Coherent States Methods for Hypoellipticity

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$X_0, X_1, [X_0, X_1]$ is $\frac{\partial}{\partial t} - x\frac{\partial}{\partial y}, \frac{\partial}{\partial x}, \frac{\partial}{\partial y}$ generates the tangent space and $\mathcal{K}$ is hypoelliptic:

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$$\mathcal{K}u \in C^\infty \Longrightarrow u \in C^\infty$$

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$$t = s,\ x = x_1,\ y = x_2 - sx_1$$

$$\frac{\partial u}{\partial s} = \frac{\partial u}{\partial t} - x\frac{\partial u}{\partial y}, \quad \frac{\partial u}{\partial x_1} + s\frac{\partial u}{\partial x_2} = \frac{\partial u}{\partial x}$$

$$\mathcal{K} = \partial_s - (\partial_{x_1} + s\partial_{x_2})^2$$

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Fourier in the x_1, x_2 variables: we get an ODE

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and, solving that ODE, we get

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Subelliptic operators

pseudodifferential operators with complex symbols $p_1 + ip_2$ satisfying some bracket condition and a geometric condition.

$p_1 = p_2 = 0 \Longrightarrow \{p_1, p_2\} > 0$: subelliptic estimate, prototype creation operator

$$D_x + ix = \frac{1}{i}\left(\frac{d}{dx} - x\right), \quad \|hD_x u + ixu\|_{L^2} \geq h^{1/2}\|u\|_{L^2}$$

$p_1 = p_2 = 0 \Longrightarrow \{p_1, p_2\} < 0$: quasi-mode, prototype annihilation operator

$$D_x - ix = \frac{1}{i}\left(\frac{d}{dx} + x\right), \quad \left(h\frac{d}{dx} + x\right)e^{-x^2/2h} = 0$$

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. . . . More brackets . . . .

Condition (P) is a geometrical condition

$\operatorname{Im} p$ *does not change sign along the oriented flow of* $H_{\operatorname{Re} p}$

Finite type assumption: a finite Poisson bracket is alive,

$$\sum_{0 \leq j \leq k} |H^{2j}_{\operatorname{Re} p}(\operatorname{Im} p)| > 0$$

then subellipticity with loss of $\frac{2k}{2k+1}$ derivatives, i.e.

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Condition $\overline{(\psi)}$ is a more general geometric condition

$$p_2 = \operatorname{Im} p \text{ does not change sign from } + \text{ to } -$$

$$\text{along the oriented flow of } H_{\operatorname{Re} p = p_1}$$

$$\text{finite type : with } I = (i_1, \dots, i_{k+1}), \quad i_l \in \{1, 2\},$$

$$p_I = H_{p_{i_1}} \dots H_{p_{i_k}}(p_{i_{k+1}}), \qquad \sum_{|I| \leq k+1} |p_I| > 0$$

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Example: $hD_t + it^2(hD_x + tx^2)$ symbol $\underbrace{\tau}_{p_1} + i\,\underbrace{t^2(\xi + tx^2)}_{p_2}$

$$p_{112} = 2\xi + 6tx^2, \quad p_{1112} = 6x^2$$

$$H^2_{p_{112}}(p_{1112}) = 24 > 0, \quad \text{length: } 3 + 3 + 4 = 10$$

$$\|(hD_t + it^2(hD_x + tx^2)u\| \gtrsim h^{\frac{9}{10}}\|u\|$$

Even that simple-looking example is pretty awkward.

What . . . if all brackets vanish ?. . .
Do not expect subellipticity but

It is possible to prove semi-classical estimates with loss of one derivative under condition (P) : this gives solvability of the adjoint

It is not possible to prove semi-classical estimates with loss of one derivative under condition $(\overline{\psi})$

It is possible to prove semi-classical estimates with loss of 3/2 derivatives under condition $(\overline{\psi})$: this gives solvability of the adjoint

Condition $(\overline{\psi})$ is necessary for any estimate to hold: when violated, existence of a quasi-mode.

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2. Subellipticity result

Theorem

Let $q(t, x, \xi)$ be a nonnegative smooth function defined on $\mathbb{R}_t \times \mathbb{R}^{2n}_{x,\xi}$, bounded as well as all its derivatives and such that

$$\partial_t^{2k} q \geq c_0 > 0. \tag{1}$$

Then

$$\| h\partial_t u + q(t, x, hD_x)u \|_{L^2} \geq h^{\frac{2k}{2k+1}} \| u \|_{L^2}$$

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Another method has to be found: do not expand the square, find a proper multiplier.

Strategy for the proof

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(3) Estimate the remainders in the approximation scheme.

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$$(Wu)(y,\eta) = \int u(x) 2^{n/4} e^{-\pi(x-y)^2} e^{-2i\pi(x-\frac{y}{2})\eta} dx.$$

$\|Wu\|_{L^2(\mathbb{R}^{2n})} = \|u\|_{L^2(\mathbb{R}^n)}$: W is isometric as the partial Fourier transform of $(x,y) \mapsto u(x)2^{n/4}e^{-\pi(x-y)^2}$ whose $L^2(\mathbb{R}^{2n}_{x,y})$ norm is $\|u\|_{L^2(\mathbb{R}^n)}$.

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$$a(x,\xi) \geq 0 \Longrightarrow a^{Wick} \geq 0$$

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Elements of Wick calculus

$$\pi_{\mathcal{H}} a \overbrace{\pi_{\mathcal{H}}\ \pi_{\mathcal{H}}}^{\pi_{\mathcal{H}}} b \pi_{\mathcal{H}} = \pi_{\mathcal{H}} c \pi_{\mathcal{H}} \quad ????$$

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For $a, b \in L^\infty(\mathbb{R}^{2n})$ real-valued with $a'' \in L^\infty(\mathbb{R}^{2n})$, we have

$$a^{Wick} b^{Wick} = \left(ab - \frac{1}{4\pi} \nabla a \cdot \nabla b + \frac{1}{4i\pi} \{a, b\} \right)^{Wick} + R,$$

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We shall use as a definition $\nabla a \cdot \nabla b = \nabla \cdot (\underbrace{b}_{L^\infty} \underbrace{\nabla a}_{\text{Lip.}}) - \underbrace{b}_{L^\infty} \underbrace{\Delta a}_{L^\infty}$.

Commutation properties. We are reduced to investigate the operator $\quad h\partial_t + \pi_{\mathcal{H}} q(t, h^{1/2} X)\pi_{\mathcal{H}} \quad$ acting on $L^2(\mathbb{R}_t; \mathcal{H})$.

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$$\int h|(\nabla_X q)(t, h^{1/2}X)|^2 |\Phi(t,X)|^2 dX$$

$$\leq 2\|q''_{XX}\|_{L^\infty(\mathbb{R}^{2n})} \int hq(t, h^{1/2}X)|\Phi(t,X)|^2 dX$$

$$\leq Ch\|L\Phi\|\|\Phi\|$$

where the last inequality follows from

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$$\leq Ch\|L\Phi\|\|\Phi\|$$

where the last inequality follows from

$$\langle h\partial_t \Phi + \pi_{\mathcal{H}} q(t, h^{1/2}X)\pi_{\mathcal{H}}\Phi, \Phi\rangle = \int q(t, h^{1/2}X)|\Phi|^2 dX$$

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Handling $L_{ODE} = \mathcal{L} = D_t + iQ(t)$, assuming say $Q \geq 0$ and

$$|\{t, Q(t) \leq \delta^k\}| \leq C\delta$$

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- Some simpler multiplier methods are available in the case $k = 1$.

Details can be found in
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Thank you for your attention