u$ is a linear equation closely related to the Laplace equation and to the **wave equation**, also linear second order,

$$\frac{1}{c^2} \frac{\partial^2 u}{\partial t^2} - \Delta_x u = 0, \quad t \in \mathbb{R}, x \in \mathbb{R}^n, \quad c > 0 \text{ is the speed of propagation.} \tag{1.1.12}$$

Note that if u has the dimension of a length $\mathtt{L}$, then $c^{-2} \partial_t^2 u$ has the dimension $\mathtt{L}^{-2} \mathtt{T}^2 \mathtt{T}^{-2} \mathtt{L} = \mathtt{L}^{-1}$ as well as $\Delta_x u$ which has dimension $\mathtt{L}^{-2} \mathtt{L} = \mathtt{L}^{-1}$. It is interesting to note that for any $\xi \in \mathbb{R}^n$ with $\sum_j \xi_j^2 = 1$, and ω of class C^2 on $\mathbb{R}$

$$u(t, x) = \omega(\xi \cdot x - ct)$$

is a solution of (1.1.12) since $c^{-2} \omega''(\xi \cdot x - ct) c^2 - \sum_{1 \le j \le n} \omega''(\xi \cdot x - ct) \xi_j^2 = 0$.

We shall study in the sequel many other linear equations, such as the **heat equation**,

$$\frac{\partial u}{\partial t} - \Delta_x u, \quad t \in \mathbb{R}_+, x \in \mathbb{R}^n,$$

and the **Schrödinger equation**,

$$\frac{1}{i} \frac{\partial u}{\partial t} - \Delta_x u, \quad t \in \mathbb{R}, x \in \mathbb{R}^n.$$

Although the two previous equations look similar, they are indeed very different. The Schrödinger[7] equation is a propagation equation which is time-reversible: assume that $u(t, x)$ solves on $\mathbb{R} \times \mathbb{R}^n$, $i \partial_t u + \Delta u = 0$, then $v(t, x) = u(-t, x)$ will satisfy

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[5] Sir WILLIAM HAMILTON (1805-1865) is an Irish mathematician, physicist and astronomer. CARL GUSTAV JACOBI (1804-1851) is a Prussian mathematician.

[6] HERMANN VON HELMHOLTZ (1821-1894) is a German mathematician.

[7] ERWIN SCHRÖDINGER (1887-1961) is an Austrian physicist, author of fundamental contributions to quantum mechanics.

$-i\partial_t v + \Delta v = 0$ on $\mathbb{R} \times \mathbb{R}^n$. The term propagation equation is due to the fact that for $\xi \in \mathbb{R}^n$ and

$$u(t,x) = e^{i(x\cdot\xi - t|\xi|^2)},$$

we have

$$\frac{1}{i}\partial_t u - \Delta u = -|\xi|^2 e^{i(x\cdot\xi - t|\xi|^2)} - \sum_j i^2 \xi_j^2 e^{i(x\cdot\xi - t|\xi|^2)} = 0,$$

so that, comparing to the transport equation (1.1.1), the Schrödinger equation behaves like a propagation equation where the speed of propagation depends on the frequency of the initial wave $\omega(x) = e^{ix\cdot\xi}$. On the other hand the heat equation is a **diffusion equation**, modelling the evolution of the temperature distribution: this equation is time-irreversible. First of all, if $u(t,x)$ solves $\partial_t u - \Delta u = 0$ on $\mathbb{R}_+ \times \mathbb{R}^n$, then $v(t,x) = u(-t,x)$ solves $\partial_t v + \Delta u = 0$ on the different domain $\mathbb{R}_- \times \mathbb{R}^n$; moreover, for $\xi \in \mathbb{R}^n$ the function

$$v(t,x) = e^{ix\cdot\xi} e^{-t|\xi|^2}$$

satisfies

$$\partial_t v - \Delta v = -|\xi|^2 v(t,x) - \sum_j i^2 \xi_j^2 v = 0, \quad \text{with } v(0,x) = e^{ix\cdot\xi}.$$

In particular $v(t = 0)$ is a bounded function in $\mathbb{R}^n$ and $v(t)$ remains bounded for $t > 0$ whereas it is exponentially increasing for $t < 0$. It is not difficult to prove that there is no bounded solution $v(t,x)$ of the heat equation on the whole real line satisfying $v(0,x) = e^{ix\cdot\xi}$ ($\xi \neq 0$).

So far, we have seen only scalar PDE, i.e. equations involving the derivatives of a single scalar-valued function $\mathbb{R}^n \ni x \mapsto u(x) \in \mathbb{R}, \mathbb{C}$. Many very important equations of mathematical physics are in fact **systems of PDE**, dealing with the partial derivatives of vector-valued functions $\mathbb{R}^n \ni x \mapsto u(x) \in \mathbb{R}^N$. A typical example is **Maxwell's equations**[8], displayed below in vacuum. For $(t,x) \in \mathbb{R} \times \mathbb{R}^3$, the electric field $E(t,x)$ belongs to $\mathbb{R}^3$ and the magnetic field $B(t,x)$ belongs to $\mathbb{R}^3$ with

$$\begin{cases} \partial_t E = \operatorname{curl} B = \begin{pmatrix} \partial_{x_1} \\ \partial_{x_2} \\ \partial_{x_3} \end{pmatrix} \times \begin{pmatrix} B_1 \\ B_2 \\ B_3 \end{pmatrix} = \begin{pmatrix} \partial_2 B_3 - \partial_3 B_2 \\ \partial_3 B_1 - \partial_1 B_3 \\ \partial_1 B_2 - \partial_2 B_1 \end{pmatrix}, \\ \partial_t B = -\operatorname{curl} E = \begin{pmatrix} \partial_3 E_2 - \partial_2 E_3 \\ \partial_1 E_3 - \partial_3 E_1 \\ \partial_2 E_1 - \partial_1 E_2 \end{pmatrix}, \\ \operatorname{div} E = \operatorname{div} B = 0, \end{cases} \tag{1.1.13}$$

with $\operatorname{div} E = \partial_1 E_1 + \partial_2 E_2 + \partial_3 E_3$. The previous system is a linear one, whereas the following, **Euler's system for incompressible fluids**[9], is non-linear: the velocity

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[8]JAMES C. MAXWELL (1831-1879) is a Scottish theoretical physicist and mathematician.

[9]LEONHARD EULER (1707 -1783) is a mathematician and physicist, born in Switzerland, who worked mostly in Germany and Russia.

field $v(t,x) = (v_1, v_2, v_3)$ and the pressure (a scalar) $p(t,x)$ should satisfy

$$\begin{cases} \partial_t v + (v \cdot \nabla)v = -\nabla(p/\rho) \\ \operatorname{div} v = 0 \\ v_{|t=0} = w \end{cases} \tag{1.1.14}$$

where $v \cdot \nabla = v_1 \partial_1 + v_2 \partial_2 + v_3 \partial_3$, ρ is the mass density, so that the system is

$$\begin{pmatrix} \partial_t v_1 + \sum_j v_j \partial_j v_1 + \partial_1 (p/\rho) \\ \partial_t v_2 + \sum_j v_j \partial_j v_2 + \partial_2 (p/\rho) \\ \partial_t v_3 + \sum_j v_j \partial_j v_3 + \partial_3 (p/\rho) \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}, \quad \sum_j \partial_j v_j = 0.$$

Note that v has dimension $\mathtt{LT}^{-1}$, so that $\partial_t v$ has dimension $\mathtt{LT}^{-2}$ (acceleration) and $v \cdot \nabla v$ has dimension $\mathtt{LT}^{-1}\mathtt{L}^{-1}\mathtt{LT}^{-1} = \mathtt{LT}^{-2}$, as well as $\nabla(p/\rho)$ which has dimension

$$\underbrace{\mathtt{L}^{-1}}_{\nabla} \underbrace{\mathtt{MLT}^{-2}}_{\text{force}} \underbrace{\mathtt{L}^{-2}}_{\text{area}^{-1}} \underbrace{\mathtt{M}^{-1}\mathtt{L}^3}_{\text{density}^{-1}} = \mathtt{LT}^{-2}$$

where M stands for the mass unit.

The **Navier-Stokes system for incompressible fluids**[10] reads

$$\begin{cases} \partial_t v + (v \cdot \nabla)v - \nu \Delta v = -\nabla(p/\rho) \\ \operatorname{div} v = 0 \\ v_{|t=0} = w \end{cases} \tag{1.1.15}$$

where ν is the kinematic viscosity expressed in *Stokes* $\mathtt{L}^2\mathtt{T}^{-1}$ so that the dimension of $\nu \Delta v$ is also

$$\underbrace{\mathtt{L}^2\mathtt{T}^{-1}}_{\nu} \underbrace{\mathtt{L}^{-2}}_{\Delta} \underbrace{\mathtt{LT}^{-1}}_{v} = \mathtt{LT}^{-2}.$$

We note that $\operatorname{curl} \operatorname{grad} = 0$ since $\begin{pmatrix} \partial_{x_1} \\ \partial_{x_2} \\ \partial_{x_3} \end{pmatrix} \times \begin{pmatrix} \partial_{x_1} f \\ \partial_{x_2} f \\ \partial_{x_3} f \end{pmatrix} = 0$ and this implies that, taking

the curl of the first line of (1.1.14), we get with the **vorticity**

$$\omega = \operatorname{curl} v \tag{1.1.16}$$

$$\partial_t \omega + \operatorname{curl}((v \cdot \nabla)v) = 0.$$

Let us compute, using Einstein's convention[11] on repeated indices (this means that $\partial_j v_j$ stands for $\sum_{1 \le j \le 3} \partial_j v_j$),

$$\operatorname{curl}((v \cdot \nabla)v) = \begin{pmatrix} \partial_{x_1} \\ \partial_{x_2} \\ \partial_{x_3} \end{pmatrix} \times \begin{pmatrix} (v \cdot \nabla)v_1 \\ (v \cdot \nabla)v_2 \\ (v \cdot \nabla)v_3 \end{pmatrix} = (v \cdot \nabla)\operatorname{curl} v + \begin{pmatrix} \partial_2 v_j \partial_j v_3 - \partial_3 v_j \partial_j v_2 \\ \partial_3 v_j \partial_j v_1 - \partial_1 v_j \partial_j v_3 \\ \partial_1 v_j \partial_j v_2 - \partial_2 v_j \partial_j v_1 \end{pmatrix}$$

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[10]CLAUDE NAVIER (1785-1836) is a French engineer and scientist. GEORGES STOKES (1819-1903) is a British mathematician and physicist.

[11]ALBERT EINSTEIN (1879-1955) is one of the greatest scientists of all times and, needless to say, his contributions to Quantum Mechanics, Brownian Motion and Relativity Theory are far more important than this convention, which is however a handy notational tool.

and since $\partial_j v_j = 0$, $\quad \omega = \begin{pmatrix} \partial_2 v_3 - \partial_3 v_2 \\ \partial_3 v_1 - \partial_1 v_3 \\ \partial_1 v_2 - \partial_2 v_1 \end{pmatrix}$, we get

$$\partial_2 v_j \partial_j v_3 - \partial_3 v_j \partial_j v_2$$
$$= [\partial_2 v_1 \partial_1 v_3] + \partial_2 v_2 \partial_2 v_3 + \partial_2 v_3 \partial_3 v_3 - [\partial_3 v_1 \partial_1 v_2] - \partial_3 v_2 \partial_2 v_2 - \partial_3 v_3 \partial_3 v_2$$
$$= [\partial_2 v_1 \partial_1 v_3] - [\partial_3 v_1 \partial_1 v_2] + \omega_1 (\partial_2 v_2 + \partial_3 v_3)$$
$$= -\omega_1 \partial_1 v_1 + \partial_2 v_1 \partial_1 v_3 - \partial_3 v_1 \partial_1 v_2$$
$$= -\omega_j \partial_j v_1 + (\partial_3 v_1 - \partial_1 v_3)\partial_2 v_1 + (\partial_1 v_2 - \partial_2 v_1)\partial_3 v_1 + \partial_2 v_1 \partial_1 v_3 - \partial_3 v_1 \partial_1 v_2$$
$$= -\omega_j \partial_j v_1 + \partial_2 v_1 \partial_3 v_1 - \partial_3 v_1 \partial_2 v_1 = -\omega_j \partial_j v_1,$$

so that, using a circular permutation, we get

$$\mathrm{curl}\big((v \cdot \nabla)v\big) = (v \cdot \nabla)\omega - (\omega \cdot \nabla)v \tag{1.1.17}$$

and (1.1.14) becomes $\partial_t \omega + (v \cdot \nabla)\omega - (\omega \cdot \nabla)v = 0, \mathrm{div}\, v = 0, \omega_{|t=0} = \mathrm{curl}\, v.$

1.2 Comments

Although the above list of examples is very limited, it is quite obvious that partial differential equations are occurring in many different domains of science: Electromagnetism with the Maxwell equations, Wave Propagation with the transport, wave, Burgers equations, Quantum Mechanics with the Schrödinger equation, Diffusion Theory with the heat equation, Fluid Dynamics with the Euler and Navier-Stokes systems. We could have mentioned Einstein's equation of General Relativity and many other examples. As a matter of fact, the law of Physics are essentially all expressed as PDE, so the domain is so vast that it is pointless to expect a useful classification of PDE, at least in an introductory chapter of a textbook on PDE.

We have already mentioned various type of questions such that the Cauchy problem for evolution equations: for that type of *Initial Value Problem*, we are given an equation of evolution $\partial_t u = F(x, u, \partial_x u, \dots)$ and the initial value $u(0)$. The first natural questions are about the existence of a solution, its uniqueness but also about the continuous dependence of the solution with respect to the data: the french mathematician JACQUES HADAMARD (1865–1963)[12] introduced the notion of *well-posedness* as one of the most important property of a PDE. After all, the data (initial or Cauchy data, various quantities occurring in the equation) in a Physics problem are known only approximatively and even if the solution were existing and proven unique, this would be useless for actual computation or applications if minute changes of the data trigger huge changes for the solution. In fact, one should try to establish some inequalities controlling the size of the norms or semi-norms of the solution u in some functional space. The lack of well-posedness is linked to instability and is also a very interesting phenomenon to study. We can quote at this point Lars Gårding's survey[13] article [10]:" When a problem about partial differential operators has been fitted into the abstract theory, all that remains is usually to prove

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[12]See [17].

[13]LARS GÅRDING (born 1919), is a Swedish mathematician.

a suitable inequality and much of our knowledge is, in fact, essentially contained in such inequalities".

On the other hand, we have seen that the solution can be submitted to *boundary conditions*, such as the Dirichlet boundary condition and we shall study other types of boundary conditions, such as the Neumann boundary[14], where the normal derivative to the boundary is given.

The questions of smoothness and regularity of the solutions are also very important: where are located the singularities of the solutions, do they "propagate"? Is it possible to consider "weak solutions", whose regularity is too limited for the equation to make "classical" sense (see our discussion above on the transport equation).

Obviously non-linear PDE are more difficult to handle than the linear ones, in particular because some singularities of the solution may occur although the initial datum is perfectly smooth (see our discussion above on the Burgers equation). The study of systems of PDE is playing a key rôle in Fluid Mechanics and the intricacies of the algebraic properties of these systems deserves a detailed examination (a simple example of calculation was given with the formula (1.1.17)).

1.3   Quotations

Let us end this introduction with a couple of quotations. First of all, we cannot avoid to quote GALILEO GALILEI (1564-1642), an Italian physicist, mathematician, astronomer and philosopher with his famous apology of Mathematics: "Nature is written in that great book which ever lies before our eyes - I mean the universe - but we cannot understand it if we do not first learn the language and grasp the symbols, in which it is written. This book is written in the mathematical language, and the symbols are triangles, circles and other geometrical figures, without whose help it is impossible to comprehend a single word of it; without which one wanders in vain through a dark labyrinth," see the translation of [4].

Our next quotation is by the physicist EUGENE P. WIGNER (1902-1995, 1963 Physics Nobel Prize) who, in his celebrated 1960 article *The Unreasonable Effectiveness of Mathematics in the Natural Sciences* [24] is unraveling part of the complex relationship between Mathematics and Physics: "The miracle of the appropriateness of the language of mathematics for the formulation of the laws of physics is a wonderful gift which we neither understand nor deserve. We should be grateful for it and hope that it will remain valid in future research and that it will extend, for better or for worse, to our pleasure, even though perhaps also to our bafflement, to wide branches of learning." It is interesting to complement that quotation by the 2009 appreciation of JAMES GLIMM[15] in [11]: "In simple terms, mathematics works. It is effective. It is essential. It is practical. Its force cannot be avoided, and the future belongs to societies that embrace its power. Its force is derived from its essential role within science, and from the role of science in technology. Wigner's observations concerning *The Unreasonable Effectiveness of Mathematics* are truer today than when they were first written in 1960."

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[14]CARL GOTTFRIED NEUMANN (1832-1925) is a German mathematician.
[15]JAMES GLIMM (born 1934) is an American mathematician.

The British physicist and mathematician Roger Penrose (born 1931), acclaimed author of popular books such as *The Emperor's new mind* and *The Road to Reality,*[16] *a complete guide to the laws of the universe [18],* should have a say with the following remarkable excerpts of the preface of [18]: "To mathematicians ...mathematics is not just a cultural activity that we have ourselves created, but it has a life of its own, and much of it finds an amazing harmony with the physical universe. We cannot get a deep understanding of the laws that govern the physical world without entering the world of mathematics...In modern physics, one cannot avoid facing up to the subtleties of much sophisticated mathematics"

Then we listen to John A. Wheeler (1911-2008), an outstanding theoretical physicist (author with Kip S. Thorne and Charles W. Misner of the landmark book *Gravitation* [16]) who deals with the aesthetics of scientific truth: "It is my opinion that everything must be based on a simple idea. And it is my opinion that this idea, once we have finally discovered it, will be so compelling, so beautiful, that we will say to one another, yes, how could it have been any different."

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[16]As a matter of fact, that extra-ordinary one-thousand-page book could not really be qualified as popular, except for the fact that it is indeed available in general bookstores.

Chapter 2

Vector Fields

We start with recalling a few basic facts on Ordinary Differential Equations.

2.1 Ordinary Differential Equations

2.1.1 The Cauchy-Lipschitz result

[1] Let I be an interval of $\mathbb{R}$ and Ω be an open set of $\mathbb{R}^n$. We consider a continuous function $F : I \times \Omega \to \mathbb{R}^n$ such that for all $(t_0, x_0) \in I \times \Omega$, there exists a neighborhood V_0 of (t_0, x_0) in $I \times \Omega$ and a positive constant L_0 such that for $(t, x_1), (t, x_2) \in V_0$

$$|F(t, x_1) - F(t, x_2)| \leq L_0 |x_1 - x_2|, \tag{2.1.1}$$

where $|\cdot|$ stands for a norm in $\mathbb{R}^n$. We shall say that F satisfies a local Lipschitz condition. Note that these assumptions are satisfied whenever $F \in C^1(I \times \Omega)$ and even if $\partial_x F(t, x) \in C^0(I \times \Omega)$.

Theorem 2.1.1 (Cauchy-Lipschitz). *Let F be as above. Then for all $(t_0, x_0) \in I \times \Omega$, there exists a neighborhood J of t_0 in I such the initial-value-problem*

$$\begin{cases} \dot{x}(t) &= F\big(t, x(t)\big) \\ x(t_0) &= x_0 \end{cases} \tag{2.1.2}$$

has a unique solution defined in J.

N.B. A solution of (2.1.2) *is a differentiable function on J, valued in Ω, and since F and x are continuous, the equation itself implies that x is C^1. One may as well consider continuous solutions of*

$$x(t) = x_0 + \int_{t_0}^t F(s, x(s))ds. \tag{2.1.3}$$

From this equation, the solution $t \mapsto x(t)$ is C^1, and satisfies (2.1.2).

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[1] See the footnote (1) for A.L. Cauchy. Rudolph Lipschitz (1832-1903) is a German mathematician.

Proof. We shall use directly the *Picard approximation scheme*[2] which goes as follows. We want to define for $k \in \mathbb{N}$, t in a neighborhood of t_0, ,

$$x_0(t) = x_0,$$

$$x_{k+1}(t) = x_0 + \int_{t_0}^{t} F(s, x_k(s))ds. \tag{2.1.4}$$

We need to prove that this makes sense, which is not obvious since F is only defined on $I \times \Omega$. Let us assume that for $t \in J_0 = \{t \in I, |t - t_0| \leq T_0\}$, $(x_l(t))_{0 \leq l \leq k}$ is such that

$$\left. \begin{array}{l} x_l(t) \in \bar{B}(x_0, R_0) \subset \Omega, \quad \text{where } T_0 \text{ and } R_0 \text{ are positive,} \\ (2.1.1) \text{ holds with } V_0 = J_0 \times \bar{B}(x_0, R_0), \end{array} \right\} \tag{2.1.5}$$

and such that

$$e^{L_0 T_0} \int_{|t-t_0| \leq T_0} |F(s, x_0)|ds \leq R_0. \tag{2.1.6}$$

The relevance of the latter condition will be clarified by the computation below, but we may note at once that, given $R_0 > 0$, there exists $T_0 > 0$ such that (2.1.6) is satisfied since the lhs goes to zero with T_0. Property (2.1.5) is true for $k = 0$; let us assume $k \geq 1$. Then we can define $x_{k+1}(t)$ as above for $t \in J_0$ and we have, with $(x_l)_{0 \leq l \leq k}$ satisfying (2.1.5)

$$|x_{k+1}(t) - x_k(t)| \leq \left| \int_{t_0}^{t} L_0 |x_k(s) - x_{k-1}(s)|ds \right|. \tag{2.1.7}$$

and inductively

$$|x_{k+1}(t) - x_k(t)| \leq L_0^k \left| \int_{t_0}^{t} |F(s, x_0)|ds \frac{|t - t_0|^k}{k!} \right|, \tag{2.1.8}$$

since (we may assume without loss of generality that $t \geq t_0$) that estimate holds true trivially for $k = 0$ and if $k \geq 1$, we have, using (2.1.7) and the induction hypothesis (2.1.8) for $k - 1$,

$$|x_{k+1}(t) - x_k(t)| \leq L_0 \int_{t_0}^{t} L_0^{k-1} \int_{t_0}^{s} |F(\sigma, x_0)|(\sigma - t_0)^{k-1}d\sigma ds \frac{1}{(k-1)!}$$

$$\leq L_0^k \frac{1}{(k-1)!} \int_{t_0}^{t} |F(\sigma, x_0)|d\sigma \int_{t_0}^{t} (s - t_0)^{k-1}ds = \frac{L_0^k(t - t_0)^k}{k!} \int_{t_0}^{t} |F(\sigma, x_0)|d\sigma.$$

As a consequence, we have for $t \in J_0$,

$$|x_{k+1}(t) - x_0| \leq \sum_{0 \leq l \leq k} |x_{l+1}(t) - x_l(t)| \leq \left| \int_{t_0}^{t} |F(\sigma, x_0)|d\sigma \right| \sum_{0 \leq l \leq k} \frac{L_0^l |t - t_0|^l}{l!}$$

$$\leq e^{L_0|t-t_0|} \left| \int_{t_0}^{t} |F(\sigma, x_0)|d\sigma \right| \underbrace{\leq}_{\text{from (2.1.6)}} R_0.$$

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[2]EMILE PICARD (1856-1941) is a French mathematician.

We have thus proven that, provided (2.1.6) holds true, then for all $k \in \mathbb{N}$ and all $t \in J_0$, $x_k(t)$ makes sense and belongs to $\bar{B}(x_0, R_0)$. Thus we have constructed a sequence $(x_k)_{k \geq 0}$ of continuous functions of $C^0(J_0; \mathbb{R}^n)$ such that, defining

$$\alpha(T_0) = \int_{J_0} |F(s, x_0)| ds, \quad J_0 = \{t \in I, |t - t_0| \leq T_0\}, \quad (2.1.9)$$

and assuming as in (2.1.6) that $\alpha(T_0) e^{L_0 T_0} \leq R_0$, we have

$$\sup_{t \in J_0} \|x_{k+1}(t) - x_k(t)\| \leq \frac{L_0^k T_0^k}{k!} \alpha(T_0). \quad (2.1.10)$$

Lemma 2.1.2. *Let J be a compact interval of $\mathbb{R}$, E be a Banach[3] space and $\mathcal{E} = \{u \in C^0(J; E)\}$ equipped with the norm $\|u\|_{\mathcal{E}} = \sup_{t \in J} |u(t)|_E$ is a Banach space.*

Proof of the lemma. Note that the continuous image $u(J)$ is a compact subset of E, thus is bounded so that the expression of $\|u\|_{\mathcal{E}}$ makes sense and is obviously a norm; let us consider now a Cauchy sequence $(u_k)_{k \geq 1}$ in $\mathcal{E}$. Then for all $t \in J$, $(u_k(t))_{k \geq 1}$ is a Cauchy sequence in E, thus converges: let us set $v(t) = \lim_k u_k(t)$, for $t \in J$. The convergence is uniform with respect to t since

$$|u_k(t) - v(t)|_E = \lim_l |u_k(t) - u_{k+l}(t)|_E \leq \limsup_l \|u_k - u_{k+l}\|_{\mathcal{E}} = \varepsilon(k) \underset{k \to +\infty}{\longrightarrow} 0.$$

The continuity of the limit follows by the classical argument: for $t, t + h \in J$, we have for all k

$$|v(t+h) - v(t)|_E \leq |v(t+h) - u_k(t+h)|_E + |u_k(t+h) - u_k(t)|_E + |u_k(t) - v(t)|_E$$
$$\leq 2\|v - u_k\|_{\mathcal{E}} + |u_k(t+h) - u_k(t)|_E,$$

and thus by continuity of u_k, $\limsup_{h \to 0} |v(t+h) - v(t)|_E \leq 2\|v - u_k\|_{\mathcal{E}}$ so that $\limsup_{h \to 0} |v(t+h) - v(t)|_E \leq 2 \inf_k \|v - u_k\|_{\mathcal{E}} = 0$. $\square$

Applying this lemma, we see that the sequence of continuous functions (x_k) is a Cauchy sequence in the Banach space $C^0(J_0; \mathbb{R}^n)$ since (2.1.10) gives

$$\sum_{k \geq 0} \|x_{k+1} - x_k\|_{C^0(J_0; \mathbb{R}^n)} \leq \alpha(T_0) e^{L_0 T_0} < +\infty.$$

Let $u = \lim_k x_k$ in the Banach space $C^0(J_0; \mathbb{R}^n)$; since $x_k(J_0) \subset \bar{B}(x_0, R_0)$, we have also $u(J_0) \subset \bar{B}(x_0, R_0)$ and from the equation (2.1.4), we get for $t \in J_0$

$$u(t) = x_0 + \int_{t_0}^t F(s, u(s)) ds,$$

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[3]STEFAN BANACH (1892-1945) is a Polish mathematician. A *Banach space* is a complete normed vector space.

since $u(t) = \lim_k x_{k+1}(t)$, $x_{k+1}(t) = x_0 + \int_0^t F(s, x_k(s))ds$ and the difference

$$\int_0^t \big(F(s, x_k(s)) - F(s, u(s))\big)ds$$

satisfies

$$\left|\int_{t_0}^t \big(F(s, u(s)) - F(t, x_k(t))\big)dt\right| \leq \left|\int_{t_0}^t L_0|u(s) - x_k(s)|ds\right| \leq L_0 T_0\|x_k - u\|_{C^0(J_0;\mathbb{R}^n)},$$

providing the local existence part of Theorem 2.1.1. Let us prove uniqueness (and even more). Let u, v be solutions of

$$\begin{cases} u(t) = x_0 + \int_{t_0}^t F(s, u(s))ds \\ v(t) = y_0 + \int_{t_0}^t F(s, v(s))ds \end{cases} \qquad \text{for } 0 \leq t - t_0 \leq T_0. \qquad (2.1.11)$$

We define $\rho(t) = |u(t) - v(t)|$ and we have

$$\rho(t) \leq |u(t_0) - v(t_0)| + \int_{t_0}^t L_0|u(s) - v(s)|ds = R(t),$$

so that $\dot{R}(t) = L_0|u(t) - v(t)| = L_0\rho(t) \leq L_0 R(t)$.

Lemma 2.1.3 (Gronwall[4]). *Let $t_0 < t_1$ be real numbers and $R : [t_0, t_1] \to \mathbb{R}$ be a differentiable function such that $\dot{R}(t) \leq LR(t)$ for $t \in [t_0, t_1]$, where $L \in \mathbb{R}$. Then for $t \in [t_0, t_1]$, $R(t) \leq e^{L(t-t_0)}R(t_0)$.*
*   More generally, if $\dot{R}(t) \leq L(t)R(t) + f(t)$ for $t \in [t_0, t_1]$ with $L, f \in L^1([t_0, t_1])$, we have for $t \in [t_0, t_1]$*

$$R(t) \leq e^{\int_{t_0}^t L(s)ds}R(t_0) + \int_{t_0}^t e^{\int_s^t L(\sigma)d\sigma} f(s)ds.$$

Remark 2.1.4. In other words a solution of the differential inequality

$$\dot{R} \leq LR + f, \ R(t_0) = R_0,$$

is smaller than the solution of the equality $\dot{R} = LR + f, R(t_0) = R_0$.

Proof of the lemma. We calculate

$$\frac{d}{dt}\big(R(t)e^{-\int_{t_0}^t L(s)ds}\big) = \big(\dot{R}(t) - L(t)R(t)\big)e^{-\int_{t_0}^t L(s)ds} \leq f(t)e^{-\int_{t_0}^t L(s)ds},$$

entailing for $t \in [t_0, t_1]$ $R(t)e^{-\int_{t_0}^t L(s)ds} \leq R(t_0) + \int_{t_0}^t f(s)e^{-\int_{t_0}^s L(\sigma)d\sigma}$.   □

Applying this lemma, we get for $0 \leq t - t_0 \leq T_0$ that

$$|u(t) - v(t)| = \rho(t) \leq R(t) \leq e^{L_0(t-t_0)}R(t_0),$$

proving uniqueness and a much better result, akin to continuous dependence on the data, summarized in the following lemma.

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[4] THOMAS GRÖNWALL (1877-1932) is a Swedish-born American mathematician

Lemma 2.1.5 (Gronwall lemma on ODE). *Let F be as in Theorem 2.1.1 with $|F(t, x_1) - F(t, x_2)| \leq L|x_1 - x_2|$ for $t \in I, x_1, x_2 \in \Omega$. Let u, v be as in (2.1.11). Then for $t_0 \leq t \in I$, $|u(t) - v(t)| \leq e^{L(t - t_0)}|u(t_0) - v(t_0)|$.*

The proof of Theorem 2.1.1 is complete. $\qquad\qquad\square$

Remark 2.1.6. We have proven a quantitatively more precise result: let F be as in Theorem 2.1.1, $(t_0, x_0) \in I \times \Omega$, $R_0 > 0$ such that $\bar{B}(x_0, R_0) \subset \Omega$ and $T_0 > 0$ such that (2.1.6) holds. Then with $J_0 = \{t \in I, |t - t_0| \leq T_0\}$, there exists a unique solution x of (2.1.2) such that $x \in C^1(J_0; \bar{B}(x_0, R_0))$. Let K be a compact subset of Ω and J be a compact nonempty subinterval of I: then

$$\sup_{\substack{t \in J \\ x_j \in K, j=1,2, x_1 \neq x_2}} \frac{|F(t, x_1) - F(t, x_2)|}{|x_1 - x_2|} < +\infty. \tag{2.1.12}$$

If it were not the case, we would find sequences (t_k) in J, $(x_{1,k}), (x_{2,k})$ in K, so that

$$|F(t_k, x_{1,k}) - F(t_k, x_{2,k})| > k|x_{1,k} - x_{2,k}|.$$

Extracting subsequences and using the compactness assumption, we may assume that the three sequences are converging to $(t, x_1, x_2) \in J \times K^2$; moreover the continuity hypothesis on F gives the convergence of the lhs to $|F(t, x_1) - F(t, x_2)|$ and the inequality gives $x_1 = x_2$ and $x_{1,k} \neq x_{2,k}$ We infer from the assumption (2.1.1) at (t, x_1) that for $k \geq k_0$,

$$k < \frac{|F(t_k, x_{1,k}) - F(t_k, x_{2,k})|}{|x_{1,k} - x_{2,k}|} \leq L_0$$

which is impossible. We have proven that for K compact subset of Ω, J compact subinterval of I, (2.1.12) holds. Let $R > 0$ such that $\cup_{x \in K} \bar{B}(x, R) = K_R \subset \Omega$. Now if t_0 is given in I, L_0 stands for the lhs of (2.1.12), and T_0 small enough so that

$$e^{L_0 T_0} \int_{t \in I, |t - t_0| \leq T_0} \sup_{y \in K} |F(s, y)| ds \leq R,$$

we know that, for all $y \in K$, there exists a unique solution of (2.1.2) defined on $J_0 = \{t \in I, |t - t_0| \leq T_0\}$ such that $x \in C^1(J_0; \bar{B}(y, R))$, $x(t_0) = y$. In particular, if the initial data y belongs to a compact subset of Ω and s belongs to a compact subset of I, the time of existence of the solution of (2.1.2) is bounded from below by a fixed constant (provided F satisfies (2.1.1)).

If we consider F as in Theorem 2.1.1, we know that, for any $(s, y) \in I \times \Omega$, the initial-value problem $\dot{x}(t) = F(t, x(t)), x(s) = y$ has a unique solution, which is defined and C^1 on a neighborhood of s in I. We may denote that solution by $x(t, s, y)$ which is characterized by

$$(\partial_t x)(t, s, y) = F\big(t, x(t, s, y)\big), \quad x(s, s, y) = y.$$

We may consider $y_1, y_2 \in \Omega, s \in I$ and the solutions $x(t, s, y_2), x(t, s, y_1)$ both defined on a neighborhood J of s in I (the intersection of the neighborhoods on which $t \mapsto x(t, s, y_j)$ are defined). We have

$$x(t, s, y_2) - x(t, s, y_1) = y_2 - y_1 + \int_s^t \big[F\big(\sigma, x(\sigma, s, y_2)\big) - F\big(\sigma, x(\sigma, s, y_2)\big) \big] d\sigma,$$

and assuming J compact, we get that $\cup_{j=1,2}\{x(\sigma, s, y_j)\}_{\sigma \in J}$ is a compact subset of Ω, so that there exists $L \geq 0$ with

$$|x(t, s, y_2) - x(t, s, y_1)| \leq |y_2 - y_1| + \left| \int_s^t L|x(\sigma, s, y_2) - x(\sigma, s, y_1)| d\sigma \right|$$

and the previous lemma implies that

$$|x(t, s, y_2) - x(t, s, y_1)| \leq e^{L|t-s|}|y_2 - y_1|. \tag{2.1.13}$$

The mapping $\Omega \ni y \mapsto x(t, s, y)$, defined for any $s \in I$ and t in a neighborhood of s is thus Lipschitz continuous. We have also proven the following

Proposition 2.1.7. *Let $F : I \times \Omega \longrightarrow \mathbb{R}^n$ be as in Theorem 2.1.1 with $0 \in I$. We define the flow ψ of the ODE, $\dot{X}(t) = F(t, X(t))$ as the unique solution of*

$$\frac{\partial \psi}{\partial t}(t, x) = F(t, \psi(t, x)), \quad \psi(0, x) = x. \tag{2.1.14}$$

The C^1 mapping $t \mapsto \psi(t, x)$ is defined on a neighborhood of 0 in I which may depend on x. However if x belongs to a compact subset K of Ω, there exists $T_0 > 0$ such that ψ is defined on $\{t \in I, |t| \leq T_0\} \times K$ and $\psi(t, \cdot)$ is Lipschitz-continous.

