Composition of Toeplitz pseudodifferential operators with rough symbols

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1. Introduction

Otto Toeplitz (1881–1940)

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The Toeplitz operator with symbol a is the operator $\mathbf{P}_+ a \mathbf{P}_+$. In particular for $u(x) = \sum_{k\in\mathbb{Z}} \hat{u}(k) e^{2i\pi kx}$, $\mathbf{P}_+ u = \sum_{k\in\mathbb{N}} \hat{u}(k) e^{2i\pi kx}$,

$$\big(\mathbf{P}_+ a \mathbf{P}_+ u\big)(x) = \sum_{j\in\mathbb{N}} e^{2i\pi jx} (\hat{a} * \widehat{\mathbf{P}_+ u})(j).$$

$\mathbf{P}_+ a \mathbf{P}_+$ is the operator

$$\left(\hat{u}(k)\right)_{k\in\mathbb{N}} \mapsto \left(\sum_{k\in\mathbb{N}} \hat{a}(j-k)\hat{u}(k)\right)_{j\in\mathbb{N}},$$

that is the operator from $\ell^2(\mathbb{N})$ into itself given by the infinite matrix

$$(m_{j,k})_{j,k\in\mathbb{N}} = \left(\hat{a}(j-k)\right)_{j,k\in\mathbb{N}}.$$

These matrices are constant on the parallels to the diagonal. Composition of this type of matrices is an interesting question:

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$$\widehat{ab}(j-k) = (\hat{a}*\hat{b})(j-k) = \sum_{l\in\mathbb{Z}} \hat{a}(j-l)\hat{b}(l-k) = \sum_{p\in\mathbb{Z}} \hat{a}(j-k-p)\hat{b}(p).$$

and there is no obvious reason for which

$$m_{j,k} = \sum_{l\in\mathbb{N}} \hat{a}(j-l)\hat{b}(l-k) = \sum_{p\geq -k} \hat{a}(j-k-p)\hat{b}(p),$$

should depend only on $j-k$.

Toeplitz matrices have been extensively studied, and the book of M. Embree & L. Trefethen, *Spectra and Pseudospectra: The Behavior of Nonnormal Matrices and Operators* (Princeton University Press, 2005), is providing many results on the spectrum (and pseudospectrum. . . to be defined later. . .) for this type of matrices.

A variation

Let us consider the Fock-Bargmann space

$$\mathbb{H} = \{u \in L^2(\mathbb{R}^{2n}_{y,\eta}) \text{ such that } u = f(z)e^{-\frac{\pi}{2}|z|^2},\ z = \eta + iy\ ,\ f \text{ entire.}\}$$

We note that $\mathbb{H} = \ker(\frac{\partial}{\partial \bar{z}} + \frac{\pi}{2} z) \cap L^2(\mathbb{C}^n)$: $\mathbb{H}$ is a closed subspace of $L^2(\mathbb{C}^n)$.

Now, let a be a bounded measurable function defined on $\mathbb{C}^n$: the operator of multiplication by a is of course bounded on $L^2(\mathbb{C}^n)$ and the Toeplitz operator with symbol a is

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There are plenty of good reasons to study these operators. Let us give a few.

REASON 1: They are directly linked to the so-called coherent states method, introduced by V. Bargmann, F.A. Berezin, V. Fock, G.-C. Wick and others, and used by manifold authors.

Maybe instead of displaying now what is the coherent states method, let us check one of its main consequences, the sharp Gårding inequality.

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Maybe instead of displaying now what is the coherent states method, let us check one of its main consequences, the sharp Gårding inequality.

We consider a Hamiltonian, i.e. a function $a(x, \xi)$ defined on the phase space $\mathbb{R}^n_x \times \mathbb{R}^n_\xi$. Let us assume that a is real-valued, smooth, bounded, with all its derivatives bounded.

