THE KK-THEORY OF AMALGAMATED FREE PRODUCTS

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ABSTRACT. Given a graph of C*-algebras as defined in [FF13], we prove a long exact sequence in KK-theory similar to the one obtained by Pimsner in [Pi86] for both the maximal and reduced fundamental C*-algebras of the graph in the presence of possibly non-GNS-faithful conditional expectations. In particular, our results give a long exact sequence in KK-theory for both maximal and reduced amalgamated free products and HNN-extensions. In the course of the proof, we established the KK-equivalence between the full amalgamated free product of two unital C*-algebras and a newly defined reduced amalgamated free product that is valid even for non GNS-faithful conditional expectations. This KK-equivalence is again true in the general context. Our results unify, simplify and generalize all the previous results obtained before by Cuntz, Pimsner, Germain and Thomsen.

1. INTRODUCTION

In 1982 J. Cuntz obtained a very elegant result about the full free product of unital C*-algebras with one-dimensional representations that leads to a conjectural long exact sequence for amalgamated free products in a general situation [Cu82]. At about the same time M. Pimsner and D. Voiculescu computation of the KK-theory for some groups C*-algebras culminated in the computation of full and reduced crossed products by groups acting on trees [Pi86] (or by the fundamental group of a graph of groups in Serre's terminology). To go over the group situation has been difficult and it relied heavily on various generalizations of Voiculescu absorption theorem (see [Th03] for the most general results in that direction). Note also that G. Kasparov and G. Skandalis had another proof of Pimsner long exact sequence when studying KK-theory for buildings [KS91]

However the results we obtain here are based on a completely different point of view. Introduced in [FF13], the full or reduced fundamental C*-algebras of a graph allows to treat on equal footings amalgamated free products and HNN extensions (and in particular cross-product by the integers). Let's describe its context. A graph of C*-algebras is a finite oriented graph with unital C*-algebras attached to its edges (B_e) and vertices (A_v) such that for any edge e there are embeddings r_e and s_e of B_e in $A_{r(e)}$ and $A_{s(e)}$ with $r(e)$ the range of e and $s(e)$ its source. As for groups, the full fundamental C*-algebra of the graph is a quotient of the universal C*-algebra generated by the A_v and unitaries u_e such that $u_e^* s_e(b) u_e = r_e(b)$ for all $b \in B_e$. In the presence of conditional expectations form $A_{s(e)}$ and $A_{r(e)}$ onto B_e, one can also construct various representations of the full fundamental C*-algebra on Hilbert modules over A_v or B_e. It is the interplay with the representations that yields the tools we need to prove our results.

In section 2, we first look at one of the simplest graphs : one edge, two different endpoints. The full fundamental C*-algebra is then the full amalgamated free products. When the conditional expectations are *not* GNS-faithful, there are two possible reduced versions: the reduced free product of D. Voiculecscu, that is often too small, which is obtained by looking at the module over the edge algebra and the "vertex" reduced free product that is obtained when looking at the two modules over the algebras attached to the vertices. In general, Voiculescu reduced free product is a quotient of the vertex reduced free product, but when the conditional expectations are GNS faithful, they coincide. As the vertex reduced free product is a new construction, we devote some time to show some of its properties.

Our first theorem in section 3 states that the full free product if always K-equivalent to the vertex reduced free product. In particular, when the conditional expectations are "extremely non GNS-faithful" i.e., when they are morphisms, we get exactly Cuntz result [Cu82]. This result also generalize and simplify the previous result obtained by the second author [Ge96]. The proof is very simple as it is a simple rotation trick. While finishing writing this paper, the authors have been aware that K. Hasegawa just obtained the same result in the very particular case of GNS-faithful conditional expectations. By a remark by Ueda ([Ue08]), this result also proves the K-equivalence between full and (vertex) reduced HNN extensions, i.e the fundamental C*-algebras associated to the graph with one edge and one vertex.

In the next section, we prove the K-equivalence, under the same hypothesis, of the full amalgamated free product $A_1 \underset{B}{*} A_2$ with the algebra D of continuous functions f from $]-1,1[$ to the full free product such that $f(]-1,0]) \subset A_1$, $f([0,1[) \subset A_2$ and $f(0) \in B$. This is done by generalizing one of the author paper ([Ge97]). Therefore, the full amalgamated free product $A_1 \underset{B}{*} A_2$ sits inside a long exact sequence for the computation of its KK-groups. Of course the vertex reduced free product has got the same long exact sequence. Again the HNN extension case follows using the isomorphism with an amalgamated free product, giving a completely new proof of Pimsner-Voiculescu exact sequence as a particular case.

At last, in section 5 we generalize the long exact sequences of the previous section to the full and reduced fundamental C*-algebras of a finite graph of C*-algebras. In section 5.1, we recall the construction of the fundamental C*-algebra of a graph of C*-algebras as it was done in [FF13]. In contrast with [FF13], we do not assume the conditional expectations to be GNS-faithful which forced us to construct a different version of the reduced fundamental C*-algebra that we called vertex-reduced. We give all the necessary background on vertex-reduced fundamental C*-algebras. In section 5.2, we construct a very natural element in KK^1 which gives, by Kasparov product, the boundary map in the long exact sequence we write in section 5.3. Explicitly, if C is any separable C*-algebra, P the full or reduced fundamental C*-algebra of the finite oriented graph (A_v, B_e) then we have the two 6-terms exact sequence, where E^+ is the set of positive edges,

$$
\begin{array}{ccc}
\bigoplus_{e \in E^+} KK^0(C, B_e) & \xrightarrow{\sum s_e^* - r_e^*} \bigoplus_{v \in V} KK^0(C, A_v) \longrightarrow & KK^0(C, P) \\
\uparrow & & \downarrow \\
KK^1(C, P) & \longleftarrow \bigoplus_{v \in V} KK^0(C, A_v) \xleftarrow{\sum s_e^* - r_e^*} & \bigoplus_{e \in E^+} KK^0(C, B_e)
\end{array}
$$

and

$$\bigoplus_{e\in E^+} KK^0(B_e,C) \xleftarrow{\sum s_{e*}-r_{e*}} \bigoplus_{v\in V} KK^0(A_v,C) \longleftarrow KK^0(P,C)$$
$$\downarrow \qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad \uparrow$$
$$KK^1(P,C) \longrightarrow \bigoplus_{v\in V} KK^0(A_v,C) \xrightarrow{\sum s_{e*}-r_{e*}} \bigoplus_{e\in E^+} KK^0(B_e,C)$$

In section 6 we give some applications of our results. A direct Corollary of our results is that the full and vertex-reduced fundamental C*-algebra of a graph of C*-algebras are K-equivalent.This generalizes and simplifies the results of Pimsner about the KK-theory of crossed-products by groups acting on trees [Pi86] as well as the results of Thomsen [Th03] about KK-theory for amalgamated free products which are valid only when the amalgam is finite dimensional.

Also, our results imply that the fundamental quantum group of a graph of discrete quantum groups is K-amenable if and only if all the vertex quantum groups are K-amenable. This generalizes and simplifies the results of [FF13]. Finally, our results also implies that a graph product of discrete quantum groups (see [CF14]) is K-amenable if and only if the initial discrete quantum groups are K-amenable.

2. Preliminaries

2.1. **Notations and conventions.** All C*-algebras are supposed to be separable. For a C*-algebra A and a Hilbert A-module H we denote by $\mathcal{L}_A(H)$ the C*-algebra a A-linear adjointable operators from H to H and by $\mathcal{K}_A(H)$ the sub-C*-algebra of $\mathcal{L}_A(H)$ consisting of A-compact operators. We write $L_A \in \mathcal{L}_A(A)$ the left multiplication operator.

2.2. **Some homotopies.** We will use the following proposition which is well known.

Proposition 2.1. *Let H be a Hilbert A-module. For any strictly continuous and norm bounded path $F : [0,1] \to \mathcal{L}_A(H)$, $t \mapsto F_t$, there exists a unique operator $F \in \mathcal{L}_{A\otimes C([0,1])}(H\otimes C([0,1]))$ for which the evaluation at t is F_t. Conversely, for any operator $F \in \mathcal{L}_{A\otimes C([0,1])}(H \otimes C([0,1]))$, the evaluation F_t at $t \in [0,1]$ defines a norm bounded strictly continuous path $[0,1] \to \mathcal{L}_A(H)$, $t \mapsto F_t$. Moreover, an operator $F \in \mathcal{L}_{A\otimes C([0,1])}(H \otimes C([0,1]))$ is actually in $\mathcal{K}_{A\otimes C([0,1])}(H \otimes C([0,1]))$ if and only if the corresponding norm bounded strictly continuous path has values in $\mathcal{K}_A(H)$.*

The previous proposition is useful to construct homotopies, as shown in the proof of the following lemma.

Lemma 2.2. *Let A, B be unital C*-algebras, H, K Hilbert B-modules, $\pi : A \to \mathcal{L}_B(H)$, $\rho : A \to \mathcal{L}_B(K)$ unital $*$-homomorphisms and $F \in \mathcal{L}_B(H,K)$ a partial isometry such that $F\pi(a) - \rho(a)F \in \mathcal{K}_B(H,K)$ for all $a \in A$ and $F^*F - 1 \in \mathcal{K}_B(H)$. Then, $[(K,\rho,V)] = 0 \in KK^1(A,B)$, where $V = 2FF^* - 1$.*

Proof. Let $\alpha := [(K,\rho,V)] \in KK^1(A,B)$. For $t \in [0,1]$, define

$$U_t = \begin{pmatrix} 1 - FF^* & 0 \\ 0 & 0 \end{pmatrix} + \cos(\pi t) \begin{pmatrix} FF^* & 0 \\ 0 & -1 \end{pmatrix} - \sin(\pi t) \begin{pmatrix} 0 & F \\ F^* & 0 \end{pmatrix} \in \mathcal{L}_B(K \oplus H).$$

We have $U_0 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$ and $U_1 = -\begin{pmatrix} V & 0 \\ 0 & 1 \end{pmatrix}$. Note that, for all $t \in [0,1]$, $U_t^* = U_t$ and,

$$
\begin{aligned}
U_t^2 &= \begin{pmatrix} 1-FF^* & 0 \\ 0 & 0 \end{pmatrix} + \cos(\pi t)^2 \begin{pmatrix} FF^* & 0 \\ 0 & 1 \end{pmatrix} + \sin(\pi t)^2 \begin{pmatrix} FF^* & 0 \\ 0 & F^*F \end{pmatrix} \\
&= \begin{pmatrix} 1-FF^* & 0 \\ 0 & 0 \end{pmatrix} + \begin{pmatrix} FF^* & 0 \\ 0 & 1 \end{pmatrix} + K_t = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} + K_t,
\end{aligned}
$$

where $K_t = \sin(\pi t)^2 \begin{pmatrix} 0 & 0 \\ 0 & F^*F-1 \end{pmatrix} \in \mathcal{K}_B(K \oplus H)$ for all $t \in [0,1]$, since $F^*F - 1 \in \mathcal{K}_B(H)$. Moreover, $U_t(\rho \oplus \pi)(a) - (\rho \oplus \pi)(a)U_t \in \mathcal{K}_B(K \oplus H)$ for all $a \in A$ since $F\pi(a) - \rho(a)F \in \mathcal{K}_B(H,K)$ for all $a \in A$. By Proposition 2.1 there exists unique operators $U \in \mathcal{L}_{B \otimes C([0,1])}(K \oplus H) \otimes C([0,1]))$ and $K \in \mathcal{K}_{B \otimes C([0,1])}(K \oplus H) \otimes C([0,1]))$ such that the evaluation of U at t is U_t and the evaluation of K at t is K_t for all $t \in [0,1]$. In particular we have $U = U^*$ and $U^2 = 1 + K$ and, since $U_t(\rho \oplus \pi)(a) - (\rho \oplus \pi)(a)U_t \in \mathcal{K}_B(K \oplus H)$ for all $a \in A$ and all $t \in [0,1]$, it follows again from Proposition 2.1 that,

$$U(\rho \oplus \pi)(a) \otimes 1_{C([0,1])} - (\rho \oplus \pi)(a) \otimes 1_{C([0,1])}U \in \mathcal{K}_{B \otimes C([0,1])}((K \oplus H) \otimes C([0,1])) \quad \text{for all } a \in A.$$

Hence we get an homotopy

$$\gamma = [((K \oplus H) \otimes C([0,1]), (\rho \oplus \pi) \otimes 1_{C([0,1])}, U)] \in KK^1(A \otimes C([0,1]), B \otimes C([0,1]))$$

between $\gamma_0 = [(K \oplus H, \rho \oplus \pi, U_0)] = [(K \oplus H, \rho \oplus \pi, \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix})] = 0$ since the triple is degenerated and $\gamma_1 = [(K \oplus H, \rho \oplus \pi, U_1)] = [(K \oplus H, \rho \oplus \pi, -\begin{pmatrix} V & 0 \\ 0 & 1 \end{pmatrix})]$. Hence, $\gamma_1 = x \oplus y$, where $x = [(K, \rho, -V)] = -\alpha$ and $y = [(H, \pi, -\mathrm{id}_H)] = 0$, since the triple is degenerated. $\qquad\square$

2.3. Conditional expectations.

Let A, B be unital C*-algebras and $\varphi : A \to B$ by a unital completely positive map (ucp). A *GNS construction* of φ is a triple (K, ρ, η), where K is a Hilbert B-module, $\eta \in K$ and $\rho : A \to \mathcal{L}_B(K)$ is a unital $*$-homomorphism such that $K = \overline{\rho(A)\eta \cdot B}$ and $\langle \eta, \rho(a)\eta \rangle = \varphi(a)$ for all $a \in A$. A GNS construction always exists and is unique, up to a canonical isomorphism.

Note that, if $B \subset A$ and $E : A \to B$ is a conditional expectation then, the Hilbert B-submodule $\eta \cdot B$ of K, where (K, ρ, η) is a GNS construction of E, is complemented. Indeed, we have $K = \eta \cdot B \oplus K^\circ$, where $K^\circ = \overline{\mathrm{Span}}\{\rho(a)\eta \cdot b : a \in A^\circ \text{ and } b \in B\}$ and $A^\circ = \mathrm{Ker}(E)$. Since E is a conditional expectation onto B we have $bA^\circ \subset A^\circ$ for all $b \in B$. It follows that $\rho(b)K^\circ \subset K^\circ$ for all $b \in B$. Hence, the restriction of ρ to B (and to K°) gives a unital $*$-homomorphism $\rho : B \to \mathcal{L}_B(K^\circ)$.

A conditional expectation is called *GNS-faithful* (or *non-degenerate*) if for a given GNS construction (and hence for all GNS constructions) (K, ρ, η), the homomorphism ρ is faithful. In this paper we will consider reduced amalgamated free product with respect to non-necessary GNS-faithful conditional expectations. Actually, the degeneracy of the conditional expectations will naturally produce different types of reduced amalgamated free products. This is why we include the next proposition, which is well known to specialists but helps to understand the extreme degenerated case: when E is an homomorphism. We include a complete proof for the convenience of the reader.

Proposition 2.3. *Let* $B \subset A$ *be a unital inclusion of unital C*-algebras and* $E : A \to B$ *be a conditional expectation with GNS construction* (K, ρ, η). *The following are equivalent.*

(1) E *is an homomorphism.*
(2) $K \simeq B$ *as Hilbert* B-*modules.*
(3) $K^\circ = \{0\}$.

Proof. Since $K = \eta \cdot B \oplus K^\circ$ the equivalence between (2) and (3) is obvious.

$(1) \Rightarrow (3)$. If E is an homomorphism from A to B then, since E is ucp, it is a unital $*$-homomorphism and we have for all $b \in B$ and all $a \in A^\circ$,

$$\langle \rho(a)\eta \cdot b, \rho(a)\eta \cdot b \rangle_K = b^* \langle \eta \cdot b, \rho(a^*a)\eta \rangle_K b = b^* E(a^*a)b = b^* E(a)^* E(a)b = 0.$$

$(3) \Rightarrow (1)$. If $K^\circ = \{0\}$ then, for all $a \in A^\circ$, we have $E(a^*a) = \langle \rho(a)\eta, \rho(a)\eta \rangle_K = 0$. Hence

$$E((a - E(a))^*(a - E(a))) = 0 = E(a^*a) - E(a^*)E(a) - E(a)^*E(a) + E(a)^*E(a) \quad \text{for all } a \in A.$$

It follows that, for all $a \in A$, we have $E(a^*a) = E(a)^*E(a)$. Hence, the multiplicative domain of the ucp map E is equal to A which implies that E is an homomorphism. $\qquad \square$

2.4. **The full and reduced amalgamated free products.** Let A_1, A_2 be two unital C*-algebras with a common C*-subalgebra $B \subset A_k$, $k = 1, 2$ and denote by A_f the full amalgamated free product. To be more precise, we sometimes write $A_f = A_1 \underset{B}{*} A_2$. It is well known that the canonical map from A_k to A_f is faithful for $k = 1, 2$ (this statement and an even more general one is proved in Remark 5.1). Hence, we will always view $A_1, A_2 \subset A_f$.

We will now construct, in the presence of conditional expectations, two different reduced amalgamated free products. One of them, that we call the *edge-reduced amalgamated free product* has been extensively studied and it is called, in the literature, the reduced amalgamated free product. The other one, that we call the *vertex-reduced amalgamated free product*, does not seem to be known, even from specialists. As it will become gradually clear, the vertex-reduced amalgamated free product is actually much more natural than the edge-reduced amalgamated free product. It is an intermediate quotient of the full amalgamated free product and it is isomorphic to the edge-reduced amalgamated free product in the presence of GNS-faithful conditional expectations. This is the reason why it has not appear before in the literature since many authors only consider amalgamated free product in the presence of GNS-faithful conditional expectations. Indeed, the GNS-faithful assumption is very convenient since it produces a universal property for the (edge) reduced amalgamated free product.

Since the vertex-reduced and the edge-reduced amalgamated free product are the foundations of our proofs we will now explain in great details the their constructions.

In the sequel, we always assume that, for $k = 1, 2$, there exists a conditional expectation $E_k : A_k \to B$. We write $A_k^\circ = \{a \in A_k : E_k(a) = 0\}$, we denote by (K_k, ρ_k, η_k) a GNS construction of E_k and by K_k° the canonical orthogonal complement of $\eta_k \cdot B$ in K_k as explain in section 2.3. Recall that the restriction of ρ_k to B (and to K_k°) gives a unital $*$-homomorphism $\rho_k : B \to \mathcal{L}_B(K_k^\circ)$.

We denote by I the subset of $\cup_{n \geq 1} \{1, 2\}^n$ defined by

$$I = \{(i_1, \dots, i_n) \in \{1, 2\}^n : n \geq 1 \text{ and } i_k \neq i_{k+1} \text{ for all } 1 \leq k \leq n - 1\},$$

Recall that an operator $x \in A_f$ is called *reduced* if $x \neq 0$ and x can be written as $x = a_1 \ldots a_n$ with $n \geq 1$ and $a_k \in A_{i_k}^\circ - \{0\}$ such that $\underline{i} = (i_1, \ldots i_n) \in I$.