Remark 2.1.8. There is essentially nothing to change in the statements and in the proofs if we wish to replace $\mathbb{R}^n$ by a Banach space (possibly infinite dimensional).

Remark 2.1.9. The local Lipschitz regularity can be replaced by a much weaker assumption related to an OSGOOD [5] modulus of continuity: let $\omega :]0, +\infty) \to]0, +\infty)$, be a continuous and non-decreasing function, such that $\omega(0_+) = 0$ and

$$\exists r_0 > 0, \quad \int_0^{r_0} \frac{dr}{\omega(r)} = +\infty. \tag{2.1.15}$$

Let I be an interval of $\mathbb{R}$, Ω be an open subset of a Banach space E and $F : I \times \Omega \to E$ such that there exists $\alpha \in L^1_{loc}(I)$ so that for all $t, x_1, x_2 \in I \times \Omega^2$

$$|F(t, x_1) - F(t, x_2)|_E \leq \alpha(t)\omega(|x_1 - x_2|_E). \tag{2.1.16}$$

Then Theorem 2.1.1 holds (see e.g. [5]). Some continuous dependence can also be proven, in general weaker than (2.1.13). Note that the Lipschitz regularity corresponds to $\omega(r) = r$ and that the integral condition above allows more general moduli of continuity such as

$$\omega(r) = r \times |\ln r| \quad \text{or} \quad r \times |\ln r| \times \ln(|\ln r|).$$

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[5]WILLIAM F. OSGOOD (1864-1943) is an American mathematician.

Naturally Hölder's regularity $\omega(r) = r^\alpha$ with $\alpha \in [0, 1[$ is excluded by (2.1.15): as a matter of fact, in that case some classical counterexamples to uniqueness are known such as the one-dimensional

$$\dot{x} = |x|^\alpha, \quad x(0) = 0, \qquad (\alpha \in [0, 1[)$$

which has the solution 0 and $x(t) = \left((1 - \alpha)t\right)^{\frac{1}{1-\alpha}}$ for $t > 0$, 0 for $t \leq 0$.

Remark 2.1.10. Going back to the finite-dimensional case, a theorem by Peano[6] is providing an existence result (without uniqueness) for the ODE (2.1.2) under a mere continuity assumption for F. That type of result is not true in the infinite dimensional case as the reader may check for instance in the exercise 18 page IV.41 of the Bourbaki's volume [2].

2.1.2 Maximal and Global Solutions

Let I be an interval of $\mathbb{R}$ and Ω be an open set of $\mathbb{R}^n$. We consider a continuous function $F : I \times \Omega \to \mathbb{R}^n$. Let $I_1 \subset I_2 \subset I$ be subintervals of I. Let $x_j : I_j \to \Omega$ $(j = 1, 2)$ be such that $\dot{x}_j = F(t, x_j)$. If $x_1 = x_{2|I_1}$ we shall say that x_2 is a continuation of x_1.

Definition 2.1.11. *We consider the ODE $\dot{x} = F(t, x)$. A maximal solution x of this ODE is a solution so that there is no continuation of x, except x itself. A global solution of this ODE is a solution defined on I.*

Note that a global solution is a maximal solution, but that the converse in not true in general. Taking $I = \mathbb{R}, \Omega = \mathbb{R}$ the equation $\dot{x} = x^2$ has the maximal solutions $t \mapsto (T_0 - t)^{-1}$ (T_0 is a real parameter) defined on the intervals $(-\infty, T_0), (T_0, +\infty)$. None of these maximal solutions can be extended globally since $|x(t)|$ goes to $+\infty$ when t approaches T_0.

$$\text{For } t < 1, \quad x(t) = \frac{1}{1 - t}, \quad x(0) = 1, \quad \text{blow-up time } t = 1,$$
$$\text{for } t \in \mathbb{R}, \quad x(t) = 0, \quad \text{the only solution not blowing-up,}$$
$$\text{for } t > 1, \quad x(t) = \frac{1}{1 - t}, \quad x(2) = -1, \quad \text{blow-up time } t = 1.$$

Note that if $x(t_0)$ is positive, then $x(t)$ is positive and blows-up in the future and if $x(t_0)$ is negative, then $x(t)$ is negative and blows-up in the past. Moreover $x(0) = T_0^{-1}$, so that the larger positive $x(0)$ is, the sooner the blow-up occurs.

Theorem 2.1.12. *Let $F : I \times \Omega \to \mathbb{R}^n$ be as in Theorem 2.1.1 and let $(t_0, x_0) \in I \times \Omega$. Then there exists a unique maximal solution $x : J \to \mathbb{R}^n$ of the initial-value-problem $\dot{x} = F(t, x), x(t_0) = x_0$, where J is a subinterval of I containing t_0.*

Proof. Let us consider all the solutions $x_\alpha : J_\alpha \to \mathbb{R}^n$ of the initial-value-problem $\dot{x}_\alpha = F(t, x_\alpha), x_\alpha(t_0) = x_0$, where J_α is a subinterval of I containing t_0. From the

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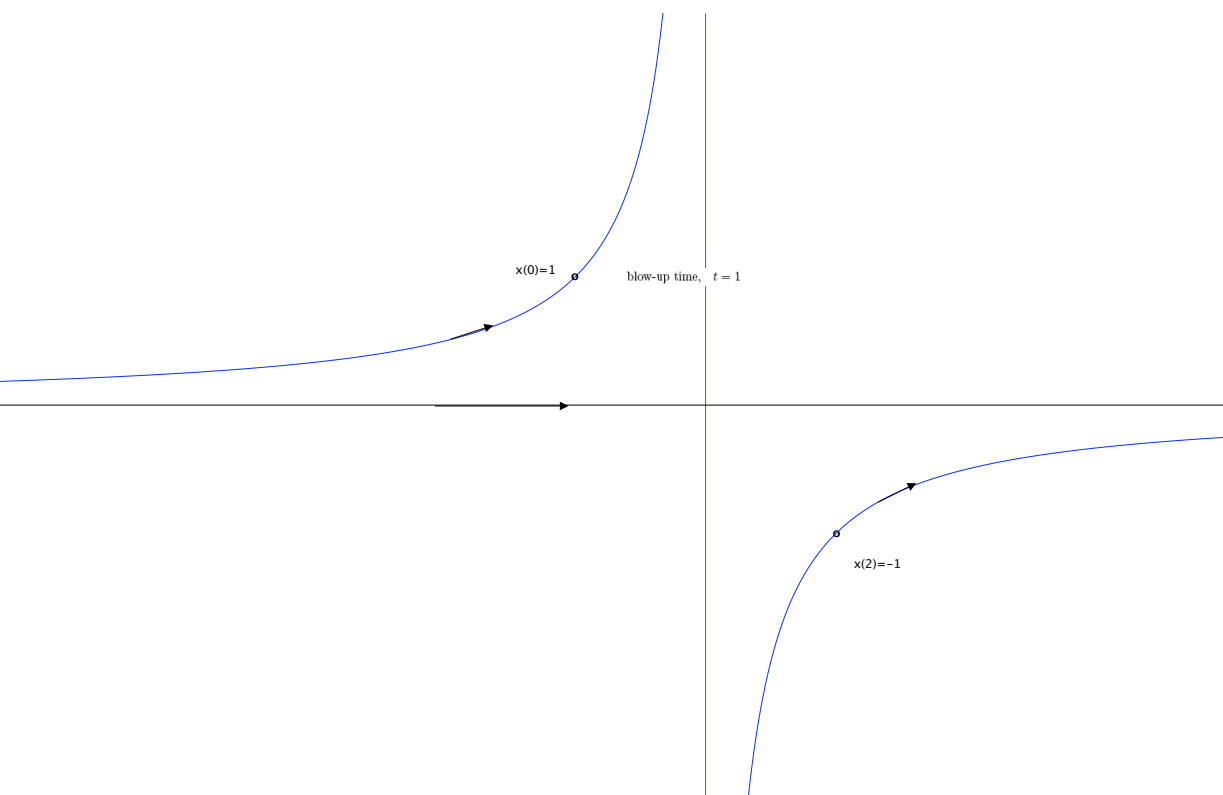[6] GIUSEPPE PEANO (1858-1932) is an Italian mathematician.

Figure 2.1: THREE SOLUTIONS OF THE ODE $\dot{x} = x^2$.

existence theorem, that family is not empty. If $\theta \in J_\alpha \cap J_\beta$, $x_\alpha(\theta) = x_\beta(\theta)$, from the uniqueness theorem on $[t_0, \theta]$ (or $[\theta, t_0]$) so that we may define for $t \in \cup_\alpha J_\alpha = J$ (J is an interval since t_0 belongs to all J_α), $x(t) = x_\alpha(t)$.

Moreover, the function x is continuous on J: take $\theta \in J$, say with $\theta > t_0$, $\theta \in J_\alpha$: the function x coincides with x_α on $[t_0, \theta]$, thus is left-continuous at θ. If $\theta = \sup J$, it is enough to prove continuity. Now if $\theta < \sup J$, $\exists \theta' \in J, \theta' > \theta$: $\theta' \in J_\alpha$ for some α and as above the function x coincides with x_α on $[t_0, \theta']$, which proves as well continuity. Note that x is continuous at t_0 since it coincides with x_α on a neighborhood of t_0 in I for all α.

For $t \in J$, we have $t \in J_\alpha$ for some α and since $t_0 \in J_\alpha$, we get

$$\int_{t_0}^{t} F(s, x(s))ds = \int_{t_0}^{t} F(s, x_\alpha(s))ds = x_\alpha(t) - x_0 = x(t) - x_0,$$

so that x is a solution of the initial-value-problem $\dot{x} = F(t, x), x(t_0) = x_0$ on J. By construction, it is a maximal solution. $\square$

Theorem 2.1.13. *Let $F : [0, +\infty) \times \Omega \to \mathbb{R}^n$ be as in Theorem 2.1.1, $x_0 \in \Omega$. The maximal solution of $\dot{x} = F(t, x), x(0) = x_0$ is defined on some interval $[0, T_0[$ and if $T_0 < +\infty$ then*

$$\sup_{0 \le t < T_0} |x(t)| = +\infty \quad or \quad \overline{x([0, T_0[)} \text{ is not a compact subset of } \Omega.$$

Proof. If the maximal solution were defined on some interval $[0, T_0]$, $T_0 > 0$, then $(T_0, x(T_0)) \in [0, +\infty) \times \Omega$ and the local existence theorem would imply the existence of a solution of $\dot{y} = F(t, y), y(T_0) = x(T_0)$ on some neighborhood of T_0: by the uniqueness theorem, that solution should coincide with x for $t \le T_0$ and provide a continuation of x, contradicting its maximality.

Let us assume that x is defined on $[0, T_0[$ with $0 < T_0 < +\infty$ and

$$\sup_{0 \le t < T_0} |x(t)| \le M < +\infty, \quad \text{as well as } \overline{x([0, T_0[)} = K \text{ compact subset of } \Omega.$$

We consider a sequence $(t_k)_{k \ge 1}$ with $0 < t_k < T_0, \lim_k t_k = T_0$. The sequence $(x(t_k))_{k \ge 1}$ belongs to K and thus has a convergent subsequence, that we shall call again $(x(t_k))_{k \ge 1}$ so that

$$\lim_k x(t_k) = \xi, \quad \xi \in K.$$

The equation $\dot{y} = F(t, y), y(T_0) = \xi$ has a unique solution defined in $[T_0 - \varepsilon_0, T_0 + \varepsilon_0]$ with $\varepsilon_0 > 0$. For $t \in [T_0 - \varepsilon_0, T_0[$, we have $x(t) \in K$ which is a compact subset of Ω and $y(t)$ in a neighborhood of ξ so that (see the remark 2.1.6 for the uniformity of the constant L)

$$|x(t) - y(t)| \le |x(t_k) - y(t_k)| + \left| \int_{t_k}^t L|x(s) - y(s)|ds \right|,$$

implying

$$\sup_{T_0 - \varepsilon_0 \le t < T_0} |x(t) - y(t)| \le |x(t_k) - y(t_k)| + L|t - t_k| \sup_{T_0 - \varepsilon_0 \le t < T_0} |x(t) - y(t)|$$

and thus, since $\sup_{0 \le t < T_0} |x(t)| \le M < +\infty$, we have $\sup_{T_0 - \varepsilon_0 \le t < T_0} |x(t) - y(t)| = 0$, i.e. $x(t) = y(t)$ on $[T_0 - \varepsilon_0, T_0[$. Considering the continuous function $X(t) = x(t)$ for $0 \le t < T_0$, $X(t) = y(t)$ for $T_0 - \varepsilon_0 \le t \le T_0 + \varepsilon_0$, we see that for $T_0 - \varepsilon_0 \le t \le T_0 + \varepsilon_0$,

$$\int_0^t F(s, X(s))ds = \int_0^{T_0 - \varepsilon_0} F(s, x(s))ds + \int_{T_0 - \varepsilon_0}^t F(s, y(s))ds$$

$$= x(T_0 - \varepsilon_0) - x_0 + y(t) - y(T_0 - \varepsilon_0) = X(t) - x_0,$$

so that X is a continuation of x, contradicting the maximality of the latter. $\quad\square$

The previous theorems have the following consequences.

Corollary 2.1.14. *We consider a continuous function $F : \mathbb{R} \times \mathbb{R}^n \to \mathbb{R}^n$ which satisfies the Lipschitz condition (2.1.1). Then for all $(t_0, x_0) \in \mathbb{R} \times \mathbb{R}^n$ the initial value problem*

$$\begin{cases} \dot{x}(t) &= F(t, x(t)) \\ x(t_0) &= x_0 \end{cases}$$

has a unique maximal solution (defined on a non-empty interval J). Then

$$\text{if } \sup J < +\infty, \text{ one has } \limsup_{t \to (\sup J)_-} |x(t)| = +\infty, \qquad (2.1.17)$$

$$\text{if } \inf J > -\infty, \text{ one has } \limsup_{t \to (\inf J)_+} |x(t)| = +\infty. \qquad (2.1.18)$$

This follows immediately from Theorem 2.1.13. In other words, maximal solutions always exist (under mild regularity assumptions for F) and the only possible obstruction for a maximal solution to be global is that $|x(t)|$ gets unbounded, or if $\Omega \neq \mathbb{R}^n$, that $x(t)$ gets close to the boundary $\partial\Omega$.

Corollary 2.1.15. *Let I be an interval of $\mathbb{R}$. We consider a continuous function $F : I \times \mathbb{R}^n \to \mathbb{R}^n$ such that (2.1.1) holds and there exists a continuous function $\alpha : I \to \mathbb{R}_+$ so that*

$$\forall t \in I, \forall x \in \mathbb{R}^n, \quad |F(t,x)| \leq \alpha(t)\big(1 + |x|\big). \qquad (2.1.19)$$

Then all maximal solutions of the ODE $\dot{x} = F(t,x)$ are global. In particular, the solutions of linear equations with C^0 coefficients are globally defined.

The motto for this result should be: *solutions of nonlinear equations may blow-up in finite time, whereas solutions of linear equations do exist globally.*

Proof. We assume that $0 \in I \subset [0, +\infty)$ and we consider a maximal solution of the ODE: we note that for $I \ni t \geq 0$

$$|x(t)| \leq |x(0)| + \int_0^t \alpha(s)\big(1 + |x(s)|\big)ds = R(t),$$

so that $\dot{R} = \alpha(1 + |x|) \leq \alpha + \alpha R, R(0) = |x(0)|$, and Gronwall's inequality gives

$$|x(t)| \leq R(t) \leq e^{\int_0^t \alpha(s)ds}|x(0)| + \int_0^t \alpha(s)e^{\int_s^t \alpha(\sigma)d\sigma}ds < +\infty, \quad \text{for all } I \ni t \geq 0,$$

implying global existence. In particular a linear equation with C^0 coefficients would be $\dot{x} = A(t)x(t) + b(t)$, with A a $n \times n$ continuous matrix, $t \mapsto b(t) \in \mathbb{R}^n$ continuous, so that (2.1.1) holds trivially and

$$|F(t,x)| = |A(t)x + b(t)| \leq \|A(t)\||x| + |b(t)|$$

satisfying the assumption of the corollary. $\qquad \square$

We can check the example $\dot{x} = x(x^2 - 1)$.
If $|x(0)| > 1$, the solutions blow-up in finite time,
If $|x(0)| \in \{\pm 1, 0\}$, stationary solutions,
If $|x(0)| \leq 1$, global solutions.
When $0 < x(0) < 1$, $x(t) \in]0, 1[$ for all $t \in \mathbb{R}$ (and thus are decreasing), otherwise at some t_0, we would have by continuity $x(t_0) \in \{0, 1\}$ and thus by uniqueness it would be a stationary solution 0 or 1, contradicting $0 < x(0) < 1$. The lines $x = 0, \pm 1$ are separating the solutions.
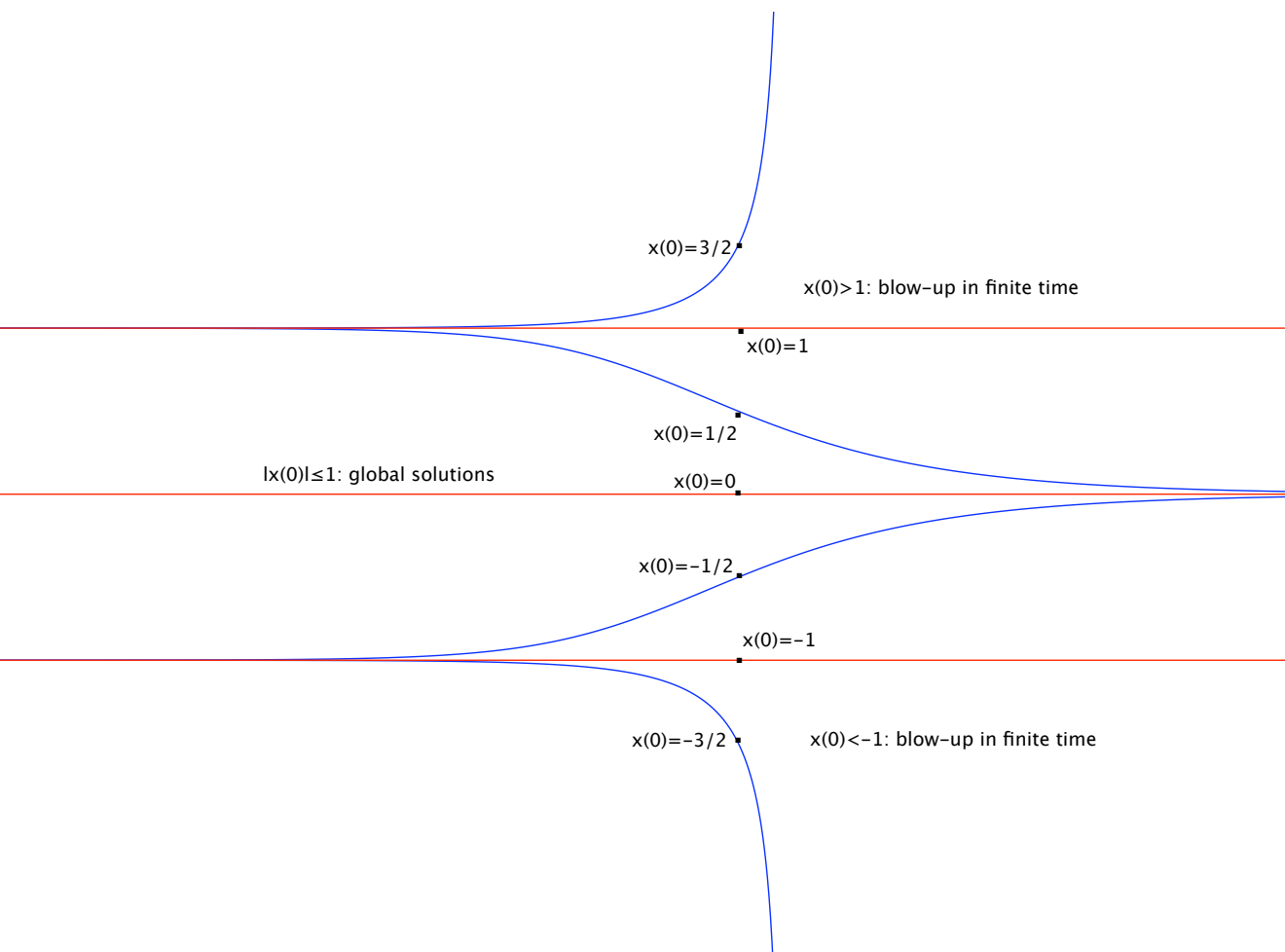
x(0)=3/2

x(0)>1: blow-up in finite time

x(0)=1

x(0)=1/2

|x(0)|≤1: global solutions

x(0)=0

x(0)=-1/2

x(0)=-1

x(0)=-3/2

x(0)<-1: blow-up in finite time

Figure 2.2: SOLUTIONS OF THE ODE $\dot{x} = x(x^2 - 1)$.

2.1.3 Continuous dependence

Theorem 2.1.16. *Let I be an interval of $\mathbb{R}$, Ω be an open set of $\mathbb{R}^n$ and U be an open set of $\mathbb{R}^m$. We consider a continuous function $F : I_t \times \Omega_x \times U_\lambda \to \mathbb{R}^n$ such that the partial derivatives $\partial F/\partial x_j, \partial F/\partial \lambda_k$ exist and are continuous. Assuming $0 \in I$, $y \in \Omega$, we define $x(t, y, \lambda)$ as the unique solution of the initial value problem*

$$\frac{\partial x}{\partial t}(t, y, \lambda) = F\big(t, x(t, y, \lambda), \lambda\big), \quad x(0, y, \lambda) = y.$$

Then the function x is a C^1 function defined on a neighborhood of $\{(0, y)\} \times U$.

Proof. We consider first the flow ψ of the ODE defined by

$$\frac{\partial \psi}{\partial t}(t, x) = F(t, \psi(t, x)), \quad \psi(0, x) = x$$

and we recall that $\psi(t, \cdot)$ is Lipschitz-continous from (2.1.13). According to Proposition 2.1.7, we may assume that ψ is defined on $[0, T_0] \times K_0$ with $T_0 > 0$ and K_0 a

compact subset of Ω with

$$|\psi(t, x_1) - \psi(t, x_2)| \leq e^{tL_0}|x_1 - x_2|. \tag{2.1.20}$$

For x given in Ω, we consider the linear ODE ($D, \partial_2 F$ are $n \times n$ matrices)

$$\dot{D}(t, x) = (\partial_2 F)\big(t, \psi(t, x)\big)D(t, x), \quad D(0, x) = \mathrm{Id}, \tag{2.1.21}$$

and we claim that $\frac{\partial \psi}{\partial x}(t, x) = D(t, x)$. In fact, we have, since $\partial_2 F$ is continuous and $\psi(t, \cdot)$ is Lipschitz-continuous,

$$\psi(t, x + h) - \psi(t, x) = h + \int_0^t \Big(F\big(s, \psi(s, x + h)\big) - F\big(s, \psi(s, x)\big)\Big)ds$$

$$= h+$$

$$\int_0^t \int_0^1 (\partial_2 F)\Big(s, \psi(s, x) + \theta\big(\psi(s, x + h) - \psi(s, x)\big)\Big)d\theta\big(\psi(s, x + h) - \psi(s, x)\big)ds.$$

As a result, with $\rho(t, x, h) = \psi(t, x + h) - \psi(t, x)$, we have

$$\dot{\rho}(t, x, h) = \int_0^1 (\partial_2 F)\big(t, \psi(t, x) + \theta\rho(t, x, h)\big)d\theta\rho(t, x, h), \quad \rho(0, x, h) = h.$$

We obtain

$$\dot{\rho}(t, x, h) = (\partial_2 F)(t, \psi(t, x))\rho(t, x, h) + \omega(t, x, \rho(t, x, h))\rho(t, x, h),$$

$$\omega(t, x, \rho) = \int_0^1 \big((\partial_2 F)(t, \psi(t, x) + \theta\rho) - (\partial_2 F)(t, \psi(t, x))\big)d\theta.$$

Using (2.1.21), (2.1.20) we have

$$\rho(0, x, h) - D(0, x)h = 0 \quad \text{and} \quad |\rho(t, x, h)| \leq e^{tL_0}|h|,$$

so that

$$\dot{\rho}(t, x, h) - \dot{D}(t, x)h$$
$$= (\partial_2 F)(t, \psi(t, x))(\rho(t, x, h) - D(t, x)h) + \omega(t, x, \rho(t, x, h))\rho(t, x, h),$$

and as a consequence with $r(t) = |\rho(t, x, h) - D(t, x)h|$ for $t \geq 0$,

$$r(t) \leq \int_0^t \|(\partial_2 F)(s, \psi(s, x))\|r(s)ds + t\eta(h)|h| \leq \int_0^t C_1 r(s)ds + t\eta(h)|h| = R(t),$$

with $\lim_{h \to 0} \eta(h) = 0$. This gives

$$\dot{R}(t) = C_1 r(t) + \eta(h)|h| \leq C_1 R(t) + \eta(h)|h|, \quad R(0) = 0,$$

and by Gronwall's inequality $R(t) \leq \int_0^t e^{C_1(t-s)}ds\eta(h)|h| = o(h)$ which gives

$$r(t) = o(h), \quad \rho(t, x, h) = D(t, x)h + o(h),$$

so that $\frac{\partial \psi}{\partial x}(t, x) = D(t, x)$. We note also that (2.1.21) and (2.1.13) imply that $D(t, x)$ is solution of the linear equation

$$\dot{D}(t, x) = \Omega(t, x)D(t, x), \quad D(0, x) = \mathrm{Id} \tag{2.1.22}$$

with Ω continuous.

Lemma 2.1.17. *Let $N \in \mathbb{N}$, I be an interval of $\mathbb{R}$ and U be an open subset of $\mathbb{R}^n$. Let Ω be a continuous function on $I \times U$ valued in $\mathcal{M}_N(\mathbb{R})$, the $N \times N$ matrices with real entries. Let $\Delta(x)$ be a continuous mapping from $\mathbb{R}^n$ into $\mathcal{M}_N(\mathbb{R})$. The unique solution of the linear ODE*

$$\dot{D}(t,x) = \Omega(t,x)D(t,x), \quad D(0,x) = \Delta(x),$$

is a continous function of its arguments.

Proof. From Theorem 2.1.1, we know that there exists a unique global solution for every $x \in U$, so that $I \ni t \mapsto D(t,x)$ is C^1 for each $x \in U$. We may assume $[0,T_0] \subset I$ with some positive T_0, and for $t \in [0,T_0], x, x+h \in K_0$, where K_0 is a compact neighborhood of x in U, we calculate

$$D(t,x+h)-D(t,x) = \Delta(x+h)-\Delta(x)+\int_0^t \big(\Omega(s,x+h)D(s,x+h)-\Omega(s,x)D(s,x)\big)ds,$$

entailing

$$|D(t,x+h) - D(t,x)| \leq |\Delta(x+h) - \Delta(x)|$$
$$+ \int_0^t |\Omega(s,x+h) - \Omega(s,x)||D(s,x+h)|ds$$
$$+ \int_0^t |\Omega(s,x)||D(s,x+h) - D(s,x)|ds.$$

We note also that

$$|D(t,x)| \leq |\Delta(x)| + \int_0^t |\Omega(s,x)||D(s,x)|ds$$
$$\leq \sup_{x \in K_0}|\Delta| + \int_0^t |D(s,x)|ds \sup_{[0,T_0] \times K_0} |\Omega| = R(t),$$

so that $\dot{R}(t) = \|\Omega\|_{[0,T_0] \times K_0}|D(t,x)| \leq \|\Omega\|_{[0,T_0] \times K_0}R(t)$ and Gronwall's inequality implies

$$|D(t,x)| \leq R(t) \leq \|\Delta\|_{K_0} \exp t\|\Omega\|_{[0,T_0] \times K_0} \leq C_0, \quad \text{for } t \in [0,T_0], x \in K_0.$$

With $\rho(t,x,h) = |D(t,x+h) - D(t,x)|$, we get thus with $C_1 = \|\Omega\|_{[0,T_0] \times K_0}$, $\lim_{h \to 0} \eta(h) = 0$,

$$\rho(t,x,h) \leq |\Delta(x+h) - \Delta(x)| + C_0 T_0 \eta(h) + C_1 \int_0^t \rho(s,x,h)ds = R_1(t).$$

We obtain $\dot{R}_1(t) = C_1\rho(t,x,h) \leq C_1 R_1(t)$ and Gronwall's inequality gives

$$\rho(t,x,h) \leq R_1(t) \leq \big(|\Delta(x+h) - \Delta(x)| + C_0 T_0 \eta(h)\big) \exp T_0 C_1.$$

For $t \in [0,T_0], x, x+h \in K_0$ we get

$$|D(t,x+h) - D(0,x)|$$
$$\leq \big(|\Delta(x+h) - \Delta(x)| + C_0 T_0 \eta(h)\big) \exp T_0 C_1 + \int_0^t |\Omega(s,x)||D(s,x)|ds$$
$$\leq \big(|\Delta(x+h) - \Delta(x)| + C_0 T_0 \eta(h)\big) \exp T_0 C_1 + tC_1 C_0,$$

which proves the continuity of D at $(0,x)$, ending the proof of the Lemma. $\square$

We may apply this lemma to get the fact that the flow is C^1, under the assumptions of Theorem 2.1.16. To handle the question with an additional parameter λ, we use the previous results, remarking that the equation

$$\dot{\psi}(t, y, z) = F(t, \psi(t, y, z), z), \quad \psi(0, y, z) = y$$

can be written as

$$\dot{\Psi}(t, y, z) = F(t, \Psi(t, y, z)), \quad \Psi(0, y, z) = (y, z)$$

with $\Psi(t, y, z) = (\psi(t, y, z), z)$. The proof of Theorem 2.1.16 is complete. $\qquad\square$

Corollary 2.1.18. *Let I be an interval of $\mathbb{R}$ containing 0, Ω be an open set of $\mathbb{R}^n$, $k \in \mathbb{N}^*$ and $F : I \times \Omega \to \mathbb{R}^n$ be a continuous function such that $\{\partial_x^\alpha F\}_{|\alpha| \le k}$ exist and are continuous on $I \times \Omega$. We denote[7] by $J_x \times \Omega \ni (t, x) \mapsto \psi(t, x) \in \mathbb{R}^n$ the maximal solution of the ODE*

$$\frac{\partial \psi}{\partial t}(t, x) = F(t, \psi(t, x)), \quad \psi(0, x) = x.$$

Then the function ψ is a C^1 function such that $\{\partial_x^\alpha \psi, \partial_t \partial_x^\alpha \psi\}_{|\alpha| \le k}$ are continuous.

Proof. For $k = 1, x_0 \in \Omega$, we get from the previous theorem that ψ is a C^1 function defined in a neighborhood of $(0, x_0)$ in $I \times \Omega$. Moreover, the proof of that theorem and (2.1.21) gives

$$\frac{\partial^2 \psi}{\partial t \partial x}(t, x) = (\partial_2 F)(t, \psi(t, x)) \cdot \frac{\partial \psi}{\partial x}(t, x), \quad \frac{\partial \psi}{\partial x}(0, x) = \mathrm{Id}, \qquad (2.1.23)$$

with $\frac{\partial \psi}{\partial x}$ continuous, according to Lemma 2.1.17, entailing from the above equation the continuity of $\frac{\partial^2 \psi}{\partial t \partial x}$. We want now to prove the theorem by induction on k with the additional statement that for any multi-index α with $|\alpha| \le k$,

$$\partial_t \partial_x^\alpha \psi = \sum_{\substack{|\rho| \ge 1 \\ \alpha_1 + \cdots + \alpha_{|\rho|} = \alpha, \ |\alpha_j| \ge 1}} c(\alpha_1, \ldots, \alpha_\rho, \rho)(\partial_2^\rho F)(t, \psi) \partial_x^{\alpha_1} \psi \ldots \partial_x^{\alpha_{|\rho|}} \psi, \qquad (2.1.24)$$

where $c(\alpha_1, \ldots, \alpha_\rho, \rho)$ are positive constants and $(\partial_x^\alpha \psi)(0, x)$ is a C^1 function. The formula (2.1.23) gives precisely the case $k = 1$ (note that $\psi(0, x), \partial_x \psi(0, x)$ are both C^1). Let us now assume that $k \ge 1$ and the assumptions of the Theorem are fulfilled for $k + 1$. For $|\alpha| = k$, (2.1.24) implies that $\partial_x^\alpha \psi$ satisfies

$$\partial_t \partial_x^\alpha \psi = (\partial_2 F)(t, \psi) \cdot \partial_x^\alpha \psi + G(t, \psi, \partial_x^\beta \psi)_{\beta < \alpha}$$

where G is a linear combination of products $(\partial_2^\rho F)(t, \psi) \partial_x^{\alpha_1} \psi \ldots \partial_x^{\alpha_{|\rho|}} \psi$, with $|\rho| \le k, 1 \le |\alpha_j| < k$. As a result the function G is a C^1 function of t, x. Since $(t, x) \mapsto \underbrace{(\partial_2 F)}_{C^k}(t, \underbrace{\psi(t, x)}_{C^1})$ is also C^1 since $k \ge 1$, $Y_\alpha = \partial_x^\alpha \psi$ is the solution of a linear ODE

$$\partial_t Y_\alpha = a(t, x) Y_\alpha + f(t, x), \quad a, f \in C^1.$$

---

[7]According to Remark 2.1.6, the time of existence of the solutions is bounded below by a positive constant T_0, provided the initial data belong to a compact subset.

