Integrating the Wigner Distribution on subsets of the phase space

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Moreover, $\mathcal{W}(u,v)$ belongs to $\mathscr{S}(\mathbb{R}^{2n})$ when u, v belong to $\mathscr{S}(\mathbb{R}^n)$.

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so that, with $z = 2x = y$, we get the Reconstruction Formula,

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There are many other properties of the Wigner distribution and it turns out that most of these properties are closely linked to Weyl quantization, named after the German mathematician HERMANN WEYL (1885–1955).

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instead of the standard

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Here is a sufficient condition for $L^2(\mathbb{R}^n)$ boundedness of $a^{\mathrm{W}\scriptsize{\mathrm{EYL}}}$: Let a be a tempered distribution on $\mathbb{R}^{2n}$. Then we have

$$\|a^{\mathrm{Weyl}}\|_{\mathcal{B}(L^2(\mathbb{R}^n))} \le \min\bigl(2^n\|a\|_{L^1(\mathbb{R}^{2n})}, \|\hat{a}\|_{L^1(\mathbb{R}^{2n})}\bigr). \tag{3}$$

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proving the estimates of (3). As a consequence, we obtain that

$$\left(a^{\mathrm{WEYL}}\right)^* = (\overline{a})^{\mathrm{WEYL}}, \quad \text{so that for } a \text{ real-valued, } (a^{\mathrm{WEYL}})^* = a^{\mathrm{WEYL}}.$$

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Integrals of the Wigner distribution

• Let E be a measurable subset with finite Lebesgue measure of the phase space $\mathbb{R}^n \times \mathbb{R}^n$ and let $\mathbf{1}_E$ be the indicator function of the set E. Then the operator with Weyl symbol $\mathbf{1}_E$ is bounded (see (3)) self-adjoint (see (4)) on $L^2(\mathbb{R}^n)$ and for any $u \in L^2(\mathbb{R}^n)$, we have

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$$\mathbf{1}_E{}^{\textrm{Weyl}} \leq 1 \quad \text{and in fact} \quad \lambda_+(E) = \sup\{\texttt{spectrum }\mathbf{1}_E^{\textrm{Weyl}}\} = 1.$$

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The results for the disk in two dimensions are readily extendable to polydisks by tensorisation.

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for any $u \in L^2(\mathbb{R}^n)$ and we have

$$\boxed{\sup\big(\texttt{spectrum } \mathbf{1}_{D_a}^{\mathrm{Weyl}}\big) = \lambda_+(D_a) = 1 - \frac{\Gamma(n,a)}{\Gamma(n)} = 1 - \frac{1}{\Gamma(n)}\int_a^{+\infty} e^{-t}t^{n-1}dt}$$

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We have with ψ_k standing for the Hermite function at level k in one dimension

$$\mathcal{W}(\psi_k,\psi_k)(x,\xi) = (-1)^k 2e^{-2\pi(x^2+\xi^2)}L_k\big(4\pi(x^2+\xi^2)\big), \quad L_k \text{ is the Laguerre polynomial.}$$

The Laguerre polynomials $\{L_k\}_{k\in\mathbb{N}}$ are defined by

$$L_k(x) = e^x \frac{1}{k!}\left(\frac{d}{dx}\right)^k\{x^k e^{-x}\} = \left(\frac{d}{dx} - 1\right)^k\{\frac{x^k}{k!}\}.$$

A result due to E. Feldheim in 1940 states that

$$\forall \mathbf{k} \in \mathbb{N}, \forall \mathbf{x} \geq \mathbf{0}, \quad \sum_{\mathbf{0}\leq \mathbf{l}\leq \mathbf{k}} (-\mathbf{1})^{\mathbf{l}}\mathbf{L}_{\mathbf{l}}(\mathbf{x}) \geq \mathbf{0}.$$

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On page 2178 of his 1988 article, *Maximum signal energy concentration in a time-frequency domain* (Proc. IEEE), P. Flandrin writes *"it is conjectured that the inequality*

$$\lambda_+(C) \le 1 \quad \textit{is true for any convex domain } C\text{''}, \tag{9}$$

a quite mild commitment for the validity of (9), although that statement was referred to later on as *Flandrin's conjecture* in the literature.

If Flandrin's conjecture were true, we would have for C convex subset of $\mathbb{R}^{2n}$ (not necessarily bounded or with finite measure) and for all $u \in \mathscr{S}(\mathbb{R}^n)$,

$$\iint_C \mathcal{W}(u,u)(x,\xi)dxd\xi \le \|u\|^2_{L^2(\mathbb{R}^n)}. \tag{10}$$

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and assuming Flandrin's conjecture, we get $\iint_C W(u,u)(x,\xi)dxd\xi \le \|u\|^2_{L^2(\mathbb{R}^n)}$.