2.4.1. *The vertex reduced amalgamated free products.* For $\underline{i} = (i_1, \ldots, i_n) \in I$, we define a A_{i_1}-A_{i_n}-bimodule $H_{\underline{i}}$. As Hilbert A_{i_n}-module we have:

$$H_{\underline{i}} = \begin{cases} K_{i_1} \underset{B}{\otimes} K_{i_2}^\circ \underset{B}{\otimes} \ldots \underset{B}{\otimes} K_{i_{n-1}}^\circ \underset{B}{\otimes} A_{i_n} & \text{if} \quad n \geq 3, \\ K_{i_1} \underset{B}{\otimes} A_{i_2} & \text{if} \quad n = 2, \\ A_{i_1} & \text{if} \quad n = 1. \end{cases}$$

The left action of A_{i_1} on $H_{\underline{i}}$ is given by the unital $*$-homomorphism defined by

$$\lambda_{\underline{i}} : A_{i_1} \to \mathcal{L}_{A_{i_n}}(H_{\underline{i}}); \quad \lambda_{\underline{i}} = \begin{cases} \rho_{i_1} \underset{B}{\otimes} \mathrm{id} & \text{if} \quad n \geq 2, \\ L_{A_{i_1}} & \text{if} \quad n = 1. \end{cases}$$

We consider, for $k, l \in \{1, 2\}$, the subset $I_{k,l} = \{\underline{i} = (i_1, \ldots, i_n) \in I : i_1 = k \text{ and } i_n = l\}$ and the A_k-A_l-bimodule defined by

$$H_{k,l} = \bigoplus_{\underline{i} \in I_{k,l}} H_{\underline{i}} \quad \text{and} \quad \lambda_{k,l} = \bigoplus_{\underline{i} \in I_{k,l}} \lambda_{\underline{i}} : A_k \to \mathcal{L}_{A_l}(H_{k,l}).$$

For $k \in \{1, 2\}$ we denote by $\overline{k}$ the unique element in $\{1, 2\} \setminus \{k\}$.

Example 2.4. If, for $k \in \{1, 2\}$, E_k is an homomorphism from A_k to B it follows from Proposition 2.3 that $K_k^\circ = \{0\}$. Hence, $H_{k,k} = A_k \oplus K_k \underset{B}{\otimes} K_{\overline{k}}^\circ \underset{B}{\otimes} A_k$ and $H_{\overline{k},k} = K_{\overline{k}} \underset{B}{\otimes} A_k$. Note that, since $K_k \simeq B$, we have $H_{k,k} \simeq A_k \oplus K_{\overline{k}}^\circ \underset{B}{\otimes} A_k \simeq K_{\overline{k}} \underset{B}{\otimes} A_k = H_{\overline{k},k}$. Also we have $H_{k,\overline{k}} = K_k \underset{B}{\otimes} A_{\overline{k}}$ and $H_{\overline{k},\overline{k}} = A_{\overline{k}}$. Again, $H_{k,\overline{k}} \simeq A_{\overline{k}} = H_{\overline{k},\overline{k}}$. Actually the isomorphism of Hilbert A_l-modules $H_{k,l} \simeq H_{\overline{k},l}$ is true in full generality as explained below.

For $k, l \in \{1, 2\}$ we define a unitary $u_{k,l} \in \mathcal{L}_{A_l}(H_{k,l}, H_{\overline{k},l})$, by the following formula. Let $\underline{i} = (i_1, \ldots, i_n) \in I$, with $i_1 = k$ and $i_l = l$. For $\xi \in H_{\underline{i}}$ we define $u_{k,l}\xi \in H_{\overline{k},l}$ in the following way.

- If $n \geq 2$, write $\underline{i} = (k, \underline{i}')$, where $\underline{i}' = (i_2, \ldots, i_n) \in I_{\overline{k},l}$. For $\xi = \rho_k(a)\eta_k \otimes \xi'$, with $a \in A_k$ and $\xi' \in H_{\underline{i}'}$, we define $u_{k,l}\xi := \begin{cases} \eta_{\overline{k}} \otimes \xi & \text{if} \quad E_k(a) = 0, \\ \lambda_{\underline{i}'}(a)\xi' & \text{if} \quad a \in B. \end{cases}$
- If $n = 1$ then $k = l$, $\underline{i} = (l)$ and $\xi \in A_l = H_{\underline{i}}$. We define $u_{k,l}\xi := \eta_{\overline{k}} \otimes \xi$.

It is easy to check that, for all $k, l \in \{1, 2\}$, the operator $u_{k,l}$ commutes with the right actions of A_l on $H_{k,l}$ and $H_{\overline{k},l}$ and extends to a unitary operators, still denoted $u_{k,l}$, in $\mathcal{L}_{A_l}(H_{k,l}, H_{\overline{k},l})$ such that $u_{k,l}^* = u_{\overline{k},l}$. Moreover, the definition of $u_{k,l}$ implies that,

$$(1) \qquad u_{k,l}^* \lambda_{\overline{k},l}(b) u_{k,l} = \lambda_{k,l}(b) \quad \text{for all } b \in B.$$

Definition 2.5. Let $k \in \{1, 2\}$. The *k-vertex-reduced amalgamated free product* is the C*-sub-algebra $A_{v,k} \subset \mathcal{L}_{A_k}(H_{k,k})$ generated by $\lambda_{k,k}(A_k) \cup u_{k,k}^* \lambda_{\overline{k},k}(A_{\overline{k}}) u_{k,k} \subset \mathcal{L}_{A_k}(H_{k,k})$. To be more precise, we use sometime the notation $A_{v,k} = A_1 \overset{k}{\underset{B}{*}} A_2$.

For a fixed $k \in \{1, 2\}$ the relations (1) imply the existence of a unique unital $*$-homomorphism $\pi_k : A_f \to A_{v,k}$ such that $\pi_k(a) = \begin{cases} \lambda_{k,k}(a) & \text{if } a \in A_k, \\ u_{k,k}^* \lambda_{\overline{k},k}(a) u_{k,k} & \text{if } a \in A_{\overline{k}}. \end{cases}$

In the sequel we will denote by ξ_k the vector $\xi_k := 1_{A_k} \in A_k \subset H_{k,k}$. We summarize the fundamental properties of $A_{v,k}$ in the following proposition.

Proposition 2.6. *For all $k \in \{1, 2\}$ the following holds.*

(1) *The morphism π_k is faithful on A_k.*
(2) *If $E_{\overline{k}}$ is GNS-faithful then π_k is faithful on $A_{\overline{k}}$.*
(3) *There exists a unique ucp map $\mathbb{E}_k : A_{v,k} \to A_k$ such that $\mathbb{E}_k(\pi_k(a)) = a \ \forall a \in A_k$ and*

$\mathbb{E}_k(\pi_k(a_1 \ldots a_n)) = 0$ *for all $a = a_1 \ldots a_n \in A_f$ reduced with $n \geq 2$ or $n = 1$ and $a = a_1 \in A_{\overline{k}}^\circ$.*

Moreover, $\mathbb{E}_k$ is GNS-faithful.

(4) *For any unital C*-algebra C with unital $*$-homomorphisms $\nu_k : A_k \to C$ such that*
  - *$\nu_1(b) = \nu_2(b)$ for all $b \in B$,*
  - *C is generated, as a C*-algebra, by $\nu_1(A_1) \cup \nu_2(A_2)$,*
  - *ν_k is faithful and there exists a GNS-faithful ucp map $E : C \to A_k$ such that $E(\nu_k(a)) = a$ for all $a \in A_k$ and*

$E(\nu_{i_1}(a_1) \ldots \nu_{i_n}(a_n)) = 0$ *for all $a = a_1 \ldots a_n \in A_f$ reduced with $n \geq 2$ or $n = 1$ and $a = a_1 \in A_{\overline{k}}^\circ$,*

there exists a unique unital $*$-isomorphism $\nu : A_{v,k} \to C$ such that $\nu \circ \pi_k(a) = \nu_k(a)$ for all $a \in A_1 \cup A_2$. Moreover, ν satisfies $E \circ \nu = \mathbb{E}_k$.

Proof. By definition of π_k we have, if $a \in A_k$, $\langle \xi_k, \pi_k(a)\xi_k \rangle = a$. It follows directly that π_k is faithful on A_k. Moreover, the map $\mathbb{E}_k : A_{v,k} \to A_k$, $x \mapsto \langle \xi_k, x\xi_k \rangle$ satisfies $\mathbb{E}_k(\pi_k(a)) = a$ $\forall a \in A_k$. By the definition of the unitaries $u_{k,l}$ we have, for all $k \in \{1, 2\}$ and all reduced operator $x = a_1 \ldots a_n$ with $a_k \in A_k^\circ$ and $\underline{i} = (i_1, \ldots, i_n) \in I$,

$$(2) \quad \pi_k(a_1 \ldots a_n)\xi_k = \begin{cases} \rho_{i_1}(a_1)\eta_1 \otimes \ldots \otimes \rho_{i_{n-1}}(a_{n-1})\eta_{n-1} \otimes a_n & \text{if } i_1 = k \text{ and } i_n = k, \\ \eta_k \otimes \rho_{i_1}(a_1)\eta_1 \otimes \ldots \otimes \rho_{i_{n-1}}(a_{n-1})\eta_{n-1} \otimes a_n & \text{if } i_1 \neq k \text{ and } i_n = k, \\ \rho_{i_1}(a_1)\eta_1 \otimes \ldots \otimes \rho_{i_n}(a_n)\eta_n \otimes 1_{A_k} & \text{if } i_1 = k \text{ and } i_n \neq k, \\ \eta_k \otimes \rho_{i_1}(a_1)\eta_1 \otimes \ldots \otimes \rho_{i_n}(a_n)\eta_n \otimes 1_{A_k} & \text{if } i_1 \neq k \text{ and } i_n \neq k. \end{cases}$$

Hence we have $\mathbb{E}_k(\pi_k(a_1, \ldots a_n)) = 0$ for all $a = a_1 \ldots a_n \in A_f$ reduced with $\underline{n \geq 2}$ or $n = 1$ and $a = a_1 \in A_{\overline{k}}^\circ$. It also follows easily from the previous set of equations that $\overline{\pi_k(A_f)\xi_k \cdot A_k} = H_{k,k}$. Hence the triple $(H_{k,k}, \mathrm{id}, \xi_k)$ is a GNS construction for $\mathbb{E}_k$. This shows that $\mathbb{E}_k$ is GNS-faithful. Note that the uniqueness statement of the third assertion is obvious since A_f is the linear span of B and the reduced operators. Also, the second statement becomes now obvious since, by the properties of $\mathbb{E}_k$ we have, for all $x \in A_{\overline{k}}$, $\mathbb{E}_k(\pi_k(x)) = \mathbb{E}_k(\pi_k(x - E_{\overline{k}}(x))) + \mathbb{E}_k(\pi_k(E_{\overline{k}}(x))) = \pi_k(E_{\overline{k}}(x))$. It follows easily from this equation that π_k is faithful on $A_{\overline{k}}$ whenever $E_{\overline{k}}$ is GNS-faithful. Indeed, let $x \in A_{\overline{k}}$ such that $\pi_k(a) = 0$. Then, for all $y \in A_{\overline{k}}$ we have $\pi_k(y^*x^*xy) = 0$. Hence, $\pi_k \circ E_{\overline{k}}(y^*x^*xy) = \mathbb{E}_k \circ \pi_k(y^*x^*xy) = 0$ for all $y \in A_{\overline{k}}$. Since π_k is faithful on A_k we find $E_{\overline{k}}(y^*x^*xy) = 0$, for all $y \in A_{\overline{k}}$. Since $E_{\overline{k}}$ is GNS-faithful we conclude that $x = 0$.

(4). The proof is a routine. We write the argument for the convenience of the reader. Let (K, ρ, η) be the GNS construction of E. Since E is GNS-faithful we may and will assume that $\rho = \mathrm{id}$ and $C \subset \mathcal{L}_{A_k}(K)$. By the properties of $\mathbb{E}_k$ and E, the map $U : H_{k,k} \to K$ defined by, for

$x = a_1 \ldots a_n \in A_f$ reduced with $a_k \in A_{i_k}^\circ$, $U(\pi_k(x)\xi_k) := \nu_{i_1}(a_1) \ldots \nu_{i_n}(a_n)\eta$ and, for $x = b \in B$, $U(\pi_k(b)\xi_k) = \nu_1(b)\eta = \nu_2(b)\eta$, is well defined and extends to a unitary $U \in \mathcal{L}_{A_k}(H_{k,k}, K)$. By construction, the map $\nu(x) := UxU^*$, for $x \in A_{v,k}$, satisfies the claimed properties. The uniqueness is obvious. $\qquad\square$

Remark 2.7. It is known that the canonical homomorphism from A_k to A_f is faithful for $k \in \{1,2\}$ without assuming the existence of conditional expectations from A_k to B. However, assertion (1) of Proposition 2.6 gives a very simple proof of this fact, since it shows that the composition of the canonical homomorphism from A_k to A_f with the homomorphism π_k is faithful, which implies that the canonical homomorphism from A_k to A_f itself is faithful.

Example 2.8. Suppose that, for a given $k \in \{1,2\}$, E_k is an homomorphism. Then, as observed in Example 2.4, we have $H_{\overline{k},\overline{k}} = A_{\overline{k}}$ (and $\lambda_{\overline{k},\overline{k}} = L_{A_{\overline{k}}}$). It follows from the definition of $\pi_{\overline{k}}$ that

$$\pi_{\overline{k}}(a) = \begin{cases} L_{A_{\overline{k}}}(a) & \text{if} \quad a \in A_{\overline{k}}, \\ 0 & \text{if} \quad a \in A_k^\circ. \end{cases}$$

Hence, since A_f the closed linear span of $A_{\overline{k}}$ and the reduced operators and $\pi_{\overline{k}} : A_f \to A_{v,\overline{k}}$ is surjective, we find that $A_{v,\overline{k}} = \overline{\pi_{\overline{k}}(A_{\overline{k}})}$. Moreover, since $\pi_{\overline{k}}$ is faithful on $A_{\overline{k}}$ we conclude that the restriction of $\pi_{\overline{k}}$ to $A_{\overline{k}}$ gives an isomorphism $A_{\overline{k}} \simeq A_{v,\overline{k}}$.

Definition 2.9. The *vertex-reduced amalgamated free product* is the C*-algebra obtained by separation and completion of A_f with respect to the C*-semi-norm $\|\cdot\|_v$ on A_f defined by

$$\|x\|_v := \operatorname{Max}\{\|\pi_1(x)\|, \|\pi_2(x)\|\} \quad \text{for all } x \in A_f.$$

We will note it $A_1 \overset{v}{\underset{B}{*}} A_2$ or A_v for simplicity in the rest of this section and let $\pi : A_f \to A_v$ be the canonical surjective unital $*$-homomorphism. Note that, by construction of A_v, for all $k \in \{1,2\}$, there exists a unique unital (surjective) $*$-homomorphism $\pi_{v,k} : A_v \to A_{v,k}$ such that $\pi_{v,k} \circ \pi = \pi_k$. We describe the fundamental properties of the vertex-reduced amalgamated free product in the following proposition. We call a family of ucp maps $\{\varphi_i\}_{i \in I}$, $\varphi_i : A \to B_i$ GNS-faithful if $\cap_{i \in I} \operatorname{Ker}(\pi_i) = \{0\}$, where (H_i, π_i, ξ_i) is a GNS-construction for φ_i. From Proposition 2.6 and the definition of A_v we deduce the following result.

Proposition 2.10. *The following holds.*
  (1) π is faithful on A_k for all $k \in \{1,2\}$.
  (2) For all $k \in \{1,2\}$, there is a unique ucp map $\mathbb{E}_{A_k} : A_v \to A_k$ such that $\mathbb{E}_{A_k} \circ \pi(a) = a$ for all $a \in A_k$ and all $k \in \{1,2\}$ and,

$$\mathbb{E}_{A_k}(\pi(a_1 \ldots a_n)) = 0 \text{ for all } a = a_1 \ldots a_n \in A_f \text{ reduced with } n \geq 2 \text{ or } n = 1 \text{ and } a = a_1 \in A_{\overline{k}}^\circ.$$

   Moreover, the family $\{\mathbb{E}_{A_1}, \mathbb{E}_{A_2}\}$ is GNS-faithful.
  (3) Suppose that C is a unital C-algebra with $*$-homomorphisms $\nu_k : A_k \to C$ such that*
   - *$\nu_1(b) = \nu_2(b)$ for all $b \in B$,*
   - *C is generated, as a C*-algebra, by $\nu_1(A_1) \cup \nu_2(A_2)$,*
   - *ν_1 and ν_2 are faithful and, for all $k \in \{1,2\}$, there exists a ucp map $E_{A_k} : C \to A_k$ such that $E_{A_k} \circ \nu_k(a) = a$ for all $a \in A_k$ and all $k \in \{1,2\}$ and,*

$$E_{A_k}(\nu_{i_1}(a_1) \ldots \nu_{i_n}(a_n)) = 0 \text{ for all } a = a_1 \ldots a_n \in A_f \text{ reduced with } n \geq 2 \text{ or } n = 1 \text{ and } a = a_1 \in A_{\overline{k}}^\circ,$$

   and the family $\{E_{A_1}, E_{A_2}\}$ is GNS-faithful.

Then, there exists a unique unital $*$-isomorphism $\nu : A_v \to C$ such that $\nu \circ \pi(a) = \nu_k(a)$ for all $a \in A_k$ and all $k \in \{1, 2\}$. Moreover, ν satisfies $E_{A_k} \circ \nu = \mathbb{E}_{A_k}$, $k \in \{1, 2\}$.

Proof. (1). It is obvious since, by Proposition 2.6, π_k is faithful on A_k for $k = 1, 2$.

(2). By Proposition 2.6, the maps $\mathbb{E}_{A_k} = \mathbb{E}_k \circ \pi_{v,k}$ satisfy the desired properties and it suffices to check that the family $\{\mathbb{E}_{A_1}, \mathbb{E}_{A_2}\}$ is GNS-faithful. Let $x_0 \in A_f$ be such that $x = \pi(x_0) \in A_v$ satisfies $\mathbb{E}_{A_k}(y^*x^*xy) = 0$ for all $y \in A_v$ and all $k \in \{1, 2\}$. Then, for all $k \in \{1, 2\}$ we have $\mathbb{E}_k(y^*\pi_{v,k}(x^*x)y) = 0$ for all $y \in A_{v,k}$. Since $\mathbb{E}_k$ is GNS-faithful, this implies that $\pi_{v,k}(x) = \pi_k(x_0) = 0$ for all $k \in \{1, 2\}$. Hence, $\|x\|_{A_v} = \mathrm{Max}(\|\pi_1(x_0)\|, \|\pi_2(x_0)\|) = 0$.