A direct integration of that ODE gives

$$Y_\alpha(t,x) = e^{\int_0^t a(\sigma,x)d\sigma} \underbrace{Y_\alpha(0,x)}_{\in C^1} + \int_0^t e^{\int_s^t a(\sigma,x)d\sigma} f(s,x)ds,$$

so that $\partial_x^\alpha \psi$ is also C^1, as well as $\partial_t \partial_x^\alpha \psi$ from the equation. Taking the derivative with respect to x_1 of (2.1.24), using the fact that $\{\partial_x^\alpha F\}_{|\alpha| \le k+1}$ exist, we get a linear combination of terms

$$(\partial_2^\rho F)(t,\psi)\partial_x^{\alpha_1}\partial_{x_1}\psi \ldots \partial_x^{\alpha_{|\rho|}}\psi, \qquad \sum_{|\rho'|=|\rho|+1} (\partial_2^{\rho'} F)(t,\psi)\partial_{x_1}\psi\partial_x^{\alpha_1}\psi \ldots \partial_x^{\alpha_{|\rho|}}\psi,$$

entailing the formula (2.1.24) for $k+1$; note also that for $|\beta| = 1+k \ge 2$, $(\partial_x^\beta \psi)(0,x) = 0$. The proof of the induction is complete as well as the proof of the theorem. $\qquad\square$

Corollary 2.1.19. *Let Ω be an open set of $\mathbb{R}^n$, $1 \le k \in \mathbb{N}$ and $F : \Omega \to \mathbb{R}^n$ be a C^k function. Then the flow of the autonomous ODE, $\dot{\psi} = F(\psi)$, is of class C^k (in both variables t,x).*

Proof. For $k = 1$, it follows from Corollary 2.1.18. Assume inductively that $k \ge 1$ and $F \in C^{k+1}$: from Corollary 2.1.18, we know that $\partial_t \partial_x^\alpha \psi, \partial_x^\alpha \psi$ are continuous functions for $|\alpha| \le k+1$. Moreover we know from (2.1.24) an explicit expression for $\partial_t \partial_x^\alpha \psi$ for $|\alpha| \le k+1$ and in particular for $|\alpha| = k$; since in (2.1.24), $|\rho| \le |\alpha| = k$, we can compute $\partial_t^2 \partial_x^\alpha \psi$, which is a linear combination of

$$\underbrace{\partial^{\rho'} F}_{|\rho'|=1+|\rho|\le k+1} \underbrace{\partial_t \psi}_{F(\psi)} \partial_x^{\alpha_1}\psi \ldots \partial_x^{\alpha_{|\rho|}}\psi, \quad \partial^\rho F \partial_x^{\alpha_1} \underbrace{\partial_t \psi}_{F(\psi)} \ldots \partial_x^{\alpha_{|\rho|}}\psi,$$

which is a continuous function. More generally, $\partial_t^l \partial_x^\alpha \psi$ for $l + |\alpha| \le k+1$ is a polynomial in $\partial_x^\beta \psi$, $(\partial^\rho F)(\psi)$, with $|\beta| \le k+1, |\rho| \le k+1$, thus a continuous function. All the partial derivatives of ψ with order $\le k+1$ are continuous, completing the induction and the proof. $\qquad\square$

2.2   Vector Fields, Flow, First Integrals

2.2.1   Definition, examples

Definition 2.2.1. *Let Ω be an open set of $\mathbb{R}^n$. A vector field X on Ω is a mapping from Ω into $\mathbb{R}^n$. The differential system associated to X is*

$$\frac{dx}{dt} = X(x). \tag{2.2.1}$$

An integral curve of X is a solution of the previous system and the flow of the vector field is the flow of that system of ODE. A singular point of X is a point $x_0 \in \Omega$ such that $X(x_0) = 0$. When for $x \in \Omega$, $X(x) = (X_j(x))_{1 \le j \le n}$, the vector field X is denoted by $\sum_{1 \le j \le n} X_j(x)\frac{\partial}{\partial x_j}$.

N.B. We have introduced the notion of flow of an ODE in Proposition 2.1.7. In the above definition, we deal with a so-called *autonomous* flow since X depends only on the variable x and not on t.

Remark 2.2.2. Let X be a C^1 vector field on some open subset Ω of $\mathbb{R}^n$. The flow of the vector field X, denoted by $\Phi_X^t(x)$ is the maximal solution of the ODE

$$\dot{\Phi}_X^t(x) = X\big(\Phi_X^t(x)\big), \quad \Phi_X^0(x) = x. \tag{2.2.2}$$

The mapping $t \mapsto \Phi_X^t(x)$ is defined in a neighborhood of 0 which may depend on x; however, thanks to Proposition 2.1.7 and Corollary 2.1.18, for each compact subset K_0 of Ω, there exists $T_0 > 0$ such that $(t, x) \mapsto \Phi_X^t(x)$ is defined and C^1 on $[-T_0, T_0] \times K_0$. We have for $x \in \Omega$ and t, s in a neighborhood of 0,

$$\frac{d}{dt}\big(\Phi_X^{t+s}(x)\big) = X\big(\Phi_X^{t+s}(x)\big), \quad \Phi_X^{t+s}(x)_{|t=0} = \Phi_X^s(x),$$

$$\frac{d}{dt}\big(\Phi_X^t\big(\Phi_X^s(x)\big)\big) = X\big(\Phi_X^t\big(\Phi_X^s(x)\big)\big), \quad \Phi_X^t\big(\Phi_X^s(x)\big)_{|t=0} = \Phi_X^s(x),$$

so that the uniqueness theorem 2.1.1 forces

$$\Phi_X^{t+s} = \Phi_X^t \Phi_X^s. \tag{2.2.3}$$

In particular the flow Φ_X^t is a local C^1 diffeomorphism with inverse Φ_X^{-t} since Φ_X^0 is the identity.

Let us give a couple of examples. The radial vector field in $\mathbb{R}^2$ is $x_1\partial_{x_1} + x_2\partial_{x_2}$, namely is the mapping $\mathbb{R}^2 \ni (x_1, x_2) \mapsto (x_1, x_2) \in \mathbb{R}^2$. The differential system associated to this vector field is

$$\begin{cases} \dot{x}_1 = x_1 \\ \dot{x}_2 = x_2 \end{cases} \quad \text{i.e.} \quad \begin{cases} x_1 = y_1 e^t, \ x_1(0) = y_1 \\ x_2 = y_2 e^t, \ x_2(0) = y_2 \end{cases}$$

so that the integral curves are straight lines through the origin. The flow $\psi(t, x)$ of the radial vector field defined on $\mathbb{R} \times \mathbb{R}^2$ is thus

$$\psi(t, x) = e^t x.$$

We can note that $\psi(t, 0_{\mathbb{R}^2}) = 0_{\mathbb{R}^2}$ for all t, expressing the fact that $0_{\mathbb{R}^2}$ is a singular point of $x_1\partial_{x_1} + x_2\partial_{x_2}$, namely a point where the vector field is vanishing.

We consider now the angular vector field in $\mathbb{R}^2$ given by $x_1\partial_{x_2} - x_2\partial_{x_1}$ (the mapping $\mathbb{R}^2 \ni (x_1, x_2) \mapsto (-x_2, x_1) \in \mathbb{R}^2$). The differential system associated to this vector field is

$$\begin{cases} \dot{x}_1 = -x_2 \\ \ \dot{x}_2 = x_1 \end{cases} \quad \text{i.e.} \quad \frac{d}{dt}(x_1 + ix_2) = i(x_1 + ix_2), \quad (x_1 + ix_2) = e^{it}(x_1(0) + ix_2(0))$$

so that the integral curves are circles centered at the origin. The flow $\psi(t, x)$ of the angular vector field defined on $\mathbb{R} \times \mathbb{R}^2$ is thus

$$\psi(t, x) = R(t)x, \quad R(t) = \begin{pmatrix} \cos t & -\sin t \\ \sin t & \cos t \end{pmatrix} \quad \text{(rotation with angle } t\text{)}.$$
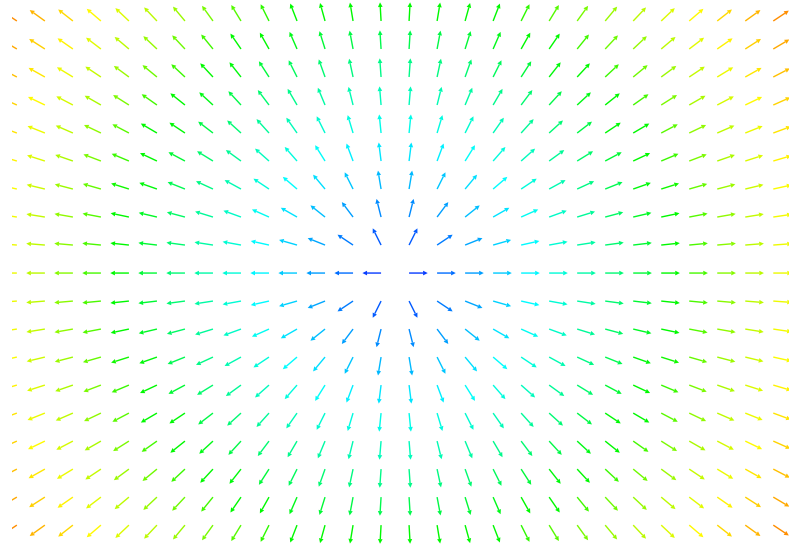
Figure 2.3: THE RADIAL VECTOR FIELD $x_1\partial_{x_1} + x_2\partial_{x_2}$.

Remark 2.2.3. We note that if a vector field X is locally Lipschitz-continuous on Ω, for all $x_0 \in \Omega$, there exists a unique maximal solution $t \mapsto x(t) = \psi(t, x_0)$ of the ODE (2.2.1) defined on $[0, T_0[$ with some positive T_0. From Theorem 2.1.13, if $x([0, T_0[)$ is contained in a compact subset of Ω, we have $T_0 = +\infty$, (otherwise $T_0 < +\infty$ and $\sup_{0 \leq t < T_0} |x(t)| = +\infty$).

Definition 2.2.4. *Let X be a vector field on Ω, open subset of $\mathbb{R}^n$. A first integral of X is a differentiable mapping $f : \Omega \to \mathbb{R}$ such that*

$$\forall x \in \Omega, \quad (Xf)(x) = \sum_{1 \leq j \leq n} X_j(x)\frac{\partial f}{\partial x_j}(x) = 0.$$

In other words, $\langle df, X \rangle = 0$, where the bracket stand for a bracket of duality between the one-form $df = \sum_{1 \leq j \leq n} \frac{\partial f}{\partial x_j} dx_j$ and the vector field $X = \sum_{1 \leq j \leq n} X_j \partial_{x_j}$. If f is of class C^1 with $df \neq 0$, the set (a level surface of f)

$$\Sigma = \{x \in \Omega, f(x) = f(x_0)\}$$

is a C^1 hypersurface to which the vector field X is tangent since it is "orthogonal" to the *gradient* of f given by $\nabla f = (\partial_{x_j} f)_{1 \leq j \leq n}$, which is the "normal" vector to Σ. The quotation marks here are important in the sense that orthogonality here must be understood in the sense of duality. The tangent bundle to the open set $\mathcal{U}$ is simply the product $\mathcal{U} \times \mathbb{R}^n$ and a vector field on $\mathcal{U}$ is a section of that bundle, i.e. a mapping $\mathcal{U} \ni x \mapsto (x, X(x)) \in \mathcal{U} \times \mathbb{R}^n$. Now if $\mathcal{V}$ is an open set of $\mathbb{R}^n$ and $\kappa : \mathcal{V} \to \mathcal{U}$ is a C^1-diffeomorphism, we can define the pull-back of the vector field X by κ as the vector field Y on $\mathcal{V}$ such that for $f \in C^1(\mathcal{U})$

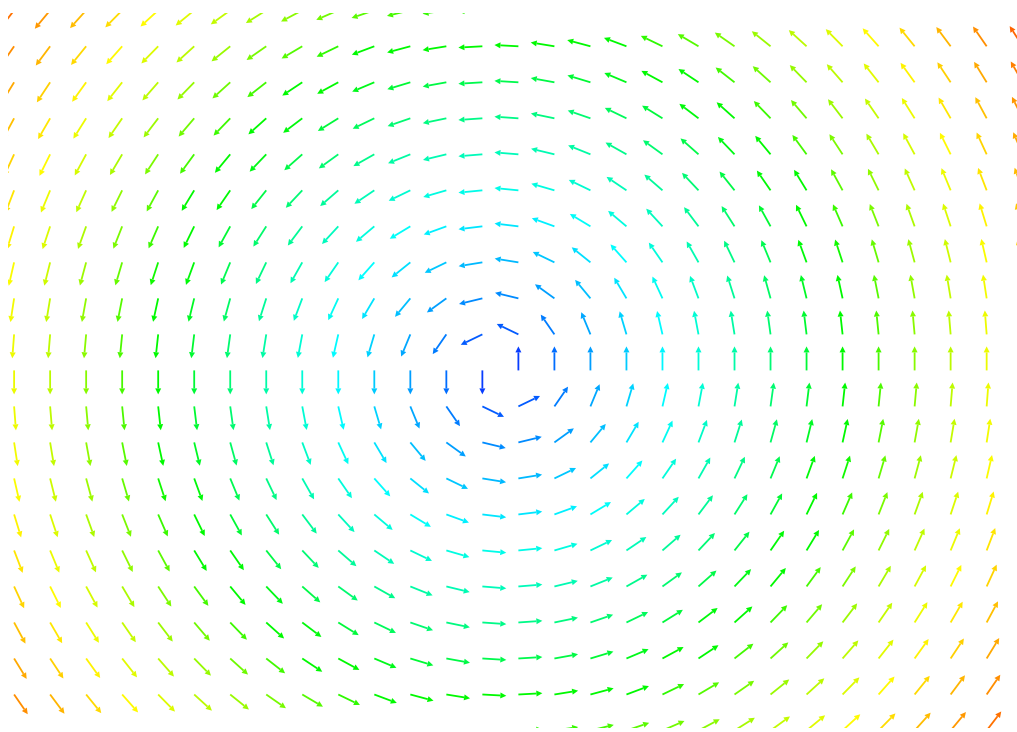$$\langle d(f \circ \kappa), Y \rangle = \langle df, X \rangle \circ \kappa, \tag{2.2.4}$$

Figure 2.4: THE ANGULAR VECTOR FIELD $x_1\partial_{x_2} - x_2\partial_{x_1}$.

i.e. for $X = \sum_{1\le j\le n} a_j(x)\partial_{x_j}$, we define $Y = \sum_{1\le k\le n} b_k(y)\partial_{y_k}$ so that

$$\sum_{1\le k\le n} b_k(y)\frac{\partial(f\circ\kappa)}{\partial y_k}(y) = \sum_{1\le j\le n} a_j(\kappa(y))\frac{\partial f}{\partial x_j}(\kappa(y))$$

which means

$$\sum_{1\le k\le n} b_k(y)\frac{\partial(f\circ\kappa)}{\partial y_k}(y) = \sum_{1\le k,j\le n} b_k(y)\frac{\partial f}{\partial x_j}(\kappa(y))\frac{\partial\kappa_j}{\partial y_k}(y) = \sum_{1\le j\le n} a_j(\kappa(y))\frac{\partial f}{\partial x_j}(\kappa(y))$$

and this gives

$$a_j(\kappa(y)) = \sum_{1\le k\le n} b_k(y)\frac{\partial\kappa_j}{\partial y_k}(y).$$

Abusing the notations, these relationships are written usually in a more convenient way as

$$\sum_k b_k\frac{\partial}{\partial y_k} = \sum_{j,k} b_k\frac{\partial x_j}{\partial y_k}\frac{\partial}{\partial x_j} = \sum_j \underbrace{\left(\sum_k b_k\frac{\partial x_j}{\partial y_k}\right)}_{a_j}\frac{\partial}{\partial x_j}.$$

Anyhow, we get immediately from these expressions that if $X(f) = 0$, i.e. f is a first integral of X, the function $f\circ\kappa$ is a first integral of Y, meaning that the notion of first integral is invariant by diffeomorphism, as well as the notion of a vector field tangent to the level surfaces of a function f. Similarly, the one-form

df has an invariant meaning as a conormal vector to Σ: the reader may remember from that discussion that no Euclidean nor Riemannian structure was involved in the definitions above.

Let us go back to our examples above: the angular vector field $x_1\partial_2 - x_2\partial_1$ has obviously the first integral $x_1^2 + x_2^2$ and we see that this vector field is tangential to the circles centered at 0. The radial vector field $x_1\partial_1 + x_2\partial_2$ has the first integral x_2/x_1 on the open set $x_1 \neq 0$ and is indeed tangential to all straight lines through the origin. We see as well that

$$\frac{x_1}{\sqrt{x_1^2 + x_2^2}}, \quad \frac{x_2}{\sqrt{x_1^2 + x_2^2}}$$

are first integrals of the radial vector field as well as all homogeneous functions of degree 0. Considering the diffeomorphism[8]

$$\begin{array}{ccc} \mathbb{R}_+^* \times] - \pi, \pi[& \to & \mathbb{C}\backslash\mathbb{R}_- \\ (r, \theta) & \mapsto & re^{i\theta} \end{array}$$

we see as well that

$$r\frac{\partial}{\partial r} = x_1\partial_1 + x_2\partial_2, \qquad \frac{\partial}{\partial \theta} = x_1\partial_2 - x_2\partial_1. \tag{2.2.5}$$

For the so-called spherical coordinates in $\mathbb{R}^3$, we have

$$\begin{cases} x_1 = r\cos\theta\sin\phi \\ x_2 = r\sin\theta\sin\phi \\ x_3 = r\cos\phi \end{cases} \tag{2.2.6}$$

(θ is the longitude, ϕ the colatitude[9]) and the diffeomorphism

$$\begin{array}{ccccccc} \kappa: &]0, +\infty[& \times &]0, \pi[& \times &] - \pi, \pi[& \to & \mathbb{R}^3\backslash\{(x_1, x_2, x_3), x_1 \leq 0, x_2 = 0\} = \mathcal{D} \\ & (r & , & \phi & , & \theta) & \mapsto & (r\cos\theta\sin\phi, r\sin\theta\sin\phi, r\cos\phi) \end{array}$$

we have

$$\frac{\partial}{\partial r} = \cos\theta\sin\phi\frac{\partial}{\partial x_1} + \sin\theta\sin\phi\frac{\partial}{\partial x_2} + \cos\phi\frac{\partial}{\partial x_3},$$
$$\frac{\partial}{\partial \phi} = r\cos\theta\cos\phi\frac{\partial}{\partial x_1} + r\sin\theta\cos\phi\frac{\partial}{\partial x_2} - r\sin\phi\frac{\partial}{\partial x_3},$$
$$\frac{\partial}{\partial \theta} = -r\sin\theta\sin\phi\frac{\partial}{\partial x_1} + r\cos\theta\sin\phi\frac{\partial}{\partial x_2}.$$

---

[8] For $z \in \mathbb{C}\backslash\mathbb{R}_-$, we define $\operatorname{Log} z = \int_{[1,z]} \frac{d\xi}{\xi}$ and we get by analytic continuation that $e^{\operatorname{Log} z} = z$; we define $\arg z = \operatorname{Im}(\operatorname{Log} z)$, so that $z = e^{i\arg z}e^{\operatorname{Re}\operatorname{Log} z} = e^{i\arg z}e^{\ln|z|}$.

[9] The latitude is $\frac{\pi}{2} - \phi$, equal to $\pi/2$ at the "north pole" $(0,0,1)$ and to $-\pi/2$ at the "south pole" $(0,0,-1)$.

Since we have an analytic determination of the argument (see footnote [8]) on $\mathbb{C}\backslash\mathbb{R}_-$, we define on $\mathcal{D}$

$$
\begin{cases}
r = \sqrt{x_1^2 + x_2^2 + x_3^2} \\
\phi = \arg\big(x_3 + i(x_1^2 + x_2^2)^{1/2}\big) \\
\theta = \arg(x_1 + ix_2)
\end{cases}
$$

The radial vector field $x \cdot \partial_x = x_1 \partial_{x_1} + x_2 \partial_{x_2} + x_3 \partial_{x_3} = r \partial_r$ is *transverse* to the spheres with center 0 since $\partial_r r = 1 \neq 0$ whereas the vector fields $\frac{\partial}{\partial \phi}, \frac{\partial}{\partial \theta}$ are tangential to the spheres with center 0 since it is easy to verify that

$$
\frac{\partial r}{\partial \phi} = \frac{\partial r}{\partial \theta} = 0.
$$

In fact the three vector fields $x_2 \partial_{x_1} - x_1 \partial_{x_2}$, $x_3 \partial_{x_2} - x_2 \partial_{x_3}$, $x_1 \partial_{x_3} - x_3 \partial_{x_1}$, are tangential to the spheres with center 0 and

$$
\frac{\partial}{\partial \theta} = x_1 \partial_{x_2} - x_2 \partial_{x_1}, \tag{2.2.7}
$$

$$
\frac{\partial}{\partial \phi} = r \cos \theta \cos \phi \, \partial_{x_1} + r \sin \theta \cos \phi \, \partial_{x_2} - r \sin \phi \, \partial_{x_3},
$$

so that

$$
\begin{aligned}
r \sin \phi \frac{\partial}{\partial \phi} &= x_3 r \frac{\partial}{\partial r} - r^2 \frac{\partial}{\partial x_3} = x_1 x_3 \frac{\partial}{\partial x_1} + x_2 x_3 \frac{\partial}{\partial x_2} - (x_1^2 + x_2^2) \frac{\partial}{\partial x_3} \\
&= x_1 \big(x_3 \frac{\partial}{\partial x_1} - x_1 \frac{\partial}{\partial x_3}\big) + x_2 \big(x_3 \frac{\partial}{\partial x_2} - x_2 \frac{\partial}{\partial x_3}\big)
\end{aligned}
$$

so that

$$
\frac{\partial}{\partial \phi} = \frac{x_1}{\sqrt{x_1^2 + x_2^2}} \big(x_3 \frac{\partial}{\partial x_1} - x_1 \frac{\partial}{\partial x_3}\big) + \frac{x_2}{\sqrt{x_1^2 + x_2^2}} \big(x_3 \frac{\partial}{\partial x_2} - x_2 \frac{\partial}{\partial x_3}\big). \tag{2.2.8}
$$

The integral curves of $\partial/\partial \theta$ are the so-called parallels, which are horizontal circles with center on the x_3-axis (e.g. the Equator, the Artic circle), whereas the integral curves of $\partial/\partial \phi$ are the meridians, which are circles (or half-circles) with diameter NS where $N = (0, 0, 1)$ is the north pole and $S = (0, 0, -1)$ is the south pole.

2.2.2   Local Straightening of a non-singular vector field

Theorem 2.2.5. *Let $k \in \mathbb{N}^*$, Ω an open set of $\mathbb{R}^n$, $x_0 \in \Omega$ and let X be C^k-vector field on Ω such that $X(x_0) \neq 0$ (X is non-singular at x_0). Then there exists a C^k diffeomorphism $\kappa : V \to U$, where U is an open neighborhood of x_0 and V is an open neighborhood of $0_{\mathbb{R}^n}$ such that*

$$
\kappa^*\big(X_{|U}\big) = \frac{\partial}{\partial y_1}.
$$

Proof. Assuming as we may $x_0 = 0$, with $X = \sum_{1 \le j \le n} a_j(x) \partial_{x_j}$, we may assume that $a_1(0) \ne 0$. The flow of the vector field X satisfies

$$\dot{\psi}(t; x) = X\big(\psi(t; x)\big), \quad \psi(0; x) = x \tag{2.2.9}$$

and thus the mapping $(y_1, y_2, \ldots, y_n) \mapsto \kappa(y_1, y_2, \ldots, y_n) = \psi(y_1; 0, y_2, \ldots, y_n)$ is of class C^k in a neighborhood of 0 (see Corollary 2.1.18) with a Jacobian determinant at 0 equal to

$$\frac{\partial \psi}{\partial t}(0) \wedge \frac{\partial \psi}{\partial y_2}(0) \wedge \cdots \wedge \frac{\partial \psi}{\partial y_n}(0) = X(0) \wedge \vec{e}_2 \wedge \cdots \wedge \vec{e}_n = a_1(0) \ne 0.$$

As a result, from the local inversion Theorem, κ is a C^k diffeomorphism between V and U, neighborhoods of 0 in $\mathbb{R}^n$. From the identity

$$\frac{\partial}{\partial t}\big\{u(\psi(t, y))\big\} = \sum_{1 \le j \le n} \frac{\partial u}{\partial x_j}\big(\psi(t, y)\big) \frac{\partial \psi_j}{\partial t}(t, y) = \sum_{1 \le j \le n} a_j\big(\psi(t, y)\big) \frac{\partial u}{\partial x_j}\big(\psi(t, y)\big),$$

we get

$$\frac{\partial}{\partial y_1}\big\{(u \circ \kappa)(y)\big\} = \frac{\partial}{\partial y_1}\big\{u\big(\psi(y_1; 0, y_2, \ldots, y_n)\big)\big\} = \sum_{1 \le j \le n} a_j(\kappa(y)) \frac{\partial u}{\partial x_j}(\kappa(y)),$$

so that

$$\langle d(u \circ \kappa), \frac{\partial}{\partial y_1}\rangle = \langle du, X \rangle \circ \kappa$$

and the identity (2.2.4) gives the result. We have with $(b_1, \ldots, b_n) = (1, 0 \ldots, 0)$

$$\kappa^*\big(X_{|U}\big) = \sum_{1 \le k \le n} b_k(y) \frac{\partial}{\partial y_k}, \quad a_j(\kappa(y)) = \sum_{1 \le k \le n} \frac{\partial \kappa_j}{\partial y_k}(y) b_k(y) = a_j(\kappa(y)) b_1(y).$$

$\square$

The previous method also gives a way to actually solve a first-order linear PDE: let us consider the following equation on some open set Ω of $\mathbb{R}^n$

$$\sum_{1 \le j \le n} a_j(x) \frac{\partial u}{\partial x_j}(x) = a_0(x) u(x) + f(x) \tag{2.2.10}$$

where the $\{a_j\}_{1 \le j \le n} \in C^1(\Omega; \mathbb{R}), a_0, f \in C^1(\Omega; \mathbb{R})$. We are seeking some C^1 solution u. That equation can be written as $Xu = a_0 u + f$ and if ψ is the flow of the vector field X, i.e., satisfies (2.2.9) and u is a C^1 function solving (2.2.10), we get

$$\frac{d}{dt}\big(u(\psi(t, x))\big) = \langle du, X \rangle(\psi(t, x)) = a_0(\psi(t, x)) u(\psi(t, x)) + f(\psi(t, x)).$$

The function $t \mapsto u(\psi(t, x))$ satisfies an ODE that we can solve explicitely: with $a(t, x) = \int_0^t a_0(\psi(s, x)) ds$

$$u(\psi(t, x)) = e^{a(t,x)} u(x) + \int_0^t e^{a(t,x) - a(s,x)} f(\psi(s, x)) ds. \tag{2.2.11}$$

In particular, if in the formula above x belongs to some C^1 hypersurface Σ so that X is tranverse to Σ, the proof of the previous theorem shows that the mapping $\mathbb{R} \times \Sigma \ni (t, x) \mapsto \psi(t, x) \in \Omega$ is a local C^1-diffeomorphism. As a result the datum $u_{|\Sigma}$ determines uniquely the C^1 solution of the equation (2.2.10).

Corollary 2.2.6. *Let Ω be an open set of $\mathbb{R}^n$, Σ a C^1 hypersurface*[10] *of Ω and X a C^1 vector field on Ω such that X is transverse to Ω. Let $a_0, f \in C^0(\Omega)$, $x_0 \in \Sigma$ and $g \in C^0(\Sigma)$. There exists a neighborhood $\mathcal{U}$ of x_0 such that the Cauchy problem*

$$\begin{cases} Xu & = a_0 u + f \quad on\ \mathcal{U}, \\ u_{|\Sigma} & = g \quad on\ \Sigma, \end{cases}$$

has a unique continuous solution.

Proof. Using Theorem 2.2.5, we may assume that $X = \frac{\partial}{\partial x_n}$. Since X is transverse to the hypersurface Σ with equation $\rho(x) = 0$, the implicit function theorem gives that on a possibly smaller neighborhood of x_0 (we take $x_0 = 0$),

$$\Sigma = \{(x', x_n) \in \mathbb{R}^{n-1} \times \mathbb{R}^n, x_n = \alpha(x')\}$$

where α is a C^1 function. We get

$$\frac{\partial u}{\partial x_n} = a_0(x', x_n) u(x', x_n) + f(x', x_n), \quad u(x', \alpha(x')) = g(x').$$

Using the notation

$$a(x', x_n) = \int_{\alpha(x')}^{x_n} a_0(x', t) dt$$

The unique solution of that ODE with respect to the variable x_n with parameters x' is given by

$$u(x', x_n) = e^{a(x', x_n)} g(x') + \int_{\alpha(x')}^{x_n} e^{a(x', x_n) - a(x', t)} f(x', t) dt.$$

$\square$

Definition 2.2.7. *Let Ω be an open set of $\mathbb{R}^n$, X a Lipschitz-continuous vector field on Ω. The divergence of X is defined as $\operatorname{div} X = \sum_{1 \le j \le n} \partial_{x_j}(a_j)$.*

Definition 2.2.8. *Let Ω be an open set of $\mathbb{R}^n$: Ω will be said to have a C^1 boundary if for all $x_0 \in \partial\Omega$, there exists a neighborhood U_0 of x_0 in $\mathbb{R}^n$ and a C^1 function $\rho_0 \in C^1(U_0; \mathbb{R})$ such that $d\rho_0$ does not vanish and $\Omega \cap U_0 = \{x \in U_0, \rho_0(x) < 0\}$.*

Note that $\partial\Omega \cap U_0 = \{x \in U_0, \rho_0(x) = 0\}$ since the implicit function theorem shows that, if $(\partial\rho_0/\partial x_n)(x_0) \ne 0$ for some $x_0 \in \partial\Omega$, the mapping $x \mapsto (x_1, \ldots, x_{n-1}, \rho_0(x))$ is a local C^1-diffeomorphism.

---

[10]We shall define a C^1 hypersurface of Ω as the set $\Sigma = \{x \in \Omega, \rho(x) = 0\}$ where the function $\rho \in C^1(\Omega; \mathbb{R})$ such that $d\rho \ne 0$ at Σ. The transversality of the vector field X means here that $X\rho \ne 0$ at Σ.

Theorem 2.2.9 (Gauss-Green formula). *Let Ω be an open set of $\mathbb{R}^n$ with a C^1 boundary, X a Lipschitz-continuous vector field on Ω. Then we have, if X is compactly supported or Ω is bounded,*

$$\int_\Omega (\operatorname{div} X) dx = \int_{\partial\Omega} \langle X, \nu \rangle d\sigma, \tag{2.2.12}$$

where ν is the exterior unit normal and $d\sigma$ is the Euclidean measure on $\partial\Omega$.

Proof. We may assume that $\Omega = \{x \in \mathbb{R}^n, \rho(x) < 0\}$, where $\rho : \mathbb{R}^n \longrightarrow \mathbb{R}$ is C^1 and such that $d\rho \neq 0$ at $\partial\Omega$. The exterior normal to the open set Ω is defined on (a neighborhood of) $\partial\Omega$ as $\nu = \|d\rho\|^{-1} d\rho$. We can reformulate the theorem as

$$\int_\Omega \operatorname{div} X \, dx = \int \langle X, \nu \rangle \delta(\rho(x)) \|d\rho(x)\| = \lim_{\epsilon \to 0_+} \int \langle X, d\rho(x) \rangle \theta(\rho(x)/\epsilon) dx/\epsilon$$

where $\theta \in C_c(\mathbb{R})$ has integral 1. Since it is linear in X, it is enough to prove it for $a(x)\partial_{x_1}$, with $a \in C_c^1$. We have, with $\psi = 1$ on $(1, +\infty)$, $\psi = 0$ on $(-\infty, 0)$,

$$\int_\Omega \operatorname{div} X \, dx = \int_{\rho(x)<0} \frac{\partial a}{\partial x_1}(x) dx = \lim_{\epsilon \to 0_+} \int \frac{\partial a}{\partial x_1}(x) \psi(-\rho(x)/\epsilon) dx$$

$$= \lim_{\epsilon \to 0_+} \int a(x) \psi'(-\rho(x)/\epsilon) \epsilon^{-1} \frac{\partial \rho}{\partial x_1}(x) dx = \lim_{\epsilon \to 0_+} \int \langle a(x)\partial_{x_1}, d\rho \rangle \psi'(-\rho(x)/\epsilon) \epsilon^{-1} dx$$

$$= \lim_{\epsilon \to 0_+} \int \langle X, d\rho \rangle \theta(\rho(x)/\epsilon) \epsilon^{-1} dx,$$

with $\theta(t) = \psi'(-t)$, $\quad \int_{-\infty}^{+\infty} \theta(t) dt = \int_{-\infty}^{+\infty} \psi'(-t) dt = \int_{-\infty}^{+\infty} \psi'(t) dt = 1$, $\qquad \square$

In two dimensions, we get the Green-Riemann formula

$$\iint_\Omega \left(\frac{\partial P}{\partial x} + \frac{\partial Q}{\partial y}\right) dx dy = \int_{\partial\Omega} P dy - Q dx, \tag{2.2.13}$$

since with $X = P\partial_x + Q\partial_y$, $\Omega \equiv \rho(x,y) < 0$, the lhs of (2.2.13) and (2.2.12) are the same, whereas the rhs of (2.2.12) is, if $\rho(x,y) = f(x) - y$ on the support of X,

$$\iint \langle X, d\rho \rangle \delta(\rho) dx dy = \lim_{\varepsilon \to 0_+} \iint \big(P(x,y) f'(x) - Q(x,y)\big) \theta((f(x) - y)/\varepsilon) dx dy/\varepsilon$$

$$= \int \big(P(x, f(x)) f'(x) - Q(x, f(x))\big) dx = \int_{\partial\Omega} P dy - Q dx.$$

Corollary 2.2.10. *Let Ω be an open subset of $\mathbb{R}^n$ with a C^1 boundary, $u, v \in C^2(\bar\Omega)$. Then*

$$\int_\Omega (\Delta u)(x) v(x) dx = \int_\Omega u(x)(\Delta v)(x) dx + \int_{\partial\Omega} \left(v \frac{\partial u}{\partial \nu} - u \frac{\partial v}{\partial \nu}\right) d\sigma, \tag{2.2.14}$$

$$\int_\Omega \nabla u \cdot \nabla v \, dx = -\int_\Omega u \Delta v \, dx + \int_{\partial\Omega} u \frac{\partial v}{\partial \nu} d\sigma. \tag{2.2.15}$$

where $\Delta = \sum_{1 \le j \le n} \partial_{x_j}^2$ is the Laplace operator and $\frac{\partial u}{\partial \nu} = \nabla u \cdot \nu$ where ν is the exterior normal.

