Hypoellipticity for a class of kinetic equations

July 13, 2010, University of Wuhan

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A simple exercise (see next page): let $\mathbb{H}$ *be a Hilbert space,* $J, K \in \mathcal{B}(\mathbb{H})$*, then* $[J,K] \neq \mathrm{Id}$*. The observables of Quantum Mechanics are unbounded operators.*

`Claim: Let` $\mathbb{E}$ `be a Banach space and let` J, K `be bounded operators on` $\mathbb{E}$`. Then` $[J, K] \neq \mathrm{Id}$.
Reductio ad absurdum. If J, K are bounded operators with $[J, K] = \mathrm{Id}$, then

$$(\ddagger) \qquad \text{for all integers } N \geq 1, \quad [J, K^N] = NK^{N-1}.$$

This is true for $N = 1$, and if true for some $N \geq 1$, then

$$[J, K^{N+1}] = JK^N K - K^{N+1} J = [J, K^N]K + K^N JK - K^{N+1} J = [J, K^N]K + K^N = (N+1)K^N, \ \textit{qed}.$$

Note that Property $(\ddagger)$ implies that for all $N \in \mathbb{N}^*$, $K^N \neq 0$: if we had $K^N = 0$ for some $N \geq 2$, then this would imply $K^{N-1} = 0$ and eventually $K = 0$, which is incompatible with $[J, K] = \mathrm{Id}$. As a result, we get from $(\ddagger)$ that for all $N \geq 2$,

$$N\|K^{N-1}\| \leq 2\|J\|\|K^N\| \leq 2\|J\|\|K\|\|K^{N-1}\| \Longrightarrow N \leq 2\|J\|\|K\|,$$

which is impossible, proving the claim.

Harmonic oscillator

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at the ground state $\phi_0 = e^{-\pi|x|^2} 2^{n/4}$ which solves

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$C^\alpha \phi_0 = C_1^{\alpha_1} \dots C_n^{\alpha_n}\phi_0$ eigenvector with eigenvalue $\frac{n}{2} + |\alpha|$, discrete spectrum $\frac{n}{2} + \mathbb{N}$ for the harmonic oscillator.

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$$\frac{h^2|D|^2}{2m} - \frac{e^2}{|x|} = \frac{h^2|D|^2}{2m} - \frac{\mu h^2}{2\pi 2m}\frac{(n-1)}{|x|} \ge -\frac{\mu^2 h^2}{2m} = -\frac{e^4 m^2 16\pi^2 h^2}{h^4(n-1)^2 2m}$$

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$$\frac{h^2|D|^2}{2m} - \frac{e^2}{|x|} \ge -\frac{me^4 8\pi^2}{(n-1)^2 h^2} > -\infty \quad \text{stability (and best constant).}$$

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$$(-\Delta) \ge \Big(\frac{n-2}{2}\Big)^2 \frac{1}{|x|^2} \qquad \text{(Hardy's inequality).}$$

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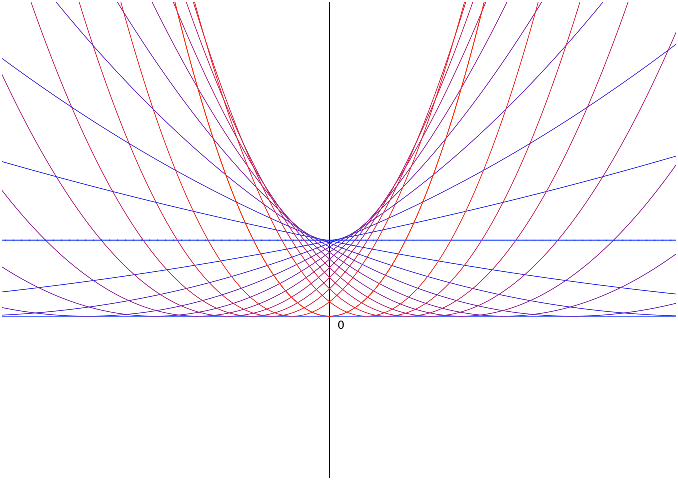
It is easy to solve explicitely that ODE with parameters: Fourier transform in the x_1, x_2 variables and we have to deal with

$$\frac{d}{ds} + (\xi_2 + s\xi_1)^2$$



Family of parabolas $\quad s \mapsto (\xi_2 + s\xi_1)^2 \quad$ for $\xi_1^2 + \xi_2^2 = 1$

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and we get the standard subelliptic $\|\widetilde{\mathcal{K}}v\| \gtrsim \lambda^{1/3}\|v\| = |\xi_1|^{2/3}\|v\|$.