(3). The proof is a routine. We include it for the convenience of the reader. Let (L_k, m_k, f_k) be the GNS construction of E_{A_k}. By the universal property of $A_{v,k}$, the C*-algebra C_k generated by $m_k(C) \subset \mathcal{L}_{A_k}(L_k)$ is canonically isomorphic with $A_{v,k}$. Hence, in the remainder of the proof we suppose that $C_k = A_{v,k}$ and, by the universal property of A_f, we have a unital surjective $*$-homomorphism $\nu_f : A_f \to C$ such that $\nu_f|_{A_k} = \nu_k$. Note that, by the identification we made, $m_k \circ \nu_f = \pi_k$. Hence, by construction of A_v, there exists a unique unital (surjective) $*$-homomorphism $\nu : C \to A_v$ such that $\pi_{v,k} \circ \nu = m_k$ for all $k \in \{1, 2\}$. The homomorphism ν satisfies all the claimed properties and it suffices to check that it is faithful. But it is obvious since, by the identity $\pi_{v,k} \circ \nu = m_k$, $k = 1, 2$, it follows that $\mathrm{Ker}(\nu) \subset \mathrm{Ker}(m_1) \cap \mathrm{Ker}(m_2) = \{0\}$, since the pair (E_{A_1}, E_{A_2}) is GNS-faithful. $\qquad\square$

Corollary 2.11. *If both E_1 and E_2 are homomorphisms then there is a canonical isomorphism $A_v \simeq A_1 \underset{B}{\oplus} A_2$, where $A_1 \underset{B}{\oplus} A_2 := \{(a_1, a_2) \in A_1 \oplus A_2 : E_1(a_1) = E_2(a_2)\}$.*

Proof. We use the universal property of A_v described in Proposition 2.10. Define $\nu_k : A_k \to A_1 \underset{B}{\oplus} A_2$ by $\nu_1(x) = (x, E_1(x))$ and $\nu_2(y) = (E_2(y), y)$. It is clear that ν_1 and ν_2 are both faithful unital $*$-homomorphisms such that $\nu_1(b) = \nu_2(b)$ for all $b \in B$. Define $E_{A_k} : A_1 \underset{B}{\oplus} A_2 \to A_k$ by $E_{A_1}(a_1, a_2) = a_1$ and $E_{A_2}(a_1, a_2) = a_2$. Then, for all $k \in \{1, 2\}$, E_k is a unital $*$-homomorphisms such that $E_{A_k} \circ \nu_k(a) = a$ for all $a \in A_k$. In particular both E_1 and E_2 are conditional expectations and, since $\mathrm{Ker}(E_{A_1}) \cap \mathrm{Ker}(E_{A_2}) = \{0\}$, the family $\{E_{A_1}, E_{A_2}\}$ is GNS-faithful. Hence, it suffices to check the condition on the reduced operators. Since $\nu_1(A_1^\circ) = \{(x, 0) : x \in A_1^\circ\}$ and $\nu_2(A_2^\circ) = \{(0, y) : y \in A_2^\circ\}$, we have $\nu_1(A_1^\circ)\nu_2(A_2^\circ) = \nu_2(A_2^\circ)\nu_1(A_1^\circ) = \{0\}$. Hence, it suffices to check the condition on elements $(a_1, a_2) \in \nu_1(A_1^\circ) \cup \nu_2(A_2^\circ)$ which is obvious. $\qquad\square$

2.4.2. *The edge reduced amalgamated free product.* In this section we show how the construction of the edge-reduced (or, in the literature, the reduced) amalgamated free product in full generality is related to the vertex reduced free product we just defined.

For $\underline{i} \in I$, we consider the B-B-module $K_{\underline{i}} = K_{i_1}^\circ \underset{B}{\otimes} \ldots \underset{B}{\otimes} K_{i_n}^\circ$ as Hilbert B-module with the left action of B given by the unital $*$-homomorphism $\rho_{\underline{i}} : B \to \mathcal{L}_B(K_{\underline{i}})$, $\rho_{\underline{i}}(b) = \rho_{i_1}(b) \underset{B}{\otimes} \mathrm{id}$ for all $b \in B$ and we define the Hilbert B-bimodule $K = B \oplus \left(\bigoplus_{\underline{i} \in I} K_{\underline{i}}\right)$.

Example 2.12. If, for some $k \in \{1, 2\}$, E_k is an homomorphism then $K = B \oplus K_{\bar{k}}^\circ \simeq K_{\bar{k}}$. Hence, if both E_1 and E_2 are homomorphisms then $K = B$.

Proposition 2.13. *There are isomorphisms between $H_{k,k} \underset{E_k}{\otimes} B$ and K for $k = 1, 2$ implemented by some unitary V_k. Moreover when we intertwines the representation $\pi_k \otimes 1$ by V_k we get the classical representation of the reduced free product on the space K.*

Proof. Note that, for $\underline{i} = (i_1, \ldots, i_n) \in I$ with $i_1 = i_n = k$ (hence n is odd) we have, if $n = 1$,

$$H_{\underline{i}} \underset{E_k}{\otimes} B = A_k \underset{E_k}{\otimes} B \simeq K_k \simeq K_k^{\circ} \oplus B,$$

and, if $n \geq 3$, $H_{\underline{i}} \underset{E_k}{\otimes} B = K_k \underset{B}{\otimes} \left(K_{\overline{k}}^{\circ} \underset{B}{\otimes} \ldots \underset{B}{\otimes} K_{\overline{k}}^{\circ} \right) \underset{B}{\otimes} K_k \simeq K_{\underline{i}} \oplus K_{\underline{i}'} \oplus K_{\underline{i}''} \oplus K_{\underline{i}'''}$, where $\underline{i}' = (i_2, \ldots, i_n)$, $\underline{i}'' = (i_1, \ldots, i_{n-1})$ and $\underline{i}''' = (i_2, \ldots, i_{n-1})$. Hence the existence of V_k.

It is easy to check that V_k satisfies $V_k(\pi_k(a) \otimes 1)V_k^* = \rho(a)$ for all $a \in A_k$ and all $k \in \{1, 2\}$ where ρ is the (classical) reduced free product representation which we recall here for convenience. For $l \in \{1, 2\}$ define $K(l) = B \oplus \left(\underset{\underline{i} \in I, \, i_1 \neq l}{\bigoplus} K_{\underline{i}} \right)$ and note that we have a unital $*$-homomorphism $\rho_l : B \to \mathcal{L}_B(K(l))$ defined by $\rho_l = \underset{\underline{i} \in I_k, \, i_1 \neq l}{\bigoplus} \rho_{\underline{i}}$. Let $U_l \in \mathcal{L}_B(K_l \underset{\rho_l}{\otimes} K(l), K)$ be the unitary operator defined by

$$
\begin{array}{rccc}
U_l : & K_l \underset{\rho_l}{\otimes} K(l) & \longrightarrow & K \\[2mm]
& \eta_l \underset{\rho_l}{\otimes} B & \overset{\simeq}{\longrightarrow} & B \\[2mm]
& K_l^{\circ} \underset{\rho_l}{\otimes} B & \overset{\simeq}{\longrightarrow} & K_l^{\circ} \\[2mm]
& \eta_l \underset{\rho_l}{\otimes} H_{\underline{i}} & \overset{\simeq}{\longrightarrow} & H_{\underline{i}} \\[2mm]
& K_l^{\circ} \underset{\rho_l}{\otimes} H_{\underline{i}} & \overset{\simeq}{\longrightarrow} & H_{(l,\underline{i})}
\end{array}
$$

where $(l, \underline{i}) = (l, i_1, \ldots, i_n) \in I$ if $\underline{i} = (i_1, \ldots, i_n) \in I$ with $i_1 \neq l$. We define the unital $*$-homomorphisms $\lambda_l : \mathcal{L}_B(K_l) \to \mathcal{L}_B(K)$ by $\lambda_l(x) = U_l(x \otimes 1)U_l^*$.

By definition we have $\lambda_1(\rho_1(b)) = \lambda_2(\rho_2(b))$ for all $b \in B$. It follows that there exists a unique unital $*$-homomorphism $\rho : A_f \to \mathcal{L}_B(K)$ such that $\rho(a) = \lambda_k(a)$ for $a \in A_k$, for all $k \in \{1, 2\}$. $\qquad \square$

Definition 2.14. The *edge-reduced* amalgamated free product is the C*-subalgebra $A_e \subset \mathcal{L}_B(K)$ generated by $\lambda_1(A_1) \cup \lambda_2(A_2) \subset \mathcal{L}_B(K)$. To be more precise, we use sometime the notation $A_e = A_1 \underset{B}{\overset{e}{*}} A_2$.

Example 2.15. If, for some $k \in \{1, 2\}$, E_k is an homomorphism then A_e is the C*-algebra $\overline{\rho_{\overline{k}}(A_{\overline{k}})} \subset \mathcal{L}_B(K_{\overline{k}})$. If both E_1 and E_2 are homomorphisms then $A_e \simeq B$.

The preceding Example shows that the edge reduced amalgamated free product may forget everything about the initial C*-algebras A_1 and A_2 in the extreme degenerated case: it only remembers B. This shows that, in general, one should consider instead the vertex-reduced amalgamated free product. Indeed, even in the extreme degenerated case, the vertex reduced amalgamated free product remembers correctly the C*-algebras A_1 and A_2, as shown in Corollary 2.11.

In the following proposition we recall the properties of A_e. The results below are well known when E_1 and E_2 are GNS-faithful. The proof is similar to the proof of Proposition 2.6 and we leave it to the reader.

Proposition 2.16. *The following holds.*

(1) ρ *is faithful on* B.
(2) *If* E_k *is GNS-faithful then* ρ *is faithful on* A_k.
(3) *There exists a unique ucp map* $\mathbb{E} : A_e \to B$ *such that* $\mathbb{E} \circ \rho(b) = b$ *for all* $b \in B$ *and,*

$$\mathbb{E}(\rho(a_1, \dots a_n)) = 0 \text{ for all } a = a_1 \dots a_n \in A_f \text{ reduced.}$$

Moreover, $\mathbb{E}$ *is GNS-faithful.*

(4) *For any unital C*-algebra* C *with unital* $*$-*homomorphisms* $\nu_k : A_k \to C$ *such that*
   - $\nu_1(b) = \nu_2(b)$ *for all* $b \in B$,
   - C *is generated, as a C*-algebra, by* $\nu_1(A_1) \cup \nu_2(A_2)$,
   - $\nu_1|_B = \nu_2|_B$ *is faithful and there exists a GNS-faithful ucp map* $E : C \to B$ *such that* $E \circ \nu_k(b) = b$ *for all* $b \in B$, $k = 1, 2$, *and,*

$$E(\nu_{i_1}(a_1) \dots \nu_{i_n}(a_n)) = 0 \text{ for all } a = a_1 \dots a_n \in A_f \text{ reduced,}$$

there exists a unique unital $*$-*isomorphism* $\nu : A_e \to C$ *such that* $\nu \circ \rho(a) = \nu_k(a)$ *for all* $a \in A_k$, $k \in \{1, 2\}$. *Moreover,* ν *satisfies* $E \circ \nu = \mathbb{E}$.

Proposition 2.17. *For all* $k \in \{1, 2\}$ *there exists a unique unital* $*$-*homomorphism*

$$\lambda_{v,k} : A_{v,k} \to A_e \quad \text{such that} \quad \lambda_{v,k} \circ \pi_k = \rho.$$

Moreover, $\lambda_{v,k}$ *is faithful on* $\pi_k(A_{\overline{k}})$ *and, if* E_k *is GNS-faithful,* $\lambda_{k,v}$ *is an isomorphism.*

Proof. The formulae $\lambda_{v,k}(x) = V_k(x \otimes 1)V_k^*$ defines a unital $*$-homomorphism $\lambda_{v,k} : A_{v,k} \to A_e$ satisfying $\lambda_{v,k} \circ \pi_k = \rho$. The uniqueness of $\lambda_{v,k}$ is obvious. Let us check that $\lambda_{v,k}$ is faithful on $\pi_k(A_{\overline{k}})$. Suppose that $x \in A_{\overline{k}}$ and $\lambda_{v,k}(\pi_k(x)) = 0$. Then, for all $y \in A_{\overline{k}}$, we have $\rho(y^*x^*xy) = \lambda_{v,k}(\pi_k(y^*x^*xy)) = 0$. Hence, $0 = \mathbb{E} \circ \rho(y^*x^*xy) = \mathbb{E} \circ \rho(E_{\overline{k}}(y^*x^*xy)) = E_{\overline{k}}(y^*x^*xy)$. It follows that $x \in \operatorname{Ker}(\rho_{\overline{k}})$ hence, $\lambda_{\overline{k},k}(x) = \oplus_{\underline{i} \in I_{\overline{k},k}} \rho_{\overline{k}}(x) \otimes 1 = 0$ which implies that $\pi_k(x) = u_{k,k}^* \lambda_{\overline{k},k}(x)u_{k,k} = 0$. The last statement follows from the universal property of A_e since the ucp map $E_k \circ \mathbb{E}_k : A_{v,k} \to B$ is GNS-faithful whenever E_k is GNS-faithful. $\square$

In the next Proposition, we study some associativity properties between the edge-reduced and the vertex-reduced amalgamated free product. The result is interesting in itself and it will be used to obtain easily ucp radial multipliers on the vertex-reduced amalgamated free product.

Proposition 2.18. *Let* A_1, A_2, A_3 *be unital C*-algebras with a common unital C*-subalgebra* B *and conditional expectations* $E_k : A_k \to B$. *After identification of* A_1 *with a C*-subalgebra of both* $A_1 \overset{1}{\underset{B}{*}} A_2$ *and* $A_1 \overset{1}{\underset{B}{*}} A_3$, *the canonical GNS-faithful ucp maps* $A_1 \overset{1}{\underset{B}{*}} A_2 \to A_1$ *and* $A_1 \overset{1}{\underset{B}{*}} A_3 \to A_1$ *become conditional expectations and, with respect to this GNS-faithful conditional expectations, we have canonical isomorphisms*

- $\left(A_1 \overset{1}{\underset{B}{*}} A_2 \right) \overset{e}{\underset{A_1}{*}} \left(A_1 \overset{1}{\underset{B}{*}} A_3 \right) \simeq A_1 \overset{1}{\underset{B}{*}} \left(A_2 \overset{e}{\underset{B}{*}} A_3 \right)$.
- $\left(A_1 \overset{2}{\underset{B}{*}} A_2 \right) \overset{e}{\underset{A_2}{*}} \left(A_3 \overset{2}{\underset{B}{*}} A_2 \right) \simeq \left(A_1 \overset{e}{\underset{B}{*}} A_3 \right) \overset{2}{\underset{B}{*}} A_2$.

Proof. We prove the first point. The proof of the second point is similar. We write $\widetilde{A} = A_1 \underset{B}{\overset{1}{*}} \left(A_2 \underset{B}{\overset{e}{*}} A_3 \right)$. Let $\rho : A_2 \underset{B}{*} A_3 \to A_1 \underset{B}{\overset{e}{*}} A_3$ and $\widetilde{\pi} : A_1 \underset{B}{*} \left(A_2 \underset{B}{\overset{e}{*}} A_3 \right) \to \widetilde{A}$ be the canonical surjections and $\widetilde{\mathbb{E}} : \widetilde{A} \to A_1$ the canonical GNS-faithful ucp map. Define, for $k = 1, 2$, $\nu_k : A_k \to D$ by $\nu_1 = \widetilde{\pi}|_D$ and $\nu_2 = \widetilde{\pi} \circ \rho|_{A_2}$. By definition, $\nu_1(b) = \nu_2(b)$ for all $b \in B$ and ν_1 is faithful. Let C be the C*-subalgebra of $\widetilde{A}$ generated by $\nu_1(A_1) \cup \nu_2(A_2)$. We claim that the exists a (unique) unital faithful $*$-homomorphism $\nu : A_1 \underset{B}{\overset{1}{*}} A_2 \to \widetilde{A}$ such that $\nu \circ \pi_1|_{A_k} = \nu_k$ for $k = 1, 2$, where $\pi_1 : A_1 \underset{B}{*} A_2 \to A_1 \underset{B}{\overset{1}{*}} A_2$ is the canonical surjection. By the universal property of the 1-vertex-reduced amalgamated free product, it suffices to show the following claim, where $E = \widetilde{\mathbb{E}}|_C : C \to A_1$.

Claim. *The ucp map E is GNS-faithful and satisfies $E \circ \nu_1 = id_{A_1}$ and, for all $a = a_1 \dots a_n \in A_f$ reduced with $a_k \in A^\circ_{i_k}$, $E(\nu_{i_1}(a_1) \dots \nu_{i_n}(a_n)) = 0$ whenever $n \geq 2$ or $n = 1$ and $a = a_1 \in A^\circ_2$.*

Proof of the Claim. The fact the E vanishes on the reduced operators (not in A°_1) is obvious, since $\widetilde{\mathbb{E}}$ satisfies the same property. The only non-trivial property to check is the fact that E is GNS-faithful: indeed, it is not true, in general, that the restriction of a GNS-faithful ucp map to a subalgebra is again GNS-faithful. So suppose that there exists $x \in C$ such that $E(y^*x^*xy) = 0$ for all $y \in C$ and let us show that x must be zero. Since $\widetilde{\mathbb{E}} : \widetilde{A} \to A_1$ is GNS-faithful, it suffices to show that $\widetilde{\mathbb{E}}(y^*x^*xy) = 0$ for all $y \in \widetilde{A}$. By hypothesis, we know that it is true for all $y \in C$. Since $\widetilde{A}$ is the closed linear span of $\widetilde{\pi}(A_1)$ and $\widetilde{\pi}(z)$, for $z \in A_1 \underset{B}{*} \left(A_2 \underset{B}{\overset{e}{*}} A_3 \right)$ a reduced operator not in A°_1 and since $\widetilde{\pi}(A_1) \cup \widetilde{\pi} \circ \rho(A_2) \subset C$, it suffices to show that $\widetilde{\mathbb{E}}(y^*x^*xy) = 0$ for $y = \widetilde{\pi}(z)$ and $z = z_1 \dots z_n \in A_1 \underset{B}{*} \left(A_1 \underset{B}{\overset{e}{*}} A_3 \right)$ a reduced operator with letters z_k alternating from A°_1, $\rho(A^\circ_2)$ and $\rho(A^\circ_3)$ and containing at least one letter in $\rho(A^\circ_3)$. Since one of the z_k is in $\rho(A^\circ_3)$ and $x \in C$ we have, by the property of $\widetilde{\mathbb{E}}$, $\widetilde{\mathbb{E}}(y^*(x^*x - \widetilde{\mathbb{E}}(x^*x))y) = 0$. Hence, $\widetilde{\mathbb{E}}(y^*x^*xy) = \widetilde{\mathbb{E}}(y^*\widetilde{\mathbb{E}}(x^*x)y) = \widetilde{\mathbb{E}}(y^*E(x^*x)y) = 0$, since $E(x^*x) = 0$.