Proof. We have $v\Delta u = \operatorname{div}(v\nabla u) - \nabla u \cdot \nabla v$ so that $v\Delta u - u\Delta v = \operatorname{div}(v\nabla u - u\nabla v)$ providing the first formula from Green's formula (2.2.12). The same formula written as $\nabla u \cdot \nabla v = -u\Delta v + \operatorname{div}(u\nabla v)$ entails the second formula. $\qquad \square$

2.2.3   2D examples of singular vector fields

Theorem 2.2.5 shows that, locally, a C^1 non-vanishing (i.e. non-singular) vector field is equivalent to $\partial/\partial x_1$. The general question of classification of vector fields with singularities is a difficult one, but we can give a pretty complete discussion for planar vector fields with non-degenerate differential: it amounts to look at the system of ODE

$$\begin{pmatrix} \dot{x}_1 \\ \dot{x}_2 \end{pmatrix} = A \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}, \quad \text{where } A \text{ is a } 2 \times 2 \text{ constant real matrix, } \det A \neq 0. \quad (2.2.16)$$

The characteristic polynomial of A is

$$p_A(X) = X^2 - X \operatorname{trace} A + \det A, \quad \Delta_A = (\operatorname{trace} A)^2 - 4 \det A.$$

Case $\Delta_A > 0$: two distinct real roots λ_1, λ_2, $\mathbb{R}^2 = E_{\lambda_1} \oplus E_{\lambda_2}$

In a basis of eigenvectors the system is

$$\begin{cases} \dot{y}_1 = \lambda_1 y_1 \\ \dot{y}_2 = \lambda_2 y_2 \end{cases} \quad \text{i.e.} \quad \begin{cases} y_1 = e^{t\lambda_1} y_{10} \\ y_2 = e^{t\lambda_2} y_{20} \end{cases}$$

· $\det A > 0, \operatorname{trace} A > 0$: $0 < \lambda_1 < \lambda_2$

$$\frac{y_2}{y_{20}} = \left(\frac{y_1}{y_{10}} \right)^{\lambda_2/\lambda_1} \qquad \text{repulsive node.}$$

· $\det A > 0, \operatorname{trace} A < 0$: $\lambda_2 < \lambda_1 < 0$, attractive node, reverse the arrows in the previous picture,

$$\frac{y_2}{y_{20}} = \left(\frac{y_1}{y_{10}} \right)^{\lambda_2/\lambda_1}.$$

· $\det A < 0$: $\lambda_1 < 0 < \lambda_2$, saddle point

$$\frac{y_2}{y_{20}} \left(\frac{y_1}{y_{10}} \right)^{\lambda_2/(-\lambda_1)} = 1$$

Case $\Delta_A = 0$: a double real root, $\lambda_1 = \lambda_2 = \frac{1}{2} \operatorname{trace} A$.

· $\dim E_\lambda = 2$ attractive node if $\operatorname{trace} A < 0$, repulsive node if $\operatorname{trace} A > 0$,
· $\dim E_\lambda = 1$ attractive node if $\operatorname{trace} A < 0$, repulsive node if $\operatorname{trace} A > 0$.

Case $\Delta_A < 0$: two distinct conjugate non-real roots, $\lambda_1 = \alpha + i\beta, \lambda_2 = \alpha - i\beta$, $\beta > 0$

· $\alpha = 0$ center,
· $\alpha > 0$ expanding spiral point,
· $\alpha < 0$ shrinking spiral point.

Exercise: draw a picture of the integral curves for each case above.

2.3 Transport equations

2.3.1 The linear case

We shall deal first with the linear equation

$$
\begin{cases}
\dfrac{\partial u}{\partial t} + \displaystyle\sum_{1 \le j \le d} a_j(t,x)\dfrac{\partial u}{\partial x_j} = a_0(t,x)u + f(t,x) & \text{on } t > 0, \ x \in \mathbb{R}^d, \\[2mm]
u_{|t=0} = u_0, & \text{for } x \in \mathbb{R}^d,
\end{cases}
\tag{2.3.1}
$$

where a_j, f are functions of class C^1 on $\mathbb{R}^{d+1}$. We claim that solving that first-order scalar linear PDE amounts to solve a non-linear system of ODE. We check with $x(t) = \big(x_j(t)\big)_{1 \le j \le d} \in \mathbb{R}^d$

$$
\dot{x}_j(t) = a_j\big(t, x(t)\big), \quad 1 \le j \le d, \quad x_j(0) = y_j,
\tag{2.3.2}
$$

and we note that if u is a C^1 solution of (2.3.1), we have

$$
\frac{d}{dt}\big\{u(t,x(t))\big\} = (\partial_t u)(t,x(t)) + \sum_j (\partial_{x_j}u)(t,x(t))a_j(t,x(t))
$$

$$
= a_0(t,x(t))u(t,x(t)) + f(t,x(t)),
$$

so that $u(t,x(t)) = u_0(y)e^{\int_0^t a_0(s,x(s))ds} + \displaystyle\int_0^t e^{\int_s^t a_0(\sigma,x(\sigma))d\sigma} f(s,x(s))ds$. As a result, the value of the solution u along the *characteristic curves* $t \mapsto x(t)$, which satisfies a linear ODE, is completely determined by u_0, a_0 and the *source term* f. We can write as well $x(t) = \psi(t,y)$ and notice that ψ is a C^1 function: following Theorem 2.1.16, ψ is the flow of the non-autonomous ODE (2.3.2) and we shall say as well that ψ is the flow of the non-autonomous vector field

$$
\frac{\partial}{\partial t} + \sum_{1 \le j \le n} a_j(t,x)\frac{\partial}{\partial x_j}.
\tag{2.3.3}
$$

We have

$$
\frac{\partial^2 \psi_j}{\partial t \partial y_k}(t,y) = \sum_{1 \le l \le d} \frac{\partial a_j}{\partial x_l}(t,\psi(t,y))\frac{\partial \psi_l}{\partial y_k}(t,y), \quad \text{i.e.} \quad \frac{d}{dt}\frac{\partial \psi}{\partial y} = \frac{\partial a}{\partial x}\frac{\partial \psi}{\partial y},
$$

so that with $D(t,y) = \det\big(\frac{\partial \psi_j}{\partial y_k}\big)_{1 \le j,k \le d}$, we have from $\psi(0,y) = y$,

$$
D(t,y) = \exp\int_0^t (\text{trace}\,\frac{\partial a}{\partial x})(s,\psi(s,y))ds = \exp\int_0^t (\text{div}\,a)(s,\psi(s,y))ds.
\tag{2.3.4}
$$

As a result, from the implicit function theorem, the equation $x = \psi(t,y)$ is locally equivalent to $y = \varphi(t,x)$ with a C^1 function φ so that

$$
\psi(t,\varphi(t,x)) = x.
$$

We obtain from

$$u(t, \psi(t,y)) = u_0(y)e^{\int_0^t a_0(s,\psi(s,y))ds} + \int_0^t e^{\int_s^t a_0(\sigma,\psi(\sigma,y))d\sigma} f(s, \psi(s,y))ds$$

that

$$u(t,x) = u_0(\varphi(t,x))e^{\int_0^t a_0(s,\psi(s,\varphi(t,x)))ds} + \int_0^t e^{\int_s^t a_0(\sigma,\psi(\sigma,\varphi(t,x)))d\sigma} f\big(s, \psi(s, \varphi(t,x))\big)ds.$$
(2.3.5)

We need to introduce a slightly more general version of the non-autonomous flow in order to compare it with the actual flow of the vector field $\partial_t + a(t,x) \cdot \partial_x$ in particular when a satisfies the assumption (2.1.19), ensuring global existence for the characteristic curves.

Definition 2.3.1. *Let $a : \mathbb{R} \times \mathbb{R}^d \longrightarrow \mathbb{R}^d$ be a C^1 function such that (2.1.19) holds. The non-autonomous flow of the vector field $\partial_t + a(t,x) \cdot \partial_x$ is defined as the mapping $\mathbb{R} \times \mathbb{R} \times \mathbb{R}^d \ni (t,s,y) \mapsto \Psi(t,s,y) \in \mathbb{R}^d$ such that*

$$\big(\frac{\partial \Psi}{\partial t}\big)(t,s,y) = a\big(t, \Psi(t,s,y)\big), \quad \Psi(s,s,y) = y. \tag{2.3.6}$$

Lemma 2.3.2. *With a as in the previous definition, we have $\Psi(t,0,y) = \psi(t,y)$, where ψ is defined by (2.3.2). The flow Φ^θ of the vector field $\partial_t + a(t,x) \cdot \partial_x$ in $\mathbb{R}^{1+d}$ satisfies*

$$\Phi^\theta(s,y) = \big(s + \theta, \Psi(s + \theta, s, y)\big). \tag{2.3.7}$$

Moreover for $s, \theta_1, \theta_2 \in \mathbb{R}, y \in \mathbb{R}^d$, we have

$$\Psi(s + \theta_1 + \theta_2, s, y) = \Psi\big(s + \theta_1 + \theta_2, s + \theta_1, \Psi(s + \theta_1, s, y)\big). \tag{2.3.8}$$

For all $t \in \mathbb{R}$, $y \mapsto \psi(t,y)$ is a C^1 diffeomorphism of $\mathbb{R}^d$.

Proof. Formula (2.3.7) follows immediately from (2.3.6). Considering the lhs (resp. rhs) $u(\theta_2)$ (resp. $v(\theta_2)$) of (2.3.8), we have

$$\dot{u}(\theta_2) = a(s + \theta_1 + \theta_2, u(\theta_2)), \quad \dot{v}(\theta_2) = a(s + \theta_1 + \theta_2, v(\theta_2)),$$

and $u(0) = \Psi(s + \theta_1, s, y)$, $v(0) = \Psi\big(s + \theta_1, s + \theta_1, \Psi(s + \theta_1, s, y)\big) = \Psi(s + \theta_1, s, y)$, so that (2.3.8) follows. Note also that (2.3.8) is equivalent to the fact that $\Phi^{\theta_1 + \theta_2} = \Phi^{\theta_1}\Phi^{\theta_2}$ proven in (2.2.3). In particular, we have

$$y = \Psi(s,s,y) = \Psi\big(s, s + \theta, \Psi(s + \theta, s, y)\big)$$

so that $y = \Psi(s,t,\Psi(t,s,y)) = \Psi(0,t,\psi(t,y)) = \Psi(s,0,\Psi(0,s,y)) = \psi(s, \Psi(0,s,y))$ and for all $t \in \mathbb{R}$, $y \mapsto \psi(t,y) = \Psi(t,0,y)$ is a C^1 global diffeomorphism of $\mathbb{R}^d$ with inverse diffeomorphism $x \mapsto \varphi(t,x) = \Psi(0,t,x)$: both mappings are C^1 from Theorem 2.1.16, and with $\varphi_t(x) = \varphi(t,x), \psi_t(y) = \psi(t,y)$ we have

$$(\varphi_t \circ \psi_t)(y) = \varphi_t(\psi(t,y)) = \Psi(0,t,\Psi(t,0,y)) = y,$$
$$(\psi_t \circ \varphi_t)(x) = \psi_t(\varphi(t,x)) = \Psi(t,0,\Psi(0,t,x)) = x.$$

$\square$

Remark 2.3.3. The group property of the non-autonomous flow is expressed by (2.3.8) and in general, $\psi(t + s, y) \neq \psi(t, \psi(s, y))$: the simplest example is given by the vector field $\partial_t + 2t\partial_x$ in $\mathbb{R}^2_{t,x}$ for which

$$\psi(t, y) = t^2 + y \Longrightarrow \begin{cases} \psi(t + s, y) = (t + s)^2 + y \\ \psi(t, \psi(s, y)) = t^2 + s^2 + y. \end{cases}$$

Note also that here $\Psi(t, s, y) = t^2 - s^2 + y$ and (2.3.8) reads

$$(s + \theta_1 + \theta_2)^2 - s^2 + y = (s + \theta_1 + \theta_2)^2 - (s + \theta_1)^2 + (s + \theta_1)^2 - s^2 + y.$$

Of course when a does not depend on t, the flow of $L = \partial_t + \underbrace{a(x) \cdot \partial_x}_{X}$ is given by

$$\Phi^\theta_L(s, y) = (s + \theta, \Phi^\theta_X(y)), \quad \Psi(t, s, y) = \Phi^{t-s}_X(y), \quad \psi(t, y) = \Phi^t_X(y)$$

and in that very particular case, $\psi(t + s, y) = \psi(t, \psi(s, y))$.

We have proven the following

Theorem 2.3.4. *Let* $a : \mathbb{R}_t \times \mathbb{R}^d_x \to \mathbb{R}^d$ *be a continuous function which satisfies* (2.1.19) *and such that* $\partial_x a$ *is continuous. Let* $a_0, f : \mathbb{R}_t \times \mathbb{R}^d_x \to \mathbb{R}$ *be continuous functions and* $u_0 : \mathbb{R}^d \to \mathbb{R}$ *be a* C^1 *function. The Initial-Value-Problem* (2.3.1) *has a unique* C^1 *solution given by* (2.3.5).

Note that, thanks to the hypothesis (2.1.19) and $\partial_x a$ continuous, the flow $x \mapsto \psi(t, x)$ and $y \mapsto \varphi(t, y)$ defined above are global C^1 diffeomorphisms of $\mathbb{R}^d$.

Remark 2.3.5. Let us assume that a_0 and f are both identically vanishing: then we have from (2.3.5)

$$u(t, x) = u_0(\varphi(t, x))$$

and since $\psi(t, y) = y + \int_0^t a(s, \psi(s, y)) ds$ we get

$$x = \psi(t, \varphi(t, x)) = \varphi(t, x) + \int_0^t a(s, \psi(s, \varphi(t, x))) ds$$

so that $|\varphi(t, x) - x| \leq |t| \|a\|_{L^\infty}$ and the solution $u(t, x)$ depends only on u_0 on the ball $B(x, |t| \|a\|_{L^\infty})$, that is a finite-speed-of-propagation property. Moreover the range of $u(t, \cdot)$ is included in the range of u_0: in particular if u_0 is valued in $[m, M]$, so is the solution u. If the vector field is autonomous, we have seen that

$$\varphi(t, x) = \Phi^{-t}_X(x), \quad X = \sum_{1 \leq j \leq d} a_j(x) \partial_{x_j},$$

so that $u(t, x) = u_0(\Phi^{-t}_X(x))$, $\quad u(t + s, x) = u_0(\Phi^{-t-s}_X(x)) = u(t, \Phi^{-s}_X(x))$.

2.3.2   The quasi-linear case

A linear companion equation

We want now to investigate a more involved case

$$
\begin{cases}
\dfrac{\partial u}{\partial t} + \displaystyle\sum_{1 \le j \le d} a_j\big(t, x, u(t,x)\big)\dfrac{\partial u}{\partial x_j} = b\big(t, x, u(t,x)\big) & \text{on } 0 < t < T, \; x \in \mathbb{R}^d, \\[2mm]
u_{|t=0} = u_0, & \text{for } x \in \mathbb{R}^d.
\end{cases}
\tag{2.3.9}
$$

That equation is a general *quasi-linear* scalar first-order equation. We know from the introduction and the discussion around Burgers equation (1.1.4) that we should not expect global existence in general for such an equation. We assume that the functions $a_j, b : \mathbb{R}_t \times \mathbb{R}_x^d \times \mathbb{R}_v \to \mathbb{R}$ are of class C^1 and we shall introduce a companion linear homogeneous equation

$$
\frac{\partial F}{\partial t} + \sum_{1 \le j \le d} a_j(t, x, v)\frac{\partial F}{\partial x_j} + b(t, x, v)\frac{\partial F}{\partial v} = 0, \quad F(0, x, v) = v - u_0(x). \tag{2.3.10}
$$

From the discussion in the previous section, we know that, provided u_0 is continuous, there exists a unique C^1 local solution of the initial value problem (2.3.10), near the point $(t, x, v) = (0, x_0, v_0 = u_0(x_0))$. We claim now that, at this point, $\partial F / \partial v \ne 0$: in fact this follows obviously from the identity $F(0, x, v) = v - u_0(x)$ which implies that $(\partial F/\partial v)(0, x, v) = 1$. We can now apply the Implicit Function Theorem, which implies that the equation $F(t, x, v) = 0$ is equivalent to $v = u(t, x)$ with a C^1 function u defined in a neighborhood of $(t = 0, x = x_0)$ with $u(0, x_0) = u_0(x_0)$. We have

$$
F(t, x, u(t,x)) \equiv 0 \tag{2.3.11}
$$

and

$$
\begin{aligned}
\frac{\partial u}{\partial t} &+ \sum_{1 \le j \le d} a_j\big(t, x, u(t,x)\big)\frac{\partial u}{\partial x_j} \\
&= -\frac{\partial F/\partial t}{\partial F/\partial v}(t, x, u) - \sum_{1 \le j \le d} a_j(t, x, u)\frac{\partial F/\partial x_j}{\partial F/\partial v}(t, x, u) = b(t, x, u),
\end{aligned}
$$

so that we have found a local solution for our quasi-linear PDE. Moreover, since $F(0, x, v) = v - u_0(x)$, we get from (2.3.11) $u(0, x) - u_0(x) = 0$, so that the initial condition is also fulfilled. We shall develop later on this discussion on the first-order scalar quasi-linear case, but it is interesting to note that finding a local solution for such an equation is not more difficult than getting a solution for a linear equation. Moreover, we shall be able to track the solution by a suitable method of characteristics adapted to this quasi-linear case, in fact following the method of characteristics for the companion linear equation (2.3.10).

The previous discussion shows that a local solution of (2.3.9) does exist. A direct method of characteristics can be devised, following the discussion above: we assume that u is a C^1 solution of (2.3.9) and we consider the ODE

$$
\dot{x}(t) = a(t, x, u(t, x(t))), \dot{v}(t) = b(t, x(t), u(t, x(t))), x(0) = x_0, v(0) = u_0(x_0). \tag{2.3.12}
$$

We calculate

$$\frac{d}{dt}\big(u(t,x(t)) - v(t)\big)$$
$$= (\partial_t u)(t,x(t)) + (\partial_x u)(t,x(t)) \cdot a(t,x(t),u(t,x(t))) - b(t,x(t),u(t,x(t))) = 0$$

so that

$$u(t,x(t)) = v(t). \tag{2.3.13}$$

2.3.3 Classical solutions of Burgers equation

We have already encountered Burgers' equation in (1.1.4). According to the previous discussion, the linear companion equation is $\partial_t F + v\partial_x F = 0$ whose flow is $\psi(t,x,v) = (x+tv,v)$ since $\dot{x} = v, \dot{v} = 0$; we have $F(t,x+tv,v) = v - u_0(x)$ and thus the identity $F(t,x+tu_0(x),u_0(x)) = 0$. Since a C^1 solution $u(t,x)$ of

$$\partial_t u + u\partial_x u = 0, \quad u(0,x) = u_0(x), \tag{2.3.14}$$

satisfies the identity (2.3.11) we have

$$u(t,x + tu_0(x)) = u_0(x). \tag{2.3.15}$$

If $u_0 \in C^1$ with u_0, u_0' bounded the mapping $x \mapsto x + tu_0(x) = f_t(x)$ for $t \geq 0$ is a C^1 diffeomorphism provided $1 + tu_0'(x) > 0$ which is satisfied whenever

$$0 \leq t < T_0 = \frac{1}{\sup(-u_0')_+}$$

since the inequality $u_0' \geq -M$ with $M \geq 0$ implies

$$1 + tu_0'(x) \geq 1 - tM > 1 - T_0 M = 0.$$

Moreover f_t and $g_t = f_t^{-1}$ are C^1 functions of t. As a result, we have for $0 \leq t < T_0$

$$u(t,x) = u_0(g_t(x)), \quad \text{so that } \sup_x |u(t,x)| = \sup_x |u_0(x)|.$$

Note that

$$(\partial_x u)(t,f_t(x))f_t'(x) = u_0'(x) \implies (\partial_x u)(t,x+tu_0(x)) = \frac{u_0'(x)}{1 + tu_0'(x)}$$

so that this quantity is unbounded when $t \to (T_0)_-$ if $-u_0'$ reaches a positive maximum at x, but nevertheless $\int |(\partial_x u)(t,x)|dx = \int |u_0'(g_t(x))|g_t'(x)dx = \int |u_0'(x)|dx$. Let us check some simple examples.

• When $u_0(x) = \alpha x$, with $\alpha \geq 0$ we do have a global solution for $t \geq 0$ given by the identity

$$u(t,x) = u_0(x - tu(t,x)) = \alpha x - \alpha t u(t,x) \implies u(t,x) = \alpha x/(1 + \alpha t).$$

• When $u_0(x) = -\alpha x$, with $\alpha > 0$ the solution blows-up at time $T = 1/\alpha$,

$$u(t,x) = u_0(x - tu(t,x)) = -\alpha x + \alpha t u(t,x) \implies u(t,x) = \alpha x/(\alpha t - 1).$$

• When $u_0(x) = (1 - x)H(x) + (1 + x)H(-x)$, with $H = \mathbf{1}_{\mathbb{R}_+}$, we find, using the method of characteristics that

$$u(t, x) = H(x - t)\frac{1 - x}{1 - t} + H(t - x)\frac{x + 1}{t + 1}, \quad 0 \le t < 1, \quad x \in \mathbb{R}. \qquad (2.3.16)$$

The function u is only Lipschitz continuous, but we may compute its distribution derivative and we get on the open set $-1 < t < 1$

$$\partial_t u = -\delta(x - t)\frac{1 - x}{1 - t} + \delta(t - x)\frac{x + 1}{t + 1} + H(x - t)\frac{1 - x}{(1 - t)^2} - H(t - x)\frac{x + 1}{(t + 1)^2}$$

$$= H(x - t)\frac{1 - x}{(1 - t)^2} - H(t - x)\frac{x + 1}{(t + 1)^2} \in L^\infty_{loc}$$

$$\partial_x u = \delta(x - t)\frac{1 - x}{1 - t} - \delta(t - x)\frac{x + 1}{t + 1} - H(x - t)\frac{1}{(1 - t)} + H(t - x)\frac{1}{(t + 1)}$$

$$= -H(x - t)\frac{1}{(1 - t)} + H(t - x)\frac{1}{(t + 1)} \in L^\infty_{loc}, \quad \text{the product } u\partial_x u \text{ makes sense}$$

$$u\partial_x u = -H(x - t)\frac{1 - x}{(1 - t)^2} + H(t - x)\frac{x + 1}{(t + 1)^2} = -\partial_t u,$$

so that Burgers equation holds for u. The following picture is helpful. In fact, the
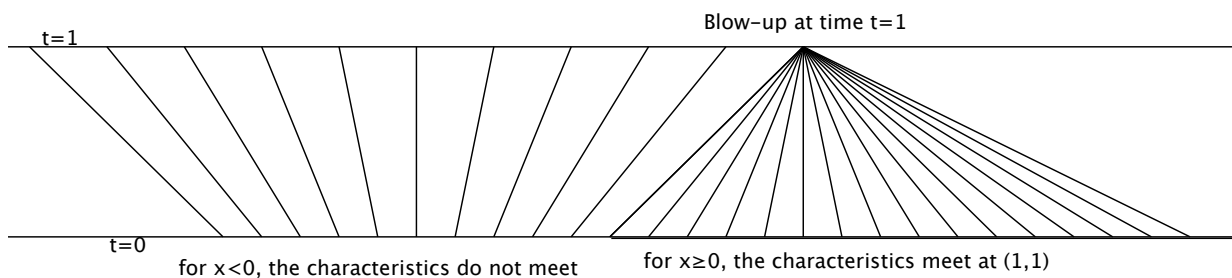

Figure 2.5: The characteristic curves with
$u_0(x) = (1 - x)H(x) + (1 + x)H(-x)$.

function u_0 is equal to $1 + x$ for $x \le 0$ and to $1 - x$ for $x \ge 0$ and we have from (2.3.15) $u(t, x + tu_0(x)) = u_0(x)$, so that u is constant along the characteristic curves $t \mapsto (x_0 + tu_0(x_0), t) \in \mathbb{R}^2_{x,t}$: these curves are straight lines starting at $(x_0, t = 0)$ with slope $1/u_0(x_0)$. In the case under scope, we have

$$\begin{cases} x(t, x_0) = x_0 + t(1 + x_0) & \text{if } x_0 \le 0, \\ x(t, x_0) = x_0 + t(1 - x_0) & \text{if } x_0 \ge 0. \end{cases}$$

We have indeed for $x_0 \neq x_1$ both ≥ 0, $0 \leq t < 1$

$$x(1, x_0) = 1, \quad x(t, x_0) - x(t, x_1) = (x_0 - x_1)(1 - t) \neq 0.$$

On the other hand for $x_0 \neq x_1$ both ≤ 0, $t \geq 0$, we have

$$x(t, x_0) - x(t, x_1) = (x_0 - x_1)(1 + t) \neq 0.$$

2.4 One-dimensional conservation laws

2.4.1 Rankine-Hugoniot condition and singular solutions

We shall consider here the following type of non-linear equation

$$\partial_t u + \partial_x \big(f(u)\big) = 0, \tag{2.4.1}$$

where (t, x) are two real variables, and u is a real-valued function, whereas f is a smooth given function, called the *flux*. We shall consider some singular solutions of this equation, with a discontinuity across a C^1 curve with equation $x = \sigma(t)$. We define

$$u(t, x) = H(x - \sigma(t))u_r(t, x) + H(\sigma(t) - x)u_l(t, x) \tag{2.4.2}$$

where H is the Heaviside function and we shall assume that u_r and u_l are C^1 functions respectively on the closure of the open subsets

$$\Omega_r = \{(x, t), x > \sigma(t)\} \text{ and } \Omega_l = \{(x, t), x < \sigma(t)\}$$

and that u_r solves the equation (2.4.1) on the open set Ω_r (resp. u_l solves the equation (2.4.1) on the open set Ω_l). We shall assume also the so-called *Rankine-Hugoniot condition*,

$$\text{at } x = \sigma(t), \quad f(u_r) - f(u_l) = \sigma'(t)(u_r - u_l). \tag{2.4.3}$$

Theorem 2.4.1. *Let σ be a C^1 function, $\Omega_{r,l}$ defined as above, let u_r, u_l be C^1 solutions of (2.4.1) respectively on Ω_r, Ω_l. Then u given by (2.4.2) is a distribution solution of (2.4.1) if and only if the Rankine-Hugoniot condition (2.4.3) is fulfilled.*

Proof. We have

$$f(u(t, x)) = H(x - \sigma(t))f(u_r(t, x)) + H(\sigma(t) - x)f(u_l(t, x))$$

and thus the distribution derivative with respect to x of $f(u)$ is equal to

$$\partial_x(f(u)) = H(x - \sigma(t))f'(u_r(t, x))(\partial_x u_r)(t, x) + H(\sigma(t) - x)f'(u_l(t, x))(\partial_x u_l)(t, x)$$
$$+ \delta(x - \sigma(t))\big(f(u_r(t, x)) - f(u_l(t, x))\big).$$

On the other hand, we have $\partial_t u = \delta(x - \sigma(t))\big(u_l(t, x) - u_r(t, x)\big)\sigma'(t)$. Since u_r and u_l are C^1 solutions respectively on Ω_r, Ω_l, we get

$$\partial_t u + \partial_x \big(f(u)\big) = \delta(x - \sigma(t))\Big(f(u_r) - f(u_l) - \sigma'(t)\big(u_r - u_l\big)\Big)$$

and the results follows. $\qquad \square$

2.4.2   The Riemann problem for Burgers equation

We consider Burgers equation and a L^∞_{loc} solution u of

$$\begin{cases} \partial_t u + \partial_x(u^2/2) = 0, & \text{on } t > 0, x \in \mathbb{R}, \\ u(0, x) = u_0(x), & x \in \mathbb{R}. \end{cases} \qquad (2.4.4)$$

It means that for all $\varphi \in C^\infty_c(\mathbb{R}^2)$,

$$\int_{\mathbb{R}} u_0(x)\varphi(0, x)dx + \iint H(t)\partial_t\varphi(t, x)u(t, x)dtdx$$

$$+ \frac{1}{2}\iint H(t)u(t, x)^2(\partial_x\varphi)(t, x)dtdx = 0, \quad (2.4.5)$$

or for $u \in L^\infty_{loc}(\mathbb{R}^2)$,

$$\partial_t(Hu) + \partial_x((Hu)^2/2) = \delta(t) \otimes u_0(x), \qquad H = H(t). \qquad (2.4.6)$$

Indeed (2.4.5) means that

$$\langle \partial_t(Hu) + \partial_x(Hu)^2/2, \varphi \rangle_{\mathscr{D}', \mathscr{D}} = \langle \delta(t) \otimes u_0(x), \varphi \rangle_{\mathscr{D}', \mathscr{D}},$$

which is (2.4.6).

Non-Physical Shock and Rarefaction Wave

Let us first assume $u_0(x) = H(x)$, i.e. $u_l(0, x) = 0, u_r(0, x) = 1$. Following the method of characteristics (2.3.15), we should have $u(t, x + tu_0(x)) = u_0(x)$, i.e.

$$\begin{cases} u(t, x) = 0, & \text{for } x < 0, \\ u(t, x + t) = 1 & \text{for } x > 0. \end{cases} \text{ that is } \begin{cases} u(t, x) = 0, & \text{for } x < 0, \\ u(t, x) = 1 & \text{for } x > t, \end{cases}$$

so we get no information in the region $0 < x < t$. We could use our knowledge on the construction of singular solutions to create a somewhat arbitrary shock at $x = t/2$ (non-physical shock)

$$u(t, x) = \begin{cases} u(t, x) = 0, & \text{for } x < t/2, \\ u(t, x) = 1 & \text{for } x > t/2. \end{cases} \qquad (2.4.7)$$

The Rankine-Hugoniot condition (2.4.3) is satisfied since $\sigma(t) = t/2, u_l^2 = u_l, u_r^2 = u_r$, so that

$$\frac{1}{2}(u_r^2 - u_l^2) = \sigma'(t)(u_r - u_l).$$
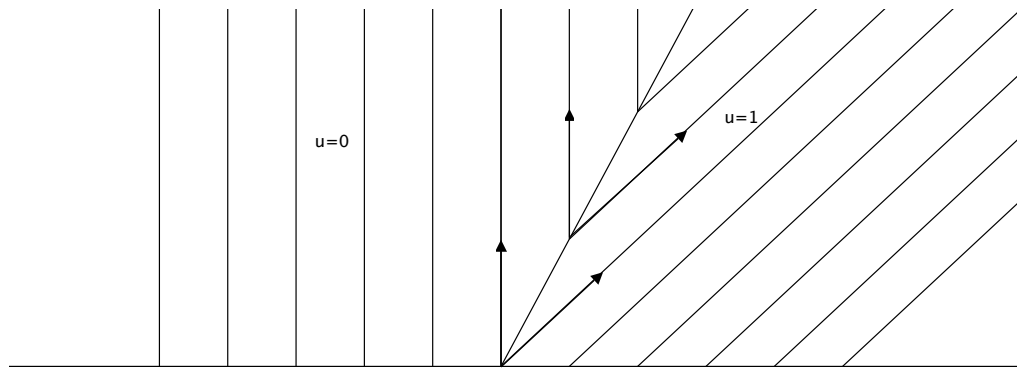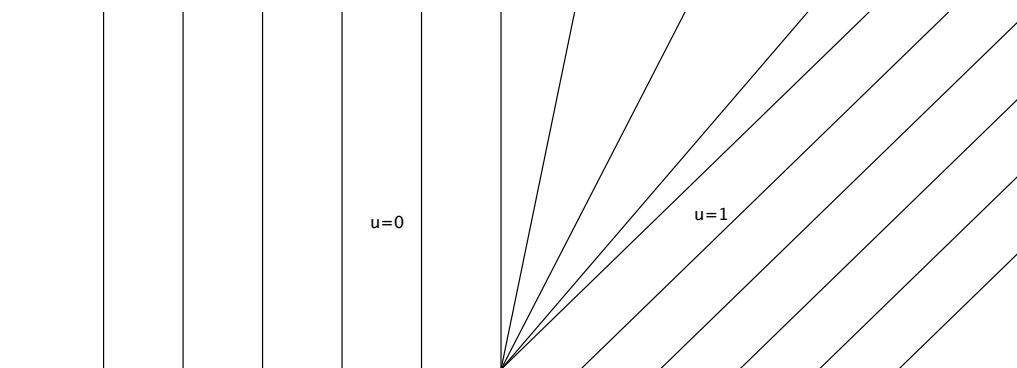
A non-physical shock along the line x=t/2

However, we can also devise another solution v by defining (rarefaction wave)

$$v(t,x) = \begin{cases} v(t,x) = 0, & \text{for } x < 0, \\ v(t,x) = x/t & \text{for } 0 < x < t, \\ v(t,x) = 1 & \text{for } x > t. \end{cases} \qquad (2.4.8)$$

We can indeed calculate

$$\partial_t \Big(H(t)H(x)\big(H(t-x)x/t + H(x-t)\big)\Big)$$
$$+ \frac{1}{2}\partial_x \Big(H(t)H(x)\big(H(t-x)x/t + H(x-t)\big)\Big)^2$$

and refer the reader to the proof of Theorem 2.4.2 to show that (2.4.8) is actually a solution.

A rarefaction wave: u=x/t in the region 0<x<t,   u=0 on x<0,   u=1 on x>t

We have thus two different weak solutions of (2.4.4) with the same inital datum! This very unnatural situation has to be modified and we have to find a criterion to select the "correct" solution.

Entropy condition

For a general one-dimensional conservation law $\partial_t u + \partial_x\big(f(u)\big) = 0$ with a strictly convex flux f (assume $f \in C^2(\mathbb{R})$ with $\inf f'' > 0$), suppose that we have a curve of discontinuity $\Gamma \equiv x = \phi(t)$ with distinct left and right limits u_l, u_r. Then nonetheless the Rankine-Hugoniot (2.4.3) should be satisfied across Γ, but also $u_l > u_r$ along Γ: this eliminates in particular the solution (2.4.7). As a geometric explanation, we may say that singularities are due to the crossing of characteristics, but we want to avoid that by moving backwards along a characteristic, we encounter a singular curve.

Theorem 2.4.2. *We consider the initial-value problem*

$$\begin{cases} \partial_t u + \partial_x\big(\frac{u^2}{2}\big) = 0, & t > 0, \\ u_0(x) = H(-x)u_l + H(x)u_r. \end{cases} \tag{2.4.9}$$

where u_l, u_r are distinct constants and we define

$$\sigma = \frac{1}{2}\frac{u_r^2 - u_l^2}{u_r - u_l} \tag{2.4.10}$$

(1) If $u_l > u_r$, the unique entropy solution is given by

$$u(t, x) = H(\sigma t - x)u_l + H(x - \sigma t)u_r. \tag{2.4.11}$$

This is a shock wave with speed σ satisfying the Rankine-Hugoniot condition (2.4.3) at the discontinuity curve $x = \sigma t$.
(2) If $u_l < u_r$, the unique entropy solution is given by

$$u(t, x) = H(tu_l - x)u_l + \frac{x}{t}\mathbf{1}_{[tu_l, tu_r]}(x) + H(x - tu_r)u_r. \tag{2.4.12}$$

The states u_l, u_r are separated by a rarefaction wave.