If Flandrin's conjecture were true, we would have for C convex subset of $\mathbb{R}^{2n}$ (not necessarily bounded or with finite measure) and for all $u \in \mathscr{S}(\mathbb{R}^n)$,

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Indeed, when C is convex with infinite Lebesgue measure, we find that for $u \in \mathscr{S}(\mathbb{R}^n)$ (implying $\mathcal{W}(u,u) \in \mathscr{S}(\mathbb{R}^{2n})$), thanks to the Lebesgue Dominated Convergence Theorem, we have

$$\iint_C W(u,u)(x,\xi)dxd\xi = \lim_{\lambda\to+\infty} \iint_{C\cap\{(x,\xi),\max(|x|,|\xi|)\le\lambda\}} W(u,u)(x,\xi)dxd\xi,$$

↑
convex

and assuming Flandrin's conjecture, we get $\iint_C W(u,u)(x,\xi)dxd\xi \le \|u\|^2_{L^2(\mathbb{R}^n)}$. Conversely, you may also apply (10) to C convex bounded and recover Flandrin's conjecture for C via the boundedness of $\mathbf{1}_C^{\mathrm{WEYL}}$.

The quarter-plane

We choose now to focus our attention on a simple-looking case, when C is the "quarter-plane" $C_0 = \{(x, \xi) \in \mathbb{R}^2,\ x \geq 0, \xi \geq 0\}$.

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Theorem

Let A_0 be the operator with Weyl symbol $H(x)H(\xi)$, where H is the Heaviside function. Then A_0 is a bounded self-adjoint operator on $L^2(\mathbb{R})$ such that

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This theorem was proven in the paper entitled *On integrals over a convex set of the Wigner distribution*, by B. Delourme, T. Duyckaerts and N.L., published by the `Journal of Fourier Analysis and Applications`, volume 26, February 2020.

Corollary (A counterexample to Flandrin's conjecture)

There exists a function $\phi_0 \in \mathscr{S}(\mathbb{R})$, *with* $L^2(\mathbb{R})$ *norm equal to 1 such that*

$$\iint_{x\geq 0, \xi \geq 0} \mathcal{W}(\phi_0, \phi_0)(x, \xi) dx d\xi > 1. \tag{12}$$

There exists $a > 0$ *such that* $\iint_{0\leq x\leq a, 0\leq \xi \leq a} \mathcal{W}(\phi_0, \phi_0)(x, \xi) dx d\xi > 1.$

Corollary (A counterexample to Flandrin's conjecture)

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There exists $a > 0$ such that $\iint_{0 \leq x \leq a, 0 \leq \xi \leq a} \mathcal{W}(\phi_0, \phi_0)(x, \xi) dx d\xi > 1$.

As a consequence, there exists $a > 0$ such that

$$\lambda_+\big([0, a] \times [0, a]\big) > 1,$$

invalidating Flandrin's conjecture.

3. More results and comments

Rethinking the whole business

We have seen a simple example where E was a half-space

$$E = \{(x, \xi) \in \mathbb{R}^n \times \mathbb{R}^n, L(x, \xi) \geq \alpha\}, \quad \text{where } L \text{ is a linear form, } \alpha \in \mathbb{R},$$

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The simplicity of that first case is misleading

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In some sense, although we have the trivial identity for functions

$$\mathbf{1}_E(x,\xi)^2 = \mathbf{1}_E(x,\xi),$$

we shall see that the quantization process by the Weyl formula is destroying that property. The Wigner distribution is not a probability density: although it takes real values and its integral is 1 (for a normalized L^2 function), it can take negative values so that

$$\iint \mathbf{1}_E(x,\xi)\mathcal{W}(u,u)(x,\xi)dxd\xi \quad \text{does not necessarily belongs to } [0,1].$$

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Understand integrals of the Wigner distribution on subsets of the phase space forces us to consider the Weyl quantization of the function $\mathbf{1}_E(x,\xi)$. The Heisenberg Uncertainty Principle shows that non-commutation properties are governing operators whose symbols actually depend on conjugate variables (say x_1,ξ_1) and these properties are of course distorting the classical identities satisfied by classical Hamiltonians.

We must point out as well that we do not have here at our disposal a semi-classical version of our quantization which could ensure some bridge between classical properties and operator-theoretic results as it is the case for the quantization of nice smooth semi-classical symbols depending on a small parameter h.

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In particular for a symbol a such that $a(x, \xi, h) = a_1(x, h\xi)$, $a_1 \in C_b^\infty(\mathbb{R}^{2n})$, we have the following result: if for all $(x, \xi, h) \in \mathbb{R}^n \times \mathbb{R}^n \times (0, 1]$ we have $a(x, \xi, h) \le 1$, then there exists a semi-norm C of the symbol a such that

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$$\mathrm{Id} - a^{\mathrm{WEYL}} + Ch^2 \ge 0 \qquad \text{i.e.} \quad a^{\mathrm{WEYL}} \le \mathrm{Id} + Ch^2, \tag{13}$$

an inequality following from the Fefferman-Phong Inequality which implies as well the following lemma.