End of the Proof of the Proposition. Define, for $k = 1, 3$, the unital $*$-homomorphism $\eta_k : A_k \to \widetilde{A}$ by $\eta_1 = \widetilde{\pi}|_{A_1} = \nu_1$ and $\eta_3 = \widetilde{\pi} \circ \rho|_{A_3}$. Using the universal property of the 1-vertex-reduced amalgamated free product one can show, using exactly the same arguments we used to construct the homomorphism ν, that there exists a (necessarily unique) unital faithful $*$-homomorphism $\eta : A_1 \underset{B}{\overset{1}{*}} A_3 \to \widetilde{A}$ such that $\eta \circ \pi'_1|_{A_k} = \eta_k$ for $k = 1, 3$, where $\pi'_1 : A_1 \underset{B}{*} A_3 \to A_1 \underset{B}{\overset{1}{*}} A_3$ is the canonical surjection. Note that $\nu(b) = \eta(b)$ for all $b \in B$ and $\widetilde{A}$ is generated, as a C*-algebra, by $\nu(A_1 \underset{B}{\overset{1}{*}} A_2) \cup \eta(A_1 \underset{B}{\overset{1}{*}} A_3)$. Since the GNS-faithful ucp map $\widetilde{\mathbb{E}} : \widetilde{A} \to A_1$ obviously satisfies the condition on the reduced operators we may use the universal property of the edge-reduced amalgamated free product to conclude that there exists a canonical $*$-isomorphism $\left(A_1 \underset{B}{\overset{1}{*}} A_2 \right) \underset{A_1}{\overset{e}{*}} \left(A_1 \underset{B}{\overset{1}{*}} A_3 \right) \to \widetilde{A}$. $\square$

Using the previous identifications one can prove the following results about completely positive radial multipliers. For $\underline{i} = (i_1, \ldots, i_n) \in I$ and $l \in \{1, 2\}$ we define the number

$$\underline{i}_l = |\{s \in \{1, \ldots, n\} : i_s = l\}|.$$

Proposition 2.19. *For all $k, l \in \{1, 2\}$ and all $0 < r \le 1$ there exists a unique ucp map $\varphi_r : A_{v,k} \to A_{v,k}$ such that $\varphi_r(\pi_k(a)) = \pi_k(a)$ for all $a \in A_l$ and,*

$$\varphi_r(\pi_k(a_1 \ldots a_n)) = r^{\underline{i}_l} \pi_k(a_1 \ldots a_n) \text{ for all } a_1 \ldots a_n \in A_f \text{ reduced with } a_k \in A_{i_k}^\circ \text{ and } \underline{i} = (i_1, \ldots, i_n).$$

Proof. We first prove the proposition for $k = 1$. We separate the proof in two cases.

Case 1: $l = 2$. Since π_1 is faithful on A_1, we may and will view $A_1 \subset A_{v,1}$. After this identification, the canonical GNS-faithful ucp map $\mathbb{E}_1 : A_{v,1} \to A_1$ becomes a conditonal expectation. Consider the conditional expectation $\tau \otimes \mathrm{id} : C[0,1] \otimes B \to B$, where τ is the integral with respect to the normalized Lebesgue measure on $[0,1]$. We will also view $A_1 \subset A_1 \underset{B}{\overset{1}{*}} (C[0,1] \otimes B)$ so that the canonical GNS-faithful ucp map $\widetilde{\mathbb{E}}_1 : A_1 \underset{B}{\overset{1}{*}} (C[0,1] \otimes B) \to A_1$ is a conditional expectation. Define $\widetilde{A} = A_{v,1} \underset{A_1}{\overset{e}{*}} \left(A_1 \underset{B}{\overset{1}{*}} (C[0,1] \otimes B) \right)$ with respect to the conditional expectations $\mathbb{E}_1$ and $\widetilde{\mathbb{E}}_1$. Since $\mathbb{E}_1$ and $\widetilde{\mathbb{E}}_1$ are GNS-faithful, the edge-reduced and the k-vertex-reduced amalgamated free products coincides for $k = 1, 2$. Hence, we may and will view $A_{v,1} \subset A_1 \overset{1}{\underset{B}{*}} (C[0,1] \otimes B) \subset \widetilde{A}$ and we have a canonical GNS-faithful conditional expectation $\widetilde{E} : \widetilde{A} \to A_{v,1}$. Also, by the first assertion of Proposition 2.18 we have a canonical identification $\widetilde{A} = A_1 \overset{1}{\underset{B}{*}} \widetilde{A}_2$, where $\widetilde{A}_2 = A_2 \overset{e}{\underset{B}{*}} (C[0,1] \otimes B)$. Let $\widetilde{\rho}_2 : A_2 \underset{B}{*} C[0,1] \otimes B \to \widetilde{A}_2$ be the canonical surjection from the full to the edge-reduced amalgamated free product and $\widetilde{\pi} : A_1 \underset{B}{*} \widetilde{A}_2 \to A_1 \overset{1}{\underset{B}{*}} \widetilde{A}_2 = \widetilde{A}$ be the canonical surjection from the full to the vertex-reduced amalgamated free product. Fix $t \in \mathbb{R}$ and define the unitary $v_t \in C[0,1]$ by $v_t(x) = e^{2i\pi t x}$. Let $\rho_t = |\tau(v_t)|^2$ and $u_t = \widetilde{\pi} \circ \widetilde{\rho}_2(v_t \otimes 1_B) \in \widetilde{A}$. Define the unital $*$-homomorphisms $\nu_1 = \widetilde{\pi}|_{A_1} : A_1 \to \widetilde{A}$ and $\nu_2 : \widetilde{A}_2 \to \widetilde{A}$ by $\nu_2(x) = u_t \widetilde{\pi}(x) u_t^*$. Note that ν_1 is faithful. To simplify the notations we put $\widetilde{A}_1 := A_1$.

Claim. *For all $x = x_1 \ldots x_n \in A_1 \underset{B}{*} \widetilde{A}_2$ reduced with $a_k \in \widetilde{A}_{i_k}^\circ$ and $\underline{i} = (i_1, \ldots, i_n)$ one has:*

$$\widetilde{\mathbb{E}}(\nu_{i_1}(x_1) \ldots \nu_{i_n}(x_n)) = \begin{cases} \rho_t^{\underline{i}_l} \widetilde{\pi}(x_1 \ldots x_n) & \text{if } \widetilde{\pi}(x) \in A_{v,1} \\ 0 & \text{if } \widetilde{\mathbb{E}}(\widetilde{\pi}(x)) = 0. \end{cases}$$

Proof of the Claim. Note that $\widetilde{\pi}(x) \in A_{v,1}$. if and only if the letters x_k of x are alternating from A_1° and $\widetilde{\rho}_2(A_2^\circ)$ and $\widetilde{\mathbb{E}}(\widetilde{\pi}(x)) = 0$ if and only if one of the letters of x comes from $\rho_2((C[0,1] \otimes B)^\circ)$. We prove the formula by induction on n. If $n = 1$ we have either $x \in A_1^\circ$ in that case $\widetilde{\mathbb{E}}(\nu_1(x)) = \widetilde{\mathbb{E}}(\widetilde{\pi}((x)) = \widetilde{\pi}(x)$ or $x \in \widetilde{\rho}_2(\widetilde{A}_2^\circ)$ and

$$\begin{aligned}
\widetilde{\mathbb{E}}(\nu_2(x)) &= \widetilde{\mathbb{E}}(u_t\widetilde{\pi}(x)u_t^*) \\
&= \widetilde{\mathbb{E}}((u_t - \tau(v_t))\widetilde{\pi}(x)(u_t^* - \overline{\tau(v_t)})) + \tau(v_t)\widetilde{\mathbb{E}}(\widetilde{\pi}(x)(u_t^* - \overline{\tau(v_t)})) \\
&\quad + \overline{\tau(v_t)}\widetilde{\mathbb{E}}((u_t - \tau(v_t))\widetilde{\pi}(x)) + |\tau(v_t)|^2\widetilde{\mathbb{E}}(\widetilde{\pi}(x)) \\
&= |\tau(v_t)|^2\widetilde{\mathbb{E}}(\widetilde{\pi}(x)) = \rho_t\widetilde{\mathbb{E}}(\widetilde{\pi}(x)).
\end{aligned}$$

Hence, $\widetilde{\mathbb{E}}(\nu_2(x)) = \begin{cases} \rho_t\widetilde{\pi}(x) & \text{if} \quad \widetilde{\pi}(x) \in A_{v,1} \\ 0 & \text{if} \quad \widetilde{\mathbb{E}}(\widetilde{\pi}(x)) = 0 \end{cases}$

This proves the formula for $n = 1$. Suppose that the formulae holds for a given $n \geq 1$. Let $x = x_1 \ldots x_{n+1}$ be reduced with $x_k \in \widetilde{A}_{i_k}^\circ$ and define $x' = x_1 \ldots x_n$ and $z = \nu_{i_1}(x_1) \ldots \nu_{i_n}(x_n)$. Let $\underline{i} = (i_1, \ldots, i_{n+1})$ and $\underline{i}' = (i_1, \ldots, i_n)$.

Suppose that $x_{n+1} \in A_1^\circ$. Then $\underline{i}_2 = \underline{i}'_2$ and,

$$\widetilde{\mathbb{E}}(\nu_{i_1}(x_1) \ldots \nu_{i_n}(x_n)\nu_{i_{n+1}}(x_{n+1})) = \widetilde{\mathbb{E}}(\nu_{i_1}(x_1) \ldots \nu_{i_n}(x_n)\widetilde{\pi}(x_{n+1})) = \widetilde{\mathbb{E}}(z)\widetilde{\pi}(x_{n+1}).$$

Hence, if $\widetilde{\pi}(x) \in A_{v,1}$ then also $\widetilde{\pi}(x') \in A_{v,1}$ and we have, by the induction hypothesis,

$$\widetilde{\mathbb{E}}(\nu_{i_1}(x_1) \ldots \nu_{i_n}(x_n)\nu_{i_{n+1}}(x_{n+1})) = \rho_t^{i'_2}\widetilde{\pi}(x')\widetilde{\pi}(x_{n+1}) = \rho_t^{i_2}\widetilde{\pi}(x).$$

If $\widetilde{\mathbb{E}}(\widetilde{\pi}(x)) = 0$ then also $\widetilde{\mathbb{E}}(\widetilde{\pi}(x')) = 0$ and we have, by the induction hypothesis, $\widetilde{\mathbb{E}}(z) = 0$ so $\widetilde{\mathbb{E}}(\nu_{i_1}(x_1) \ldots \nu_{i_n}(x_n)\nu_{i_{n+1}}(x_{n+1})) = 0$.

Suppose now that $x_{n+1} \in \widetilde{A}_2^\circ$ then $x_n \in A_1^\circ$ and we have,

$$\begin{aligned}
\widetilde{\mathbb{E}}(z\nu_{i_{n+1}}(x_{n+1})) &= \widetilde{\mathbb{E}}(zu_t\widetilde{\pi}(x_{n+1})u_t^*) \\
&= \widetilde{\mathbb{E}}(z(u_t - \tau(v_t))\widetilde{\pi}(x_{n+1})(u_t^* - \overline{\tau(v_t)})) + \tau(v_t)\widetilde{\mathbb{E}}(z\widetilde{\pi}(x_{n+1})(u_t^* - \overline{\tau(v_t)})) \\
&\quad + \overline{\tau(v_t)}\widetilde{\mathbb{E}}(z(u_t - \tau(v_t))\widetilde{\pi}(x_{n+1})) + |\tau(v_t)|^2\widetilde{\mathbb{E}}(z\widetilde{\pi}(x_{n+1})) \\
&= |\tau(v_t)|^2\widetilde{\mathbb{E}}(z\widetilde{\pi}(x_{n+1})) = \rho_t\widetilde{\mathbb{E}}(z\widetilde{\pi}(x_{n+1})).
\end{aligned}$$

Hence, if $\widetilde{\pi}(x) \in A_{v,1}$ then also $\widetilde{\pi}(x') \in A_{v,1}$ and $x_{n+1} \in A_2^\circ$ so $\widetilde{\pi}(x_{n+1}) \in A_{v,1}$ and $\underline{i}_2 = \underline{i}'_2 + 1$. By the preceding computation and the induction hypothesis we find:

$$\widetilde{\mathbb{E}}(z\nu_{i_{n+1}}(x_{n+1})) = \rho_t\widetilde{\mathbb{E}}(z\widetilde{\pi}(x_{n+1})) = \rho_t\widetilde{\mathbb{E}}(z)\widetilde{\pi}(x_{n+1}) = \rho_t\rho_t^{i'_2}\widetilde{\pi}(x')\widetilde{\pi}(x_{n+1}) = \rho_t^{i_2}\widetilde{\pi}(x).$$

Finally, if $\widetilde{\mathbb{E}}(\widetilde{\pi}(x)) = 0$, we need to prove that $\widetilde{\mathbb{E}}(z\widetilde{\pi}(x_{n+1})) = 0$. Note that $z = \nu_{i_1}(x_1) \ldots \nu_{i_{n-1}}(x_{n-1})x_n$ since $x_n \in A_1^\circ$. Hence, if $\widetilde{\mathbb{E}}(\widetilde{\pi}(x')) = 0$ so by the induction hypothesis we have $\widetilde{\mathbb{E}}(z) = 0$, z may be written as a sum of reduced operators, containing at least one letter from $\widetilde{\rho}_2((C[0,1] \otimes B)^\circ)$ and ending with a letter from A_1°. It follows that $z\widetilde{\pi}(x_{n+1})$ may be written as a sum of reduced operators, containing at least one letter from $\widetilde{\rho}_2((C[0,1] \otimes B)^\circ)$. Hence, $\widetilde{\mathbb{E}}(z\widetilde{\pi}(x_{n+1})) = 0$. Eventually, if $\widetilde{\mathbb{E}}(\widetilde{\pi}(x)) = 0$ and $\widetilde{\mathbb{E}}(\widetilde{\pi}(x')) \in A_{v,1}$ then, $x_1, \ldots x_n \in A_1^\circ \cup A_2^\circ$ but $\widetilde{\mathbb{E}}(\widetilde{\pi}(x_{n+1})) = 0$. It follows that $z = \nu_{i_1}(x_1) \ldots \nu_{i_{n-1}}(x_{n-1})x_n$ may be written as a sum of reduced operators ending with a letter from A_1°. Hence, $z\widetilde{\pi}(x_{n+1})$ be be written as a sum of reduced operators containing at least one letter from $\widetilde{\rho}_2((C[0,1] \otimes B)^\circ)$. Hence, $\widetilde{\mathbb{E}}(z\widetilde{\pi}(x_{n+1})) = 0$.

End of the proof of the Proposition. By the Claim, $\mathbb{E}_1 \circ \widetilde{\mathbb{E}}(\nu_{i_1}(x_1) \ldots \nu_{i_n}(x_n)) = 0$ for all reduced operators $x = x_1 \ldots x_n \in A_1 \underset{B}{*} \widetilde{A}_2$ which are not in A_1 and, we obviously have, $\mathbb{E}_1 \circ \widetilde{\mathbb{E}} \circ \nu_1 = \mathrm{id}_{A_1}$.

Viewing $\widetilde{A} = A_1 \overset{1}{\underset{B}{*}} \widetilde{A}_2$ and using the universal property of the vertex-reduced amalgamated free product, there exists, for all $t \in \mathbb{R}$, a unique unital $*$-isomorphism $\alpha_t : \widetilde{A} \to \widetilde{A}$ such that $\alpha_t(\widetilde{\pi}(a)) = \widetilde{\pi}(a)$ if $a \in A_1$ and $\alpha_t((\widetilde{\pi}(x)) = u_t \widetilde{\pi}(x) u_t^*$ if $x \in A_2 \overset{e}{\underset{B}{*}} (C[0,1] \otimes B)$. In particular, it follows from the Claim that $\widetilde{\mathbb{E}} \circ \alpha_t|_{A_{v,1}} : A_{v,1} \to A_{v,1}$, which is a ucp map, satisfies the properties of the map φ_r described in the statement of the Proposition, with $r = \rho_t = \left| \frac{\sin(\pi t)}{\pi t} \right|^2$. This concludes the proof.

Case 2: $l = 1$. The proof is similar. This time, the automorphism $\alpha_t : \widetilde{A} \to \widetilde{A}$ is defined, by the universal property, starting with the maps $\nu_1 = \widetilde{\pi}|_{A_1} : A_1 \to \widetilde{A}$ and $\nu_2 : \widetilde{A}_2 \to \widetilde{A}$ defined by $\nu_1(a) = u_t \widetilde{\pi}(a) u_t^*$ and $\nu_2(x) = \widetilde{\pi}(x)$. The reminder of the proof is the same.

The proof for $k = 2$ is the same, using the second assertion of Proposition 2.18. $\qquad \square$

3. K-EQUIVALENCE BETWEEN THE FULL AND REDUCED AMALGAMATED FREE PRODUCTS

Let A_1, A_2 be two unital C*-algebra with a common C*-subalgebra $B \subset A_k$, $k = 1, 2$ and denote by A_f the full amalgamated free product.

Let $A := A_1 \overset{v}{\underset{B}{*}} A_2$ be the vertex-reduced amalgamated free product. For $k = 1, 2$, let E_{A_k} (resp. E_B) be the canonical conditional expectation from A to A_k (reps. from A to B). We will denote by the same symbol $\mathcal{A}$ the set of reduced operators viewed in A or in A_f. Recall that the linear span of $\mathcal{A}$ and B is a weakly dense unital $*$-subalgebra of A (resp. A_f).

We denote by $\lambda : A_f \to A$ the canonical surjective unital $*$-homomorphism which is the identity on $\mathcal{A}$. In this section we prove the following result.

Theorem 3.1. $[\lambda] \in \mathrm{KK}(A_f, A)$ *is invertible.*

The following Lemma is well known (see [Ve04, Lemma 3.1]). We include a proof for the convenience of the reader.

Lemma 3.2. *Let* $n \geq 1$, $a_k \in A_{l_k}^{\circ}$ *for* $1 \leq k \leq n$, *and* $a = a_1 \ldots a_n \in \mathcal{A}$ *a reduced word. One has*

$$E_{A_k}(a^*a) = E_B(a^*a) \quad whenever \quad l_n \neq k.$$

Proof. We prove it for $k = 1$ by induction on n. The proof for $k = 2$ is the same.

It's obvious for $n = 1$. Suppose that $n \geq 2$, define $b = E_B(a_1^*a_1)^{\frac{1}{2}}$, $x = (ba_2) \ldots a_n$. One has:

$$E_{A_1}(a^*a) = E_{A_1}(a_n^* \ldots a_1^* a_1 \ldots a_n) = E_{A_1}(a_n^* \ldots a_2^* E_B(a_1^*a_1) a_2 \ldots a_n) = E_{A_1}(x^*x) = E_B(x^*x),$$

where we applied the induction hypothesis to get the last equality. Since the same computation gives $E_B(a^*a) = E_B(x^*x)$, this concludes the proof. $\qquad \square$

We denote by (H_k, π_k, ξ_k) (resp. (K, ρ, η)) the GNS construction of E_{A_k} (resp. E_B). We may and will assume that $A \subset \mathcal{L}_{A_k}(H_k)$ and $\pi_k = \mathrm{id}$.