Proof. In the case $u_l > u_r$, we have a singular solution according to Theorem 2.4.1 satisfying our entropy condition $u_l > u_r$. In the other case, we must avoid a shock curve and we check directly, with u given by (2.4.12),

$$\partial_t u = \delta(tu_l - x)u_l^2 - \frac{x}{t^2}\mathbf{1}_{[tu_l, tu_r]}(x) + \frac{x}{t}\big(-\delta(x - tu_l)u_l + \delta(tu_r - x)u_r\big)$$
$$- \delta(x - tu_r)u_r^2,$$

and

$$\partial_x(u^2) = \partial_x\Big(H(tu_l - x)u_l^2 + \frac{x^2}{t^2}\mathbf{1}_{[tu_l, tu_r]}(x) + H(x - tu_r)u_r^2\Big)$$
$$= -u_l^2\delta(tu_l - x) + \frac{2x}{t^2}\mathbf{1}_{[tu_l, tu_r]}(x) + \frac{x^2}{t^2}\big(\delta(x - tu_l) - \delta(x - tu_r)\big)$$
$$+ \delta(x - tu_r)u_r^2$$

so that

$$\partial_t u + \partial_x (u^2/2) = \delta(tu_l - x) \overbrace{\left(u_l^2/2 - xu_l/t + x^2/2t^2\right)}^{=u_l^2\left(\frac{1}{2}-1+\frac{1}{2}\right)=0}$$
$$+ \delta(x - tu_r) \underbrace{\left(-u_r^2/2 + xu_r/t - x^2/2t^2\right)}_{=u_r^2\left(-\frac{1}{2}+1-\frac{1}{2}\right)=0} = 0.$$

$\square$

Chapter 3

Five classical equations

3.1 The Laplace and Cauchy-Riemann equations

3.1.1 Fundamental solutions

We define the Laplace operator Δ in $\mathbb{R}^n$ as

$$\Delta = \sum_{1 \le j \le n} \partial_{x_j}^2. \tag{3.1.1}$$

In one dimension, we have that $\frac{d^2}{dt^2}(t_+) = \delta_0$ and for $n \ge 2$ the following result describes the fundamental solutions of the Laplace operator. In $\mathbb{R}^2_{x,y}$, we define the operator $\bar{\partial}$ (a.k.a. the Cauchy-Riemann operator) by

$$\bar{\partial} = \frac{1}{2}(\partial_x + i\partial_y). \tag{3.1.2}$$

Theorem 3.1.1. *We have $\Delta E = \delta_0$ with $\| \cdot \|$ standing for the Euclidean norm,*

$$E(x) = \frac{1}{2\pi} \ln \|x\|, \quad \text{for } n = 2, \tag{3.1.3}$$

$$E(x) = \|x\|^{2-n} \frac{1}{(2-n)|S^{n-1}|}, \quad \text{for } n \ge 3, \text{ with } |S^{n-1}| = \frac{2\pi^{n/2}}{\Gamma(n/2)}, \tag{3.1.4}$$

$$\bar{\partial}\left(\frac{1}{\pi z}\right) = \delta_0, \quad \text{with } z = x + iy \text{ (equality in } \mathscr{D}'(\mathbb{R}^2_{x,y})). \tag{3.1.5}$$

Proof. We start with $n \ge 3$, noting that the function $\|x\|^{2-n}$ is L^1_{loc} and homogeneous with degree $2-n$, so that $\Delta\|x\|^{2-n}$ is homogeneous with degree $-n$ (see section 3.4.3 in [15]). Moreover, the function $\|x\|^{2-n} = f(r^2), r^2 = \|x\|^2, f(t) = t_+^{1-\frac{n}{2}}$ is smooth outside 0 and we can compute there

$$\Delta(f(r^2)) = \sum_j \partial_j(f'(r^2)2x_j) = \sum_j f''(r^2)4x_j^2 + 2nf'(r^2) = 4r^2 f''(r^2) + 2nf'(r^2),$$

so that with $t = r^2$,

$$\Delta(f(r^2)) = 4t(1-\frac{n}{2})(-\frac{n}{2})t^{-\frac{n}{2}-1} + 2n(1-\frac{n}{2})t^{-\frac{n}{2}} = t^{-\frac{n}{2}}(1-\frac{n}{2})(-2n+2n) = 0.$$

As a result, $\Delta \|x\|^{2-n}$ is homogeneous with degree $-n$ and supported in $\{0\}$. From Theorem 3.3.4 in [15], we obtain that

$$\underbrace{\Delta \|x\|^{2-n} = c\delta_0}_{\substack{\text{homogeneous} \\ \text{degree } -n}} + \sum_{1 \leq j \leq m} \underbrace{\sum_{|\alpha|=j} c_{j,\alpha} \delta_0^{(\alpha)}}_{\substack{\text{homogeneous} \\ \text{degree } -n-j}} .$$

Lemma 3.4.8 in [15] implies that for $1 \leq j \leq m$, $0 = \sum_{|\alpha|=j} c_{j,\alpha} \delta_0^{(\alpha)}$ and $\Delta \|x\|^{2-n} = c\delta_0$. It remains to determine the constant c. We calculate, using the previous formulas for the computation of $\Delta(f(r^2))$, here with $f(t) = e^{-\pi t}$,

$$c = \langle \Delta \|x\|^{2-n}, e^{-\pi \|x\|^2} \rangle = \int \|x\|^{2-n} e^{-\pi \|x\|^2} (4\|x\|^2 \pi^2 - 2n\pi) dx$$

$$= |S^{n-1}| \int_0^{+\infty} r^{2-n+n-1} e^{-\pi r^2} (4\pi^2 r^2 - 2n\pi) dr$$

$$= |S^{n-1}| \left(\frac{1}{2\pi} [e^{-\pi r^2} (4\pi^2 r^2 - 2n\pi)]_{+\infty}^0 + \frac{1}{2\pi} \int_0^{+\infty} e^{-\pi r^2} 8\pi^2 r \, dr \right)$$

$$= |S^{n-1}| (-n+2),$$

giving (3.1.4). For the convenience of the reader, we calculate explicitly $|S^{n-1}|$. We have indeed

$$1 = \int_{\mathbb{R}^n} e^{-\pi \|x\|^2} dx = |S^{n-1}| \int_0^{+\infty} r^{n-1} e^{-\pi r^2} dr$$

$$\underbrace{=}_{r=t^{1/2}\pi^{-1/2}} |S^{n-1}| \pi^{(1-n)/2} \int_0^{+\infty} t^{\frac{n-1}{2}} e^{-t} \frac{1}{2} t^{-1/2} dt \, \pi^{-1/2} = |S^{n-1}| \pi^{-n/2} 2^{-1} \Gamma(n/2).$$

Turning now our attention to the Cauchy-Riemann equation, we see that $1/z$ is also $L^1_{\text{loc}}(\mathbb{R}^2)$, homogeneous of degree -1, and satisfies $\bar{\partial}(z^{-1}) = 0$ on the complement of $\{0\}$, so that the same reasoning as above shows that

$$\bar{\partial}(\pi^{-1} z^{-1}) = c\delta_0.$$

To check the value of c, we write $c = \langle \bar{\partial}(\pi^{-1} z^{-1}), e^{-\pi z\bar{z}} \rangle = \int_{\mathbb{R}^2} e^{-\pi z\bar{z}} \pi^{-1} z^{-1} \pi z \, dx \, dy = 1$, which gives (3.1.5). We are left with the Laplace equation in two dimensions and we note that with $\frac{\partial}{\partial z} = \frac{1}{2}(\partial_x - i\partial_y)$, $\frac{\partial}{\partial \bar{z}} = \frac{1}{2}(\partial_x + i\partial_y)$, we have in two dimensions

$$\Delta = 4 \frac{\partial}{\partial z} \frac{\partial}{\partial \bar{z}} = 4 \frac{\partial}{\partial \bar{z}} \frac{\partial}{\partial z}. \tag{3.1.6}$$

Solving the equation $4\frac{\partial E}{\partial z} = \frac{1}{\pi z}$ leads us to try $E = \frac{1}{2\pi} \ln |z|$ and we check directly[1] that $\frac{\partial}{\partial z} \left(\ln(z\bar{z}) \right) = z^{-1}$

$$\Delta(\frac{1}{2\pi} \ln |z|) = \pi^{-1} 2^{-2} 4 \frac{\partial}{\partial \bar{z}} \frac{\partial}{\partial z} \left(\ln(z\bar{z}) \right) = \pi^{-1} \frac{\partial}{\partial \bar{z}} \left(z^{-1} \right) = \delta_0.$$

---

[1]Noting that $\ln(x^2 + y^2)$ and its first derivatives are $L^1_{\text{loc}}(\mathbb{R}^2)$, we have for $\varphi \in C_c^\infty(\mathbb{R}^2)$, $\langle \frac{\partial}{\partial z} \left(\ln |z|^2 \right), \varphi \rangle =$

$$\frac{1}{2} \iint_{\mathbb{R}^2} (-\partial_x \varphi + i\partial_y \varphi) \ln(x^2 + y^2) dx \, dy = \iint \varphi(x, y)(xr^{-2} - iyr^{-2}) dx \, dy = \iint (x - iy)^{-1} \varphi(x, y) dx \, dy.$$

□

3.1.2 Hypoellipticity

Definition 3.1.2. *We consider a constant coefficients differential operator*

$$P = P(D) = \sum_{|\alpha| \le m} a_\alpha D_x^\alpha, \quad where \ a_\alpha \in \mathbb{C}, D_x^\alpha = \frac{1}{(2i\pi)^{|\alpha|}} \partial_x^\alpha. \tag{3.1.7}$$

A distribution $E \in \mathscr{D}'(\mathbb{R}^n)$ is called a fundamental solution of P when $PE = \delta_0$.

Definition 3.1.3. *Let P be a linear operator of type* (3.1.7). *We shall say that P is hypoelliptic when for all open subsets Ω of $\mathbb{R}^n$ and all $u \in \mathscr{D}'(\Omega)$, we have*

$$\text{singsupp } u = \text{singsupp } Pu. \tag{3.1.8}$$

We note that if $f \in \mathscr{E}'(\mathbb{R}^n)$ and E is a fundamental solution of P, we have from (3.5.13), (3.5.14) in [15],

$$P(E * f) = PE * f = \delta_0 * f = f,$$

which allows to find a solution of the Partial Differential Equation $P(D)u = f$, at least when f is a compactly supported distribution.

Examples. We have on the real line already proven (see (3.2.2) in [15]) that $\frac{dH}{dt} = \delta_0$, so that the Heaviside function is a fundamental solution of d/dt (note that from Lemma 3.2.4 in [15], the other fundamental solutions are $C + H(t)$). This also implies that

$$\partial_{x_1}\big(H(x_1) \otimes \delta_0(x_2) \otimes \cdots \otimes \delta_0(x_n)\big) = \delta_0(x), \quad \text{(the Dirac mass at 0 in } \mathbb{R}^n).$$

Let $N \in \mathbb{N}$. With x_+^λ defined in (3.4.8) of [15], we get, since $\partial_{x_1}^{N+1}(x_{1,+}^{N+1}) = H(x_1)(N+1)!$, that

$$(\partial_{x_1} \ldots \partial_{x_n})^{N+2}\Big(\prod_{1 \le j \le n} \Big(\frac{x_{j,+}^{N+1}}{(N+1)!}\Big)\Big) = \delta_0(x).$$

It is obvious that $\text{singsupp } Pu \subset \text{singsupp } u$, so the hypoellipticity means that $\text{singsupp } u \subset \text{singsupp } Pu$, which is a very interesting piece of information since we can then determine the singularities of our (unknown) solution u, which are located at the same place as the singularities of the source f, which is known when we try to solve the equation $Pu = f$.

Theorem 3.1.4. *Let P be a linear operator of type* (3.1.7) *such that P has a fundamental solution E satisfying*

$$\text{singsupp } E = \{0\}. \tag{3.1.9}$$

Then P is hypoelliptic. In particular the Laplace and the Cauchy-Riemann operators are hypoelliptic.

N.B. The condition (3.1.9) appears as an iff condition for the hypoellipticity of the operator P since it is also a consequence of the hypoellipticity property.

Proof. Assume that (3.1.9) holds, let Ω be an open subset of $\mathbb{R}^n$ and $u \in \mathscr{D}'(\Omega)$. We consider $f = Pu \in \mathscr{D}'(\Omega)$, $x_0 \notin \text{singsupp } f$, $\chi_0 \in C_c^\infty(\Omega)$, $\chi_0 = 1$ near x_0. We have from Proposition 3.5.7 in [15] that

$$\chi u = \chi u * PE = (P\chi u) * E = ([P, \chi]u) * E + \underbrace{\overbrace{(\chi f)}^{\in C_c^\infty(\mathbb{R}^n)} * E}_{\in C^\infty(\mathbb{R}^n)}$$

and thus, using Proposition 3.5.7 in [15] for singular supports, we get

$$\text{singsupp}(\chi u) \subset \text{singsupp}([P, \chi]u) + \text{singsupp } E = \text{singsupp}([P, \chi]u) \subset \text{supp}(u\nabla\chi),$$

and since χ is identically 1 near x_0, we get that $x_0 \notin \text{supp}(u\nabla\chi)$, implying $x_0 \notin \text{singsupp}(\chi u)$, proving that $x_0 \notin \text{singsupp } u$ and the result. $\square$

A few words on the Gamma function

The gamma function Γ is a meromorphic function on $\mathbb{C}$ given for $\text{Re } z > 0$ by the formula

$$\Gamma(z) = \int_0^{+\infty} e^{-t} t^{z-1} dt. \tag{3.1.10}$$

For $n \in \mathbb{N}$, we have $\Gamma(n+1) = n!$; another interesting value is $\Gamma(1/2) = \sqrt{\pi}$. The functional equation

$$\Gamma(z+1) = z\Gamma(z) \tag{3.1.11}$$

is easy to prove for $\text{Re } z > 0$ and can be used to extend the Γ function into a meromorphic function with simple poles at $-\mathbb{N}$ and $\text{Res}(\Gamma, -k) = \frac{(-1)^k}{k!}$. For instance, for $-1 < \text{Re } z \le 0$ with $z \neq 0$ we define

$$\Gamma(z) = \frac{\Gamma(z+1)}{z}, \quad \text{where we can use (3.1.10) to define } \Gamma(z+1).$$

More generally for $k \in \mathbb{N}$, $-1 - k < \text{Re } z \le -k$, $z \neq -k$, we can define

$$\Gamma(z) = \frac{\Gamma(z+k+1)}{z(z+1)\dots(z+k)}.$$

There are manifold references on the Gamma function. One of the most comprehensive is certainly the chapter VII of the Bourbaki volume *Fonctions de variable réelle* [2].

3.1.3   Polar and spherical coordinates

The polar coordinates in $\mathbb{R}^2$ are $]0, +\infty) \times]-\pi, \pi[\ni (r, \theta) \mapsto (r\cos\theta, r\sin\theta) = (x, y)$ which is a C^1 diffeomorphism from $]0, +\infty) \times]-\pi, \pi[$ onto $\mathbb{R}^2 \backslash (\mathbb{R}_- \times \{0\})$ with inverse mapping given by

$$r = (x^2 + y^2)^{1/2}, \ \theta = \text{Im}(\text{Log}(x + iy)), \quad \text{where for } z \in \mathbb{C} \backslash \mathbb{R}_-, \text{Log } z = \int_{[1,z]} \frac{d\xi}{\xi}.$$

We have in two dimensions

$$r^2 \Delta = (r\partial_r)^2 + \partial_\theta^2, \tag{3.1.12}$$

since

$$(x\partial_x + y\partial_y)^2 + (x\partial_y - y\partial_x)^2$$
$$= x^2\partial_x^2 + y^2\partial_y^2 + 2xy\partial_{xy}^2 + x\partial_x + y\partial_y + x^2\partial_y^2 + y^2\partial_x^2 - 2xy\partial_{xy}^2 - x\partial_x - y\partial_y$$
$$= (x^2 + y^2)(\partial_x^2 + \partial_y^2).$$

In three dimensions, the spherical coordinates are given by

$$\begin{cases} x &= r\cos\theta\sin\phi \\ y &= r\sin\theta\sin\phi \\ z &= r\cos\phi \end{cases}, \quad \begin{cases} r &= (x^2+y^2+z^2)^{1/2} \\ \theta &= \mathrm{Im}\big(\mathrm{Log}(x+iy)\big) \\ \phi &= \mathrm{Im}\big(\mathrm{Log}(z+i(x^2+y^2)^{1/2})\big) \end{cases} \tag{3.1.13}$$

defining a C^1 diffeomorphism

$$]0,+\infty)\times]-\pi,\pi[\times]0,\pi[\ni (r,\theta,\phi) \mapsto (x,y,z) \in \mathbb{R}^3\backslash\big(\mathbb{R}_- \times \{0\} \times \mathbb{R}\big).$$

The expression of the Laplace operator in spherical coordinates is

$$r^2\Delta = (r\partial_r)^2 + r\partial_r + \partial_\phi^2 + \frac{1}{\sin^2\phi}\partial_\theta^2 + \frac{1}{\tan\phi}\partial_\phi. \tag{3.1.14}$$

To prove the above formula, we use (3.1.12), with

$$z = r\cos\phi, \quad \rho = r\sin\phi, \quad r^2(\partial_z^2 + \partial_\rho^2) = (r\partial_r)^2 + \partial_\phi^2,$$
$$(\rho\partial_\rho)^2 + \partial_\theta^2 = \rho^2(\partial_x^2 + \partial_y^2),$$

so that $(r\partial_r)^2 + \partial_\phi^2 = r^2\partial_z^2 + r^2\rho^{-2}\big(\rho^2(\partial_x^2 + \partial_y^2) - \rho\partial_\rho - \partial_\theta^2\big)$ and thus

$$r^2\Delta = (r\partial_r)^2 + \partial_\phi^2 + \frac{1}{\sin^2\phi}\partial_\theta^2 + r^2\rho^{-1}\partial_\rho. \tag{3.1.15}$$

We have also, using the change of variables $(r,\phi) \mapsto (z,\rho)$

$$r^2\rho^{-1}\partial_\rho = r^2\rho^{-1}\big(\frac{r\cos\phi}{z^2+\rho^2}\partial_\phi + \rho r^{-1}\partial_r\big) = \frac{1}{\tan\phi}\partial_\phi + r\partial_r$$

and with (3.1.15), this provides the sought formula (3.1.14).

3.2 The heat equation

The heat operator is the following constant coefficient differential operator on $\mathbb{R}_t\times\mathbb{R}_x^n$

$$\partial_t - \Delta_x, \tag{3.2.1}$$

where the Laplace operator Δ_x on $\mathbb{R}^n$ is defined by (3.1.1).

Theorem 3.2.1. *We define on $\mathbb{R}_t \times \mathbb{R}_x^n$ the L_{loc}^1 function*

$$E(t, x) = (4\pi t)^{-n/2} H(t) e^{-\frac{|x|^2}{4t}}. \tag{3.2.2}$$

The function E is C^∞ on the complement of $\{(0,0)\}$ in $\mathbb{R} \times \mathbb{R}^n$. The function E is a fundamental solution of the heat equation, i.e. $\partial_t E - \Delta_x E = \delta_0(t) \otimes \delta_0(x)$.

Proof. To prove that $E \in L_{\mathrm{loc}}^1(\mathbb{R}^{n+1})$, we calculate for $T \geq 0$,

$$\int_0^T \int_0^{+\infty} t^{-n/2} r^{n-1} e^{-\frac{r^2}{4t}} dt dr \underbrace{=}_{r=2t^{1/2}\rho} \int_0^T \int_0^{+\infty} t^{-n/2} 2^{n-1} t^{(n-1)/2} \rho^{n-1} e^{-\rho^2} 2t^{1/2} dt d\rho$$

$$= 2^n T \int_0^{+\infty} \rho^{n-1} e^{-\rho^2} d\rho < +\infty.$$

Moreover, the function E is obviously analytic on the open subset of $\mathbb{R}^{1+n}$ $\{(t, x) \in \mathbb{R} \times \mathbb{R}^n, t \neq 0\}$. Let us prove that E is C^∞ on $\mathbb{R} \times (\mathbb{R}^n \backslash \{0\})$. With ρ_0 defined in (3.1.1) of [15], the function ρ_1 defined by $\rho_1(t) = H(t) t^{-n/2} \rho_0(t)$ is also C^∞ on $\mathbb{R}$ and

$$E(t, x) = H\left(\frac{|x|^2}{4t}\right) \left(\frac{|x|^2}{4t}\right)^{n/2} e^{-\frac{|x|^2}{4t}} |x|^{-n} \pi^{-n/2} = |x|^{-n} \pi^{-n/2} \rho_1\left(\frac{4t}{|x|^2}\right),$$

which is indeed smooth on $\mathbb{R}_t \times (\mathbb{R}_x^n \backslash \{0\})$. We want to solve the equation $\partial_t u - \Delta_x u = \delta_0(t)\delta_0(x)$. If u belongs to $\mathscr{S}'(\mathbb{R}^{n+1})$, we can consider its Fourier transform v with respect to x (well-defined by transposition as the Fourier transform in (4.1.10) of [15], and we end-up with the simple ODE with parameters on v,

$$\partial_t v + 4\pi^2 |\xi|^2 v = \delta_0(t). \tag{3.2.3}$$

It remains to determine a fundamental solution of that ODE: we have

$$\frac{d}{dt} + \lambda = e^{-t\lambda} \frac{d}{dt} e^{t\lambda}, \quad \left(\frac{d}{dt} + \lambda\right)(e^{-t\lambda} H(t)) = \left(e^{-t\lambda} \frac{d}{dt} e^{t\lambda}\right)(e^{-t\lambda} H(t)) = \delta_0(t), \tag{3.2.4}$$

so that we can take $v = H(t) e^{-4\pi^2 t |\xi|^2}$, which belongs to $\mathscr{S}'(\mathbb{R}_t \times \mathbb{R}_\xi^n)$. Taking the inverse Fourier transform with respect to ξ of both sides of (3.2.3) gives[2] with $u \in \mathscr{S}'(\mathbb{R}_t \times \mathbb{R}_\xi^n)$

$$\partial_t u - \Delta_x u = \delta_0(t) \otimes \delta_0(x). \tag{3.2.5}$$

To compute u, we check with $\varphi \in \mathscr{D}(\mathbb{R}), \psi \in \mathscr{D}(\mathbb{R}^n)$,

$$\langle u, \varphi \otimes \check{\psi} \rangle = \langle \widehat{v}^x, \varphi \otimes \psi \rangle = \langle v, \varphi \otimes \hat{\psi} \rangle = \int_0^{+\infty} \int_{\mathbb{R}^n} \varphi(t) \hat{\psi}(\xi) e^{-4\pi^2 t |\xi|^2} dt d\xi.$$

We can use the Fubini theorem in that absolutely converging integral and use (4.1.2) in [15] to get

$$\langle u, \varphi \otimes \check{\psi} \rangle = \int_0^{+\infty} \varphi(t) \left(\int_{\mathbb{R}^n} (4\pi t)^{-n/2} e^{-\pi \frac{|x|^2}{4\pi t}} \psi(x) dx \right) dt = \langle E, \varphi \otimes \check{\psi} \rangle,$$

where the last equality is due to the Fubini theorem and the local integrability of E. We have thus $E = u$ and E satisfies (3.2.5). The proof is complete. $\qquad\square$

---

[2]The Fourier transformation obviously respects the tensor products.

Corollary 3.2.2. *The heat equation is C^∞ hypoelliptic (see the definition 3.1.3) , in particular for $w \in \mathscr{D}'(\mathbb{R}^{1+n})$,*

$$\operatorname{singsupp} w \subset \operatorname{singsupp}(\partial_t w - \Delta_x w),$$

where singsupp *stands for the C^∞ singular support as defined by (3.1.9) in [15].*

Proof. It is an immediate consequence of the theorem 3.1.4, since E is C^∞ outside zero from the previous theorem. □

Remark 3.2.3. It is also possible to define the *analytic singular support* of a distribution T in an open subset Ω of $\mathbb{R}^n$: we define

$$\operatorname{singsupp}_{\mathcal{A}} T = \{x \in \Omega, \forall U \text{open} \in \mathscr{V}_x,\ T_{|U} \notin \mathcal{A}(U)\}, \tag{3.2.6}$$

where $\mathcal{A}(U)$ stands for the analytic[3] functions on the open set U. It is a consequence[4] of the proof of theorem 3.2.1 that

$$\operatorname{singsupp}_{\mathcal{A}} E = \{0\} \times \mathbb{R}_x^n. \tag{3.2.7}$$

In particular this implies that the heat equation is *not* analytic-hypoelliptic since

$$\{0\} \times \mathbb{R}_x^n = \operatorname{singsupp}_{\mathcal{A}} E \not\subset \operatorname{singsupp}_{\mathcal{A}}(\partial_t E - \Delta_x E) = \operatorname{singsupp}_{\mathcal{A}} \delta_0 = \{0_{\mathbb{R}^{1+n}}\}.$$

3.3 The Schrödinger equation

We move forward now with the Schrödinger equation,

$$\frac{1}{i}\frac{\partial}{\partial t} - \Delta_x \tag{3.3.1}$$

which looks similar to the heat equation, but which is in fact drastically different.

Lemma 3.3.1.

$$\mathscr{D}(\mathbb{R}^{n+1}) \mapsto \int_0^{+\infty} e^{-i(n-2)\frac{\pi}{4}} (4\pi t)^{-n/2} \left(\int_{\mathbb{R}^n} \Phi(t,x) e^{i\frac{|x|^2}{4t}} \, dx \right) dt = \langle E, \Phi \rangle \tag{3.3.2}$$

is a distribution in $\mathbb{R}^{n+1}$ of order $\leq n+2$.

---

[3]A function f is said to be analytic on an open subset U of $\mathbb{R}^n$ if it is $C^\infty(U)$, and for each $x_0 \in U$ there exists $r_0 > 0$ such that $\bar{B}(x_0, r_0) \subset U$ and

$$\forall x \in \bar{B}(x_0, r_0), \quad f(x) = \sum_{\alpha \in \mathbb{N}^n} \frac{1}{\alpha!} \partial_x^\alpha f(x_0)(x - x_0)^\alpha.$$

[4]In fact, in the theorem, we have noted the obvious inclusion $\operatorname{singsupp}_{\mathcal{A}} E \subset \{0\} \times \mathbb{R}_x^n$, but since E is C^∞ in $t \neq 0$, vanishes identically on $t < 0$, is positive (it means > 0) on $t > 0$, it cannot be analytic near any point of $\{0\} \times \mathbb{R}_x^n$.

Proof. Let $\Phi \in \mathscr{D}(\mathbb{R} \times \mathbb{R}^n)$; for $t > 0$ we have, using (4.6.7) iin [15],

$$e^{-i(n-2)\frac{\pi}{4}}(4\pi t)^{-n/2} \int_{\mathbb{R}^n} \Phi(t,x)e^{i\frac{|x|^2}{4t}}dx = i \int_{\mathbb{R}^n} \hat{\Phi}^x(t,\xi)e^{-4i\pi^2 t|\xi|^2}d\xi,$$

so that with $\mathbb{N} \ni \tilde{n}$ even $> n$, using (4.1.7) and (4.1.14) in [15],

$$\sup_{t>0}\left|e^{-i(n-2)\frac{\pi}{4}}(4\pi t)^{-n/2}\int_{\mathbb{R}^n}\Phi(t,x)e^{i\frac{|x|^2}{4t}}dx\right| \leq \sup_{t>0}\int_{\mathbb{R}^n}|\hat{\Phi}^x(t,\xi)|d\xi$$

$$\leq \sup_{t>0}\int (1+|\xi|^2)^{-\tilde{n}/2}|\underbrace{(1+|\xi|^2)^{\tilde{n}/2}}_{\text{polynomial}}\hat{\Phi}(t,\xi)|d\xi \leq C_n \max_{|\alpha|\leq\tilde{n}}\|\partial_x^\alpha\Phi\|_{L^\infty(\mathbb{R}^{n+1})}.$$

As a result the mapping

$$\mathscr{D}(\mathbb{R}^{n+1}) \mapsto \int_0^{+\infty} e^{-i(n-2)\frac{\pi}{4}}(4\pi t)^{-n/2}\left(\int_{\mathbb{R}^n}\Phi(t,x)e^{i\frac{|x|^2}{4t}}dx\right)dt = \langle E, \Phi\rangle$$

is a distribution of order $\leq n+2$. $\hfill\square$

Theorem 3.3.2. *The distribution E given by* (3.3.2) *is a fundamental solution of the Schrödinger equation, i.e. $\frac{1}{i}\partial_t E - \Delta_x E = \delta_0(t) \otimes \delta_0(x)$. Moreover, E is smooth on the open set $\{t \neq 0\}$ and equal there to*

$$e^{-i(n-2)\frac{\pi}{4}}H(t)(4\pi t)^{-n/2}e^{i\frac{|x|^2}{4t}}. \tag{3.3.3}$$

The distribution E is the partial Fourier transform with respect to the variable x of the $L^\infty(\mathbb{R}^{n+1})$ function

$$\tilde{E}(t,\xi) = iH(t)e^{-4i\pi^2 t|\xi|^2}. \tag{3.3.4}$$

Proof. We want to solve the equation $-i\partial_t u - \Delta_x u = \delta_0(t)\delta_0(x)$. If u belongs to $\mathscr{S}'(\mathbb{R}^{n+1})$, we can consider its Fourier transform v with respect to x (well-defined by transposition as the Fourier transform in (4.1.10) of [15]), and we end-up with the simple ODE with parameters on v,

$$\partial_t v + i4\pi^2|\xi|^2 v = i\delta_0(t). \tag{3.3.5}$$

Using the identity (3.2.4), we see that we can take $v = iH(t)e^{-i4\pi^2 t|\xi|^2}$, which belongs to $\mathscr{S}'(\mathbb{R}_t \times \mathbb{R}_\xi^n)$. Taking the inverse Fourier transform with respect to ξ of both sides of (3.3.5) gives with $u \in \mathscr{S}'(\mathbb{R}_t \times \mathbb{R}_\xi^n)$

$$\partial_t u - i\Delta_x u = i\delta_0(t) \otimes \delta_0(x) \quad \text{i.e.} \quad \frac{1}{i}\partial_t u - \Delta_x u = \delta_0(t) \otimes \delta_0(x). \tag{3.3.6}$$

To compute u, we check with $\varphi \in \mathscr{D}(\mathbb{R}), \psi \in \mathscr{D}(\mathbb{R}^n)$,

$$\langle u, \varphi \otimes \psi\rangle = \langle \hat{v}^x, \varphi \otimes \check{\psi}\rangle = \langle v, \varphi \otimes \check{\hat{\psi}}\rangle = i\int_0^{+\infty}\varphi(t)\left(\int_{\mathbb{R}^n}\hat{\psi}(\xi)e^{i\pi(-4\pi t)|\xi|^2}d\xi\right)dt. \tag{3.3.7}$$

We note now that, using (4.6.7) and (4.1.10) in [15], for $t > 0$,

$$i \int_{\mathbb{R}^n} \hat{\psi}(\xi) e^{i\pi(-4\pi t)|\xi|^2} d\xi = i \int_{\mathbb{R}^n} \psi(x)(4\pi t)^{-n/2} e^{i\frac{|x|^2}{4t}} dx e^{-n\frac{i\pi}{4}}$$

$$= e^{-i(n-2)\frac{\pi}{4}}(4\pi t)^{-n/2} \int_{\mathbb{R}^n} e^{i\frac{|x|^2}{4t}} \psi(x) dx.$$

As a result, u is a distribution on $\mathbb{R}^{n+1}$ defined by

$$\langle u, \Phi \rangle = e^{-i(n-2)\frac{\pi}{4}}(4\pi)^{-n/2} \int_0^{+\infty} t^{-n/2} \left(\int_{\mathbb{R}^n} \Phi(t,x) e^{i\frac{|x|^2}{4t}} dx \right) dt$$

and coincides with E, so that E satisfies (3.3.6). The identity (3.3.7) is proving (3.3.4). The proof of the theorem is complete. □

Remark 3.3.3. The fundamental solution of the Schrödinger equation is unbounded near $t = 0$ and, since E is smooth on $t \neq 0$, its C^∞ singular support is equal to $\{0\} \times \mathbb{R}^n_x$. In particular, the Schrödinger equation is *not* hypoelliptic. We shall see that it looks like a propagation equation with an infinite speed, or more precisely with a speed depending on the frequency of the wave.

3.4 The Wave Equation

3.4.1 Presentation

The wave equation in d dimensions with speed of propagation $c > 0$, is given by the operator on $\mathbb{R}_t \times \mathbb{R}^d_x$

$$\Box_c = c^{-2}\partial_t^2 - \Delta_x. \tag{3.4.1}$$

We want to solve the equation $c^{-2}\partial_t^2 u - \Delta_x u = \delta_0(t)\delta_0(x)$. If u belongs to $\mathscr{S}'(\mathbb{R}^{d+1})$, we can consider its Fourier transform v with respect to x, and we end-up with the ODE with parameters on v,

$$c^{-2}\partial_t^2 v + 4\pi^2|\xi|^2 v = \delta_0(t), \quad \partial_t^2 v + 4\pi^2 c^2|\xi|^2 v = c^2\delta_0(t). \tag{3.4.2}$$

Lemma 3.4.1. *Let* $\lambda, \mu \in \mathbb{C}$. *A fundamental solution of* $P_{\lambda,\mu} = (\frac{d}{dt} - \lambda)(\frac{d}{dt} - \mu)$ *(on the real line) is*

$$\begin{cases} \left(\dfrac{e^{t\lambda} - e^{t\mu}}{\lambda - \mu}\right) H(t) & \text{for } \lambda \neq \mu, \\ te^{t\lambda} H(t) & \text{for } \lambda = \mu. \end{cases} \tag{3.4.3}$$

Proof. If $\lambda \neq \mu$, to solve $(\frac{d}{dt} - \lambda)(\frac{d}{dt} - \mu) = \delta_0(t)$, the method of variation of parameters gives a solution $a(t)e^{\lambda t} + b(t)e^{\mu t}$ with

$$\begin{pmatrix} e^{t\lambda} & e^{t\mu} \\ \lambda e^{t\lambda} & \mu e^{t\mu} \end{pmatrix} \begin{pmatrix} \dot{a} \\ \dot{b} \end{pmatrix} = \begin{pmatrix} 0 \\ \delta \end{pmatrix} \Longrightarrow \begin{pmatrix} \dot{a} \\ \dot{b} \end{pmatrix} = \frac{1}{\lambda - \mu} \begin{pmatrix} \delta \\ -\delta \end{pmatrix} \Longrightarrow (3.4.3) \text{ for } \lambda \neq \mu,$$

which gives also the result for $\lambda = \mu$ by differentiation with respect to λ of the identity $P_{\lambda,\mu}(e^{t\lambda} - e^{t\mu}) = (\lambda - \mu)\delta$. □

Going back to the wave equation, we can take v as the temperate distribution[5] given by

$$v(t, \xi) = c^2 H(t) \frac{e^{2i\pi ct|\xi|} - e^{-2i\pi ct|\xi|}}{4i\pi c|\xi|} = c^2 H(t) \frac{\sin(2\pi ct|\xi|)}{2\pi c|\xi|}. \qquad (3.4.4)$$

Taking the inverse Fourier transform with respect to ξ of both sides of (3.4.2) gives with $u \in \mathscr{S}'(\mathbb{R}_t \times \mathbb{R}_\xi^d)$

$$c^{-2}\partial_t^2 u - \Delta_x u = \delta_0(t) \otimes \delta_0(x). \qquad (3.4.5)$$

To compute u, we check with $\Phi \in \mathscr{D}(\mathbb{R}^{1+d})$,

$$\langle u, \Phi \rangle = \langle \widehat{v}^x(t, \xi), \Phi(t, -\xi) \rangle = \int_0^{+\infty} \int_{\mathbb{R}^n} \widehat{\Phi}^x(t, \xi) c \frac{\sin(2\pi ct|\xi|)}{2\pi|\xi|} d\xi dt. \qquad (3.4.6)$$

We have found an expression for a fundamental solution of the wave equation in d space dimensions and proven the following proposition.