The first thing that you can do is to quantize that Hamiltonian, i.e. to associate linearly to a a bounded operator on $L^2(\mathbb{R}^n)$.

Using for instance, H. Weyl formula, we define with $h \in (0, 1]$,

$$(a^{Weyl_h} u)(x) = \iint_{\mathbb{R}^n \times \mathbb{R}^n} e^{2i\pi\langle x-y,\xi\rangle} a(\frac{x+y}{2}, h\xi) u(y) dy d\xi.$$

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It is easy to prove that, with the strong assumptions made on a, the operator a^{Weyl_h} is bounded selfadjoint on $L^2(\mathbb{R}^n)$.

Let us assume now that a is a non-negative function: then $\exists C > 0, \forall h \in (0,1]$,

$$a^{Weyl_h} + Ch \geq 0 \quad \text{as an operator.}$$

This is true also for matrix-valued Hamiltonian and even for $a(x,\xi) \in \mathcal{B}_{selfadjoint}(H)$, H Hilbert space; then non-negativity of $a(x,\xi)$ means non-negativity of the selfadjoint "matrix" $a(x,\xi)$.

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The proof of that sharp Gårding inequality is simple: defining a semi-classical version of the projection $\mathbf{P}_h$, we get

(1) The Toeplitz operator $\mathbf{P}_h a \mathbf{P}_h \geq 0$ since $a \geq 0$.

(2) The difference $a^{Weyl_h} - \mathbf{P}_h a \mathbf{P}_h$ is $O(h)$ in operator norm.

As a result

$$a^{Weyl_h} = a^{Weyl_h} - \mathbf{P}_h a \mathbf{P}_h + \mathbf{P}_h a \mathbf{P}_h \geq O(h).$$

Although (2) is not completely obvious, it means simply that the Toeplitz quantization $\mathbf{P}_h a \mathbf{P}_h$ of the Hamiltonian a is close to the Weyl quantization and it is not difficult to check directly the Weyl symbols.

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REASON 2. Minimization of the Gross-Pitaevski energy. For $\psi \in L^2(\mathbb{R}^2)$,

$$E_{GP}(\psi) = \frac{1}{2}\langle q^{Weyl} u, u\rangle + \frac{g}{2}\int_{\mathbb{R}^2} |\psi|^4 dx,$$

$$q = (\xi_1 + \omega x_2)^2 + (\xi_2 - \omega x_1)^2 + \epsilon^2(x_1^2 + x_2^2)$$

$$2E_{GP}(\psi) = \|\frac{1}{i\pi}(\bar\partial + \pi\omega z)\psi\|^2 + \frac{\omega}{\pi}\|\psi\|^2 + \epsilon^2\||x|\psi\|^2 + g\int |\psi|^4 dx.$$

We define the Lowest Landau Level space with parameter ω as

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L. Boutet de Monvel & V. Guillemin in their 1981 book, *The spectral theory of Toeplitz operators* (Princeton University Press) have already studied that question extensively and in geometric terms when the symbols a, b are smooth functions.

Well, we want to understand that composition formula when one of the symbols is quite singular, say no better than L^∞, and the other one has a couple of derivatives bounded. This question is interesting per se and also is useful to prove some energy estimates.

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2. Statements

Framework

For $X, Y \in \mathbb{R}^{2n}$ we set

$$\Pi_H(X, Y) = e^{-\frac{\pi}{2}|X-Y|^2} e^{-i\pi[X,Y]}$$

where $[X, Y]$ is the symplectic form (if $X = (x, \xi)$, $Y = (y, \eta)$, $[X, Y] = \xi \cdot y - \eta \cdot x$).