Observe that the Hilbert A_k-module $\xi_k.A_k \subset H_k$ is orthogonally complemented i.e. $H_k = \xi_k.A_k \oplus H_k^{\circ}$, as Hilbert A_k-modules, where H_k° is the closure of $\{a\xi_k : a \in A, E_{A_k}(a) = 0\}$.

We now define a partial isometry $F_k \in \mathcal{L}_{A_k}(H_k, K \underset{B}{\otimes} A_k)$ in the following way. First we put $F_k(\xi_k.a) = 0$ for all $a \in A_k$. Then, it follows from Lemma 3.2 that we can define an isometry $F_k : H_k^\circ \to K \underset{B}{\otimes} A_k$ by the following formula:

$$F_k(a_1 \ldots a_n \xi_k) = \begin{cases} \rho(a_1 \ldots a_n)\eta \underset{B}{\otimes} 1 & \text{if} \quad l_n \neq k \\ \rho(a_1 \ldots a_{n-1})\eta \underset{B}{\otimes} a_n & \text{if} \quad l_n = k \end{cases} \quad \text{for all} \quad a_1 \ldots a_n \in A \text{ a reduced operator.}$$

Hence, $F_k \in \mathcal{L}_{A_k}(H_k, K \underset{B}{\otimes} A_k)$ is a well defined partial isometry such that $1 - F_k^* F_k$ is the orthogonal projection onto $\xi_k.A_k$ and, $1 - F_k F_k^*$ is the orthogonal projection onto

$$(\eta \otimes 1).A_k \oplus \overline{\text{Span}}\{\rho(a_1 \ldots a_n)\eta \otimes 1 : a = a_1 \ldots a_n \in A \text{ reduced with } l_n = k\}.A_k.$$

We will denote in the sequel q_0 the orthogonal projection of K onto $\eta.B$, and for $l = 1, 2$ q_l the projection in K such that $F_l F_l^* = q_l \otimes_{A_l} 1$. It is clear that $1 = q_1 + q_2 + q_0$ and that all the projections commutes. Define also $\overline{F}_l = F_l + \theta_{\eta \otimes_B 1, \xi_l}$. It is again clear that $\overline{F}_l$ is an isometry and $\overline{F}_l \overline{F}_l^* = q_l + q_0 = 1 - q_k$ for $k \neq l$.

Lemma 3.3. *For $k = 1, 2$ the following holds.*

    (1) $\rho(a)F_k = F_k a \in \mathcal{L}_{A_k}(H_k, K \underset{B}{\otimes} A_k)$ for all $a \in A_k$.

    (2) $\text{Im}(\rho(a)aF_k - F_k a) \subset (\rho(a)\eta \underset{B}{\otimes} 1).A_k \oplus (\eta \underset{B}{\otimes} 1).A_k$ for all $a \in A_l^\circ$ with $l \neq k$.

    (3) $\rho(x)F_k - F_k x \in \mathcal{K}_{A_k}(H_k, K \underset{B}{\otimes} A_k)$ for all $x \in A$.

    (4) $\rho(a)\overline{F}_k = \overline{F}_k a$ for all $a \in A_l$ with $l \neq k$ and $\rho(x)\overline{F}_k - \overline{F}_k x \in \mathcal{K}_{A_k}(H_k, K \underset{B}{\otimes} A_k)$ for all
        $x \in A$.

Proof. We prove the lemma for $k = 1$. The proof for $k = 2$ is the same.

(1). When $a \in B$ the commutation is obvious hence we may and will assume that $a \in A_1^\circ$. One has $F_1 a \xi_1 = 0 = \rho(a)F_1 \xi_1$. Let now $n \geq 1$ and $x = a_1 \ldots a_n \in A$, $a_k \in A_{l_k}^\circ$, be a reduced operator with $E_{A_1}(x) = 0$. It suffices to show that $F_1 a x \xi_1 = \rho(a)F_1 x \xi_1$. If $n = 1$ we must have $x \in A_2^\circ$ and $F_1 a x \xi_1 = \rho(ax)\eta \otimes 1 = \rho(a)F_1 x \xi_1$. Suppose that $n \geq 2$. If $l_1 = 2$ then ax is reduced and ends with a letter from $A_{l_n}^\circ$. It follows that $F_1 a x \xi_2 = \rho(a)F_1 x \xi_2$. It $l_1 = 1$ then we can write $ax = (aa_1)^\circ a_2 \ldots a_n + E_B(aa_1)a_2 \ldots a_n$. Since $a_2 \ldots a_n$ is reduced and ends with l_n we find again that $F_1 a x \xi_1 = \rho(a)F_1 x \xi_1$.

(2). Let $a \in A_2^\circ$ and put $X_a = (\rho(a)\eta \underset{B}{\otimes} 1).A_k \oplus (\eta \underset{B}{\otimes} 1).A_k$. We have $F_1 a \xi_1 = \rho(a)\eta \otimes 1$ and $\rho(a)F_1 \xi_1 = 0$ hence, $(\rho(a)F_1 - F_1 a)\xi_1 = -\rho(a)\eta \otimes 1 \in X_a$. Let now $n \geq 1$ and $x = a_1 \ldots a_n \in A$, $a_k \in A_{l_k}^\circ$, be a reduced operator with $E_{A_1}(x) = 0$. If $n = 1$ we must have $x \in A_2^\circ$. It follows that $F_1 a x \xi_1 = F_1(ax)^\circ \xi_1 + F_1 E_B(ax)\xi_1 = \rho((ax)^\circ)\eta \otimes 1$ and $\rho(a)F_1 x \xi_1 = \rho(ax)\eta \otimes 1$. Hence, $(\rho(a)F_1 - F_1 a)x\xi_1 = E_B(ax)\eta \otimes 1 = (\eta \otimes 1).E_B(ax) \in X_a$. If $n \geq 2$, arguing as in the proof of (1), we see that $F_1 a x \xi_1 = \rho(a)F_1 x \xi_1$. Hence, $\text{Im}(\rho(a)F_k - F_k a) \subset X_a$.

(3). It is obvious since A is generated, as a C*-algebra, by A_1 and A_2 and, by assertions (1) and (2), $\rho(a)F_k - F_k a \in \mathcal{K}_{A_k}(H_k, K \underset{B}{\otimes} A_k)$ for all $a \in A_1 \cup A_2$.

(4). The second part is obvious in view of (3), so let's concentrate on the exact commutation. Let $a \in A_2^\circ$. Clearly $\overline{F}_1 a \xi_1 = F_1 a \xi_1 = \rho(a)\eta \otimes 1$ and $\rho(a)\overline{F}_1 \xi_1 = \rho(a)\eta \otimes 1$. Let now $n \geq 1$ and

$x = a_1 \ldots a_n \in A$, $a_k \in A^\circ_{l_k}$, be a reduced operator with $E_{A_1}(x) = 0$. If $n = 1$ we must have $x \in A^\circ_2$. It follows that $\overline{F}_1 ax\xi_1 = F_1(ax)^\circ\xi_1 + \theta_{\eta\otimes 1,\xi_l}E_B(ax)\xi_1 = \rho((ax)^\circ)\eta \otimes 1 + E_B(ax)\eta \otimes 1$ and $\rho(a)\overline{F}_1 x\xi_1 = F_1 x\xi_1 = \rho(ax)\eta \otimes 1$. If $n \geq 2$, arguing as in the proof of (1), we see that $\overline{F}_1 ax\xi_1 = F_1 ax\xi_1 = \rho(a)F_1 x\xi_1 = \rho(a)\overline{F}_1 x\xi_1$.

$\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\square$

We define the following Hilbert A_f-modules:

$$H_m = H_1 \underset{A_1}{\otimes} A_f \oplus H_2 \underset{A_2}{\otimes} A_f \quad \text{and} \quad K_m = K \underset{B}{\otimes} A_f = \left(K \underset{B}{\otimes} A_k\right) \underset{A_k}{\otimes} A_f,$$

with the canonical representations $\pi : A \to \mathcal{L}_{A_f}(H_m)$, $\pi(x) = x \underset{A_1}{\otimes} 1_{A_f} \oplus x \underset{A_2}{\otimes} 1_{A_f}$ and $\bar{\rho} : A \to \mathcal{L}_{A_f}(K)$, $\bar{\rho}(x) = \rho(x) \underset{B}{\otimes} 1_{A_f}$. We consider, for $k = 1, 2$, the partial isometry

$$F_k \underset{A_k}{\otimes} 1_{A_f} \in \mathcal{L}_{A_f}(H_k \underset{A_k}{\otimes} A_f, (K \underset{B}{\otimes} A_k) \underset{A_k}{\otimes} A_f).$$

Observe that $F_1 \underset{A_1}{\otimes} 1_{A_f}$ and $F_2 \underset{A_2}{\otimes} 1_{A_f}$ have orthogonal images. Indeed, the image of $F_k \underset{A_k}{\otimes} 1_{A_f}$ is the closed linear span of $\{\rho(a_1 \ldots a_n)\eta \underset{B}{\otimes} y : y \in A_f \text{ and } a_1 \ldots a_n \in A \text{ reduced with } a_n \notin A^\circ_k\}$. Hence the operator $F \in \mathcal{L}_{A_f}(H_m, K_m)$ defined by $F = F_1 \underset{A_1}{\otimes} 1_{A_f} \oplus F_2 \underset{A_2}{\otimes} 1_{A_f}$ is a partial isometry such that $1 - FF^*$ is the orthogonal projection onto $(\eta \underset{B}{\otimes} 1_{A_f}).A_f$ and $1 - F^*F$ is the orthogonal projection onto $(\xi_1 \underset{A_1}{\otimes} 1_{A_f}).A_f \oplus (\xi_2 \underset{A_2}{\otimes} 1_{A_f}).A_f$. In particular $1 - F^*F, 1 - FF^* \in \mathcal{K}_{A_f}(H_m, K_m)$. Moreover, it follows from Lemma 3.3 that $F\pi(x) - \bar{\rho}(x)F \in \mathcal{K}_{A_f}(H_m, K_m)$ for all $x \in A$. Hence, we get an element $\alpha = [(H_m \oplus K_m, \pi \oplus \bar{\rho}, F)] \in \mathrm{KK}(A, A_f)$.

To prove Theorem 3.1 it suffices to prove that $\alpha \underset{A_f}{\otimes} [\lambda] = [\mathrm{id}_A]$ in $\mathrm{KK}(A, A)$ and $[\lambda] \underset{A}{\otimes} \alpha = [\mathrm{id}_{A_f}]$ in $\mathrm{KK}(A_f, A_f)$. We prove the easy part in the next Proposition.

Proposition 3.4. *One has* $[\lambda] \underset{A}{\otimes} \alpha = [id_{A_f}]$ *in* $\mathrm{KK}(A_f, A_f)$.

Proof. Observe that $[\lambda] \underset{A}{\otimes} \alpha = [(H_m \oplus K_m, \pi_m \oplus \rho_m, F)]$ where $\pi_m = \pi \circ \lambda : A_f \to \mathcal{L}_{A_f}(H_m)$ and $\rho_m = \bar{\rho} \circ \lambda : A_f \to \mathcal{L}_{A_f}(K_m)$. Hence, $[\lambda] \underset{A}{\otimes} \alpha - [\mathrm{id}_{A_f}]$ is represented by the Kasparov triple $(H_m \oplus \widetilde{K}_m, \pi_m \oplus \widetilde{\rho}_m, \widetilde{F})$, where $\widetilde{K}_m = K_m \oplus A_f$ and $\widetilde{\rho}_m(x) = \rho_m(x) \oplus x$, where we view $A_f = \mathcal{L}_{A_f}(A_f)$ by left multiplication. Finally, $\widetilde{F} \in \mathcal{L}_{A_f}(H_m, \widetilde{K}_m)$ is the unitary defined by

$$\widetilde{F}(\xi_1 \underset{A_1}{\otimes} 1_{A_f}) = \eta \underset{B}{\otimes} 1_{A_f}, \quad \widetilde{F}(\xi_2 \underset{A_2}{\otimes} 1_{A_f}) = 1_{A_f} \quad \text{and,}$$

$$\widetilde{F}(\xi) = F(\xi) \text{ for all } \xi \in H_m \ominus \left((\xi_1 \underset{A_1}{\otimes} 1_{A_f}).A_f \oplus (\xi_2 \underset{A_2}{\otimes} 1_{A_f}).A_f\right).$$

We collect some computations in the following claim.

Claim. *Let* $v \in \mathcal{L}_{A_f}(H_m)$ *be the self-adjoint unitary defined by the identity on* $H_m \ominus ((\xi_1 \underset{A_1}{\otimes} 1_{A_f}).A_f \oplus (\xi_2 \underset{A_2}{\otimes} 1_{A_f}).A_f)$ *and* $v(\xi_1 \underset{A_1}{\otimes} 1_{A_f}) = \xi_2 \underset{A_2}{\otimes} 1_{A_f}$, $v(\xi_2 \underset{A_2}{\otimes} 1_{A_f}) = \xi_1 \underset{A_1}{\otimes} 1_{A_f}$. *One has:*

   (1) $\widetilde{F}^*\widetilde{\rho}_m(b)\widetilde{F} = \pi_m(b)$ *and* $v^*\pi_m(b)v = \pi_m(b)$ *for all* $b \in B$.

(2) $\widetilde{F}^*\widetilde{\rho}_m(a)\widetilde{F} = v^*\pi_m(a)v$ for all $a \in A_1$.

(3) $\widetilde{F}^*\widetilde{\rho}_m(a)\widetilde{F} = \pi_m(a)$ for all $a \in A_2$.

Proof of the claim. The proof of (1) is obvious and we leave it to the reader.

(2). By (1), it suffices to prove (2) for $a \in A_1^\circ$. Let $a \in A_1^\circ$. One the one hand:

$$\widetilde{F}^*\widetilde{\rho}_m(a)\widetilde{F}\xi_1 \underset{A_1}{\otimes} 1_{A_f} = \widetilde{F}^*(\rho(a)\eta \underset{B}{\otimes} 1_{A_f}) = a\xi_2 \underset{A_2}{\otimes} 1_{A_f} \quad \text{and} \quad \widetilde{F}^*\widetilde{\rho}_m(a)\widetilde{F}\xi_2 \underset{A_2}{\otimes} 1_{A_f} = \widetilde{F}^*(a) = \xi_2 \underset{A_2}{\otimes} a.$$

One the other hand:

$$v^*\pi_m(a)v\xi_1 \underset{A_1}{\otimes} 1_{A_f} = v^*(a\xi_2 \underset{A_2}{\otimes} 1_{A_f}) = a\xi_2 \underset{A_2}{\otimes} 1_{A_f} \quad \text{and} \quad v^*\pi_m(a)v\xi_2 \underset{A_2}{\otimes} 1_{A_f} = v^*(a\xi_1 \underset{A_1}{\otimes} 1_{A_f}) = \xi_2 \underset{A_2}{\otimes} a.$$

Let now $x = a_1 \ldots a_n \in A$ be reduced operator with $a_k \in A_{l_k}^\circ$. We prove by induction on n that $\widetilde{F}^*\widetilde{\rho}_m(a)\widetilde{F}x\xi_k \underset{A_k}{\otimes} 1_{A_f} = v^*\pi_m(a)vx\xi_k \underset{A_k}{\otimes} 1_{A_f}$ for all $k \in \{1, 2\}$. Suppose that $n = 1$ so $x \in A_1^\circ \cup A_2^\circ$ and let $k \in \{1, 2\}$ such that $x \notin A_k^\circ$ (the case $x \in A_k^\circ$ has been done before). We have:

$$\widetilde{F}^*\widetilde{\rho}_m(a)\widetilde{F}x\xi_k \underset{A_k}{\otimes} 1_{A_f} = \widetilde{F}^*(\rho(ax)\eta \underset{B}{\otimes} 1_{A_f}) = \begin{cases} (ax)^\circ\xi_2 \underset{A_2}{\otimes} 1_{A_f} + \xi_1 \underset{A_1}{\otimes} E_B(ax) & \text{if} \quad x \in A_1^\circ, \\ ax\xi_1 \underset{A_1}{\otimes} 1_{A_f} & \text{if} \quad x \in A_2^\circ. \end{cases}$$

One the other hand we have:

$$v^*\pi_m(a)vx\xi_k \underset{A_k}{\otimes} 1_{A_f} = v^*(ax\xi_k \underset{A_k}{\otimes} 1_{A_f}) = \begin{cases} (ax)^\circ\xi_2 \underset{A_2}{\otimes} 1_{A_f} + \xi_1 \underset{A_1}{\otimes} E_B(ax) & \text{if} \quad x \in A_1^\circ \ (k = 2), \\ ax\xi_1 \underset{A_1}{\otimes} 1_{A_f} & \text{if} \quad x \in A_2^\circ \ (k = 1). \end{cases}$$

Finally, suppose that $n \geq 2$ and the formula holds for $n - 1$. Write $ax = y + z$, where, if $l_1 = 1$, $y = (aa_1)^\circ a_2 \ldots a_n$ and $z = E_B(aa_1)a_2 \ldots a_n$ and, if $l_1 = 2$, $y = ax$ and $z = 0$. Observe that, in both cases, y is a reduced operator ending with a letter from $A_{l_n}^\circ$ and z is either 0 or a reduced operator ending with a letter from $A_{l_n}^\circ$. By the induction hypothesis, we may and will assume that $k \neq l_n$. We have:

$$\begin{aligned} \widetilde{F}^*\widetilde{\rho}_m(a)\widetilde{F}x\xi_k \underset{A_k}{\otimes} 1_{A_f} &= \widetilde{F}^*(\rho(ax)\eta \underset{B}{\otimes} 1_{A_f}) = \widetilde{F}^*(\rho(y)\eta \underset{B}{\otimes} 1_{A_f}) + \widetilde{F}^*(\rho(z)\eta \underset{B}{\otimes} 1_{A_f}) \\ &= y\xi_k \underset{A_k}{\otimes} 1_{A_f} + z\xi_k \underset{A_k}{\otimes} 1_{A_f} = ax\xi_k \underset{A_k}{\otimes} 1_{A_f}. \end{aligned}$$

Moreover,

$$\begin{aligned} v^*\pi_m(a)vx\xi_k \underset{A_k}{\otimes} 1_{A_f} &= v^*(ax\xi_k \underset{A_k}{\otimes} 1_{A_f}) = v^*(y\xi_k \underset{A_k}{\otimes} 1_{A_f}) + v^*(z\xi_k \underset{A_k}{\otimes} 1_{A_f}) \\ &= y\xi_k \underset{A_k}{\otimes} 1_{A_f} + z\xi_k \underset{A_k}{\otimes} 1_{A_f} = ax\xi_k \underset{A_k}{\otimes} 1_{A_f}. \end{aligned}$$

The proof of (3) is similar. $\square$

End of the proof of Proposition 3.4. Let $t \in \mathbb{R}$ and define $v_t = \cos(t) + iv\sin(t) \in \mathcal{L}_{A_f}(H_m)$. Since $v = v^*$ is unitary, v_t is a unitary for all $t \in \mathbb{R}$. Moreover, assertion (1) of the Claim implies

that $v_t \pi_m(b) v_t^* = \pi_m(b)$ for all $b \in B$. It follows from the universal property of A_f that there exists a unique unital $*$-homomorphism $\pi_t : A_f \to \mathcal{L}_{A_f}(H_m)$ such that:

$$\pi_t(a) = \begin{cases} v_t^* \pi_m(a) v_t & \text{if} \quad a \in A_1, \\ \pi_m(a) & \text{if} \quad a \in A_2. \end{cases}$$

Then the triple $\alpha_t = (H_m \oplus \widetilde{K}_m, \pi_t \oplus \widetilde{\rho}_m, \widetilde{F})$ gives an homotopy between α_0 which represents $[\lambda] \underset{A}{\otimes} \alpha - [\mathrm{id}_{A_f}]$ and α_1 which is degenerated by the Claim. $\qquad \square$

We finish the proof of Theorem 3.1 in the next Proposition.