Lemma

Let a be a semi-classical symbol of order 0, e.g. $a(x,\xi,h) = a_1(x, h\xi)$, $a_1 \in C_b^\infty(\mathbb{R}^{2n})$, *such that for all* $(x,\xi,h) \in \mathbb{R}^n \times \mathbb{R}^n \times (0,1]$ *we have*

$$0 \le a(x,\xi,h) \le 1.$$

Then there exists a semi-norm C of the symbol a such that

$$-Ch^2 \le a^{\mathrm{WEYL}} \le \mathrm{Id} + Ch^2.$$

Managing the quarter-plane. We want an explicit spectral decomposition for the operator

$$A_0 = \big(H(x)H(\xi)\big)^{\mathrm{Weyl}}.$$

The kernel of A_0 is

$$k_0(x,y) = H(x+y)\hat{H}(y-x) = H(x+y)\frac{1}{2}\Big(\delta_0(y-x) + \frac{1}{i\pi}\mathrm{pv}\frac{1}{y-x}\Big).$$

First tool: use logarithmic coordinates on each half-line: The mapping Ψ defined by

$$\begin{array}{rccc} \Psi: & L^2(\mathbb{R}) & \longrightarrow & L^2(\mathbb{R};\mathbb{C}^2) \\ & u & \mapsto & \Big(\phi_1(t) = u(e^t)e^{t/2}, \phi_2(t) = u(-e^t)e^{t/2}\Big) \end{array}$$

is an isometric isomorphism of Hilbert spaces.

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so that the new kernel for the operator HKH in logarithmic coordinates is $\tilde{k}(s, t)$.

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&=\frac{1}{2}\delta_0(t-s)+\frac{1}{2i\pi}\mathrm{pv}\frac{1}{2\sinh(\frac{t-s}{2})} \quad \text{(pretty explicit stuff)}
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Next step: study the explicit 2×2 matrix multiplier. We work indeed on $L^2(\mathbb{R}_t; \mathbb{C}^2)$ one space dimension (the t variable) but acting on vectors of $\mathbb{C}^2$. We have

$$\mathcal{N}(\tau) = \begin{pmatrix} a_{11}(\tau) & a_{12}(\tau) \\ \overline{a_{12}(\tau)} & 0 \end{pmatrix}.$$

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$$g_0(t) = H(t) \operatorname{sech} t$$

has a singularity at $t = 0$ so that its Fourier transform $a_{12}(\tau)$ cannot go to 0 rapidly when $\tau \to +\infty$. On the other hand $1 - a_{11}(\tau)$ belongs to the Schwartz space and decays rapidly when $\tau \to +\infty$.

Final comments and questions

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- Hermite functions and Laguerre polynomials for ellipses,
- Airy functions for parabolas,
- Homogeneous distributions for hyperbolas and so on.

Final comments and questions

The study of $\mathbf{1}_E^{\mathrm{WEYL}}$ for a subset E of the phase space is highly correlated to some particular set of special functions related to E:

- Hermite functions and Laguerre polynomials for ellipses,
- Airy functions for parabolas,
- Homogeneous distributions for hyperbolas and so on.

It is quite likely that the "shape" of E will determine the type of special functions to be studied to getting a diagonalization of the operator $\mathbf{1}_E^{\mathrm{WEYL}}$.

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• For a general convex polygon P_N with N vertices, it is possible to prove that

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where σ_N does not depend on the area of the polygon but only on N. Is it true that $\sup_N \sigma_N < +\infty$?

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where σ_N does not depend on the area of the polygon but only on N. Is it true that $\sup_N \sigma_N < +\infty$?

• Does there exist $\sigma > 1$ such that for all convex compact subsets K of the plane $\mathbf{1}_K^{\textrm{WEYL}} \le \sigma$?

- In $2n$ dimensions, defining

$$E(a_1, \dots, a_n) = \{(x, \xi) \in \mathbb{R}^n \times \mathbb{R}^n, 2\pi \sum_{1 \le j \le n} \frac{x_j^2 + \xi_j^2}{a_j} \le 1\}.$$

Find

$$\sup \text{SPECTRUM}(\mathbf{1}_{E(a_1,\dots,a_n)})^{\text{WEYL}}.$$

Done by E. LIEB & Y. OSTROVER for $a_1 = \dots = a_n$, but in $2n$ dimensions with $n \ge 2$, ellipsoids are not symplectically equivalent to the Euclidean ball.

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Best wishes to Jorge