Proposition 3.4.2. *Let E_+ be the temperate distribution on $\mathbb{R}^{d+1}$ such that*

$$\widehat{E_+}^x(t, \xi) = cH(t) \frac{\sin(2\pi ct|\xi|)}{2\pi|\xi|}. \qquad (3.4.7)$$

Then E_+ is a fundamental solution of the wave equation (3.4.1), i.e. satisfies $\Box_c E_+ = \delta_0(t) \otimes \delta_0(x)$.

Remark 3.4.3. Defining the forward-light-cone $\Gamma_{+,c}$ as

$$\Gamma_{+,c} = \{(t, x) \in \mathbb{R} \times \mathbb{R}^d, ct \geq |x|\}, \qquad (3.4.8)$$

one can prove more precisely that E_+ is the only fundamental solution with support in $\{t \geq 0\}$ and that

$$\text{supp } E_+ = \Gamma_+, \text{ when } d = 1 \text{ and } d \geq 2 \text{ is even,} \qquad (3.4.9)$$
$$\text{supp } E_+ = \partial\Gamma_+, \text{ when } d \geq 3 \text{ is odd,} \qquad (3.4.10)$$
$$\text{singsupp } E_+ = \partial\Gamma_+, \text{ in any dimension.} \qquad (3.4.11)$$

Lemma 3.4.4. *Let E_1, E_2 be fundamental solutions of the wave equation such that $\text{supp } E_1 \subset \Gamma_{+,c}, \text{supp } E_2 \subset \{t \geq 0\}$. Then $E_1 = E_2$.*

Proof. Defining $u = E_1 - E_2$, we have $\text{supp } u \subset \{t \geq 0\}$ and the mapping

$$\{t \geq 0\} \times \Gamma_{+,c} \ni \big((t, x), (s, y)\big) \mapsto (t + s, x + y) \in \mathbb{R}^{d+1}$$

is proper since

$$t, s \geq 0, cs \geq |y|, |t + s| \leq T, |x + y| \leq R \Longrightarrow t, s \in [0, T], |x| \leq R + cT, |y| \leq cT,$$

so that Section 3.5.3 in [15] allows to perform the following calculations

$$u = u * \delta_0 = u * \Box_c E_1 = \Box_c u * E_1 = 0.$$

$\square$

---

[5]The function $\mathbb{R} \ni s \mapsto \frac{\sin s}{s} = \sum_{k \geq 0} (-1)^k \frac{s^{2k}}{(2k+1)!} = S(s^2)$ is a smooth bounded function of s^2, so that $v(t, \xi) = c^2 H(t) t S(4\pi^2 c^2 t^2 |\xi|^2)$ is continuous and such that $|v(t, \xi)| \leq CtH(t)$, thus a tempered distribution.

3.4.2 The wave equation in one space dimension

Theorem 3.4.5. *On $\mathbb{R}_t \times \mathbb{R}_x$, the only fundamental solution of the wave equation supported in $\Gamma_{+,c}$ is*

$$E_+(t,x) = \frac{c}{2}H(ct - |x|). \tag{3.4.12}$$

where E_+ is defined in (3.4.7). That fundamental solution is bounded and the properties (3.4.9), (3.4.11) are satisfied.

Proof. We have $c^{-2}\partial_t^2 - \partial_x^2 = (c^{-1}\partial_t - \partial_x)(c^{-1}\partial_t + \partial_x)$ and changing (linearly) the variables with $x_1 = ct + x, x_2 = ct - x$, we have $t = \frac{1}{2c}(x_1 + x_2), x = \frac{1}{2}(x_1 - x_2)$, using the notation

$$(x_1, x_2) \mapsto (t, x) \mapsto u(t, x) = v(x_1, x_2),$$

$$\frac{\partial u}{\partial t} = \frac{\partial v}{\partial x_1}c + \frac{\partial v}{\partial x_2}c, \quad \frac{\partial u}{\partial x} = \frac{\partial v}{\partial x_1} - \frac{\partial v}{\partial x_2}, \quad c^{-1}\partial_t - \partial_x = 2\partial_{x_2}, c^{-1}\partial_t + \partial_x = 2\partial_{x_1},$$

and thus $\square_c = 4\frac{\partial^2}{\partial x_1 \partial x_2}$, so that a fundamental solution is $v = \frac{1}{4}H(x_1)H(x_2)$. We have now to pull-back this distribution by the linear mapping $(t, x) \mapsto (x_1, x_2)$: we have the formula

$$\varphi(0,0) = \langle 4\frac{\partial^2 v}{\partial x_1 \partial x_2}(x_1, x_2), \varphi(x_1, x_2) \rangle = \langle (\square_c u)(t, x), \varphi(ct + x, ct - x) \rangle 2c$$

which gives the fundamental solution $\frac{2c}{4}H(ct + x)H(ct - x) = \frac{c}{2}H(ct - |x|)$. Moreover that fundamental solution is supported in $\Gamma_{+,c}$ and since E_+ is supported in $\{t \geq 0\}$, we can apply the lemma 3.4.4 to get their equality. $\square$

3.4.3 The wave equation in two space dimensions

We consider (3.4.1) with $d = 2$, i.e. $\square_c = c^{-2}\partial_t^2 - \partial_{x_1}^2 - \partial_{x_2}^2$.

Theorem 3.4.6. *On $\mathbb{R}_t \times \mathbb{R}_x^2$, the only fundamental solution of the wave equation supported in $\Gamma_{+,c}$ is*

$$E_+(t,x) = \frac{c}{2\pi}H(ct - |x|)(c^2t^2 - |x|^2)^{-1/2}, \tag{3.4.13}$$

where E_+ is defined in (3.4.7). That fundamental solution is L^1_{loc} and the properties (3.4.9), (3.4.11) are satisfied.

Proof. From the lemma 3.4.4, it is enough to prove that the rhs of (3.4.13) is indeed a fundamental solution. The function $E(t, x) = \frac{c}{2\pi}H(ct - |x|)(c^2t^2 - |x|^2)^{-1/2}$ is locally integrable in $\mathbb{R} \times \mathbb{R}^2$ since

$$\int_0^T \int_0^{ct} (c^2t^2 - r^2)^{-1/2}r\,dr\,dt = \int_0^T [(c^2t^2 - r^2)^{1/2}]_{r=ct}^{r=0}dt = cT^2/2 < +\infty.$$

Moreover E is homogeneous of degree -1, so that $\square_c E$ is homogeneous with degree -3 and supported in $\Gamma_{+,c}$. We use now the independently proven three-dimensional

case (Theorem 3.4.7). We define with $E_{+,3}$ given by (3.4.15), $\varphi \in \mathscr{D}(\mathbb{R}^3_{t,x_1,x_2})$, $\chi \in \mathscr{D}(\mathbb{R})$ with $\chi(0) = 1$,

$$
\begin{aligned}
\langle u, \varphi \rangle_{\mathscr{D}'(\mathbb{R}^3), \mathscr{D}(\mathbb{R}^3)} &= \lim_{\epsilon \to 0} \langle E_{+,3}, \varphi(t, x_1, x_2) \otimes \chi(\epsilon x_3) \rangle_{\mathscr{D}'(\mathbb{R}^4), \mathscr{D}(\mathbb{R}^4)} \\
&= \lim_{\epsilon \to 0} \frac{1}{4\pi} \iiint_{\mathbb{R}^3} \frac{\varphi(c^{-1}\sqrt{x_1^2 + x_2^2 + x_3^2}, x_1, x_2)}{\sqrt{x_1^2 + x_2^2 + x_3^2}} \chi(\epsilon x_3) dx_1 dx_2 dx_3 \\
&= \frac{1}{4\pi} 2 \iiint_{\mathbb{R}^2_{x_1,x_2} \times \{x_3 \geq 0\}} \frac{\varphi(c^{-1}\sqrt{x_1^2 + x_2^2 + x_3^2}, x_1, x_2)}{\sqrt{x_1^2 + x_2^2 + x_3^2}} dx_1 dx_2 dx_3 \\
&\hspace{4cm} (t = c^{-1}\sqrt{x_1^2 + x_2^2 + x_3^2}) \\
&= \frac{1}{2\pi} \iiint_{\mathbb{R}^2_{x_1,x_2} \times \{ct \geq \sqrt{x_1^2+x_2^2}\}} \frac{\varphi(t, x_1, x_2)}{ct} \frac{1}{2}(c^2 t^2 - x_1^2 - x_2^2)^{-1/2} 2c^2 t dx_1 dx_2 dt \\
&= \frac{c}{2\pi} \iiint_{\mathbb{R}^2_{x_1,x_2} \times \{ct \geq \sqrt{x_1^2+x_2^2}\}} \varphi(t, x_1, x_2)(c^2 t^2 - x_1^2 - x_2^2)^{-1/2} dx_1 dx_2 dt \\
&= \langle E, \varphi \rangle_{\mathscr{D}'(\mathbb{R}^3), \mathscr{D}(\mathbb{R}^3)}, \qquad \text{so that } E_+ = u.
\end{aligned}
$$

With $\square_{c,d}$ standing for the wave operator in d dimensions with speed c, we have, since

$$
\square_{c,3}\big(\varphi(t, x_1, x_2) \otimes \chi(\epsilon x_3)\big) = \square_{c,2}\big(\varphi(t, x_1, x_2)\big) \otimes \chi(\epsilon x_3) - \varphi(t, x_1, x_2)\epsilon^2 \chi''(\epsilon x_3)
$$

$$
\begin{aligned}
\langle \square_{c,2} u, \varphi \rangle &= \lim_{\epsilon \to 0} \langle E_{+,3}, (\square_{c,2}\varphi)(t, x_1, x_2) \otimes \chi(\epsilon x_3) \rangle \\
&= \lim_{\epsilon \to 0} \Big(\langle E_{+,3}, \square_{c,3}\big(\varphi(t, x_1, x_2) \otimes \chi(\epsilon x_3)\big) \rangle \rangle + \langle E_{+,3}, \varphi(t, x_1, x_2)\epsilon^2 \chi''(\epsilon x_3) \rangle \Big) \\
&= \varphi(0, 0, 0),
\end{aligned}
$$

which gives $\square_{c,2} E = \square_{c,2} u = \delta_{0,\mathbb{R}^3}$ and the result. $\qquad \square$

3.4.4   The wave equation in three space dimensions

We consider (3.4.1) with $d = 3$, i.e. $\square_c = c^{-2}\partial_t^2 - \partial_{x_1}^2 - \partial_{x_2}^2 - \partial_{x_3}^2$.

Theorem 3.4.7. *On $\mathbb{R}_t \times \mathbb{R}_x^3$, the only fundamental solution of the wave equation supported in $\Gamma_{+,c}$ is*

$$
E_+(t, x) = \frac{1}{4\pi|x|} \delta_{0,\mathbb{R}}(t - c^{-1}|x|), \tag{3.4.14}
$$

i.e. for $\Phi \in \mathscr{D}(\mathbb{R}_t \times \mathbb{R}_x^3)$, $\langle E_+, \Phi \rangle = \displaystyle\int_{\mathbb{R}^3} \frac{1}{4\pi|x|} \Phi(c^{-1}|x|, x) dx.$ $\tag{3.4.15}$

where E_+ is defined in (3.4.7). The properties (3.4.10), (3.4.11) are satisfied.

Proof. The formula (3.4.15) is defining a Radon measure E with support $\partial\Gamma_{+,c}$, so that the last statements of the lemmas are clear. From the lemma 3.4.4, it is

enough to prove that (3.4.15) defines indeed a fundamental solution. We check for $\varphi \in \mathscr{D}(\mathbb{R}), \psi \in \mathscr{D}(\mathbb{R}^3)$

$$\langle \Box_c E, \varphi(t) \otimes \psi(x) \rangle = \langle E, \Box_c(\varphi \otimes \psi) \rangle$$
$$= \frac{1}{4\pi} \int_{\mathbb{R}^3} |x|^{-1} \Big(c^{-2} \varphi''(c^{-1}|x|) \psi(x) - \varphi(c^{-1}|x|)(\Delta\psi)(x) \Big) dx.$$

If we assume that $\operatorname{supp} \varphi \subset \mathbb{R}_+^*$, we get

$$\int_{\mathbb{R}^3} |x|^{-1} \varphi(c^{-1}|x|)(\Delta\psi)(x) dx = \int_{\mathbb{R}^3} \Delta\big(|x|^{-1}\varphi(c^{-1}|x|)\big)\psi(x) dx$$
$$= \int_{\mathbb{R}^3} \Big(\big(r^{-1}\varphi(c^{-1}r)\big)'' + 2r^{-1}\big(r^{-1}\varphi(c^{-1}r)\big)' \Big) \psi(x) dx \qquad (r = |x|)$$
$$= \int \psi(x) \Big(r^{-1}\varphi''(c^{-1}r)c^{-2} + 2(-r^{-2})\varphi'(c^{-1}r)c^{-1} + 2r^{-3}\varphi(c^{-1}r)$$
$$+ 2r^{-1}r^{-1}\varphi'(c^{-1}r)c^{-1} + 2r^{-1}(-r^{-2})\varphi(c^{-1}r) \Big) dx,$$

which gives $\langle \Box_c E, \varphi(t) \otimes \psi(x) \rangle = 0$. As a result,

$$\operatorname{supp}(\Box_c E) \subset \partial\Gamma_{+,c} \cap \{t \le 0\} = \{(0_{\mathbb{R}}, 0_{\mathbb{R}^3})\},$$

and since E is homogeneous with degree -2, the distribution $\Box_c E$ is homogeneous with degree -4 with support at the origin of $\mathbb{R}^4$: Lemma 3.4.8 and Theorem 3.3.4 in [15] imply that $\Box_c E = \kappa\delta_{0,\mathbb{R}^4}$. To check that $\kappa = 1$, we calculate for $\varphi \in \mathscr{D}(\mathbb{R})$ (noting that $|t| \le C$ and $|x| \le c|t| + 1$ implies $|x| \le cC + 1$)

$$\langle \Box_c E, \varphi(t) \otimes 1 \rangle = \frac{1}{4\pi} \int_0^{+\infty} r^{-1}c^{-2}\varphi''(c^{-1}r)r^2 dr 4\pi = \int_0^{+\infty} \varphi''(r)r\, dr$$
$$= [\varphi'(r)r]_0^{+\infty} - \int_0^{+\infty} \varphi'(r) dr = \varphi(0),$$

so that $\kappa = 1$ and the theorem is proven. $\qquad\square$

Chapter 4

Analytic PDE

4.1 The Cauchy-Kovalevskaya theorem

Let $m \in \mathbb{N}^*$. We consider the Cauchy problem in $\mathbb{R}_t \times \mathbb{R}_x^d$

$$\begin{cases} \partial_t^m u = F\big(t, x, (\partial_t^k \partial_x^\alpha u)_{|\alpha|+k \leq m, k < m}\big), \\ (\partial_t^j u)(0, x) = v_j(x), \quad 0 \leq j < m. \end{cases} \tag{4.1.1}$$

where F, v_j are all analytic of their arguments.

Theorem 4.1.1. *Let F be an analytic function in a neighborhood of $(0, x_0, y_0) \in \mathbb{R}_t \times \mathbb{R}_x^d \times \mathbb{R}^N$ with $y_0 = \big((\partial_x^\alpha v_k)(x_0)\big)_{|\alpha|+k \leq m, k < m}$, $N = C_{d+m+1}^{d+1} - 1$ (see (7.3.3) in the appendix) and let $(v_j)_{0 \leq j < m}$ be analytic functions in a neighborhood of x_0. Then there exists a neighborhood of $(0, x_0)$ on which the Cauchy problem (4.1.1) has a unique analytic solution.*

Proof. The uniqueness part is a consequence of the following lemma.

Lemma 4.1.2. *Let $m, m' \in \mathbb{N}^*$. We consider the Cauchy problem in $\mathbb{R}_t \times \mathbb{R}_x^d$*

$$\begin{cases} \partial_t^m u = G\big(t, x, (\partial_t^k \partial_x^\alpha u)_{|\alpha| \leq m', k < m}\big), \\ (\partial_t^j u)(0, x) = v_j(x), \quad 0 \leq j < m. \end{cases} \tag{4.1.2}$$

where G, v_j are all analytic of their arguments. The problem (4.1.2) has a unique analytic solution.

Proof of the lemma. Let u be an analytic solution of (4.1.2): we prove by induction on l that

$$\forall l \in \mathbb{N}, \exists m_l' \in \mathbb{N}, \quad \partial_t^{m+l} u = G_l(t, x, (\partial_t^k \partial_x^\alpha u)_{|\alpha| \leq m_l', k < m}), \tag{4.1.3}$$

where G_l depends on a finite number of derivatives of G. It is true for $l = 0$ and if true for some $l \geq 0$, we get

$$\partial_t^{m+l+1} u = \frac{\partial G_l}{\partial t} + \sum_{\substack{k, \alpha \\ |\alpha| \leq m', k < m}} \frac{\partial G_l}{\partial w_{k\alpha}} \underbrace{\partial_t^{k+1} \partial_x^\alpha u}_{\text{expected term if } k < m-1} \quad .$$

If $k = m - 1$ in the sum above, we have $\partial_t^{k+1}\partial_x^\alpha u = \partial_x^\alpha\Big(G(t, x, (\partial_t^k\partial_x^\alpha u)_{|\alpha|\leq m', k<m})\Big)$, and this concludes the induction proof. As a result, we get that

$$\forall l \in \mathbb{N}, \quad \partial_t^{m+l}u(0, x) = G_l(0, x, (\partial_x^\alpha v_k)_{|\alpha|\leq m'_l, k<m}) \quad (\text{and } \partial_t^j u(0, x) = v_j(x), 0 \leq j < m).$$

This implies that for all k, α, $(\partial_t^k\partial_x^\alpha u)(0, x)$ are determined by the equation (4.1.2) and by analyticity of u, gives the uniqueness result. The proof of the lemma is complete. $\qquad\square$

Let us now prove the existence part of Theorem 4.1.1. Introducing $U(t, x) = u(t, x) - \sum_{0\leq j<m} v_j(x)\frac{t^j}{j!}$ we see that (4.1.1) is equivalent to

$$\begin{cases} \partial_t^m U = F\big(t, x, (\partial_t^k\partial_x^\alpha(U + \sum_{0\leq j<m} v_j(x)\frac{t^j}{j!}))_{\substack{|\alpha|+k\leq m \\ k<m}}\big) = G\big(t, x, (\partial_t^k\partial_x^\alpha U)_{\substack{|\alpha|+k\leq m \\ k<m}}\big), \\ (\partial_t^j U)(0, x) = 0, \quad 0 \leq j < m. \end{cases}$$
$$(4.1.4)$$

with G analytic. To prove the theorem, we may thus assume that the v_j in (4.1.1) are all identically 0. Let us notice that if u is a smooth function satisfying (4.1.1), then for $k + |\alpha| \leq m, k < m$, we have with $w_{k,\alpha} = \partial_t^k\partial_x^\alpha u$,

- if $k + 1 + |\alpha| \leq m$ and $k + 1 < m$, $\partial_t w_{k,\alpha} = w_{k+1,\alpha}$,
- if $k = m - 1, |\alpha| = 0$, $\partial_t w_{k,\alpha} = \partial_t^m u = F(t, x, (w_{l\beta})_{l+|\beta|\leq m, l<m})$,
- if $k = m - 1, |\alpha| = 1, \alpha = e_j$, $\partial_t w_{k,\alpha} = \partial_{x_j}\partial_t^m u = \dfrac{\partial F}{\partial x_j} + \displaystyle\sum_{\substack{l+|\beta|\leq m \\ l<m}}\dfrac{\partial F}{\partial w_{l\beta}}\dfrac{\partial w_{l\beta}}{\partial x_j}$,
- if $k < m - 1, k + 1 + |\alpha| \geq 1 + m \implies k + |\alpha| = m, k \leq m - 2, |\alpha| \geq 2$,

$\exists j$ with $\alpha_j \geq 1$, $\quad\partial_t w_{k,\alpha} = \partial_t^{k+1}\partial_x^\alpha u = \partial_{x_j}\partial_t^{k+1}\partial_x^{\alpha-e_j}u = \partial_{x_j}w_{k+1,\alpha-e_j}$,

with $k + 1 < m, k + 1 + |\alpha - e_j| = k + |\alpha| = m$,

- $w_{k\alpha}(0, x) = \partial_x^\alpha v_k(x) \equiv 0$.

Conversely, if the functions $(w_{k,\alpha})_{k+|\alpha|\leq m, k<m}$ satisfy

- if $k + 1 + |\alpha| \leq m$ and $k + 1 < m$, $\partial_t w_{k,\alpha} = w_{k+1,\alpha}$,   (4.1.5)
- if $k = m - 1, |\alpha| = 0$, $\partial_t w_{k,\alpha} = F(t, x, (w_{l\beta})_{l+|\beta|\leq m, l<m})$,   (4.1.6)
- if $k = m - 1, |\alpha| = 1, \alpha = e_j$, $\partial_t w_{k,\alpha} = \dfrac{\partial F}{\partial x_j} + \displaystyle\sum_{\substack{l+|\beta|\leq m \\ l<m}}\dfrac{\partial F}{\partial w_{l\beta}}\dfrac{\partial w_{l\beta}}{\partial x_j}$,   (4.1.7)
- if $k + |\alpha| = m, k \leq m - 2, |\alpha| \geq 2$, $\quad\partial_t w_{k,\alpha} = \partial_{x_j}w_{k+1,\alpha-e_j}$,   (4.1.8)

  where j is the smallest integer in $[1, d]$ such that $\alpha_j \geq 1$,
- $w_{k\alpha}(0, x) \equiv 0$,   (4.1.9)

we have

$$w_{k,\alpha} = \partial_t^k\partial_x^\alpha w_{00}, \quad k + |\alpha| \leq m, k < m.\qquad(4.1.10)$$

In fact, if $|\alpha| = 0$, we have for $k < m$ from (4.1.5)

$$\partial_t w_{00} = w_{10}, \ \ldots \ , \ \partial_t w_{m-2,0} = w_{m-1,0} \implies \partial_t^k w_{00} = w_{k0} \quad \text{for } 0 \leq k < m,$$

and the property (4.1.10) for $|\alpha| = 0$. We perform now an induction on $|\alpha|$. If $k + |\alpha| = m, k \leq m - 2$, (4.1.8), (4.1.9) imply

$$\partial_t w_{k,\alpha} = \partial_{x_j} w_{k+1,\alpha-e_j}$$

$$\implies w_{k,\alpha} = \int_0^t \partial_{x_j} w_{k+1,\alpha-e_j} ds \underbrace{=}_{\substack{\text{induction} \\ \text{since} |\alpha - e_j| = |\alpha| - 1}} \int_0^t \partial_{x_j} \partial_x^{\alpha-e_j} \partial_t^{k+1} w_{00} ds = \partial_t^k \partial_x^\alpha w_{00}.$$

$$(4.1.11)$$

Moreover, if $k = m - 1, |\alpha| = 1, \alpha = e_j$, (4.1.7), (4.1.9) imply

$$w_{m-1,e_j} = \int_0^t \Big(\frac{\partial F}{\partial x_j} + \sum_{\substack{l+|\beta|\leq m \\ l<m}} \frac{\partial F}{\partial w_{l\beta}} \frac{\partial w_{l\beta}}{\partial x_j} \Big) ds \qquad (4.1.12)$$

whereas from (4.1.6), (4.1.9),

$$w_{m-1,0} = \int_0^t F(s, x, (w_{l\beta})_{l+|\beta|\leq m, l<m}) ds$$

and thus (4.1.12) gives $\partial_{x_j} w_{m-1,0} = w_{m-1,e_j}$ so that

$$w_{m-1,e_j} = \partial_t^{m-1} \partial_{x_j} w_{00}, \qquad (4.1.13)$$

from the case $|\alpha| = 0$. Assume now that $k + 1 + |\alpha| \leq m, k + 1 < m$, i.e. $k \leq m - 2$, $k + |\alpha| \leq m - 1$.

If $k = m - 2, |\alpha| = 1, \alpha = e_j$, we have from (4.1.5), (4.1.9) and (4.1.13)

$$w_{m-2,\alpha} = \int_0^t w_{m-1,e_j} ds = \int_0^t \partial_{x_j} w_{m-1,0} ds = \partial_{x_j} \int_0^t \partial_t w_{m-2,0} ds$$

$$= \partial_{x_j} w_{m-2,0} = \partial_t^{m-2} \partial_x^\alpha w_{00}.$$

If $k = m - 3, |\alpha| \geq 1, \alpha_j \geq 1$, we have from (4.1.5), (4.1.9) and the case $k = m - 2$,

$$w_{m-3,\alpha} = \int_0^t w_{m-2,\alpha} ds = \int_0^t \partial_{x_j} w_{m-2,\alpha-e_j} ds = \partial_{x_j} \int_0^t \partial_t w_{m-3,\alpha-e_j} ds$$

$$= \partial_{x_j} w_{m-3,\alpha-e_j} = \partial_{x_j} \partial_t^{m-3} \partial_x^{\alpha-e_j} w_{00} = \partial_t^{m-3} \partial_x^\alpha w_{00}.$$

$$\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots$$

If $k = m - l, l \geq 2, |\alpha| \geq 1, \alpha_j \geq 1$, we have from (4.1.5), (4.1.9) and the case $k = m - l + 1$,

$$w_{m-l,\alpha} = \int_0^t w_{m-l+1,\alpha} ds = \int_0^t \partial_{x_j} w_{m-l+1,\alpha-e_j} ds = \partial_{x_j} \int_0^t \partial_t w_{m-l,\alpha-e_j} ds$$

$$= \partial_{x_j} w_{m-l,\alpha-e_j} = \partial_{x_j} \partial_t^{m-l} \partial_x^{\alpha-e_j} w_{00} = \partial_t^{m-l} \partial_x^\alpha w_{00},$$

proving (4.1.10). Property (4.1.10) and (4.1.6) give

$$\partial_t^m w_{00} = \partial_t w_{m-1,0} = F(t, x, (w_{l\beta})_{l+|\beta|\leq m, l<m}) = F(t, x, (\partial_t^l \partial_x^\beta w_{00})_{l+|\beta|\leq m, l<m})$$

so that $u = w_{00}$ satisfies the equation (4.1.1) with $\partial_t^j u(0, x) = 0$ for $0 \leq j < m$. As a result, considering the vector-valued function $Y = (w_{k,\alpha})_{k+|\alpha| \leq m, k < m}$, we have

$$\begin{cases} \partial_t Y = \sum_{1 \leq j \leq d} A_j(t, x, Y) \partial_{x_j} Y + T(t, x, Y), \\ Y(0, x) = 0, \end{cases} \tag{4.1.14}$$

where each A_j is a $N \times N$ matrix and T belongs to $\mathbb{R}^N$. Moreover if F is analytic, it is also the case of the functions A_j, T. Adding a dimension to the vector Y, we may take t as a first component and deal finally with the existence of an analytic solution for the quasi-linear system

$$\begin{cases} \partial_t Y = \sum_{1 \leq j \leq d} A_j(x, Y) \partial_{x_j} Y + T(x, Y), \\ Y(0, x) = 0, \end{cases} \tag{4.1.15}$$

where each A_j is a $N' \times N'$ matrix and T belongs to $\mathbb{R}^{N'}$ (this new $N' = N + 1$, where N is given in (4.1.14): nevertheless, we shall call it N in the sequel). If $Y = \sum_{j,\alpha} x^\alpha t^j Y_{\alpha,j}$ is an analytic solution of (4.1.15), we have

$$\partial_t Y = \sum_{j \geq 1, \alpha} j t^{j-1} x^\alpha Y_{\alpha,j} = \sum_{j \geq 0, \alpha} (j+1) t^j x^\alpha Y_{\alpha,j+1}$$

and since $A(x, Y) \cdot \partial_x Y + T(x, Y) = \sum_{j,\alpha} \mathcal{P}_{\alpha,j}\big((Y_{\beta,l})_{l \leq j}, \text{coeff.} A, T\big) x^\alpha t^j$, where $\mathcal{P}_{\alpha,j}$ is a polynomial with non-negative coefficients, we get

$$Y_{\alpha,j+1} = \frac{1}{j+1} \mathcal{P}_{\alpha,j}\big((Y_{\beta,l})_{l \leq j}, \text{coeff.} A, T\big).$$

As a result, the knowledge of $(Y_{\beta,l})_{l \leq j}$ provides the knowledge of $(Y_{\beta,l})_{l \leq j+1}$, and since we know $(Y_{\beta,0})$ from the initial condition in (4.1.15), the above formula determines the power series coefficients of Y and we have

$$Y_{\alpha,j} = \mathcal{Q}_{\alpha,j}\big(\text{coeff.} A, T\big),$$

where $\mathcal{Q}_{\alpha,j}$ is a polynomial with non-negative coefficients. We consider the Cauchy problem

$$\partial_t Z = \sum_j B_j(x, Z) \partial_{x_j} Z + S(x, Z), \quad Z(x, 0) = 0$$

with analytic functions B_j, S, majorizing A_j, T: if we find an analytic solution Z, then its Taylor coefficients $Z_{\alpha,j}$ will satisfy

$$Z_{\alpha,j} = \mathcal{Q}_{\alpha,j}\big(\text{coeff.} B, S\big)$$

and since $\mathcal{Q}_{\alpha,j}$ is a polynomial with non-negative coefficients, we get

$$|Y_{\alpha,j}| = |\mathcal{Q}_{\alpha,j}\big(\text{coeff.} A, T\big)| \leq \mathcal{Q}_{\alpha,j}\big(\text{coeff.} B, S\big) = Z_{\alpha,j}$$

Using (7.2.7), we see that the power series of A, T are majorized by those of

$$\frac{Mr}{r - \sum_{1 \leq j \leq d} x_j - \sum_{1 \leq l \leq N} y_l},$$

provided, M is large enough and $r > 0$ is small enough. We consider now the Cauchy problem

$$\begin{cases} \partial_t z_m = \dfrac{Mr}{r - \sum_{1 \le j \le d} x_j - \sum_{1 \le l \le N} z_l} \left(\sum_{j,l} \partial_{x_j} z_l + 1 \right) , 1 \le m \le N. \quad (4.1.16) \\ z_m(0, x) = 0. \end{cases}$$

It is enough to prove the existence of an analytic solution for this problem. If we consider the scalar equation $(t, s) \in \mathbb{R}^2 \mapsto u(t, s) \in \mathbb{R}$,

$$\partial_t u = \frac{Mr}{r - s - Nu} (Nd \partial_s u + 1), \quad u(0, s) = 0, \qquad (4.1.17)$$

and if we define

$$z_m(t, x) = u(t, x_1 + \cdots + x_d), 1 \le m \le N,$$

we get a solution of (4.1.16). The remaining task is to solve (4.1.17). To simplify the algebra we solve

$$\partial_t u = \frac{1}{1 - s - u} (\partial_s u + 1), \quad u(0, s) = 0.$$

Using the method of characteristics, we obtain

$$\begin{cases} \dot{t} = 1 - s - u, \ \dot{s} = -1, \ \dot{u} = 1, \\ t(0) = 0, \ s(0) = \sigma, \ u(0) = 0 \end{cases}$$

so that $u = \tau$, $s = \sigma - \tau$, $t = \tau - \sigma\tau$ and thus $\sigma = s + u, \tau = u, t = u(1 - s - u)$, that is $u = \frac{1 - s \pm \sqrt{(1-s)^2 - 4t}}{2}$. To satisfy the initial condition $u(0, s) = 0$, we find

$$u = \frac{1 - s - \sqrt{(1 - s)^2 - 4t}}{2}, \qquad (4.1.18)$$

which is indeed analytic near the origin. The proof of Theorem 4.1.1 is complete. $\quad\square$

Chapter 5

Elliptic Equations

5.1 Some simple facts on the Laplace operator

5.1.1 The mean-value theorem

Definition 5.1.1. *Let Ω be an open subset of $\mathbb{R}^n$ and $u \in \mathscr{D}'(\Omega)$. We shall say that u is an harmonic function on Ω if $\triangle u = 0$ on Ω.*

Remark 5.1.2. Note that from Theorem 3.1.4, an harmonic function is C^∞ and we shall see below that it is even analytic.

Proposition 5.1.3. *Let Ω be an open subset of $\mathbb{R}^n$ and u be an harmonic function on Ω. Then u is a smooth function and for all $x \in \Omega$ and all $r > 0$ such that $\bar{B}(x,r) \subset \Omega$, we have*

$$u(x) = \frac{1}{|B(x,r)|} \int_{B(x,r)} u(y)dy = \frac{1}{|\partial B(x,r)|} \int_{\partial B(x,r)} u(y)d\sigma(y). \qquad (5.1.1)$$

We shall use the notation $\fint_A f(y)dy = \frac{1}{|A|} \int_A f(y)dy$.

Proof. For x, r as in the statement above, we define

$$\varphi(r) = \fint_{\partial B(x,r)} u(y)d\sigma(y) = \fint_{\partial B(0,1)} u(x + r\omega)d\sigma(\omega)$$

and we have $\varphi'(r) = \fint_{\mathbb{S}^{n-1}} u'(x+r\omega)\cdot\omega d\sigma(\omega)$ so that with $X(y) = \sum_j (\partial_j u)(x+ry)\partial_{y_j}$

$$\varphi'(r)|\mathbb{S}^{n-1}| = \int_{\mathbb{S}^{n-1}} \sum_{1 \leq j \leq n} (\partial_j u)(x + r\omega)\omega_j d\sigma(\omega) = \int_{\mathbb{S}^{n-1}} \langle X, \nu \rangle d\sigma$$

$$(\nu \text{ is the exterior normal to } \mathbb{S}^{n-1}) \qquad = \int_{\mathbb{B}^n} \text{div } X dy = \int_{\mathbb{B}^n} (\Delta u)(x + ry)r dy = 0,$$

and φ is constant so that $\fint_{\partial B(x,r)} u(y)d\sigma(y) = \lim_{r\to 0+} \fint_{\partial B(x,r)} u(y)d\sigma(y) = u(x)$. On the other hand we have

$$\int_{B(x,r)} u(y)dy = \int_0^r \rho^{n-1} \int_{\mathbb{S}^{n-1}} u(x + \rho\omega)d\sigma(\omega)d\rho$$

$$= \int_0^r \rho^{n-1}|\mathbb{S}^{n-1}|d\rho u(x) = |B(x,r)|u(x),$$

concluding the proof.   □

Remark 5.1.4. Note that, defining a *subharmonic* function u as a C^2 function such that $\Delta u \geq 0$, we get, using the same proof, that a subharmonic function u on an open subset Ω of $\mathbb{R}^n$ satisfies

$$\forall x \in \Omega \text{ with } \bar{B}(x,r) \subset \Omega, \quad u(x) \leq \fint_{\partial B(x,r)} u(y) d\sigma(y). \qquad (5.1.2)$$

In fact the function φ above is proven non-decreasing, and thus such that $u(x) = \lim_{r \to 0} \varphi(r) \leq \fint_{\partial B(x,r)} u(y) d\sigma(y)$.