Proposition 3.5. *One has* $\alpha \underset{A_f}{\otimes} [\lambda] = [id_A]$ *in* $\mathrm{KK}(A, A)$.

Proof. Observe that $\alpha \underset{A_f}{\otimes} [\lambda] = [(H_r \oplus K_r, \pi_r \oplus \rho_r, F_r)]$ where

$$H_r = H_m \underset{\lambda}{\otimes} A = H_1 \underset{A_1}{\otimes} A \oplus H_2 \underset{A_2}{\otimes} A \quad \text{and} \quad K_r = K_m \underset{\lambda}{\otimes} A = K \underset{B}{\otimes} A = \left(K \underset{B}{\otimes} A_k \right) \underset{A_k}{\otimes} A,$$

with the canonical representations $\pi_r : A \to \mathcal{L}_A(H_r)$, $\pi_r(x) = \pi(x) \underset{\lambda}{\otimes} 1 = x \underset{A_1}{\otimes} 1_A \oplus x \underset{A_2}{\otimes} 1_A$ and $\rho_r : A \to \mathcal{L}_A(K_r)$, $\rho_r(x) = \bar{\rho}(x) \underset{\lambda}{\otimes} 1 = \rho(x) \underset{B}{\otimes} 1_A$ and with the operator $F_r = F \underset{\lambda}{\otimes} 1 \in \mathcal{L}_A(H_r, K_r)$. Hence, $\alpha \underset{A_f}{\otimes} [\lambda] - [\mathrm{id}_A]$ is represented by the Kasparov triple $(H_r \oplus \widetilde{K}_r, \pi_r \oplus \widetilde{\rho}_r, \widetilde{F}_r)$, where $\widetilde{K}_r = K_r \oplus A$ and $\widetilde{\rho}_r(x) = \rho_r(x) \oplus x$, where we view $A = \mathcal{L}_A(A)$ by left multiplication. Finally, $\widetilde{F}_r \in \mathcal{L}_A(H_r, \widetilde{K}_r)$ is the unitary defined by

$$\widetilde{F}_r(\xi_1 \underset{A_1}{\otimes} 1_A) = \eta \underset{B}{\otimes} 1_A, \quad \widetilde{F}_r(\xi_2 \underset{A_2}{\otimes} 1_A) = 1_A \quad \text{and},$$

$$\widetilde{F}(\xi) = F(\xi) \text{ for all } \xi \in H_r \ominus \left((\xi_1 \underset{A_1}{\otimes} 1_A).A \oplus (\xi_2 \underset{A_2}{\otimes} 1_A).A \right).$$

The Claim in the proof of Proposition 3.4 implies the following Claim.

Claim. *Let* $u \in \mathcal{L}_A(H_r)$ *be the self-adjoint unitary defined by the identity on* $H_r \ominus ((\xi_1 \underset{A_1}{\otimes} 1_A).A \oplus (\xi_2 \underset{A_2}{\otimes} 1_A).A)$ *and* $u(\xi_1 \underset{A_1}{\otimes} 1_A) = \xi_2 \underset{A_2}{\otimes} 1_A$, $u(\xi_2 \underset{A_2}{\otimes} 1_A) = \xi_1 \underset{A_1}{\otimes} 1_A$. *One has:*

(1) $\widetilde{F}^* \widetilde{\rho}_r(b) \widetilde{F} = \pi_r(b)$ *and* $u^* \pi_r(b) u = \pi_r(b)$ *for all* $b \in B$.
(2) $\widetilde{F}^* \widetilde{\rho}_r(a) \widetilde{F} = u^* \pi_r(a) u$ *for all* $a \in A_1$.
(3) $\widetilde{F}^* \widetilde{\rho}_r(a) \widetilde{F} = \pi_r(a)$ *for all* $a \in A_2$.

Let $t \in \mathbb{R}$ and define the unitary $u_t = \cos(t) + iu \sin(t) \in \mathcal{L}_A(H_r)$. Assertion (1) of the Claim implies that $u_t^* \pi_r(b) u_t = \pi_r(b)$ for all $b \in B$. By the universal property of full amalgamated free products, for all $t \in \mathbb{R}$, there exists a unique unital $*$-homomorphism $\pi_t : A_f \to \mathcal{L}_A(H_r)$ such that:

$$\pi_t(a) = \begin{cases} u_t^* \pi_r(a) u_t & \text{if} \quad a \in A_1, \\ \pi_r(a) & \text{if} \quad a \in A_2. \end{cases}$$

Arguing as in the end of the proof of Proposition 3.4, we see that it suffices to show that, for all $t \in [0, 1]$, π_t factorizes through A i.e. $\ker(\lambda) \subset \ker(\pi_t)$. To do that, we need the following Claim.

Claim. *For all* $t \in \mathbb{R}$ *and all* $a = a_1 \ldots a_n \in \mathcal{A}$ *a reduced operator with* $a_k \in A_{l_k}^\circ$ *one has*

(1) $\pi_t(a)u_t^*(\xi_2 \underset{A_2}{\otimes} 1_A) = e^{-it}(a\xi_2 \underset{A_2}{\otimes} 1_A)$ *if* $l_n = 1$ *and* $\pi_t(a)(\xi_1 \underset{A_1}{\otimes} 1_A) = a\xi_1 \underset{A_1}{\otimes} 1_A$ *if* $l_n = 2$.

(2) $\langle u_t^*(\xi_1 \underset{A_1}{\otimes} 1_A), \pi_t(a)u_t^*(\xi_1 \underset{A_1}{\otimes} 1_A)\rangle = \sin^{2k}(t)a$ *where* $k = \begin{cases} \frac{n}{2} & \text{if } n \text{ is even,} \\ \frac{n-1}{2} & \text{if } n \text{ is odd and } l_n = 1, \\ \frac{n+1}{2} & \text{if } n \text{ is odd and } l_n = 2. \end{cases}$

(3) $\langle \xi_2 \underset{A_2}{\otimes} 1_A, \pi_t(a)\xi_2 \underset{A_2}{\otimes} 1_A\rangle = \sin^{2k}(t)a$ *where* $k = \begin{cases} \frac{n}{2} & \text{if } n \text{ is even,} \\ \frac{n+1}{2} & \text{if } n \text{ is odd and } l_n = 1, \\ \frac{n-1}{2} & \text{if } n \text{ is odd and } l_n = 2. \end{cases}$

Proof of the Claim. (1) is obvious by induction on n once observed that $u_t\xi = e^{it}\xi$ (and $u_t^*\xi = e^{-it}\xi$) for all $\xi \in H_r \ominus (\xi_1 \underset{A_1}{\otimes} 1_A.A \oplus \xi_2 \underset{A_2}{\otimes} 1_A.A)$

(2). Define, for $a_1 \ldots a_n \in \mathcal{A}$, $F(a_1, \ldots, a_n) = \langle u_t^*(\xi_1 \underset{A_1}{\otimes} 1_A), \pi_t(a)u_t^*(\xi_1 \underset{A_1}{\otimes} 1_A)\rangle$. First suppose that $a \in A_1^\circ$ then, $F(a) = \langle u_t^*(\xi_1 \underset{A_1}{\otimes} 1_A), u_t^*\pi_r(a)(\xi_1 \underset{A_1}{\otimes} 1_A)\rangle = \langle \xi_1 \underset{A_1}{\otimes} 1_A, \xi_1 \underset{A_1}{\otimes} a\rangle = a$. Now, let $a = a_1 \ldots a_n \in \mathcal{A}$ with $n \geq 2$ and $l_n = 1$. We have:

$$F(a_1, \ldots, a_n) = \langle u_t^*(\xi_1 \underset{A_1}{\otimes} 1_A), \pi_t(a_1 \ldots a_{n-1})u_t^*(\xi_1 \underset{A_1}{\otimes} a_n)\rangle = F(a_1, \ldots, a_{n-1})a_n.$$

Hence, it suffices to show the formula for $l_n = 2$. Suppose $a \in A_2^\circ$, we have:

$$\begin{aligned} F(a) &= \langle u_t^*(\xi_1 \underset{A_1}{\otimes} 1_A), \pi_r(a)u_t^*(\xi_1 \underset{A_1}{\otimes} 1_A)\rangle \\ &= \langle \cos(t)\xi_1 \underset{A_1}{\otimes} 1_A - i\sin(t)\xi_2 \underset{A_2}{\otimes} 1_A, \cos(t)a\xi_1 \underset{A_1}{\otimes} 1_A - i\sin(t)\xi_2 \underset{A_2}{\otimes} a\rangle = \sin^2(t)a. \end{aligned}$$

Now suppose $a_1 a_2 \in \mathcal{A}$, with $l_2 = 2$, $l_1 = 1$. We have:

$$\begin{aligned} F(a_1, a_2) &= \langle \xi_1 \underset{A_1}{\otimes} 1_A, \pi_r(a_1)u_t\pi_r(a_2)u_t^*(\xi_1 \underset{A_1}{\otimes} 1_A)\rangle \\ &= \langle \xi_1 \underset{A_1}{\otimes} 1_A, \pi_r(a_1)u_t(\cos(t)a_2\xi_1 \underset{A_1}{\otimes} 1_A - i\sin(t)\xi_2 \underset{A_2}{\otimes} a_2)\rangle \\ &= \langle \xi_1 \underset{A_1}{\otimes} 1_A, \cos(t)e^{it}a_1a_2\xi_1 \underset{A_1}{\otimes} 1_A - i\cos(t)\sin(t)a_1\xi_2 \underset{A_2}{\otimes} a_2 + \sin^2(t)\xi_1 \underset{A_1}{\otimes} a_1a_2\rangle \\ &= \sin^2(t)a_1a_2. \end{aligned}$$

Finally, suppose that $n \geq 3$ and $a_1 \ldots a_n \in \mathcal{A}$ with $l_n = 2$. Define $x = a_1 \ldots a_{n-2}$. We have

$$\begin{aligned} F(a_1, \ldots, a_n) &= \langle u_t^*(\xi_1 \underset{A_1}{\otimes} 1_A), \pi_t(x)u_t^*\pi_r(a_{n-1})u_t\pi_r(a_n)u_t^*(\xi_1 \underset{A_1}{\otimes} 1_A)\rangle \\ &= \langle u_t^*(\xi_1 \underset{A_1}{\otimes} 1_A), \pi_t(x)u_t^*\pi_r(a_{n-1})u_t(\cos(t)a_n\xi_1 \underset{A_1}{\otimes} 1_A - i\sin(t)\xi_2 \underset{A_2}{\otimes} a_n)\rangle \end{aligned}$$

$$= \langle u_t^*(\xi_1 \underset{A_1}{\otimes} 1_A), \pi_t(x)u_t^*(\cos(t)e^{it}a_{n-1}a_n\xi_1 \underset{A_1}{\otimes} 1_A - i\cos(t)\sin(t)a_{n-1}\xi_2 \underset{A_2}{\otimes} a_n + \sin^2(t)\xi_1 \underset{A_1}{\otimes} a_{n-1}a_n)\rangle$$

$$= \langle u_t^*(\xi_1 \underset{A_1}{\otimes} 1_A), \cos(t)a_1 \ldots a_n\xi_1 \underset{A_1}{\otimes} 1_A - ie^{-it}\cos(t)\sin(t)a_1 \ldots a_{n-1}\xi_2 \underset{A_2}{\otimes} a_n\rangle$$

$$+ \langle u_t^*(\xi_1 \underset{A_1}{\otimes} 1_A), \sin^2(t)\pi_t(x)u_t^*\xi_1 \underset{A_1}{\otimes} a_{n-1}a_n\rangle.$$

Hence we find:

$$F(a_1, \ldots, a_n) = \sin^2(t)\langle u_t^*(\xi_1 \underset{A_1}{\otimes} 1_A), \pi_t(x)u_t^*\xi_1 \underset{A_1}{\otimes} a_{n-1}a_n\rangle = \sin^2(t)F(a_1, \ldots, a_{n-2})a_{n-1}a_n.$$

The result now follows by an obvious induction on n. The proof of (3) is similar. $\square$

End of the proof of Proposition 3.5. Fix $t \in [0,1]$ and let A_t be the C*-subalgebra of $\mathcal{L}_A(H_r)$ generated by $\pi_t(A_1) \cup \pi_t(A_2)$. Hence, $\pi_t : A_f \to A_t$ is surjective. Consider the ucp map $\varphi_t : A_t \to A$ defined by

$$\varphi_t(x) = \frac{1}{2} \left(\langle u_t^*(\xi_1 \underset{A_1}{\otimes} 1_A), x u_t^*(\xi_1 \underset{A_1}{\otimes} 1_A) \rangle + \langle \xi_2 \underset{A_2}{\otimes} 1_A, x \xi_2 \underset{A_2}{\otimes} 1_A \rangle \right).$$

Note that φ_t is GNS faithful. Indeed, let $x \in A_t$ such that $\varphi_t(y^* x^* xy) = 0$ for all $y \in A_t$. Then $L \subset \ker(x)$ where,

$$
\begin{aligned}
L &= \overline{\mathrm{Span}} \left(A_t u_t^*(\xi_1 \underset{A_1}{\otimes} 1_A).A \cup A_t(\xi_2 \underset{A_2}{\otimes} 1_A).A \right) = \overline{\mathrm{Span}} \left(A_t(\xi_1 \underset{A_1}{\otimes} 1_A).A \cup A_t(\xi_2 \underset{A_2}{\otimes} 1_A).A \right) \\
&= \overline{\mathrm{Span}} \left(A_t(\xi_1 \underset{A_1}{\otimes} 1_A).A \cup A_t u_t^*(\xi_2 \underset{A_2}{\otimes} 1_A).A \right) = H_r,
\end{aligned}
$$

where we used assertion (3) of the Claim for the last equality. Hence, $x = 0$.

Let also $A_{v,k}$ for $k = 1, 2$ be the k-vertex-reduced free product and call i_k the natural inclusion of A in $A_{v,k}$ and $\pi_k = i_k \circ \lambda$ the natural map from A_f to $A_{v,k}$. Clearly $||x||_A = \max(||i_1(x)||, ||, i_2(x)||)$ for any x in the vertex reduced free product A. From the assertions (1) and (2) of the Claim and 2.19 with $r = \sin^2(t)$ we deduced that for any $k = 1, 2$ there exists two ucp maps ψ_1^k and ψ_2^k from $A_{v,k}$ to itself such that $i_k(\varphi_t(\pi_t(a))) = \frac{1}{2}(\psi_1^k(\pi_k(a)) + \psi_2^k(\pi_k(a)))$ for all $a \in A_f$. Therefore $||\varphi_t(\pi_t(a))||_A \leq \max(||\pi_1(a)||, ||\pi_2(a)||) = ||\lambda(a)||$ for all $a \in A_f$.

Let us show that $\ker(\lambda) \subset \ker(\pi_t)$. Let $x \in \ker(\lambda)$. Then, for all $y \in A_f$ we have $\lambda(y^* x^* xy) = 0$. Therefore $\varphi_t \circ \pi_t(y^* x^* xy) = 0$ for all $y \in A_f$. Since π_t is surjective we deduced that $\varphi_t(y^* \pi_t(x)^* \pi_t(x) y) = 0$ for all $y \in A_t$. Using that φ_t is GNS faithful we deduce that $\pi_t(x) = 0$. $\qquad \square$

Corollary 3.6 ([Cu82])**.** *If we have conditional expectations $E_k : A_k \to B$ which are also unital $*$-homomorphism, then the canonical surjection $A_1 \underset{B}{*} A_2 \to A_1 \oplus_B A_2$ is K-invertible*

Obvious with Theorem 3.1 and Corollary 2.11.

4. A LONG EXACT SEQUENCE IN KK-THEORY FOR FULL AMALGAMATED FREE PRODUCT

Let A_1 and A_2 two unital C*-algebras with a common C*-subalgebra B. We will denote by i_l the inclusion of B in A_l for $l = 1, 2$. The algebra A_f is the (full) amalgamated free product. To simplify notation we will denote by S the algebra $C_0(]-1,1[)$.

Let D be the subalgebra of $S \otimes A_f$ consisting of functions f such that $f(]-1,0[) \subset A_1$, $f(]0,1[) \subset A_2$ and $f(0) \in B$. This algebra is of course isomorphic to the cone of $i_1 \oplus i_2$ from B to $A_1 \oplus A_2$. We call j the inclusion of D in the suspension of A_f.

Theorem 4.1. *Suppose that there exist unital conditional expectations from A_l to B for $l = 1, 2$, then the map j, seen as an element $[j]$ of $KK^0(D, S \otimes A_f)$, is invertible.*

The proof of this result will be done in several steps. We will start with the construction of an element x of $KK^1(A_f, D)$. As $KK^1(A_f, D)$ is isomorphic to $KK^0(S \otimes A_f, D)$ this will produce a candidate y for the inverse of j. The proof that $y \otimes_D [j]$ is the identity of the suspension of A_f will use 3.4. Finally the proof that $[j] \otimes_{S \otimes A_f} y$ is the identity of D will be done indirectly by using a short exact sequence for D.

4.1. **An inverse in KK-theory.** In order to present the inverse, we need some additional notations and preliminaries.

Let κ_1 be the inclusion of $C_0(]-1,0[;A_1)$ in D and κ_2 the inclusion of $C_0(]0,1[;A_2)$ in D. There is also κ_0 the obvious map from $S \otimes B$ in D. As K of the preceding section is a B-module, we can define $K_0 = (K \otimes S) \otimes_{\kappa_0} D$ but also $K_1 = (K \otimes_{i_1} A_1 \otimes C_0(]-1,0[)) \otimes_{\kappa_1} D$ and similarily $K_2 = (K \otimes_{i_2} A_2 \otimes C_0(]0,1[)) \otimes_{\kappa_2} D$.

If one defines I_l as the images of κ_l in D for $l = 1,2$, it is clear that these are ideals in D.