The operator Π_H with kernel $\Pi_H(X, Y)$ is the orthogonal projection in $L^2(\mathbb{R}^{2n})$ on a proper closed subspace H, canonically isomorphic to $L^2(\mathbb{R}^n)$. In fact, one may define $W : L^2(\mathbb{R}^n) \longrightarrow L^2(\mathbb{R}^{2n})$ by the formula

$$(Wu)(y, \eta) = \langle u, \varphi_{y,\eta} \rangle_{L^2(\mathbb{R}^n)}, \quad \varphi_{y,\eta}(x) = 2^{n/4} e^{-\pi(x-y)^2} e^{2i\pi(x-\frac{y}{2})\eta}.$$

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which is the isotropic LLL, up to some normalization constant.

The Toeplitz operator with symbol $a(x,\xi)$ is

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$$W^*W = \mathrm{Id}_{L^2(\mathbb{R}^n)} \text{ (reconstruction formula } u(x) = \int_{\mathbb{R}^{2n}} Wu(Y)\varphi_Y(x)dY),$$

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which is the isotropic LLL, up to some normalization constant.

The Toeplitz operator with symbol $a(x,\xi)$ is

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Composition result

Theorem

Let a, b be in $L^\infty(\mathbb{R}^{2n})$ with $a'' \in L^\infty(\mathbb{R}^{2n})$, we have

$$\Pi_H a \Pi_H b \Pi_H = \Pi_H \Big(ab - \frac{1}{4\pi}\nabla a \cdot \nabla b + \frac{1}{4i\pi}\{a, b\}\Big)\Pi_H + R, \quad (1)$$

$$\|R\|_{\mathcal{B}(L^2(\mathbb{R}^{2n}))} \le C(n)\|a''\|_{L^\infty}\|b\|_{L^\infty}. \quad (2)$$

The product $\nabla a \cdot \nabla b$ as well as the Poisson bracket $\{a, b\}$ above make sense as tempered distributions since ∇a is a Lipschitz continuous function and ∇b is the derivative of an L^∞ function: in fact, we shall use as a definition

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Comments

$$\Pi_H a \Pi_h b \Pi_H = \Pi_H \Big(\overbrace{ab - \frac{1}{4\pi}(\nabla a \cdot \nabla b)}^{\text{symmetric in } a, b} + \frac{1}{4i\pi} \overbrace{\{a, b\}}^{\text{anti-symmetric in } a, b} \Big) \Pi_H + R,$$

$$\|R\|_{\mathcal{L}(L^2(\mathbb{R}^{2n}))} \leq C(n) \|a''\|_{L^\infty} \|b\|_{L^\infty}.$$

As a result, we have, modulo $\mathcal{B}(L^2(\mathbb{R}^{2n}))$,

$$[\Pi_H a \Pi_H, \Pi_H b \Pi_H] \equiv \frac{1}{2i\pi} \Pi_H \{a, b\} \Pi_H,$$

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Moreover there are some versions of theses equalities for matrix-valued Hamiltonians a, b

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3. Proof

A direct calculation

We have

$$W^* a W = \int_{\mathbb{R}^{2n}} a(Y)\Sigma_Y dY, \quad (\Sigma_Y u)(x) = \langle u, \varphi_{y,\eta}\rangle_{L^2(\mathbb{R}^n)}\varphi_{y,\eta}(x),$$

$$\text{with} \quad \varphi_{y,\eta}(x) = 2^{n/4} e^{-\pi(x-y)^2} e^{2i\pi(x-\frac{y}{2})\eta}.$$

Thus

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REMARK. *Let ω be a measurable function defined on $\mathbb{R}^{2n} \times \mathbb{R}^{2n}$ such that*

$$|\omega(Y,Z)| \le \gamma_0 \big(1 + |Y - Z|\big)^{N_0}.$$

Then the operator $\iint \omega(Y,Z)\Sigma_Y \Sigma_Z dYdZ$ is bounded on $L^2(\mathbb{R}^n)$ with $\mathcal{B}(L^2(\mathbb{R}^n))$ norm bounded above by a constant depending on γ_0, N_0.