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which is an optimal estimate.

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2. Pseudodifferential techniques

Wick quantization. $X, Y \in \mathbb{R}^{2n}$, $\quad \Pi(X, Y) = e^{-\frac{\pi}{2}|X-Y|^2} e^{-i\pi[X,Y]}$,
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We define

$$(Wu)(y, \eta) = \langle u, \varphi_{y,\eta} \rangle_{L^2(\mathbb{R}^n)}, \quad \varphi_{y,\eta}(x) = 2^{n/4} e^{-\pi|x-y|^2} e^{2i\pi(x-\frac{y}{2})\cdot\eta}.$$

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We have $W : L^2(\mathbb{R}^n) \longrightarrow L^2(\mathbb{R}^{2n})$ isometric, not onto,

$W^*W = \mathrm{Id}_{L^2(\mathbb{R}^n)}$: reconstruction formula, W isometric,

$WW^* = \Pi_0$: projection operator onto *ran*W with oper-kernel Π.

Let a be a Hamiltonian: we can use the Weyl quantization with the formula

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If a is a semi-classical symbol, i.e. such that

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then $a^{Wick} - a^w \in \mathcal{B}(L^2(\mathbb{R}^n))$ so the change is harmless if we expect to prove some subelliptic estimate.

Subellipticity for pseudodifferential equations. We consider an evolution equation

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We start with the study of the ODE $D_t + iq$: not so difficult but we need a

Lemma A.

Lemma A. Let $k \geq 1, \delta > 0$, $C > 0$, I be an interval of $\mathbb{R}$, $f : I \to \mathbb{R}$ such that

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Proof by induction on k and we note that the conclusion can be fulfilled for k non-integer for some f merely continuous (e.g. fractional powers).

Theorem A. $q \in S^1_{semiclas}$ real-valued such that $q = 0 \Longrightarrow d_{x,\xi} q = 0$ (e.g. $q \geq 0$). Then, if $|\partial_t^k q| h \geq \delta > 0$

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we need to handle that commutator.

Commutator argument

The unwanted term here is with $\Phi = Wu$

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$$g = \frac{dx^2 + d\xi^2}{\lambda(t, x, \xi)}, \quad \lambda(t, x, \xi) = 1 + |q| + |d_{x,\xi} q|^2$$

is such that $q \in S(\lambda, g)$, and $\frac{\lambda}{1+|q|} \sim 1$ and the energy method will provide for free a term $\langle |q|\Phi, \Phi\rangle$.

3. A kinetic equation

Presentation. Boltzmann equation: $0 \leq f(t, x, v)$ probability density, $x \in \mathbb{R}^d, v \in \mathbb{R}^d, t \geq 0$,

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$$Q(f,f) = \int_{\mathbb{R}^3} \int_{\mathbb{S}^2} B(v - v_*, \sigma) \Big\{ f(v'_*) f(v') - f(v_*) f(v) \Big\} d\sigma dv_*$$

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$$v' = \frac{v + v_*}{2} + \frac{|v - v_*|}{2} \sigma, \quad v'_* = \frac{v + v_*}{2} - \frac{|v - v_*|}{2} \sigma.$$

Conservation of momentum: $v + v_* = v' + v'_*$,
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The *cross-section* $B(z,\sigma)$ depends only on $|z|$ and $\cos\theta = \langle \frac{z}{|z|}, \sigma \rangle$.

$$B(v-v_*,\sigma)=\Phi(|v-v_*|)b(\cos\theta),\quad \cos\theta=\langle\frac{v-v_*}{|v-v_*|},\sigma\rangle$$

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$$\|(-\Delta)^{\alpha/2} f\|^2 \lesssim \langle -Q(f, f), f \rangle + \|f\|^2$$