Lemma 4.2. *K_l is isomorphic to $\overline{K_0.I_l}$ for $l = 1,2$ as D Hilbert module.*

Proof. Lets do if for $l = 1$. Indeed as $I_1 = \overline{C_0(]-1,0[).I_1}$ because an approximate unit for $C_0(]-1,0[)$ is also one for I_1, it is easy to see that $\overline{K_0.I_1}$ is isomorphic to $\overline{(K \otimes S).C_0(]-1,0[)} \otimes_{\kappa_0} D.I_1$, i.e. $(K \otimes C_0(]-1,0[)) \otimes_{\kappa_0} D.I_1$. Considering that $C_0(]-1,0[;A_1) \otimes_{\kappa_1} D$ is $D.I_1$, one gets that $\overline{K_0.I_1}$ is nothing but $(K \otimes C_0(]-1,0[)) \otimes_{\tilde{\kappa}_0} C_0(]-1,0[;A_1) \otimes_{\kappa_1} D$ where $\tilde{\kappa}_0$ is the natural inclusion of $C_0(]-1,0[;B)$ in $C_0(]-1,0[;A_1)$, i.e. $i_1 \otimes Id_{C_0(]-1,0[)}$. Therefore $(K \otimes_{i_1} A_1) \otimes C_0(]-1,0[)$ is $(K \otimes C_0(]-1,0[)) \otimes_{\tilde{\kappa}_0} C_0(]-1,0[;A_1)$ and $\overline{K_0.I_1}$ is K_1. □

We will also need the following lemmas

Lemma 4.3.   *(1) If $f \in C([-1,1];\mathbb{R})$, then f is a self-adjoint element in $Z(M(D))$ and more generally for any D-Hilbert module $\mathcal{E}$ then the right multiplication by f induces a morphism $\hat{f} \in Z(\mathcal{L}_D(\mathcal{E}))$ such that the map $f \mapsto \hat{f}$ is a algebra morphism.*
   (2) Let f in $C_0(]-1,0[;\mathbb{R})$. Then $f \in I_1 \cap Z(D)$ and the right multiplication by f induces a morphism $\hat{f}$ of $\mathcal{L}_D(K_0,K_1)$ such that $\hat{f}^*\hat{f} = \hat{f}^2$ in $\mathcal{L}_D(\mathcal{K}_0)$ and $\hat{f}\hat{f}^* = \hat{f}^2$ in $\mathcal{L}_D(\mathcal{K}_1)$
   (3) Let f in $C_0(]0,1[;\mathbb{R})$. Then $f \in I_2 \cap Z(D)$ and the right multiplication by f induces a morphism $\hat{f}$ of $\mathcal{L}_D(K_0,K_2)$ such that $\hat{f}^*\hat{f} = \hat{f}^2$ in $\mathcal{L}_D(\mathcal{K}_0)$ and $\hat{f}\hat{f}^* = \hat{f}^2$ in $\mathcal{L}_D(\mathcal{K}_2)$

The first point is pretty obvious and (2) and (3) are also clear in view of lemma 4.2.

Lemma 4.4.   *(1) If $f \in C_0(]-1,1[;\mathbb{R})$ then for any B-module $\mathcal{E}$ and $F \in \mathcal{K}_B(\mathcal{E})$, we have $(F \otimes 1_S) \otimes_{\kappa_0} 1_D\hat{f}$ is a compact operator of $(\mathcal{E} \otimes S) \otimes_{\kappa_0} D$*
   (2) If $f \in C_0(]-1,0[;\mathbb{R})$ then for any A_1-module $\mathcal{E}$ and $F \in \mathcal{K}_{A_1}(\mathcal{E})$, we have $F \otimes 1_{C_0(]-1,0[;\mathbb{R})} \otimes_{\kappa_1} 1_D\hat{f}$ is a compact operator of $(\mathcal{E} \otimes C_0(]-1,0[)) \otimes_{\kappa_1} D$
   (3) Similarily for $f \in C_0(]0,1[;\mathbb{R})$ and A_2-modules.

Proof. Point (2) and (3) are similar to 1. To do (1), let F be the rank one operator $\theta_{\xi,\eta}$ for ξ and η vectors in $\mathcal{E}$ which is defined as $\theta_{\xi,\eta}(x) = \xi <\eta,x>$ for all x in $\mathcal{E}$. Then $(F \otimes 1_S) \otimes_{\kappa_0} 1_D\hat{f}$ is $\theta_{\xi \otimes f_2 \otimes f_2, \eta \otimes f_2 \otimes f_2}\hat{f}_1$ and therefore compact for any function $f = f_1 f_2^4$ with f_1 and f_2 in $C_0(]-1,1[;\mathbb{R})$. As any function can be written like that, use for example the polar decomposition, we get our result. □

Define now two functions in $C([-1,1];\mathbb{R})$: $C^+(t)$ is $\cos(\pi t)$ if $t \geq 0$ and 1 if $t \leq 0$, the function $C^-(t)$ is $\cos(\pi t)$ if $t \leq 0$ and 1 if $t \geq 0$. Similarly, we have two functions in S ; S^+ is $\sin(\pi t)$ if $t \geq 0$ and 0 if $t \leq 0$, the function $S^-(t)$ is $\sin(\pi t)$ if $t \leq 0$ and 0 if $t \geq 0$. And finally T is the identity function of $C([-1,1];\mathbb{R})$.

With the notation of the first part, we have a natural D-module

$$H = (H_1 \otimes C_0(]-1,0[)) \otimes_{\kappa_1} D \oplus (H_2 \otimes C_0(]0,1[)) \otimes_{\kappa_2} D \oplus (K \otimes S) \otimes_{\kappa_0} D.$$

It is also clear that H is endowed with a natural (left) action of A_f as H_1, H_2 and K have it.

Let G be the operator of $\mathcal{L}_D(H)$ defined in matrix form by

$$
G = \begin{pmatrix}
\widehat{C^-} & 0 & -((F_1 \otimes 1_{C_0(]-1,0[)})^* \otimes_{\kappa_1} 1)\widehat{S^-} \\
0 & -\widehat{C^+} & ((F_2 \otimes 1_{C_0(]0,1[)})^* \otimes_{\kappa_2} 1)\widehat{S^+} \\
-\widehat{S^-}^*((F_1 \otimes 1_{C_0(]-1,0[)}) \otimes_{\kappa_1} 1) & \widehat{S^+}^*((F_2 \otimes 1_{C_0(]0,1[)}) \otimes_{\kappa_2} 1) & Z
\end{pmatrix}
$$

where $Z = -\widehat{C^-}(q_1 \otimes 1_S) \otimes_{\kappa_0} 1 + \widehat{C^+}(q_2 \otimes 1_S) \otimes_{\kappa_0} 1 - \widehat{T}(q_0 \otimes 1_S) \otimes_{\kappa_0} 1$.
Thanks to lemma 4.3, G is well-defined. Moreover

Proposition 4.5. *The operator G verifies $G^2 - 1$ is a compact operator of H and G commutes modulo compact operators with the action of A_f.*

Proof. Computing G^2 one gets as upper left 2×2 corner :

$$
\begin{pmatrix}
\widehat{C^-}^2 + F_1^* \otimes_{\kappa_1} 1\widehat{S^-}\widehat{S^-}^*F_1 \otimes_{\kappa_1} 1 & F_1^* \otimes_{\kappa_1} 1\widehat{S^-}\widehat{S^+}^*F_2 \otimes_{\kappa_2} 1 \\
F_1^* \otimes_{\kappa_1} 1\widehat{S^-}\widehat{S^+}^*F_2 \otimes_{\kappa_2} 1 & \widehat{C^+}^2 + F_2^* \otimes_{\kappa_1} 1\widehat{S^+}\widehat{S^+}^*F_2 \otimes_{\kappa_1} 1
\end{pmatrix}
$$

As $F_1^*F_1$ is the identity modulo compact operator, using 4.4 (the function $(S^-)^2$ is in $C_0(]-1,1[)$) one has that $F_1^* \otimes_{\kappa_1} 1\widehat{(S^-)^2}F_1 \otimes_{\kappa_1} 1$ is $\widehat{(S^-)^2}$ modulo compact operators.

Recalling also that $F_1^*F_2 = 0$, one gets that this matrix is the identity modulo compact operators.

Let's focus now on the last row of G^2. We get first $-\widehat{C^-}F_1^* \otimes_{\kappa_1} 1\widehat{S^-} - F_1^* \otimes_{\kappa_1} 1\widehat{S^-}Z$. As $F_1^*q_1 \otimes_{i_1} 1 = F_1^*$ and $F_1^*q_2 \otimes_{i_1} 1 = 0$ along with $F_1^*q_0 \otimes_{i_1} 1 = 0$, $F_1^* \otimes_{\kappa_1} 1\widehat{S^-}Z$ is $-F_1^* \otimes_{\kappa_1} 1\widehat{S^-}\widehat{C^-}$. The second composant of that row is treated in the same way. Finally the last composant is $\widehat{S^-}^2(F_1F_1^*) \otimes_{\kappa_1} 1 + \widehat{S^+}^2(F_2F_2^*) \otimes_{\kappa_1} 1 + \widehat{C^-}^2(q_1 \otimes 1_S) \otimes_{\kappa_0} 1 + \widehat{C^+}^2(q_2 \otimes 1_S) \otimes_{\kappa_0} 1 + \widehat{T}^2(q_0 \otimes 1_S) \otimes_{\kappa_0} 1$ as q_0, q_1, q_2 are commuting projections. But $F_lF_l^*$ is $q_l \otimes_{i_l} 1$ so $\widehat{S^-}^2(F_1F_1^*) \otimes_{\kappa_1} 1$ is $\widehat{S^-}^2(q_1 \otimes 1_S) \otimes_{\kappa_0} 1$. Hence as $q_1 + q_2 + q_0 = 1$, the last component is $1 + \widehat{T^2 - 1}(q_0 \otimes 1_S) \otimes_{\kappa_0} 1$. As $T^2 - 1$ is in $C_0(]-1,1[)$ and q_0 is compact, this composant is then 1 modulo compact operator.

Addressing now the compact commutation with the left action of A_f, it is very obvious using 4.4 for every composant of G except Z as it contains multiplication with functions not in $C_0(]-1,1[)$. So let a be in A_1. We need to compute $[Z, \rho(a) \otimes_{\kappa_0} 1]$. But we know that $[q_1, \rho(a)] = 0$. As $q_2 = 1 - q_1 - q_0$ we get that $[Z, \rho(a) \otimes_{\kappa_0} 1] = -(\widehat{C^+ + T})[q_0, \rho(a)] \otimes_{\kappa_0} 1$ which is compact as $C^+ + T$ is a function that vanishes on -1 and 1. The case when a is in A_2 is treated in a similar way, hence the compact commutation property is proved for all a in A_f. $\square$

Aa a consequence, the couple (H, G) defines an element of $KK^1(A_f, D)$ which we will call x in the sequel.

4.2. **K equivalence.** In all the following proofs we will very often use the external tensor product of Kasparov elements. Instead of the traditional notation $\tau_C(x)$ for the tensorisation with the algebra C of an element x in $KK^*(A, B)$, we will write $1_C \otimes x$ for the element in $KK^*(C \otimes A, C \otimes B)$ or $x \otimes 1_C$ for the element in $KK^*(A \otimes C, B \otimes C)$. Of course $B \otimes C$ is (non canonically) isomorphic to $C \otimes B$, but as we will perform several times this operation, the order will matter. Note that we do not specify the tensor norm as the algebra C we will be using is alway nuclear.

Also when π is a morphism between A and B, we will write $[\pi]$ for the canonical element in $KK^0(A,B)$.

We will denote by b the element of $KK^1(\mathbb{C},S)$ which is defined on the S Hilbert module S itself by the operator $\widehat{T}$. It is well known that b is invertible.

Proposition 4.6. *With the hypothesis of 4.1, one has in $KK^1(A_f, A_f \otimes S)$ that $x \otimes_D [j]$ is homotopic to $(1_{A_f} \otimes b) \otimes_{A_f \otimes S} ([Id_{A_f}] \otimes 1_S)$*

Proof. To prove that we will choose the representant of $[Id_{A_f}]$ that appear in 3.4 and show that its Kasparov product with b is homotopic to $x \otimes_D [j]$. Call j_l for $l = 1, 2$ the inclusions of A_l in A_f and $j_0 = j_1 \circ i_1 = j_2 \circ i_2$ the inclusion of B in A_f.

First it is obvious that $H \otimes_j (A \otimes S)$ is $H_1 \otimes_{j_1} A_f \otimes C_0(]-1,0[) \oplus H_2 \otimes_{j_2} A_f \otimes C_0(]0,1[) \oplus K \otimes_{j_0} A_f \otimes S$ which is not quite the same as $(H_1 \otimes_{j_1} A_f \oplus H_2 \otimes_{j_2} A_f \otimes \oplus K \otimes_{j_0} A_f) \otimes S$ So we will realize now an homotopy to fix that.

Lemma 4.7. *Consider the following two spaces : $\Delta_1 = \{(t,s) \in \mathbb{R}^2, 0 \le s \le 1, -1 < t < s\}$ and $\Delta_2 = \{(t,s) \in \mathbb{R}^2, 0 \le s \le 1, -s < t < 1\}$. The Hilbert module $\overline{H} = H_1 \otimes_{j_1} A_f \otimes C_0(\Delta_1) \oplus H_2 \otimes_{j_2} A_f \otimes C_0(\Delta_2) \oplus K \otimes_{j_0} A_f \otimes S \otimes C([0,1])$ is endowed with a natural structure of $A_f \otimes S \otimes C([0,1])$ Hilbert module and A_f left action. Moreover the operator*

$$\overline{G} = \begin{pmatrix} \widehat{C^-} \otimes 1_{C([0,1])} & 0 & -F_1^* \otimes_{j_1} 1 \otimes 1_{\Delta_1} \widehat{S^-} \otimes 1_{C([0,1])} \\ 0 & -\widehat{C^+} \otimes 1_{C([0,1])} & F_2^* \otimes_{j_2} 1 \otimes 1_{\Delta_2} \widehat{S^+} \otimes 1_{C([0,1])} \\ -\widehat{S^-}^* \otimes 1_{C([0,1])} F_1 \otimes_{j_1} 1 \otimes 1_{\Delta_1} & \widehat{S^+}^* \otimes 1_{C([0,1])} F_2 \otimes_{j_2} 1 \otimes_{\Delta_2} & \overline{Z} \end{pmatrix}$$

with $\overline{Z} = \widetilde{Z} \otimes 1_{C([0,1])}$ where $\widetilde{Z} = -\widehat{C^-}q_1 \otimes_{j_0} 1 \otimes 1_S + \widehat{C^+}q_2 \otimes_{j_0} 1 \otimes 1_S - \widehat{T}q_0 \otimes_{j_0} 1 \otimes 1_S$ makes the pair $(\overline{H}, \overline{G})$ into an element of $KK^1(A_f, A \otimes S \otimes C([0,1]))$ for which the evaluation at $t = 0$ is $x \otimes_D [j]$ and the evaluation at $t = 1$ has $(H_1 \otimes_{j_1} A_f \oplus H_2 \otimes_{j_2} A_f \otimes \oplus K \otimes_{j_0} A_f) \otimes S$ as module and $\widetilde{G} = \begin{pmatrix} \widehat{C^-} & 0 & -F_1^* \otimes_{j_1} 1 \otimes 1_S \widehat{S^-} \\ 0 & -\widehat{C^+} & F_2^* \otimes_{j_2} 1 \otimes 1_S \widehat{S^+} \\ -\widehat{S^-}^* F_1 \otimes_{j_1} 1 \otimes 1_S & \widehat{S^+}^* F_2 \otimes_{j_2} 1 & \widetilde{Z} \end{pmatrix}$ as operator

Proof. As it is a straightforward check, details will be omitted. $\square$

Then one easily checks that $\widetilde{G}$ is an $\widehat{T} \otimes_{A_f} 1$ connection. Indeed as $H_1 \otimes_{j_1} A_f \oplus H_2 \otimes_{j_2} A_f$ is of grading 0 and $K \otimes j_0 A_f$ of grading -1, one need to check that when evaluating on -1, $\widetilde{G}$ does the same thing as $\widehat{T}$ i.e. is the matrix $\begin{pmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$ and when evaluating on 1, G is the opposite matrix. It is indeed the case as $q_1 + q_2 + q_0 = 1$.

Lastly one need the following lemma where the operator F of 3.4 appears.

Lemma 4.8. *The anti-commutator of $\widetilde{G}$ and $F \otimes 1_S$ is positive.*

Proof. To do that, we will decompose $\widetilde{G}$ in its diagonal and anti-diagonal part. It is clear that $\begin{pmatrix} \widehat{C^-} & 0 & 0 \\ 0 & -\widehat{C^+} & 0 \\ 0 & 0 & \widetilde{Z} \end{pmatrix}$ and $\begin{pmatrix} 0 & 0 & F_1^* \otimes_{i_1} 1 \otimes 1_S \\ 0 & 0 & F_2^* \otimes_{ß_2} 1 \otimes 1_S \\ F_1 \otimes_{i_1} 1 \otimes 1_S & F_2 \otimes_{i_2} 1 \otimes 1_S & 0 \end{pmatrix}$ anti-commutes modulo compact operator as we have (modulo compact operator) $q_1 F_1 = F_1$ and $q_2 F_1 = q_0 F_1 = 0$.

On the other hand the anti-commutator with the anti-diagonal part is

$$\begin{pmatrix} -2(F_1^*F_1) \otimes_{j_1} 1 \otimes 1_S \widehat{S^-} & 0 & 0 \\ 0 & 2(F_2^*F_2) \otimes_{j_2} 1 \otimes 1_S \widehat{S^+} & 0 \\ 0 & 0 & -2q_1 \otimes_{j_0} 1 \otimes 1_S \widehat{S^-} + 2q_2 \otimes_{j_2} 1 \otimes 1_S \widehat{S^+} \end{pmatrix}$$

As $-S^-$ and S^+ are positive functions and q_1 and q_2 commutes, the previous matrix is a diagonal matrix of positive operators hence positive. $\qquad\square$

Using Connes- Skandalis characterization of the Kasparov product, we have established that $\widetilde{G}$ is a representant of the Fredholm operator for the product $(1_{A_f} \otimes b) \otimes_{A_f \otimes S} ([Id_{A_f}] \otimes 1_S)$. Our proposition is henceforth proven.

$\qquad\square$

We need now the following two lemmas to get some information about $[j] \otimes_{A_f \otimes S} (x \otimes 1_S)$ as an element of $KK^1(D, D \otimes S)$.