This is an immediate consequence of Cotlar's lemma and of the estimate

$$\|\Sigma_Y \Sigma_Z\|_{\mathcal{B}(L^2(\mathbb{R}^n))} \le 2^n e^{-\frac{\pi}{2}|Y-Z|^2}.$$

Using that remark, we obtain that

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Classical applications

- Hypoellipticity for fractional operators such as

$$\partial_t + v \cdot \partial_x + (-\Delta_v)^\alpha,$$

coming from the linearization of the Boltzmann equation.

- The composition formula was used by F. Hérau & K. Pravda-Starov to prove some anisotropic hypoelliptic estimates for Landau-type operators (J. Math. Pures Appl., 2011).

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Questions

Paving Conjecture. There exists $r \in \mathbb{N}$, such that for any separable Hilbert space H, for any family of rank-one orthogonal projections $(p_j)_{j\in\mathbb{N}}$ with $\sum_{j\in\mathbb{N}} p_j = \mathrm{Id}$, $p_j p_k = \delta_{j,k} p_k$, for all $A \in \mathcal{B}(H)$, with $\|A\| = 1$ such that for all j, $p_j A p_j = 0$, there exists $P_1, \dots, P_r$ such that

$$\max_{1\le j\le r} \|P_j A P_j\| \le 1/2, \quad P_j = \sum_{l\in J_j} p_l, \quad \sum_{1\le j\le r} P_j = \mathrm{Id}\,.$$

The universal status of the integer r (let's call it r_{KS}) above is quite scaring and it is tempting to doubt that such a universal integer could exist.

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The only known general cases supporting the conjecture are cases where the diagonal is dominant or where the coefficients of the matrix are all non-negative.

The general Toeplitz case (matrices (a_{jk}) with $a_{jk} = \phi(j-k)$) is not known, nor is the pseudodifferential case, say on the circle. However when

$$a(x) = \sum_{j\in\mathbb{Z}} \hat{a}(j) e^{2i\pi xj}$$

is Riemann integrable, H. Halpern, V. Kaftal & G. Weiss proved that the Toeplitz operator with matrix $(\hat{a}(j-k))$ is uniformly pavable, i.e. there exists $N \in \mathbb{N}$ such that

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One may conjecture, following the result on Laurent operators with Riemann integrable symbols that classical pseudodifferential operators on the circle are uniformly pavable, as should be classical pseudodifferential operators on $\mathbb{R}^n$, or on an open subset of $\mathbb{R}^n$.

A pseudodifferential operator on the circle with symbol $a(x, k)$ ($x \in \mathbb{T}^1, k \in \mathbb{Z}$) is

$$(Au)(x) = \sum_{j \in \mathbb{Z}} e^{2i\pi xj} \sum_{k \in \mathbb{Z}} \hat{a}(j-k, k)\hat{u}(k),$$

so that A is identified with the matrix

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When a does not depend on the second variable, it is the operator of multiplication by a, Toeplitz operator with symbol a.

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$$m_{j,k} = \hat{a}(j-k,k).$$

When a does not depend on the second variable, it is the operator of multiplication by a, Toeplitz operator with symbol a.

The diagonal is 0 means that $\forall k \in \mathbb{Z}, \quad \int_0^1 a(x,k)dx = 0$. A semi-classical pseudodifferential operator on the circle is given by the matrix

$$m_{j,k}(h) = \hat{a}^1(j-k, hk), \quad h \in (0,1],$$

where the symbol a is defined on $\mathbb{T}^1 \times \mathbb{R}$. The diagonal of such a matrix is given by

$$\hat{a}^1(0, hj) = \int_0^1 a(x, hj)dx.$$

Instead of assuming that the diagonal is 0, it would be natural to assume that the diagonal is $O(h)$ and maybe formulate some semi-classical version of the paving conjecture.

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More information on the topic of Toeplitz operators and their calculus (Wick calculus) is included in Section 2.4 of my book,

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Thank you for your attention