Remark 5.1.5. If $\bar{B}(x,r) \subset \Omega$, we have defined for $u \in C^2(\Omega)$,

$$\varphi(r) = \fint_{\partial B(x,r)} u(y) d\sigma(y)$$

and we have seen that

$$\varphi(r)|\mathbb{S}^{n-1}| = \int_{\mathbb{S}^{n-1}} u(x + \omega r) d\sigma(\omega), \quad \text{so that}$$

$$\varphi'(r)|\mathbb{S}^{n-1}| = \int_{\mathbb{B}^n} (\Delta u)(x + ry) r dy = \int_{B(x,r)} (\Delta u)(z) dz r^{1-n}$$

$$= \fint_{B(x,r)} (\Delta u)(z) dz r^{1-n} \frac{r^n}{n} |\mathbb{S}^{n-1}| \quad \text{and thus}$$

$$\varphi'(r) = \fint_{B(x,r)} (\Delta u)(z) dz \frac{r}{n}. \qquad (5.1.3)$$

Theorem 5.1.6. *Let Ω be an open subset of $\mathbb{R}^n$ and $u \in C^2(\Omega)$. The function u is harmonic in Ω if and only if for all $x \in \Omega$ and all $r > 0$ such that $\bar{B}(x,r) \subset \Omega$, we have*

$$u(x) = \fint_{\partial B(x,r)} u(y) d\sigma(y),$$

that is u satisfies the mean value property.

Proof. We have seen in Proposition 5.1.3 the "only if" part. On the other hand, if u satisfies the mean-value property and $x_0 \in \Omega$ with $(\Delta u)(x_0) > 0$, φ as above, we get

$$0 = \varphi'(r) = \fint_{B(x_0,r)} (\Delta u)(y) d\sigma(y) \frac{r}{n} > 0$$

with $B(x_0, r_0) \subset \Omega$, $r_0 \geq r > 0$, $\Delta u > 0$ continuous on $B(x, r_0)$, which is impossible; the same occurs with a negative sign for Δu.   □

5.1.2 The maximum principle

Theorem 5.1.7. *Let Ω be an open bounded subset of $\mathbb{R}^n$, u an harmonic function on Ω continuous on $\overline{\Omega}$. Then $\max_{\overline{\Omega}} u = \max_{\partial\Omega} u$, and if Ω is connected and $\exists x_0 \in \Omega$ with $u(x_0) = \max_{\overline{\Omega}} u$, then u is constant on Ω.*

Proof. Note that u is continuous on the compact set $\overline{\Omega}$. Let us assume $\exists x_0 \in \Omega$ with $u(x_0) = \max_{\overline{\Omega}} u = M$. Then if $0 < r < d(x_0, \partial\Omega)$,

$$M = u(x_0) = \fint_{\partial B(x_0, r)} u(y) d\sigma(y) \leq M,$$

and this implies that $u = M$ on $B(x_0, r)$, so that the set $\mathcal{A} = \{x \in \Omega, u(x) = M\}$ is closed and open in Ω. If Ω is connected, we get the sought result. In the general case, we get that $\mathcal{A}$ contains the closure of the connected component of x_0. $\square$

Remark 5.1.8. If Ω is a connected open subset of $\mathbb{R}^n$, u is a continuous function on $\overline{\Omega}$, such that

$$\begin{cases} \Delta u = 0 & \text{in } \Omega, \\ u = g & \text{on } \partial\Omega, \end{cases}$$

with $g \geq 0$, then $u(x) > 0$ for all $x \in \Omega$ if there exists $x_0 \in \partial\Omega$ such that $g(x_0) > 0$. In fact, the function g is valued in $[m, M] \subset \mathbb{R}_+$ with $M > 0, m \geq 0$. From the previous result, the function u is also valued in $[m, M]$. If $m > 0$, we are done and if $m = 0$, we define

$$\mathcal{B} = \{x \in \Omega, u(x) = 0\}$$

we get that it is closed and open and cannot be all Ω since u must be positive near the point x_0. As a result, $\mathcal{B} = \emptyset$, proving the result.

Theorem 5.1.9. *Let Ω be an open subset of $\mathbb{R}^n$, f be a continuous function on Ω and g a continuous function on $\partial\Omega$. There exists at most one solution $u \in C(\overline{\Omega}) \cap C^2(\Omega)$ to the Dirichlet problem*

$$\begin{cases} \Delta u = f & \text{in } \Omega, \\ u = g & \text{on } \partial\Omega. \end{cases}$$

Proof. If u_1, u_2 are two solutions, the function $u_1 - u_2$ is harmonic on Ω with boundary value 0 and the maximum principle entails $u_1 - u_2 = 0$. $\square$

Theorem 5.1.10 (Harnack's inequality). *Let $U \Subset \Omega$ be open subsets of $\mathbb{R}^n$, with U connected ($U \Subset \Omega$ means $\overline{U}$ compact $\subset \Omega$). There exists $C > 0$ such that for any u nonnegative harmonic function on Ω,*

$$\sup_U u \leq C \inf_U u. \tag{5.1.4}$$

This implies that for all $x, y \in U$, $C^{-1}u(y) \leq u(x) \leq Cu(y)$.

Proof. Since $\bar{U}$ is a compact subset of Ω, dist $(\bar{U}, \partial\Omega) > 0$, and with $x_1, x_2 \in U, |x_1 - x_2| \leq r$ with $r = \text{dist}(\bar{U}, \partial\Omega)/4$, we have $\bar{B}(x_2, r) \subset \bar{B}(x_1, 2r) \subset \Omega$, and

$$u(x_1) = \fint_{B(x_1, 2r)} u(y) dy \underbrace{\geq}_{u \geq 0} 2^{-n} \fint_{B(x_2, r)} u(y) dy = u(x_2),$$

implying that for $x_1, x_2 \in U, |x_1 - x_2| \leq r$, we have $2^{-n} u(x_2) \leq u(x_1) \leq 2^n u(x_2)$. Using the compactness of $\bar{U}$, and $\bar{U} \subset \cup_{y \in \bar{U}} B(y, r/2)$, we can find a finite number N of balls such that $\bar{U} \subset \cup_{1 \leq j \leq N} B(y_j, r/2)$.

Lemma 5.1.11. *Let U be an open connected subset of $\mathbb{R}^n$ such that $U \subset \cup_{1 \leq j \leq N} B_j$ where the B_j are open balls. Then for $x_0, x_1 \in U$, there exists a continuous curve $\gamma : [0, 1] \to U$ such that $\gamma(0) = x_0, \gamma(1) = x_1$, and there exists $0 \leq T_0 \leq T_1 \leq \cdots \leq T_{\nu-1} \leq 1$ with $\nu \leq N$ and*

$$\gamma([0, T_1)) \subset B_{j_1}, \gamma([T_1, T_2) \subset B_{j_2}, \ldots, \gamma([T_{\nu-1}, 1] \subset B_{j_\nu}.$$

Proof of the lemma. Note first that since U is an open connected subset of $\mathbb{R}^n$, it is also pathwise connected and we can find a continuous curve Γ in U joining x_0 to x_1. If x_0, x_1 belong to the same ball B_j, there is nothing to prove. If $x_0 \in B_{j_1}, x_1 \notin B_{j_1}$, we define

$$T_1 = \sup\{t \in [0, 1], \Gamma(t) \in B_{j_1}\}.$$

We get that $T_1 \in (0, 1)$ and $\Gamma(T_1) \in \partial B_{j_1}$. We define γ on $[0, T_1]$ as the segment $[\gamma(0), \gamma(T_1)]$. We know now that $\Gamma(t) \notin B_{j_1}$ for all $t \geq T_1$. Since $\Gamma(T_1) \in B_{j_2}$ we can now define

$$T_2 = \sup\{t \in [T_1, 1], \Gamma(t) \in B_{j_2}\}.$$

We get that $T_2 \in (T_1, 1]$ and $\Gamma(T_2) \in \partial B_{j_2}$. We define γ on $[T_1, T_2]$ as the segment $[\gamma(T_1), \gamma(T_2)]$. And so on. $\square$

This implies that for $x, y \in U, u(x) \leq 2^{nN} u(y)$ and the result. $\square$

5.1.3   Analyticity of harmonic functions

We have seen in Theorem 3.1.1 that the fundamental solution of the Laplace operator is nonetheless C^∞ outside of the origin, but also analytic outside of the origin. We could use that result to prove directly the analytic-hypoellipticity of the Laplace equation, that is the property Δu analytic on the open set Ω implies u analytic on Ω. However, we have chosen a more direct approach, relying on the maximum principle.

Proposition 5.1.12. *Let Ω be an open set of $\mathbb{R}^n$ and u be an harmonic function on Ω. Then $u \in C^\infty(\Omega)$ and for $x_0 \in \Omega$ with $\bar{B}(x_0, r) \subset \Omega$,*

$$\beta_n |\partial_x^\alpha u(x_0)| \leq \frac{C_k}{r^{n+k}} \|u\|_{L^1(B(x_0, r))}, \quad |\alpha| = k, \quad C_0 = 1, C_k = (2^{n+1} nk)^k \text{ for } k \geq 1,$$

where

$$\beta_n = |\mathbb{S}^{n-1}|/n = \frac{2\pi^{n/2}}{n\Gamma(n/2)} = \frac{\pi^{n/2}}{\Gamma(1 + \frac{n}{2})} \tag{5.1.5}$$

is the volume of the unit ball in $\mathbb{R}^n$.

Proof. We have from the mean-value property if $\bar{B}(x, \rho) \subset \Omega$,

$$|u(x)|\rho^n \beta_n \leq \|u\|_{L^1(B(x,\rho))} \tag{5.1.6}$$

and in particular the estimate is true for $k = 0$. From the mean-value property, we have from the harmonicity of each $\partial_{x_j} u$ that, if $\bar{B}(x, \rho) \subset \Omega$,

$$\partial_{x_j} u(x) = \fint_{B(x,\rho)} \partial_{x_j} u(y) dy = \frac{n}{\rho^n |\mathbb{S}^{n-1}|} \int_{B(x,\rho)} \text{div}\,(u\partial_{x_j}) dy$$
$$= \frac{n}{\rho^n |\mathbb{S}^{n-1}|} \int_{\partial B(x,\rho)} u\nu_j d\sigma,$$

so that

$$|\partial_{x_j} u(x)| \leq \frac{n}{\rho} \|u\|_{L^\infty(\partial B(x,\rho))}. \tag{5.1.7}$$

As a result, we have, using (5.1.6)-(5.1.7) with $\rho = r/2$,

$$\beta_n |\partial_{x_j} u(x_0)| \leq \frac{2n}{r} (r/2)^{-n} \|u\|_{L^1(B(x_0,r))} = \frac{\|u\|_{L^1(B(x_0,r))}}{r^{n+1}} n 2^{n+1}$$

and the property is true for $k = 1$. Let us consider now a multi-index α with $|\alpha| = k \geq 1$ and from the harmonicity of $\partial_x^\alpha u$ and (5.1.7)

$$|\partial_{x_j} \partial_x^\alpha u(x_0)| \leq \frac{n(k+1)}{r} \|\partial_x^\alpha u\|_{L^\infty(\partial B(x_0, r/(k+1)))}$$

so that, inductively,

$$\beta_n |\partial_{x_j} \partial_x^\alpha u(x_0)| \leq \frac{n(k+1)}{r} \frac{\|u\|_{L^1(B(x_0,r))}}{\left(\frac{rk}{k+1}\right)^{n+k}} (n 2^{n+1} k)^k.$$

We check now

$$(n 2^{n+1} k)^k n(k+1)\left(\frac{k+1}{k}\right)^{n+k} = (k+1)^{k+1}(k+1)^n k^{-n} n^{k+1}(2^{n+1})^k$$
$$= \left(n 2^{n+1}(k+1)\right)^{k+1} 2^{-n-1}\left(1 + \frac{1}{k}\right)^n \leq \left(n 2^{n+1}(k+1)\right)^{k+1},$$

completing the proof of the proposition. $\qquad\square$

Theorem 5.1.13 (Liouville theorem)**.** *Let u be a bounded harmonic function on $\mathbb{R}^n$. Then u is a constant.*

Proof. From the previous proposition, we have for all $x \in \mathbb{R}^n, r > 0$

$$|\nabla u(x)| \leq \frac{2^{n+1} n}{\beta_n r^{n+1}} \|u\|_{L^1(B(x,r))} \leq \frac{2^{n+1} n}{r} \|u\|_{L^\infty(\mathbb{R}^n)},$$

implying $\nabla u \equiv 0$ so that u is constant. $\qquad\square$

Corollary 5.1.14. *Let $f \in L^\infty_{comp}(\mathbb{R}^n)$ with $n \geq 3$. The bounded solutions of $\Delta u = f$ on $\mathbb{R}^n$ are $E * f + \text{constant}$, where E is the fundamental solution of the Laplace operator given by Theorem 3.1.1*

Proof. If u is a bounded solution of $\Delta u = f$, the distribution $u - E * f$ makes sense and is harmonic on $\mathbb{R}^n$. It is also bounded since u is bounded and

$$|(E * f)(x)| \leq c_n \int \frac{|f(y)|}{|x - y|^{n-2}} dy \leq c_n \|f\|_{L^\infty(\mathbb{R}^n)} \int_{|y| \leq R_0} |x - y|^{2-n} dy.$$

If $|x| \geq R_0 + 1$, we have

$$\int_{|y| \leq R_0} |x - y|^{2-n} dy \leq \int_{|y| \leq R_0} (|x| - R_0)^{2-n} dy = \frac{R_0^n \beta_n}{(|x| - R_0)^{n-2}} \leq R_0^n \beta_n,$$

and if $|x| \leq R_0 + 1$, we have

$$\int_{|y| \leq R_0} |x - y|^{2-n} dy \leq \int_{|z-x| \leq R_0} |z|^{2-n} dz \leq \int_{|z| \leq R_0 + |x| \leq 2R_0 + 1} |z|^{2-n} dz$$

entailing that $E * f$ is bounded. By Liouville Theorem, $u - E * f$ is constant. Conversely $E * f + C$ is indeed a bounded solution of $\Delta u = f$. $\qquad \square$

Note that in two dimensions, the fundamental solution of the Laplace operator is unbounded; in particular a solution of $\Delta u = f$ with $f \in C_c^\infty(B(0,1)), f \not\equiv 0$ is given by $\frac{1}{2\pi} \int_{|y-x| \leq 1} f(x - y) \ln |y| dy$. Since $|y - x| \geq \big||y| - |x|\big|$, $|y - x| \leq 1$ implies $|x| - 1 \leq |y| \leq |x| + 1$ and if $|x| > 2$,

$$|y - x| \leq 1 \Longrightarrow 0 < \ln(|x| - 1) \leq \ln |y| \leq \ln(1 + |x|)$$

so that if $f \geq 0$, for $|x| > 2$,

$$\int_{|y-x| \leq 1} f(x - y) \ln |y| dy \geq \ln(|x| - 1) \int f(z) dz$$

which is unbounded.

Theorem 5.1.15 (Analytic-hypoellipticity of the Laplace operator). *Let u be an harmonic function on some open subset Ω of $\mathbb{R}^n$. Then u is an analytic function on Ω.*

Proof. Let x_0 be a point of Ω and $r_0 > 0$ with $\bar{B}(x_0, 4r_0) \subset \Omega$. We have proven in Proposition 5.1.12 that u is C^∞ and such that

$$\beta_n \|\partial_x^\alpha u\|_{L^\infty(B(x_0, r_0))} \leq r_0^{-n-k} k^k (2^{n+1} n)^k \|u\|_{L^1(B(x_0, 2r_0))}, \quad |\alpha| = k.$$

Furthermore Stirling's formula (7.3.5) gives for $k \geq k_0$, $k^k \leq k! 2(2\pi k)^{-1/2} e^k$ and since $n^k = \sum_{\alpha \in \mathbb{N}^n, |\alpha| = k} \frac{k!}{\alpha!}$ which implies $k! \leq \alpha! n^k$, we obtain for $k \geq k_0$,

$$\beta_n \|\partial_x^\alpha u\|_{L^\infty(B(x_0, r_0))} \leq r_0^{-n-k} (2^{n+1} n)^k \|u\|_{L^1(B(x_0, 2r_0))} k! 2(2\pi k)^{-1/2} e^k \leq C_0 \rho_0^{-|\alpha|} \alpha!$$

yielding analyticity from Theorem 7.2.4. $\qquad \square$

5.1.4 Green's function

Lemma 5.1.16. *Let Ω be a bounded open set of $\mathbb{R}^n$ with a C^1 boundary, $u \in C^2(\bar{\Omega})$. Then we have for all $x \in \Omega$,*

$$u(x) = \int_\Omega E(x-y)(\Delta u)(y)dy + \int_{\partial\Omega} u(y)\frac{\partial E}{\partial \nu}(y-x)d\sigma(y) - \int_{\partial\Omega} \frac{\partial u}{\partial \nu}(y)E(y-x)d\sigma(y) \tag{5.1.8}$$

where E is the fundamental solution of the Laplace operator (see Theorem 3.1.1). Since $E \in L^1_{loc}(\mathbb{R}^n) \cap C^\infty(\mathbb{R}^n\backslash\{0\})$, the formula above makes sense.

Proof. We consider $u \in C^\infty_c(\mathbb{R}^n)$ and we write $u = u * \delta = u * \Delta E = \Delta u * E$ so that

$$u(x) = \int \Delta u(y)E(x-y)dy = \int_\Omega \Delta u(y)E(y-x)dy + \int_{\Omega^c} \text{div}_y\big(E(y-x)\nabla u(y)\big)dy$$
$$- \int_{\Omega^c} (\nabla E)(y-x) \cdot \nabla u(y)dy,$$

entailing with Green's formula for $x \in \Omega$,

$$u(x) = \int_\Omega \Delta u(y)E(x-y)dy - \int_{\partial\Omega} E(y-x)\frac{\partial u}{\partial \nu}(y)d\sigma(y)$$
$$+ \int_{\partial\Omega} \frac{\partial E}{\partial \nu}(y-x)u(y)d\sigma(y) + \int_{\Omega^c} u(y)\underbrace{(\Delta E)(y-x)}_{=0}dy,$$

which is the result. $\qquad\square$

Remark 5.1.17. Let Ω be a bounded set of $\mathbb{R}^n$ with a C^1 boundary. Assume that for each $x \in \Omega$, we are able to find a function $y \mapsto \phi_x(y)$ such that

$$\begin{cases} \Delta\phi_x = 0 & \text{in } \Omega, \\ \phi_x(y) = -E(x-y) & \text{on } \partial\Omega, \end{cases} \tag{5.1.9}$$

As a consequence, we have $\int_\Omega \phi_x(y)(\Delta u)(y)dy = \int_\Omega \left(\phi_x\frac{\partial u}{\partial \nu} - u\frac{\partial\phi_x}{\partial \nu}\right)d\sigma(y)$. We define then the Green function for the open set as

$$G(x,y) = E(y-x) + \phi_x(y). \tag{5.1.10}$$

Using formula (5.1.8), we get for $x \in \Omega$,

$$u(x) = \int_\Omega E(x-y)(\Delta u)(y)dy + \int_{\partial\Omega} u(y)\frac{\partial E}{\partial \nu}(y-x)d\sigma(y) + \int_{\partial\Omega} \frac{\partial u}{\partial \nu}(y)\phi_x(y)d\sigma(y)$$
$$= \int_\Omega G(x,y)(\Delta u)(y)dy + \int_{\partial\Omega} u(y)\left(\frac{\partial\phi_x}{\partial \nu} + \frac{\partial E}{\partial \nu}(y-x)\right)d\sigma(y)$$
$$= \int_\Omega G(x,y)(\Delta u)(y)dy + \int_{\partial\Omega} u(y)\frac{\partial G}{\partial\nu_y}(x,y)d\sigma(y). \tag{5.1.11}$$

Note that we may symbolically write that for $x \in \Omega$,

$$\begin{cases} (\Delta_y G)(x,y) = \delta(x-y), & y \in \Omega \\ G(x,y) = 0, & y \in \partial\Omega. \end{cases}$$

As a matter of fact, we have

$$\int_\Omega G(x,y)(\Delta u)(y)dy$$

$$= \int_\Omega u(y)(\Delta_y G)(x,y)dy + \int_{\partial\Omega} \left(G(x,y)\frac{\partial u}{\partial \nu}(y) - \frac{\partial G}{\partial \nu_y}(x,y)u(y) \right) d\sigma(y)$$

$$= u(x) - \int_{\partial\Omega} \frac{\partial G}{\partial \nu_y}(x,y)u(y)d\sigma(y),$$

which is (5.1.11).

An immediate consequence of Lemma 5.1.16 and Formula (5.1.10) is the following theorem.

Theorem 5.1.18. *Let Ω be a bounded open set of $\mathbb{R}^n$ with a C^1 boundary. If $u \in C^2(\bar{\Omega})$ is such that*

$$\begin{cases} \Delta u = f & in\ \Omega, \\ u = g & on\ \partial\Omega, \end{cases} \tag{5.1.12}$$

then for all $x \in \Omega$,

$$u(x) = \int_\Omega G(x,y)f(y)dy + \int_{\partial\Omega} \frac{\partial G}{\partial \nu_y}(x,y)g(y)d\sigma(y) \tag{5.1.13}$$

where G is the Green function given by (5.1.10).

Green's function for a half-space

We consider first the following simple problem on $\mathbb{R}^n_+ = \mathbb{R}^{n-1} \times \mathbb{R}^*_+$. Let $g \in \mathscr{S}(\mathbb{R}^{n-1})$: we are looking for u defined on $\mathbb{R}^n_+$ such that

$$\begin{cases} \Delta u = 0 & in\ x_n > 0, \\ u(\cdot, 0) = g & on\ \mathbb{R}^{n-1}. \end{cases}$$

Defining $v(\xi', x_n)$ as the Fourier transform of u with respect to x', we get the ODE

$$\partial^2_{x_n} v(\xi', x_n) - 4\pi^2 |\xi'|^2 v(\xi', x_n) = 0, \quad v(\xi', 0) = \hat{g}(\xi'),$$

that we solve readily, obtaining $v(\xi', x_n) = e^{-2\pi x_n |\xi'|}\hat{g}(\xi')$, so that, at least formally,

$$u(x', x_n) = \iint e^{2i\pi(x'-y')\xi'} e^{-2\pi x_n |\xi'|} g(y')dy'd\xi'.$$

Using Formula (7.1.1), we get for $x_n > 0$,

$$u(x', x_n) = \frac{\Gamma(n/2)}{\pi^{n/2}} \int (1 + |x' - y'|^2 x_n^{-2})^{-n/2} g(y') dy' x_n^{1-n}$$

$$= \frac{2x_n}{|\mathbb{S}^{n-1}|} \int_{\mathbb{R}^{n-1}} \frac{g(y') dy'}{\left(x_n^2 + |x' - y'|^2\right)^{n/2}} = \frac{2x_n}{n\beta_n} \int_{\partial \mathbb{R}^n_+} \frac{g(y) dy}{|x - y|^n}. \quad (5.1.14)$$

We define the *Poisson kernel* for $\mathbb{R}^n_+$ as

$$k(x, y) = \frac{2x_n}{n\beta_n |x - y|^n}, \quad x \in \mathbb{R}^n_+, y \in \partial \mathbb{R}^n_+, \quad (5.1.15)$$

and we note right away that

$$\forall x \in \mathbb{R}^n_+, \quad \int_{\partial \mathbb{R}^n_+} \frac{2x_n}{n\beta_n |x - y|^n} dy = 1 \quad \text{(for a proof see Section 7.3.3).} \quad (5.1.16)$$

Theorem 5.1.19. *Let $g \in C^0(\mathbb{R}^{n-1}) \cap L^\infty(\mathbb{R}^{n-1})$ and u defined on $\mathbb{R}^n_+$ by (5.1.14). Then the function $u \in C^\infty(\mathbb{R}^n_+) \cap L^\infty(\mathbb{R}^n_+)$, is harmonic on $\mathbb{R}^n_+$ and such that for each $x_0 \in \partial \mathbb{R}^n_+$*

$$\lim_{\substack{x \to x_0 \\ x \in \mathbb{R}^n_+}} u(x) = g(x_0). \quad (5.1.17)$$

The function u is thus continuous up to the boundary.

Proof. Formula (5.1.14) is well-defined for $x_n > 0$, $g \in L^\infty(\mathbb{R}^{n-1})$, defines a smooth function which satisfies as well $|u(x)| \leq \|g\|_{L^\infty(\mathbb{R}^{n-1})}$ since the Poisson kernel k given by (5.1.15) is non-negative with integral 1 from (5.1.16). On the other hand, u is harmonic on $\mathbb{R}^n_+$ since with $\rho_{y'}(x', x_n) = (|x' - y'|^2 + x_n^2)^{-n/2}$

$$x_n \Delta(\rho^{-n}) + 2\partial_{x_n}(\rho^{-n})$$
$$= x_n \left[(-n)(-n-1)\rho^{-n-2} + (n-1)\rho^{-1}(-n)\rho^{-n-1} \right] + 2(-n)\rho^{-n-1} x_n \rho^{-1}$$
$$= x_n \rho^{-n-2} \left(n(n+1) - n(n-1) - 2n \right) = 0.$$

We now consider for $x_n > 0$, $u(x', x_n) - g(x') = \frac{2x_n}{n\beta_n} \int_{\mathbb{R}^{n-1}} \frac{(g(y') - g(x'))dy'}{(x_n^2 + |x' - y'|^2)^{n/2}}$ and we obtain with $r > 0$

$$n\beta_n |u(x', x_n) - g(x')| \leq 2x_n \sup_{y' \in B(x', r)} |g(y') - g(x')| \int_{B(x', r)} \frac{dy'}{\left(x_n^2 + |x' - y'|^2\right)^{n/2}}$$

$$+ 4x_n \|g\|_{L^\infty(\mathbb{R}^{n-1})} \int_{|x' - y'| \geq r} \frac{dy'}{(x_n^2 + |x' - y'|^2)^{n/2}}$$

so that from (5.1.16) and $k \geq 0$,

$$|u(x', x_n) - g(x')| \leq \sup_{y' \in B(x', r)} |g(y') - g(x')| + \frac{4x_n \|g\|_{L^\infty(\mathbb{R}^{n-1})}}{n\beta_n} \int_r^{+\infty} \rho^{n-2-n} d\rho |\mathbb{S}^{n-2}|$$

$$\leq \sup_{y' \in B(x', r)} |g(y') - g(x')| + \frac{4x_n \|g\|_{L^\infty(\mathbb{R}^{n-1})} |\mathbb{S}^{n-2}|}{n\beta_n r}.$$

As a result, we get

$$\limsup_{x_n \to 0_+} |u(x', x_n) - g(x')| \le \inf_{r>0} \Big(\sup_{y' \in B(x',r)} |g(y') - g(x')| \Big) = 0.$$

Since

$$|u(x', x_n) - g(z')| \le |u(x', x_n) - g(x')| + |g(x') - g(z')|$$

$$\le \sup_{y' \in B(x',r)} |g(y') - g(x')| + \frac{4x_n \|g\|_{L^\infty(\mathbb{R}^{n-1})} |\mathbb{S}^{n-2}|}{n\beta_n r} + |g(x') - g(z')|.$$

we get

$$\limsup_{\substack{(x', x_n) \to (z', 0) \\ x_n > 0}} |u(x', x_n) - g(z')| \le \inf_{r>0} \Big(\limsup_{x' \to z'} \Big(\sup_{y' \in B(x',r)} |g(y') - g(x')| \Big) \Big) = 0$$

from the continuity of g.   □

Proposition 5.1.20. *The Green function for the half-space* $\{x \in \mathbb{R}^n, x_n > 0\}$ *is*

$$G(x, y) = E(y - x) - E(y - \check{x}), \tag{5.1.18}$$

where E is the fundamental solution of the Laplace operator given by (3.1.4) *and for* $x = (x', x_n) \in \mathbb{R}^{n-1} \times \mathbb{R}$, *we have defined* $\check{x} = (x', -x_n)$.

Proof. According to (5.1.10), we have to verify for $x \in \mathbb{R}^n_+$ that $\phi_x(y) = -E(y - \check{x})$ does satisfy $(\Delta \phi_x)(y) = 0$ in Ω and $\phi_x(y) = -E(x - y)$ for $y \in \partial\mathbb{R}^n_+$. Both points are obvious. Note also that Formula (5.1.13) gives for u satisfying (5.1.12) with $f = 0$ and $x \in \mathbb{R}^n_+$,

$$u(x) = \int_{\mathbb{R}^{n-1}} \left(-\frac{\partial}{\partial y_n} \right) \Big(E(y - x) - E(y - \check{x}) \Big) g(y) dy$$

which gives for $n \ge 3$, $u(x) =$

$$\frac{1}{(2-n)n\beta_n} \int g(y) \Big(|y - x|^{1-n} (2-n) \frac{x_n - y_n}{|y - x|} + |y - \check{x}|^{1-n} (2-n) \frac{x_n + y_n}{|y - x|} \Big) dy$$

so that we recover Formula (5.1.14) for the Poisson kernel of the half-space.   □

Green's function for a ball

We want now to solve

$$\begin{cases} \Delta u = 0 & \text{in } |x| < 1, \\ u = g & \text{on } \mathbb{S}^{n-1}. \end{cases}$$

The Green function for the ball is $G(x, y) = E(y - x) + \phi_x(y)$ and with $\tilde{x} = x/|x|^2$ we define

$$\phi_x(y) = -E\big((y - \tilde{x})|x|\big).$$

We note that for $|x| < 1$, the function $y \mapsto \phi_x(y)$ is harmonic and for $y \in \mathbb{S}^{n-1}$, we have

$$\phi_x(y) = -\frac{|y - \tilde{x}|^{2-n}|x|^{2-n}}{(2-n)n\beta_n} = -E(y - x)$$

since $|x|^2|y - \tilde{x}|^2 = |x|^2 - 2y \cdot x + 1 = |x - y|^2$. We calculate now for $|y| = 1$

$$(2-n)n\beta_n\frac{\partial G}{\partial \nu_y}(x, y) = |y - x|^{-n}(2-n)(y - x) \cdot y - |y - \tilde{x}|^{-n}(2-n)|x|^{2-n}(y - \tilde{x}) \cdot y,$$

so that

$$\frac{\partial G}{\partial \nu_y}(x, y) = \frac{|y - x|^{-n}}{n\beta_n}\left(1 - x \cdot y - |x|^2(1 - \tilde{x} \cdot y)\right) = \frac{1 - |x|^2}{n\beta_n|y - x|^n}.$$

As a result the Poisson kernel for the ball $B_R = B(0, R)$ is

$$k(x, y) = \frac{R^2 - |x|^2}{n\beta_n R|x - y|^n}.$$

Theorem 5.1.21. *Let $g \in C^0(\partial B_R)$ and u defined on B_R by*

$$u(x) = \frac{R^2 - |x|^2}{n\beta_n R}\int_{\partial B_R}\frac{g(y)}{|x - y|^n}d\sigma(y) \tag{5.1.19}$$

Then the function $u \in C^\infty(B_R)$, is harmonic on B_R and such that for each $x_0 \in \partial B_R$ $\lim_{\substack{x \to x_0 \\ x \in B_R}} u(x) = g(x_0)$. The function u is thus continuous up to the boundary.

Proof. The proof is similar to that for Theorem 5.1.19. We may also compare this formula to (1.1.7) in the introduction: we have

$$u(z, \bar{z}) = c_0 + \sum_{k \geq 1}(c_k z^k + c_{-k}\bar{z}^k), \quad g(e^{i\theta}) = \sum_{k \in \mathbb{Z}} c_k e^{ik\theta},$$

so that $u(z, \bar{z}) = \frac{1}{2\pi}\int_0^{2\pi} g(e^{i\theta})d\theta + \sum_{k \geq 1}\frac{1}{2\pi}\int_0^{2\pi}(z^k e^{-ik\theta} + \bar{z}^k e^{ik\theta})d\theta$. Since we have also for $|\zeta| = 1 > |z|$,

$$1 + \sum_{k \geq 1}(z\bar{\zeta})^k + (\bar{z}\zeta)^k = 1 + \frac{z\bar{\zeta}}{1 - z\bar{\zeta}} + \frac{\bar{z}\zeta}{1 - \bar{z}\zeta} = 1 + 2\operatorname{Re}\frac{z\zeta^{-1}}{1 - z\zeta^{-1}}$$

$$= \operatorname{Re}\left(1 + \frac{2z}{\zeta - z}\right) = \operatorname{Re}\left(\frac{\zeta + z}{\zeta - z}\right) = \frac{\operatorname{Re}(\zeta + z)(\bar{\zeta} - \bar{z})}{|\zeta - z|^2} = \frac{1 - |z|^2}{|\zeta - z|^2},$$

we obtain

$$u(z, \bar{z}) = \frac{1 - |z|^2}{2\pi}\int_0^{2\pi}\frac{g(e^{i\theta})}{|z - e^{i\theta}|^2}d\theta = \frac{1 - |z|^2}{2\beta_1}\int_{\mathbb{S}^1}\frac{g(y)}{|z - y|^2}d\sigma(y),$$

which is indeed the $2D$ case of (5.1.19). $\qquad\square$

Chapter 6

Hyperbolic Equations

6.1 Energy identities for the wave equation

6.1.1 A basic identity

In Section 3.4, we have found the fundamental solution of the wave equation and provided explicit formulas when the space dimension is less than 3. Here, we want to consider Ω a bounded open subset of $\mathbb{R}^d$, with a smooth boundary. For $T > 0$, we define the cylinder $\Omega_T = (0, T] \times \Omega$ and noting that

$$\overline{\Omega}_T = [0, T] \times \overline{\Omega} = \big((0, T] \times \Omega\big) \cup \big(\{0\} \times \Omega\big) \cup \big([0, T] \times \partial\Omega\big)$$

we see that

$$\Gamma_T = \overline{\Omega}_T \backslash \Omega_T = \big(\{0\} \times \Omega\big) \cup \big([0, T] \times \partial\Omega\big).$$

With $c > 0$, we define the wave operator with speed c by the formula (3.4.1)

$$\square_c = c^{-2}\partial_t^2 - \Delta_x.$$

We consider the problem

$$\begin{cases} \square_c u = f, & \text{on } \Omega_T = (0, T] \times \Omega, \\ u = g, & \text{on } \Gamma_T = \big(\{0\} \times \Omega\big) \cup \big([0, T] \times \partial\Omega\big), \\ \partial_t u = h, & \text{on } \{0\} \times \Omega, \end{cases} \qquad (6.1.1)$$

and we want to start by proving that if a C^2 solution exists, it is unique. We calculate

$$\begin{aligned} \langle \square_c u, \partial_t u \rangle_{L^2(\Omega)} &= \frac{1}{2}\frac{d}{dt}c^{-2}\|\partial_t u\|_{L^2(\Omega)}^2 - \langle \Delta u, \partial_t u \rangle_{L^2(\Omega)} \\ &= \frac{1}{2}\frac{d}{dt}c^{-2}\|\partial_t u\|_{L^2(\Omega)}^2 - \int_{\partial\Omega} \partial_t u \frac{\partial u}{\partial \nu}d\sigma + \langle \partial_t(\nabla u), \nabla u \rangle_{L^2(\Omega)}^2 \\ &= \frac{1}{2}\frac{d}{dt}\left(c^{-2}\|\partial_t u\|_{L^2(\Omega)}^2 + \|\nabla u\|_{L^2(\Omega)}^2\right) - \int_{\partial\Omega} \partial_t u \frac{\partial u}{\partial \nu}d\sigma. \end{aligned}$$

Defining the energy of u on Ω at time t as

$$E(t) = \frac{1}{2}\left(c^{-2}\|\partial_t u\|_{L^2(\Omega)}^2 + \|\nabla u\|_{L^2(\Omega)}^2\right), \qquad (6.1.2)$$

we see that

$$\dot{E}(t) = \langle \Box_c u, \partial_t u \rangle_{L^2(\Omega)} + \int_{\partial \Omega} \partial_t u \frac{\partial u}{\partial \nu} d\sigma. \tag{6.1.3}$$

As a first result, if u_1, u_2 are two solutions of (6.1.1), the function $u = u_2 - u_1$ satisfies (6.1.1) with f, g, h all 0 and consequently $\dot{E} = 0$ for that u. Since $E(0) = 0$ as well, we get that $E(t) = 0$ for all times, and u is 0.