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We have

$$\|(-\Delta)^{\alpha/2} f\|^2 \lesssim \langle -Q(f,f), f \rangle + \|f\|^2$$

and the subelliptic properties of the Boltzmann equation are closely related to the properties of the equation

studied in the Y. Morimoto – C.-J. Xu paper (J. Math. Kyoto Univ., 2007),

$$\mathcal{P}u \equiv \partial_t u + x \cdot \nabla_y u + \sigma_0(-\Delta_x)^\alpha u = f,$$
$$(x, y) \in \mathbb{R}^n \times \mathbb{R}^n, \quad 0 < \alpha < 1, \quad \sigma_0 > 0.$$

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To avoid the singularity at $\xi = 0$, we define, with $\chi \in C_c^\infty(\mathbb{R}^n)$, $\chi \equiv 1$ near 0, $\omega = 1 - \chi$,

$$M(\xi) = |\xi|^{2\alpha}\omega(\xi) + |\xi|^2\chi(\xi)$$

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$$\mathcal{P}u \equiv \partial_t u + x \cdot \nabla_y u + \sigma_0(-\Delta_x)^\alpha u = f,$$
$$(x, y) \in \mathbb{R}^n \times \mathbb{R}^n, \quad 0 < \alpha < 1, \quad \sigma_0 > 0.$$

To avoid the singularity at $\xi = 0$, we define, with $\chi \in C_c^\infty(\mathbb{R}^n)$, $\chi \equiv 1$ near 0, $\omega = 1 - \chi$,

$$M(\xi) = |\xi|^{2\alpha}\omega(\xi) + |\xi|^2\chi(\xi)$$

Theorem B. $\|\mathcal{P}u\| + \|u\| \gtrsim \||D_x|^{2\alpha}u\| + \||D_y|^{\frac{2\alpha}{2\alpha+1}}u\|$

$$\mathcal{P} = \partial_t + x \cdot \nabla_y + \sigma_0(-\Delta_x)^\alpha$$

Proof. Fourier transform with respect to (x, y),

$$P = \partial_t - i\eta \cdot D_\xi + \sigma_0|\xi|^{2\alpha}$$

$$P = \partial_t - \eta \cdot \partial_\xi + \sigma_0|\xi|^{2\alpha} : \quad \text{following the flow of } \partial_t - \eta \cdot \partial_\xi,$$

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$$\begin{cases} s &= t \\ x_1 &= \eta \\ x_2 &= \xi + t\eta \end{cases} \qquad \begin{cases} t &= s \\ \eta &= x_1 \\ \xi &= x_2 - s x_1 \end{cases}$$

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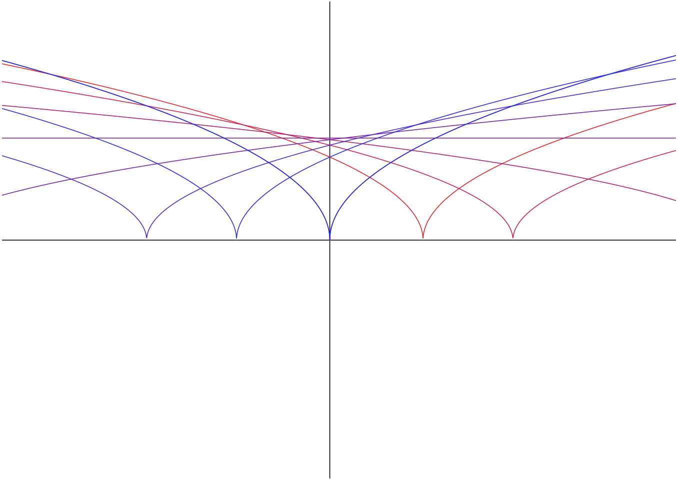
$$\frac{\partial}{\partial t} = \frac{\partial}{\partial s} + x_1 \frac{\partial}{\partial x_2}, \quad \eta \cdot \frac{\partial}{\partial \xi} = x_1 \cdot \frac{\partial}{\partial x_2}, \quad \partial_t - \eta \cdot \partial_\xi = \partial_s$$

$$P = \partial_s + |x_2 - sx_1|^{2\alpha}$$



Family of curves $s \mapsto |x_2 - sx_1|^{2\alpha}$ for $x_1^2 + x_2^2 = 1, \quad \alpha = 1/4$

$$P = \partial_s + |x_2 - s x_1|^{2\alpha} = \partial_s + |x_1|^{2\alpha} |x_2/x_1 - s|^{2\alpha}$$

$$P = \partial_s + |x_2 - sx_1|^{2\alpha} = \partial_s + |x_1|^{2\alpha}|x_2/x_1 - s|^{2\alpha}$$

$$|x_1|^{\frac{2\alpha}{2\alpha+1}} \int |u(s)|^2 ds = |x_1|^{\frac{2\alpha}{2\alpha+1}} \int_{|x_2 - sx_1|^{2\alpha} \geq |x_1|^{\frac{2\alpha}{2\alpha+1}}} |u(s)|^2 ds + |x_1|^{\frac{2\alpha}{2\alpha+1}} \int_{|x_2 - sx_1|^{2\alpha} \leq |x_1|^{\frac{2\alpha}{2\alpha+1}}} |u(s)|^2 ds$$

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We use a fractional version of Lemma A: the Lebesgue measure of $\{s, |x_2 - sx_1|^{2\alpha} \leq |x_1|^{\frac{2\alpha}{2\alpha+1}}\} \leq 2|x_1|^{-\frac{2\alpha}{2\alpha+1}}$ since

$$|x_2 - sx_1| \leq |x_1|^{\frac{1}{2\alpha+1}} \Longrightarrow |x_2/x_1 - s| \leq |x_1|^{\frac{1}{2\alpha+1}-1=-\frac{2\alpha}{2\alpha+1}}$$

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$$\le \operatorname{Re}\langle Pu, u\rangle + 2 \sup |u(s)|^2.$$

$$P = \partial_s + |x_2 - s x_1|^{2\alpha}, \quad |x_1|^{\frac{2\alpha}{2\alpha+1}} \|u\|^2 \le \operatorname{Re}\langle Pu, u\rangle + 2 \sup |u(s)|^2$$

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We get

$$\||\eta|^{\frac{2\alpha}{2\alpha+1}} u\| \lesssim \|(\partial_t - \eta\cdot\partial_\xi + |\xi|^{2\alpha})u\|$$

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$$(1+\beta)\|(\partial_t - \eta \cdot \partial_\xi + |\xi|^{2\alpha})u\|^2 + \|u\|^2$$
$$\geq \iiint |u|^2 \big(\beta c_0^2 |\eta|^{\frac{4\alpha}{2\alpha+1}} + |\xi|^{4\alpha} + \underbrace{2\alpha\eta \cdot \frac{\xi}{|\xi|}|\xi|^{2\alpha-1}}_{\text{bad term}} + 1\big) dt d\eta d\xi.$$

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However, Hölder's inequality implies

$$|\eta||\xi|^{2\alpha-1} = \left(|\eta|^{\frac{4\alpha}{2\alpha+1}}\right)^{\frac{2\alpha+1}{4\alpha}} \left(|\xi|^{4\alpha}\right)^{\frac{2\alpha-1}{4\alpha}} \leq \frac{2\alpha+1}{4\alpha}|\eta|^{\frac{4\alpha}{2\alpha+1}} + \frac{2\alpha-1}{4\alpha}|\xi|^{4\alpha}$$

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and for $\beta c_0^2 \geq \frac{2\alpha+1}{4\alpha}$, we get

$$\| |\xi|^{2\alpha} u \| + \| |\eta|^{\frac{2\alpha}{2\alpha+1}} u \| \lesssim \| (\partial_t - \eta \cdot \partial_\xi + |\xi|^{2\alpha}) u \|,$$

which is Theorem B.

Geometry of the characteristics. We consider an operator

$$\mathcal{L} = X_0 + Q, \quad X_0^* = -X_0, \quad Q \geq 0,$$

so that X_0 is the skew-adjoint part (e.g. a divergence-free vector field) and Q is the self-adjoint part (e.g. a Laplacean in some of the variables).

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The End