6.1.2   Domain of dependence for the wave equation

We consider now a C^2 solution u of the wave equation on $\mathbb{R}_+ \times \mathbb{R}^d$, $c^{-2}\partial_t^2 u - \Delta_x u = 0$, and we introduce for $t_0 \geq t \geq 0$ the following energy

$$F(t) = \frac{1}{2} \int_{B(x_0, c(t_0 - t))} \left(c^{-2}|\partial_t u|^2 + |\nabla u|^2 \right) dx$$

and we calculate its derivative, using the identity of the previous subsection,

$$\dot{F}(t) = \int_{|x - x_0| = c(t_0 - t)} \partial_t u \frac{\partial u}{\partial \nu} d\sigma - \frac{1}{2} \int_{|x - x_0| = c(t_0 - t)} \left(c^{-1}|\partial_t u|^2 + c|\nabla u|^2 \right) d\sigma.$$

We note that $2\partial_t u \frac{\partial u}{\partial \nu} \leq c^{-1}|\partial_t u|^2 + c|\nabla u|^2$ so that $\dot{F} \leq 0$. As a result, for $0 \leq t \leq t_0$, we have

$$\int_{B(x_0, c(t_0 - t))} \left(c^{-2}|\partial_t u|^2 + |\nabla u|^2 \right) dx \leq \int_{B(x_0, ct_0)} \left(c^{-2}|\partial_t u|^2 + |\nabla u|^2 \right) dx. \tag{6.1.4}$$

In particular, if u and $\partial_t u$ both vanish at time 0 on the ball $B(x_0, ct_0)$ then u vanishes on the cone

$$C_{t_0, x_0} = \{ (t, x) \in [0, t_0] \times \mathbb{R}^d, |x - x_0| \leq c(t_0 - t) \}.$$

Rephrasing that, we can say that, if both $u(t = 0)$ and $\partial_t u(t = 0)$ are supported in $\bar{B}(X_0, R_0)$, then for $t_0 \geq 0$,

$$\operatorname{supp} u(t_0, \cdot) \subset \bar{B}(X_0, R_0 + ct_0). \tag{6.1.5}$$

In fact, if $|x_0 - X_0| > R_0 + ct_0$, we have $C_{t_0, x_0} \cap \{t = 0\} = B(x_0, ct_0)$ and

$$\bar{B}(x_0, ct_0) \subset \bar{B}(X_0, R_0)^c \quad \text{since } |y - x_0| \leq ct_0 \implies |y - X_0| \geq |x_0 - X_0| - |y - x_0| > R_0.$$

As a consequence, both $u(t = 0)$ and $\partial_t u(t = 0)$ vanish on $B(x_0, ct_0)$ so that $u(t_0, x_0) = 0$ and the result (6.1.5). In other words, the value $u(T, X)$ for some positive T depends only on the values of $u(t = 0), \partial_t u(t = 0)$ on the *backward light-cone* with vertex (T, X) intersected with $t = 0$, i.e. $C_{T,X} \cap \{t = 0\} = B(X, cT)$. The cone $C_{T,X}$ is the cone of dependence at (T, X). If both $u(t = 0), \partial_t u(t = 0)$ are supported in the ball $\bar{B}(X, R)$, then

$$\operatorname{supp} u(T, \cdot) \subset \bar{B}(X, R + cT).$$

These properties bear the name of *finite propagation speed*.

Chapter 7

Appendix

7.1 Fourier transform

Lemma 7.1.1. *Let $n \in \mathbb{N}^*$ and $\mathbb{R}^n \ni x \mapsto u(x) = \exp -2\pi|x|$, where $|x|$ stands for the Euclidean norm of x. The function u belongs to $L^1(\mathbb{R}^n)$ and its Fourier transform is*

$$\hat{u}(\xi) = \pi^{-(\frac{n+1}{2})}\Gamma\big(\frac{n+1}{2}\big)\big(1+|\xi|^2\big)^{-(\frac{n+1}{2})}. \tag{7.1.1}$$

Proof. We note first that in one dimension

$$\int_{\mathbb{R}} e^{-2i\pi x\xi}e^{-2\pi|x|}dx = 2\operatorname{Re}\int_0^{+\infty} e^{-2\pi x(1+i\xi)}dx = \frac{1}{\pi(1+\xi^2)}$$

corroborating the above formula in $1D$. We want to take advantage of this to write $e^{-2\pi|x|}$ as a superposition of Gaussian functions; doing this will be very helpful since it is easy to calculate the Fourier transform of Gaussian functions (this quite natural idea seems to be used only in the wonderful textbook by Robert Strichartz [22] and we follow his method). For $t \in \mathbb{R}_+$, we have

$$e^{-2\pi t} = \int_{\mathbb{R}} e^{2i\pi t\tau}\frac{d\tau}{\pi(1+\tau^2)} = \iint_{\mathbb{R}^2} e^{2i\pi t\tau}e^{-s\pi(1+\tau^2)}H(s)dsd\tau = \int_{\mathbb{R}_+} e^{-\pi s}s^{-1/2}e^{-\frac{\pi}{s}t^2}ds$$

so that for $x \in \mathbb{R}^n$, $e^{-2\pi|x|} = \int_{\mathbb{R}_+} e^{-\pi s}s^{-1/2}e^{-\frac{\pi}{s}|x|^2}ds$ and thus

$$\hat{u}(\xi) = \iint_{\mathbb{R}^n \times \mathbb{R}_+} e^{-2i\pi x\xi}e^{-\pi s}s^{-1/2}e^{-\frac{\pi}{s}|x|^2}dxds = \int_{\mathbb{R}_+} e^{-\pi s}s^{-1/2}e^{-\pi s|\xi|^2}s^{n/2}ds$$

so that $\hat{u}(\xi) = \int_0^{+\infty} e^{-s}s^{(n-1)/2}\big(\pi(1+|\xi|^2)\big)^{-(n+1)/2}ds$, which is the sought result. $\square$

7.2 Spaces of functions

7.2.1 On the Faà de Bruno formula

The following useful formula is known as Faà de Bruno's[1] , dealing with the iterated chain rule. We write here all the coefficients explicitly.

---

[1] One could find a version of theorem 7.2.1 on pages 69-70 of the thesis of "Chevalier François FAÀ DE BRUNO, Capitaine honoraire d'État-Major dans l'armée Sarde". This thesis was

Theorem 7.2.1. *Let $k \geq 1$ be an integer and U, V, W open sets in Banach spaces. Let a and b be k times differentiable fonctions $b : U \longrightarrow V$ and $a : V \longrightarrow W$. Then the k- multilinear symmetric mapping $(a \circ b)^{(k)}$ is given by ($\mathbb{N}^* = \mathbb{N}\backslash\{0\}$)*

$$\frac{(a \circ b)^{(k)}}{k!} = \sum_{\substack{1 \leq j \leq k \\ (k_1,\ldots,k_j) \in \mathbb{N}^{*j} \\ k_1+\cdots+k_j=k}} \frac{a^{(j)} \circ b}{j!} \quad \frac{b^{(k_1)}}{k_1!} \cdots \frac{b^{(k_j)}}{k_j!}. \tag{7.2.1}$$

Remark 7.2.2. One can note that a symmetric $k-$multilinear mapping L is determined by its value on "diagonal" k-vectors $(T,\ldots,T)$, so that formula (7.2.1) means that $(a \circ b)^{(k)}$ is the only symmetric k-multilinear mapping such that, if T is a tangent vector to U, and x a point in U

$$\frac{1}{k!}(a \circ b)^{(k)}(x) \overbrace{(T,\ldots,T)}^{k \text{ times}} = \sum_{\substack{1 \leq j \leq k \\ (k_1,\ldots,k_j) \in \mathbb{N}^{*j} \\ k_1+\cdots+k_j=k}} \frac{a^{(j)}[b(x)]}{j!} \quad \left(\frac{b^{(k_1)}(x)T^{k_1}}{k_1!},\ldots,\frac{b^{(k_j)}(x)T^{k_j}}{k_j!}\right),$$

where $b^{(l)}(x)T^l$ stands for the tangent vector to V given by $b^{(l)}(x) \overbrace{(T,\ldots,T)}^{l \text{ times}}$. Since $a^{(j)}[b(x)]$ is a j-multilinear mapping from the product of j copies of the tangent space to V into the tangent space to W, the formula makes sense with both sides tangent vectors to W. Note also that the sum in (7.2.1) is extended to all the multi-indices $(k_1,\ldots,k_j) \in \mathbb{N}^{*j}$ such that $k_1 + \cdots + k_j = k$.

Proof. Let's now prove the theorem. Using the same notations as in the remark above, we see, that for $t \in \mathbb{R}$, $x \in U$ and h a tangent vector to U,

$$c^{(k)}(x)h^k = \left(\frac{d}{dt}\right)^k c(x+th)_{|t=0} \quad ,$$

so that it is enough to prove the theorem for U neighborhood of 0, $U \subset \mathbb{R}$ and $b(0) = 0$. Moreover, one can assume by regularization that $b \in C_c^\infty(\mathbb{R})$. Taylor-Young's formula gives then with a continuous ε with $\varepsilon(0) = 0$

$$(a \circ b)(t) = \sum_{0 \leq j \leq k} \frac{a^{(j)}(0)}{j!} \, b(t)^j + t^k \varepsilon(t) \tag{7.2.2}$$

and thus

$$[\partial_t^k (a \circ b)](0) = \sum_{0 \leq j \leq k} \frac{a^{(j)}(0)}{j!} \, \partial_t^k [b(t)^j]_{|t=0}.$$

Since for tensor products we have (the inverse Fourier formula comes from the usual one for $\langle \xi, b(t) \rangle$, where ξ is a linear form)

$$b(t)^j = \int_{\mathbb{R}^j} e^{2i\pi \; t(\tau_1+\cdots+\tau_j)} \; \hat{b}(\tau_1)\ldots\hat{b}(\tau_j)d\tau_1\ldots d\tau_j,$$

---

defended in 1856, in the Faculté des Sciences de Paris in front of the following jury : Cauchy (chair) , Lamé and Delaunay.

we obtain

$$\partial_t^k [b(t)^j]_{|t=0} = \int_{\mathbb{R}^j} (2i\pi)^k (\tau_1 + \cdots + \tau_j)^k \, \hat{b}(\tau_1) \ldots \hat{b}(\tau_j) d\tau_1 \ldots d\tau_j$$

$$= \sum_{\substack{(k_1,\ldots,k_j)\in\mathbb{N}^j \\ k_1+\cdots+k_j=k}} \frac{k!}{k_1! \ldots k_j!} \, b^{(k_1)}(0) \ldots b^{(k_j)}(0) \quad,$$

which gives the result of the theorem, since $b(0) = 0$ so that all the $k_1, \ldots, k_j$ above should be larger than 1. $\qquad\square$

It is now easy to derive the following

Corollary 7.2.3. *Let a and b be functions satisfying the assumptions of Theorem 7.2.1 so that $U \subset \mathbb{R}_x^m$, $V \subset \mathbb{R}_y^n$, $W \subset \mathbb{R}$ and α is a multi-index $\in \mathbb{N}^m$. Then (using the standard notation for a multi-index $\beta \in \mathbb{N}^n$, $a^{(\beta)} = \partial_y^\beta a$ and if $\gamma \in \mathbb{N}^l$, $\gamma! = \gamma_1! \ldots \gamma_l!$) we get for $|\alpha| \geq 1$,*

$$\frac{\partial_x^\alpha (a \circ b)}{\alpha!} = \sum_{\substack{1\leq j\leq|\alpha| \\ (\alpha^{(1)},\ldots,\alpha^{(j)})\in\mathbb{N}^m\times\cdots\times\mathbb{N}^m=\mathbb{N}^{mj} \\ \alpha^{(1)}+\cdots+\alpha^{(j)}=\alpha \\ \min_{1\leq r\leq j}|\alpha^{(j)}|\geq 1}} \frac{a^{(j)} \circ b}{j!} \, \frac{b^{(\alpha^{(1)})} \ldots b^{(\alpha^{(j)})}}{\alpha^{(1)}! \ldots \alpha^{(j)}!} \qquad (7.2.3)$$

We can remark that, although corollary 7.2.3 follows actually from Theorem 7.2.1, it is easier to prove it directly, along the lines of the proof above using the Fourier inversion formula to compute $\partial_x^\alpha [b(x)]^j$.

7.2.2   Analytic functions

Let Ω be an open subset of $\mathbb{R}^n$ and $f : \Omega \to \mathbb{R}$ be a C^∞ function. The function f is said to be analytic[2] on Ω if for all $x_0 \in \Omega$, there exists $R_0 > 0$ such that

$$\forall x \in B(x_0, R_0), \quad f(x) = \sum_{k\geq 0} \frac{f^{(k)}(x_0)}{k!}(x - x_0)^k. \qquad (7.2.4)$$

Note that when $n > 1$, $f^{(k)}(x_0)$ is a k-th multilinear symmetric form and that[3]

$$\frac{f^{(k)}(x)}{k!}h^k = \sum_{|\alpha|=k} \frac{(\partial_x^\alpha f)(x)}{\alpha!}h^\alpha, \quad \alpha! = \alpha_1! \ldots \alpha_n!, \quad h^\alpha = h^{\alpha_1} \ldots h^{\alpha_n}, \qquad (7.2.5)$$

so that formula (7.2.4) can be written

$$f(x) = \sum_{\alpha\in\mathbb{N}^n} \frac{f^{(\alpha)}(x_0)}{\alpha!}(x - x_0)^\alpha. \qquad (7.2.6)$$

---

[2]Real-analytic would be more appropriate.

[3]The summation is taking place on multi-indices $\alpha \in \mathbb{N}^n$ such that $|\alpha| = \sum_{1\leq j\leq n} \alpha_j = k$.

There are plenty of examples of C^∞ functions which are not analytic such as (3.1.1-2) in [15]. One should also keep in mind that the convergence of the Taylor series $\sum_{k\geq 0} \frac{f^{(k)}(0)}{k!} h^k$ is not enough to ensure analyticity as shown by the example on the real line

$$f(t) = e^{-1/t^2} \text{ for } t \neq 0, \ f(0) = 0,$$

which is easily seen to be C^∞ and is *flat* at the origin, i.e. for all $k \in \mathbb{N}$, $f^{(k)}(0) = 0$. That function is not analytic near 0 (otherwise it would be 0 near 0, which is not the case), but the Taylor series at 0 does converge.

On the other hand, there is no difficulty to extend Formula (7.2.4) to a ball with same radius in $\mathbb{C}^n$. In particular the restriction to $\mathbb{R}^n$ of an entire function (holomorphic function on the whole $\mathbb{C}^n$) is indeed analytic. However, all analytic functions on $\mathbb{R}^n$ are not restrictions of entire function: an example is given by $\mathbb{R} \ni t \mapsto 1/(1 + t^2)$ which is analytic on $\mathbb{R}$ but is not the restriction of an entire function to $\mathbb{R}$ (exercise: if it were the restriction of an entire function, that function would coincide with $1/(1 + z^2)$ which has poles at $\pm i$). This type of example is a good reason to use the terminology real-analytic for analytic functions on an open subset of $\mathbb{R}^n$.

Going back to (7.2.4), we define the k-th multi-linear symmetric form $a_k = \frac{f^{(k)}(x_0)}{k!}$ and

$$\frac{1}{R} = \limsup_k \|a_k\|^{1/k}, \quad \text{with} \quad \|a_k\| = \sup_{|T|=1} |a_k T^k|.$$

Assuming $R > 0$, we have for $|h| \leq R_2 < R_1 < R$, provided that for $k > k_0$, $\|a_k\|^{1/k} \leq 1/R_1$,

$$\sum_{k\geq 0} \sup_{|h|\leq R_2} |a_k h^k| \leq \sup_{0\leq k\leq k_0} \|a_k\| \sum_{0\leq k\leq k_0} R_2^k + \sum_{k_0<k} \sup_{|h|\leq R_2} (\|a_k\|^{1/k}|h|)^k$$

$$\leq \sup_{0\leq k\leq k_0} \|a_k\| \sum_{0\leq k\leq k_0} R_2^k + \sum_{k_0<k} (R_1^{-1}R_2)^k < +\infty,$$

so that the series $\sum_{k\geq 0} a_k h^k$ converges normally on each compact subset of $B(0, R)$. As a consequence the convergence is uniform on each compact subset of $B(0, R)$ and the series can be differentiated termwise.

If $n = 1$ and $|h| \geq R_2 > R_1 > R$, we have, extracting a subsequence, $|a_{k_j}|^{1/k_j} R_1 \geq R_1/R_2$, $\forall j \geq j_0$. As a result, for $j \geq j_0$,

$$|a_{k_j} h^{k_j}| = \big(|a_{k_j}|^{1/k_j} R_1 |h| R_2^{-1}\big)^{k_j} (R_2/R_1)^{k_j}$$

$$\geq \big(\frac{R_1}{R_2}|h|R_2^{-1}\big)^{k_j} (R_2/R_1)^{k_j} = (|h|/R_2)^{k_j} \geq 1$$

and the series $\sum a_k h^k$ cannot converge. This proves also in n dimensions that, if

$$|h| > \frac{1}{\sup_{|T|=1}\big(\limsup_k |a_k T^k|^{1/k}\big)}$$

the series $\sum a_k h^k$ cannot converge everywhere. Note that

$$1/\tilde{R} = \sup_{|T|=1} \big(\limsup_k |a_k T^k|^{1/k}\big) \leq \limsup_k \|a_k\|^{1/k} = 1/R.$$

and the series $\sum a_k h^k$ does not converge on the whole $|h| > \tilde{R}$ and converges when $|h| < R$ (note that we have indeed $R \leq \tilde{R}$ and $R = \tilde{R}$ in one dimension).

The following example is a good illustration of what may happen with the domain of convergence of multiple power series: we consider

$$\sum_{k \geq 0} x_1^k x_2^k, \quad \text{which is convergent on } |x_1 x_2| \leq 1.$$

We have $\|a_{2k}\| = \sup_{x_1^2 + x_2^2 = 1} |x_1^k x_2^k| = \sup_{\theta \in \mathbb{R}} |\cos\theta \sin\theta|^k = 2^{-k}$, so that $R = \sqrt{2}$, which is indeed the largest circle to fit between the hyperbolas $x_1 x_2 = 1$. On the other hand, with $T_0 = (\cos\theta_0, \sin\theta_0)$,

$$|a_{2k} T_0^{2k}|^{1/2k} = (\cos\theta_0 \sin\theta_0)^{1/2} = 2^{-1/2}\sqrt{\sin(2\theta_0)} \implies \tilde{R} = \sqrt{2}.$$

The radius $\sqrt{2}$ is indeed the largest possible ball in which convergence takes place. However the region of convergence is unbounded. The picture below may help the reader to understand the various regions. [4]
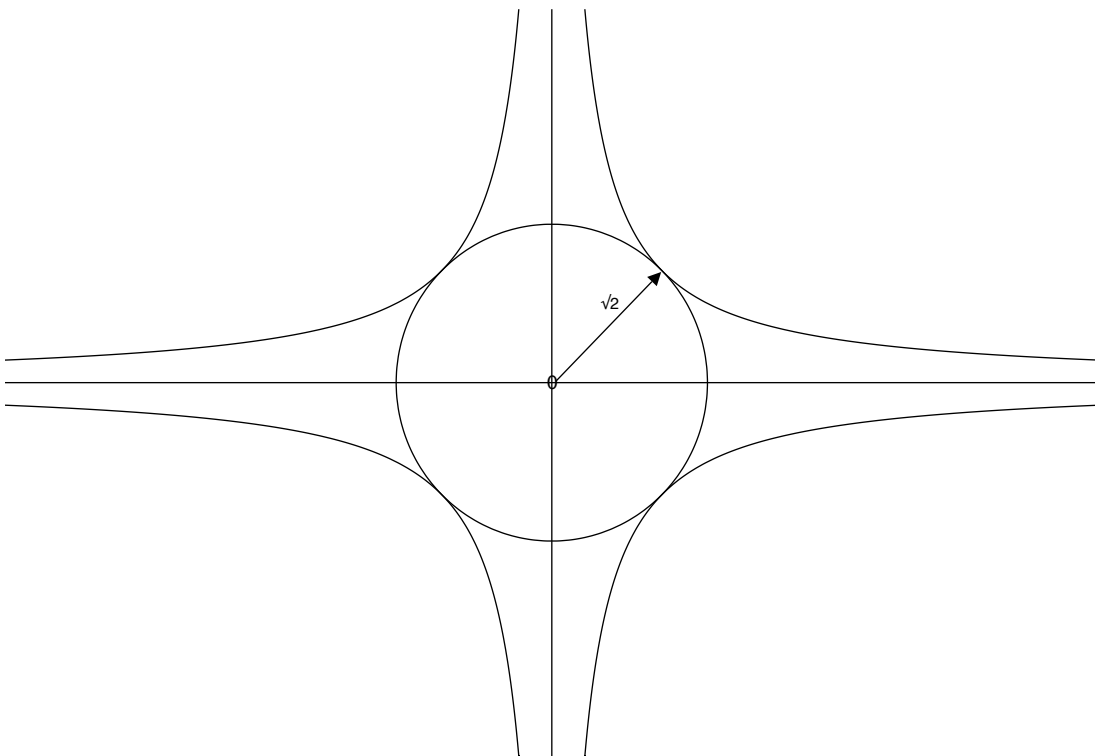
We have the following characterization of analytic functions.

---

[4]For multiple power series, it would be more natural, but also more complicated, to introduce the notion of polydisc: a polydisc with center ζ and (positive) radii $r_1, \ldots, r_n$ in $\mathbb{C}^n$ is the set

$$D(\zeta, r_1, \ldots, r_n) = \{z \in \mathbb{C}^n, \forall j, |z_j - \zeta_j| < r_j\}.$$

The interior of the set where absolute convergence of a multiple power series takes place is called the domain of convergence $\mathcal{D}$. With $r = (r_1, \ldots, r_n)$, the polydisc $D(\zeta, r)$ is called the polydisc of convergence at $\zeta \in \mathbb{C}^n$ if $D(\zeta, r) \subset \mathcal{D}$ and $D(\zeta, \rho) \not\subset \mathcal{D}$ if $\max(\rho_j - r_j) > 0$. We have then the

Theorem 7.2.4. *Let Ω be an open subset of $\mathbb{R}^n$ and $f \in C^\infty(\Omega; \mathbb{R})$. The function f is analytic on Ω if and only for all compact subsets K of Ω, there exists $C \geq 0, \rho > 0$ such that*

$$\forall \alpha \in \mathbb{N}^n, \quad \|\partial_x^\alpha f\|_{L^\infty(K)} \leq C \rho^{-|\alpha|} \alpha!$$

We leave the proof as an exercise for the reader.

Remark 7.2.5. *A consequence of Corollary 7.2.3 and is that the composition of analytic functions is analytic and that the power series coefficients of $a \circ b$ are universal polynomials with positive rational coefficients of the power series coefficients of a, b: in fact we have the explicit formula*

$$\frac{\partial_x^\alpha(a \circ b)}{\alpha!} = \sum_{\substack{1 \leq j \leq |\alpha| \\ (\alpha^{(1)},\dots,\alpha^{(j)}) \in \mathbb{N}^m \times \cdots \times \mathbb{N}^m = \mathbb{N}^{mj} \\ \alpha^{(1)} + \cdots + \alpha^{(j)} = \alpha \\ \min_{1 \leq r \leq j} |\alpha^{(j)}| \geq 1}} \frac{a^{(j)} \circ b}{j!} \frac{b^{(\alpha^{(1)})} \dots b^{(\alpha^{(j)})}}{\alpha^{(1)}! \dots \alpha^{(j)}!}$$

Definition 7.2.6. *Let $A = \sum_{\alpha \in \mathbb{N}^n} a_\alpha x^\alpha$ be a power series with non-negative coefficients and $B = \sum_{\alpha \in \mathbb{N}^n} b_\alpha x^\alpha$ be a power series with complex coefficients. The power series A is said to majorize B if for all $\alpha \in \mathbb{N}^n$, $|b_\alpha| \leq a_\alpha$. We shall write $B \ll A$. In particular, if A converges absolutely, then B converges absolutely.*

To provide simple examples, we start noting that, for $R > 0$, the function $\mathbb{R}^d \ni x \mapsto \frac{1}{R - \sum_{1 \leq j \leq d} x_j}$ is analytic on $\{x, \sum_{1 \leq j \leq d} |x_j| < R\}$ since it is equal to

$$R^{-1} \sum_{k \geq 0} \left(R^{-1} \sum_{1 \leq j \leq d} x_j\right)^k = \sum_\alpha R^{-1-|\alpha|} \frac{|\alpha|!}{\alpha!} x^\alpha,$$

and with $|x|_1 = \sum_{1 \leq j \leq n} |x_j|$, we have from the multinomial formula[5]

$$\sum_\alpha R^{-1-|\alpha|} \frac{|\alpha|!}{\alpha!} |x^\alpha| = R^{-1} \sum_{k \geq 0} \left(R^{-1} \sum_{1 \leq j \leq d} |x_j|\right)^k = R^{-1} \frac{1}{1 - (|x|_1/R)} = \frac{1}{R - |x|_1}.$$

We remark now that if the power series $C = \sum_\alpha c_\alpha x^\alpha$ converges on

$$|x|_\infty = \max_{1 \leq j \leq n} |x_j| \leq R \quad \text{for some positive } R,$$

then $c_\alpha(R, \dots, R)^\alpha = c_\alpha R^{|\alpha|}$ must be bounded, i.e. $|c_\alpha| \leq MR^{-|\alpha|} \leq MR^{-|\alpha|} \frac{|\alpha|!}{\alpha!}$, so that

$$C \ll \frac{MR}{R - \sum_{1 \leq j \leq n} x_j}. \tag{7.2.7}$$

---

relation

$$\limsup_{|k| \to +\infty} \left(|a_k r_1^{k_1} \dots r_n^{k_n}|\right)^{1/|k|} = 1,$$

analogous to the Cauchy-Hadamard relation for the radius of convergence in one dimension.

[5]The multinomial formula is $(t_1 + \cdots + t_n)^k = \sum_{\alpha \in \mathbb{N}^n} \frac{k!}{\alpha!} t^\alpha$.

7.3 Some computations

7.3.1 On multi-indices

Let $n \in \mathbb{N}^*, m \in \mathbb{N}$. We have

$$\text{Card}\underbrace{\{(\alpha_1,\ldots,\alpha_n) \in \mathbb{N}^n, \alpha_1 + \cdots + \alpha_n = m\}}_{E_{n,m}} = C_{m+n-1}^{n-1} = \frac{(m+n-1)!}{(n-1)!m!}. \quad (7.3.1)$$

In fact, defining

$$\beta_1 = \alpha_1 + 1, \ \beta_2 = \alpha_1 + \alpha_2 + 2, \ldots, \beta_{n-1} = \alpha_1 + \cdots + \alpha_{n-1} + n - 1,$$

we have $1 \leq \beta_1 < \beta_2 < \cdots < \beta_{n-1} \leq m + n - 1$, and we find a bijection between the set $E_{n,m}$ and the set of strictly increasing mappings from $\{1,\ldots,n-1\}$ to $\{1,\ldots,m+n-1\}$; the latter set has cardinality C_{m+n-1}^{n-1} since it amounts to choosing a subset with $n-1$ elements among a set of $m+n-1$ elements. On the other hand, defining $q_{n,m} = \text{Card}\, E_{n,m}$, we have obviously

$$q_{n+1,m} = \sum_{j=0}^{m} q_{n,j}$$

and we can check that $C_{m+n}^n = \sum_{j=0}^m C_{n+j-1}^{n-1}$: it is true for $m = 0$ and if verified for $m \geq 0$, we get indeed

$$C_{m+1+n}^n = C_{m+n}^n + C_{m+n}^{n-1} = \sum_{j=0}^{m} C_{n+j-1}^{n-1} + C_{m+n}^{n-1} = \sum_{j=0}^{m+1} C_{n+j-1}^{n-1}.$$

We have also obviously from the above discussion

$$\text{Card}\underbrace{\{(\alpha_1,\ldots,\alpha_n) \in \mathbb{N}^n, \alpha_1 + \cdots + \alpha_n \leq m\}}_{F_{n,m}} = q_{n+1,m} = C_{n+m}^n, \quad (7.3.2)$$

as well as

$$\text{Card}\{(\alpha_1,\ldots,\alpha_{n+1}) \in \mathbb{N}^{n+1}, \alpha_1+\cdots+\alpha_{n+1} \leq m, \alpha_{n+1} < m\} = C_{n+m+1}^{n+1}-1. \quad (7.3.3)$$

7.3.2 Stirling's formula

Let $k \in \mathbb{N}$. We have

$$k! = \left(\frac{k}{e}\right)^k \sqrt{2\pi k}\left(1 + \frac{1}{12k} + O(k^{-2})\right), \quad k \to +\infty. \quad (7.3.4)$$

and in particular

$$k! \sim \left(\frac{k}{e}\right)^k \sqrt{2\pi k} \quad k \to +\infty. \quad (7.3.5)$$

7.3.3   On the Poisson kernel for a half-space

We consider for $x_n > 0, x' \in \mathbb{R}^{n-1}, n \geq 1, n\beta_n = \frac{2\pi^{n/2}}{\Gamma(n/2)} = |\mathbb{S}^{n-1}|,$

$$
I = \frac{2x_n}{n\beta_n} \int_{\mathbb{R}^{n-1}} \frac{dy'}{\left(|x'-y'|^2 + x_n^2\right)^{n/2}} = \frac{x_n\Gamma(n/2)}{\pi^{n/2}} \int_{\mathbb{R}^{n-1}} \frac{dy'}{(x_n^2 + |y'|^2)^{n/2}}
$$

$$
= \frac{x_n\Gamma(n/2)}{\pi^{n/2}} \int_0^{+\infty} \frac{\rho^{n-2}d\rho}{(x_n^2 + \rho^2)^{n/2}} |\mathbb{S}^{n-2}| = \frac{\Gamma(n/2)}{\pi^{n/2}} \frac{2\pi^{(n-1)/2}}{\Gamma((n-1)/2)} \int_0^{+\infty} \frac{\rho^{n-2}}{(1+\rho^2)^{n/2}} d\rho.
$$

We have

$$
\int_0^{+\infty} \frac{\rho^{n-2}}{(1+\rho^2)^{n/2}} d\rho = \int_0^{\pi/2} (\tan\theta)^{n-2}(\cos\theta)^{n-2} d\theta = \int_0^{\pi/2} (\sin\theta)^{n-2} d\theta = W_{n-2},
$$

the so-called Wallis integrals. It is easy and classical to get for $k \in \mathbb{N}$,

$$
W_{2k} = \frac{\pi(2k)!}{(k!)^2 2^{2k+1}}, \quad W_{2k+1} = \frac{(k!)^2 2^{2k}}{(2k+1)!}. \tag{7.3.6}
$$

As a result, for $n = 2 + 2k$,

$$
\frac{\Gamma(n/2)}{\pi^{n/2}} \frac{2\pi^{(n-1)/2}}{\Gamma((n-1)/2)} \int_0^{+\infty} \frac{\rho^{n-2}}{(1+\rho^2)^{n/2}} d\rho = \frac{k!}{\pi^{1/2}} \frac{2}{\Gamma(k+\frac{1}{2})} \frac{\pi(2k)!}{(k!)^2 2^{2k+1}} = \frac{\pi^{\frac{1}{2}}(2k)!}{2^{2k}k!\Gamma(k+\frac{1}{2})}
$$

and for $n = 3 + 2k$,

$$
\frac{\Gamma(n/2)}{\pi^{n/2}} \frac{2\pi^{(n-1)/2}}{\Gamma((n-1)/2)} \int_0^{+\infty} \frac{\rho^{n-2}}{(1+\rho^2)^{n/2}} d\rho
$$

$$
= \frac{(k+\frac{1}{2})\Gamma(k+\frac{1}{2})}{\pi^{1/2}} \frac{2}{k!} \frac{(k!)^2 2^{2k}}{(2k+1)!} = \frac{2^{2k}k!\Gamma(k+\frac{1}{2})}{\pi^{\frac{1}{2}}(2k)!}.
$$

on the other hand we have

$$
\Gamma(k+\frac{1}{2}) = (k-\frac{1}{2})(k-\frac{3}{2})\ldots(k-\frac{(2k-1)}{2})\Gamma(1/2) = \pi^{1/2}\frac{(2k)!}{2^{2k}k!},
$$

entailing $\frac{\pi^{\frac{1}{2}}(2k)!}{2^{2k}k!\Gamma(k+\frac{1}{2})} = 1 = \frac{2^{2k}k!\Gamma(k+\frac{1}{2})}{\pi^{\frac{1}{2}}(2k)!}$. We have thus proven

$$
\forall (x', x_n) \in \mathbb{R}^{n-1} \times \mathbb{R}_+^*, \quad \frac{2x_n}{n\beta_n} \int_{\mathbb{R}^{n-1}} \frac{dy'}{\left(|x'-y'|^2 + x_n^2\right)^{n/2}} = 1. \tag{7.3.7}
$$

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