Lemma 4.9. *Call ev_0 the morphism from D to B that evaluates a function at 0. Then we have in $KK^1(D, B \otimes S)$ that $[j] \otimes_{A_f \otimes S} ((x \otimes_D [ev_0]) \otimes 1_S) = -[ev_0] \otimes_B (1_B \otimes b)$*

Proof. Let's first describe the left hand side. The Hilbert module is $K \otimes 1_S$ as the module $(H_1 \otimes C_0(]-1,0[)) \otimes_{\kappa_1} D \otimes_{ev_0} B$ is 0. The left D action is given by $(\rho \otimes 1_S) \circ j$ and the operator is just $(-q_1 + q_2) \otimes 1_S$. We can replace this operator with $G_0 = (-q_1 + q_2) \otimes 1_S - \widehat{T}q_0 \otimes 1_S$ as for any f in D, $(\rho \otimes 1_S) \circ j(f)\widehat{T}q_0 \otimes 1_S$ is compact. Note now that the evaluation at -1 of G_0 is $(1 - 2q_1)$ and at -1 is $2q_2 - 1$. It then enables us to do an homotopy. Consider the pair $(K \otimes S \otimes C([0,1]), G_0 \otimes 1_{C([0,1])})$ where the left action of D is defined now for any f in D and $k \in C(]-1,1[\times[0,1]; K)$ as $(f.k)(t,s) = \rho(f(t(1-s)))k(t,s)$. This is still a Kasparov element as $(G_0^2 - 1) \otimes 1_{C([0,1])} = ((\widehat{T^2 - 1})q_0 \otimes 1_S) \otimes 1_{C([0,1])}$ hence compact. Also the commutator of the left action with the operator $G_0 \otimes 1$ is compact. Indeed as q_0 is compact, it is only necessary to check that the evaluation at -1 or 1 of any commutator is 0. But this is true as $[q_1, \rho(A_1)] = 0$ and $[q_2, \rho(A_2)] = 0$.

Therefore $[j] \otimes_{A_f \otimes S} ((x \otimes_D [ev_0]) \otimes 1_S)$ is homotopic to an element of $KK^1(D, B \otimes S)$ wich is described with the pair $(K \otimes S, G_0)$ where D acts on $K \otimes S$ as the constant morphism $\rho \circ ev_0$. So it is $[ev_0] \otimes_B z$ with z an element of $KK^1(B, B \otimes S)$ which is only non trivial on $q_0 K \otimes S \simeq B \otimes S$ where G_0 acts as $-\widehat{T}$. Thus $z = -1_B \otimes b$. $\qquad\square$

Recall that for $l = 1, 2$, κ_l is the inclusion of $A_l \otimes C(]-1,0[)$ in D. To be precise we will use $\bar\kappa_l$ for the induced map from $A_l \otimes S$ to D via the isomorphism of $C(]-1,0[)$ with S.

Lemma 4.10. *For $l = 1, 2$, one has that in $KK^1(A_l, D)$ the element $[j_l] \otimes_{A_f} x$ is $([Id_{A_l}] \otimes b) \otimes_{A_l \otimes S} [\bar\kappa_l]$.*

Proof. We will do the lemma for $l = 1$. The element $[j_1] \otimes_{A_f} x$ as the same module and operator that x, the only change is that we only consider a left action of A_1. We first perform a compact perturbation of the operator G. With the operators $\overline{F}_l$ defined before 3.3, consider

$$G_1 = \begin{pmatrix} \widehat{C^-} & 0 & -F_1^* \otimes_{\kappa_1} 1\widehat{S^-} \\ 0 & -\widehat{C^+} & \overline{F}_2^* \otimes_{\kappa_2} 1\widehat{S^+} \\ -\widehat{S^-}^* F_1 \otimes_{\kappa_1} 1 & \widehat{S^+}^* \overline{F}_2 \otimes_{\kappa_2} 1 & \overline{Z} \end{pmatrix}$$

where $\overline{Z} = -\widehat{C^-}(q_1 \otimes 1_S) \otimes_{\kappa_0} 1 + \widehat{C^+}(1 - q_1 \otimes 1_S) \otimes_{\kappa_0} 1$.

As $F_2 - \overline{F}_2$ is compact (see 3.3) and $\overline{Z} - Z = \widehat{C^+ + T}(q_0 \otimes 1_S) \otimes_{\kappa_0} 1$ is compact as $C^+ + T$ is in S, we get the same element of $KK^1(A_1, D)$.

Observe now that when evaluating at any positive t, G_1^2 is the identity because $\overline{F}_2$ is an isometry and $\widehat{S^-} F_1 \otimes_{\kappa_1} 1$ vanishes and that for any t, G_1 commutes exactly with the left action of A_1 as F_1 and $\overline{F}_2$ does.

We will now construct an homotopy to remove the $[0, 1[$ part of our module. Consider the space $\Delta_3 = \{(t, s) \in \mathbb{R} \, 0 < s \le 1, \, 0 < t < s\}$ and $\Delta_4 = \{(t, s) \in \mathbb{R} \, 0 \le s \le 1, \, -1 < t < s\}$ which are open in $] - 1, 1[\times [0, 1]$. Hence we also have a natural imbedding δ_4 of $C_0(\Delta_4; B)$ in $D \otimes C([0, 1])$ and δ_3 of $C_0(\Delta_3; A_2)$ in $D \otimes C([0, 1])$. Then $\widetilde{H} = (H_1 \otimes C_0(] - 1, 0[)) \otimes_{\kappa_1} D \otimes C([0, 1]) \oplus (H_2 \otimes C_0(\Delta_3)) \otimes_{\delta_3} D \otimes C([0, 1]) \oplus (K \otimes C_0(\Delta_4) \otimes_{\delta_4} D \otimes C([0, 1])$ is well defined and the pair $(\widetilde{H}, \widetilde{G}_1)$ is a Kasparov element in $KK^1(A_1, D \otimes C([0, 1]))$. Indeed the only thing to check is whether $\widetilde{G}_1^2$ is the identity modulo compact operator as $\widetilde{G}_1$ has exact commutation with the action of A_1. But this is true by the previous observation.

Therefore $[j_l] \otimes_{A_f} x$ can be represented by the evaluation at 0 of this Kasparov element. Let's describe it: the module part is $(H_1 \oplus K \otimes_{i_1} A_1) \otimes C_0(] - 1, 0[) \otimes_{\kappa_1} D$ with obvious left A_1 action as $(K \otimes C_0(] - 1, 0[)) \otimes_{\kappa_0} D$ is isomorphic to $(K \otimes_{i_1} A_1) \otimes C_0(] - 1, 0[) \otimes_{\kappa_1} D$. With this identification, the operator is

$$E_1 = \begin{pmatrix} \widehat{C^-} & -F_1^* \otimes 1_{C_0(]-1,0[)} \otimes_{\kappa_1} 1\widehat{S^-} \\ -\widehat{S^-}^* F_1 \otimes 1_{C_0(]-1,0[)} \otimes_{\kappa_1} 1 & -\widehat{C^-}(q_1 \otimes_{i_1} 1 \otimes 1_{C_0(]-1,0[)} \otimes_{\kappa_1} 1 + (1 - q_1 \otimes_{i_1} 1 \otimes 1_{C_0(]-1,0[)} \otimes_{\kappa_1} 1) \end{pmatrix}$$

.

It is then clear, after identifying $C_0(] - 1, 0[)$ with S, that $[j_l] \otimes_{A_f} x$ is $z \otimes_{A_1} [\overline{\kappa}_1]$ with z in $KK^1(A_1, A_1 \otimes S)$.

By recalling that $1 - q_1$ commutes with the left action of A_1, it is obvious that z is represented by the pair $((H_1 \oplus q_1 K \otimes_{i_1} A_1) \otimes S, \overline{E}_1)$ with $\overline{E}_1 = \begin{pmatrix} \widehat{C_1} & -F_1^* \otimes 1_S \widehat{S_1} \\ -\widehat{S_1}^* F_1 \otimes 1_S & -\widehat{C_1}(q_1 \otimes_{i_1} 1 \otimes 1_S) \end{pmatrix}$ where C_1 is the function $\cos(\pi(t/2 - 1/2))$ and S_1 the function $\sin(\pi(t/2 - 1/2))$.

Following the proof of 4.6, z is obviously the product $z' \otimes b$ where z' is the element of $KK^0(A_1, A_1)$ given by the module $H_1 \oplus q_1 K \otimes_{i_1} A_1$ with H_1 positively graded and the obvious left action of A_1 and the operator $\begin{pmatrix} 0 & F_1^* \\ F_1 & 0 \end{pmatrix}$.

We will be finish when we prove that z' is $[Id_{A_1}]$. To do this we represent $z' \oplus -[Id_{A_1}]$ by the module $H_1 \oplus q_1 K \otimes_{i_1} A_1 \oplus q_0 K \otimes_{i_1} A_1 \simeq H_1 \oplus (1 - q_2) K \otimes_{i_1} A_1$ and the previous operator. But it is a compact perturbation of $\begin{pmatrix} 0 & \overline{F}_1^* \\ \overline{F}_1 & 0 \end{pmatrix}$. This last operator is homotopic via a simple rotation (see 2.2) to $\begin{pmatrix} \overline{F}_1^* \overline{F}_1 & 0 \\ 0 & \overline{F}_1 \overline{F}_1^* \end{pmatrix}$ hence trivial as $\overline{F}_1^* \overline{F}_1 = 1$ and $\overline{F}_1 \overline{F}_1^* = 1$ modulo compact operators as observed before 3.3. $\square$

We are now ready to prove our theorem 4.1

Proof. Call $a \in KK^1(S, \mathbb{C})$ the inverse of b.

Then $y = (1_{A_f} \otimes a) \otimes_{A_f} x$ is an element of $KK^0(A \otimes S, D)$. We claim that this is the inverse of $[j]$.

Indeed thanks to 4.6 we have that

$$y \otimes_D [j] = (1_{A_f} \otimes a) \otimes_{A_f} x \otimes_D [j] = (1_{A_f} \otimes a) \otimes_{A_f} (1_{A_f} \otimes b) \otimes_{A_f \otimes S} ([Id_{A_f}] \otimes 1_S).$$

As $a \otimes_{\mathbb{C}} b = [Id_S]$ we get that $y \otimes_D [j] = (1_A \otimes [Id_S]) \otimes_{A_f \otimes S} ([Id_{A_f}] \otimes 1_S)$ is $[Id_{A_f \otimes S}]$.

To prove the reverse equality, we will need a trick that can be found already in [Pi86]. Observe first that for any $l = 1, 2$ and using 4.10

$$
\begin{aligned}
[\bar{\kappa}_l] \otimes_D \otimes [j] \otimes_{A_f \otimes S} y &= [j \circ \bar{\kappa}_l] \otimes_{A_f \otimes S} y = ([j_l] \otimes 1_S) \otimes_{A_f \otimes S} (1_{A_f} \otimes a) \otimes_{A_f} x \\
&= (1_{A_l} \otimes a) \otimes_{A_l} [j_l] \otimes_{A_f} x \\
&= (1_{A_l} \otimes a) \otimes_{A_l} (1_{A_l} \otimes b) \otimes_{A_l} ([Id_{A_l}] \otimes 1_S) \otimes_{A_l \otimes S} [\bar{\kappa}_l] \\
&= [\bar{\kappa}_l]
\end{aligned}
$$

We need now to compute $[j] \otimes_{A_f \otimes S} y \otimes_D [ev_0]$. To do this carefully we will use the following lemma

Lemma 4.11. *In $KK^1(D \otimes S, A \otimes S)$, one has $([j] \otimes_{A_f \otimes S} (1_{A_f} \otimes a)) \otimes 1_S = -(1_D \otimes a) \otimes_D [j]$.*

Proof. Indeed

$$(1_D \otimes b) \otimes_{D \otimes S} ([j] \otimes_{A_f \otimes S} (1_{A_f} \otimes a)) \otimes 1_S = [j] \otimes_{A_f \otimes S} (1_{A_f} \otimes (1_S \otimes b) \otimes_{S \otimes S} (a \otimes 1_S))$$

.

If Σ is the flip automorphism of $S \otimes S$ then clearly $[\Sigma] = -[Id_{S \otimes S}]$ in $KK^0(S \otimes S, S \otimes S)$. As a consequence $(1_S \otimes b) \otimes_{S \otimes S} (a \otimes 1_S) = -1_S \otimes (b \otimes_{\mathbb{C}} a) = -[Id_S]$. Hence

$$(1_D \otimes b) \otimes_{D \otimes S} ([j] \otimes_{A_f \otimes S} (1_{A_f} \otimes a)) \otimes 1_S) = -[j].$$

Multiplying both side by $1_D \otimes a$ gives the result. $\qquad \square$

In view of the lemma and 4.9

$$
\begin{aligned}
([j] \otimes_{A_f \otimes S} y \otimes_D [ev_0]) \otimes 1_S &= -(1_D \otimes a) \otimes_D ([j] \otimes_{A_f \otimes S} (x \otimes_D [ev_0]) \otimes 1_S) \\
&= +(1_D \otimes a) \otimes_D [ev_0] \otimes_B (1_B \otimes b) \\
&= (1_D \otimes a) \otimes_D (1_D \otimes b) \otimes_{D \otimes S} ([ev_0] \otimes 1_S) \\
&= [ev_0] \otimes 1_S
\end{aligned}
$$

As $- \otimes 1_S$ from $KK(B_1, B_2)$ to $KK(B_1 \otimes S, B_2 \otimes S)$ is an isomorphism for any B_1 and B_2, we get

$$[j] \otimes_{A_f \otimes S} y \otimes_D [ev_0] = [ev_0]$$

.

Denote now $q = [Id_D] - [j] \otimes_{A_f \otimes S} y$. As $y \otimes_D [j] = [Id_{A_f \otimes S}]$, q is an idempotent in the ring $KK^0(D, D)$.

On the other end, D fits into a short exact sequence

$$0 \to A_1 \otimes S \oplus A_2 \otimes S \xrightarrow{\bar{\kappa}_1 \oplus \bar{\kappa}_2} D \xrightarrow{ev_0} B \to 0.$$

The induced six terms exact sequence for the functor $KK^0(D, -)$ then shows that, as $q \otimes_D [ev_0] = 0$, there exist q_l in $KK^0(D, A_l)$ for $l = 1, 2$ such that $q = (q_1 \oplus q_2) \otimes_{A_1 \oplus A_2} ([\bar{\kappa}_1] \oplus [\bar{\kappa}_2])$

So $q = q \otimes_D q = (q_1 \oplus q_2) \otimes_{A_1 \oplus A_2} ([\bar{\kappa}_1] \oplus [\bar{\kappa}_2]) \otimes_D q = 0$. because $[\bar{\kappa}_l] \otimes_D q = 0$ for $l = 1, 2$ as observed before 4.11. Therefore $[Id_D] = [j] \otimes_{A_f \otimes S} y$ and the K-equivalence between A_f and D is established.

$\square$

5. EXACT SEQUENCE IN KK-THEORY FOR FUNDAMENTAL C*-ALGEBRAS

5.1. **Preliminaries on fundamental C*-algebras.** In this section we recall the results and notations of [FF13].

If $\mathcal{G}$ is a graph in the sense of [Se77, Def 2.1], its vertex set will be denoted $\mathrm{V}(\mathcal{G})$ and its edge set will be denoted $\mathrm{E}(\mathcal{G})$. We will always assume that $\mathcal{G}$ is at most countable. For $e \in \mathrm{E}(\mathcal{G})$ we denote by $s(e)$ and $r(e)$ respectively the source and range of e and by $\bar{e}$ the inverse edge of e. An *orientation* of $\mathcal{G}$ is a partition $\mathrm{E}(\mathcal{G}) = \mathrm{E}^+(\mathcal{G}) \sqcup \mathrm{E}^-(\mathcal{G})$ such that $e \in \mathrm{E}^+(\mathcal{G})$ if and only if $\bar{e} \in \mathrm{E}^-(\mathcal{G})$. We call $\mathcal{G}' \subset \mathcal{G}$ a *connected subgraph* if $V(\mathcal{G}') \subset V(\mathcal{G})$, $E(\mathcal{G}') \subset E(\mathcal{G})$ such that $e \in E(\mathcal{G}')$ if and only if $\bar{e} \in E(\mathcal{G}')$ and the graph $\mathcal{G}'$ with the source map and inverse map given map the ones of $\mathcal{G}$ restricted to $E(\mathcal{G}')$ is a connected graph.

Let $(\mathcal{G}, (A_q)_q, (B_e)_e)$ be a *graph of unital C*-algebras*. This means that

- $\mathcal{G}$ is a connected graph.
- For every $q \in \mathrm{V}(\mathcal{G})$ and every $e \in \mathrm{E}(\mathcal{G})$, A_q and B_e are unital C*-algebras.
- For every $e \in \mathrm{E}(\mathcal{G})$, $B_{\bar{e}} = B_e$.
- For every $e \in \mathrm{E}(\mathcal{G})$, $s_e : B_e \to A_{s(e)}$ is a unital faithful $*$-homomorphism.

For every $e \in \mathrm{E}(\mathcal{G})$, we set $r_e = s_{\bar{e}} : B_e \to A_{r(e)}$, $B_e^s = s_e(B_e)$ and $B_e^r = r_e(B_e)$. Given a maximal subtree $\mathcal{T} \subset \mathcal{G}$ the *maximal fundamental C*-algebra with respect to $\mathcal{T}$* is the universal C*-algebra generated by the C*-algebras A_q, $q \in \mathrm{V}(\mathcal{G})$, and by unitaries u_e, $e \in \mathrm{E}(\mathcal{G})$, such that

- For every $e \in \mathrm{E}(\mathcal{G})$, $u_{\bar{e}} = u_e^*$.
- For every $e \in \mathrm{E}(\mathcal{G})$ and every $b \in B_e$, $u_{\bar{e}} s_e(b) u_e = r_e(b)$.
- For every $e \in \mathrm{E}(\mathcal{T})$, $u_e = 1$.

This C*-algebra will be denoted by P or $P_{\mathcal{G}}$. We will always view $A_p \subset P$ since, as explain in the following remark, the canonical unital $*$-homomorphisms from A_p to P are faithful.

Remark 5.1. The C*-algebra P is not zero and the canonical maps $\nu_p : A_p \to P$ are injective for all $p \in V(\mathcal{G})$. Indeed, this follows easily from the Voiculescu's absorption Theorem and since we did assume all our C*-algebras separable and the graph $\mathcal{G}$ countable. Indeed, since A_p is separable for all $p \in V(\mathcal{G})$ and since $\mathcal{G}$ is at most countable we can representation faithfully all the A_p on the same separable Hilbert space H. Write $\pi_p : A_p \to \mathcal{L}(H)$ the faithful representation. Replacing H by $H \otimes H$ and π_p by $\pi_p \otimes \mathrm{id}$ if necessary, we may and will assume that $\pi_p(A_p) \cap \mathcal{K}(H) = \{0\}$ for all $p \in V(\mathcal{G})$. Write $C = \mathcal{L}(H)/\mathcal{K}(H)$ the Calkin algebra and $Q : \mathcal{L}(H) \to C$ the canonical surjection. For $e \in E(\mathcal{G})$ we have two faithful representation $\pi_{s(e)} \circ s_e$ and $\pi_{r(e)} \circ r_e$ of B_e on H with trivial intersection with $K(H)$. By Voiculescu's absorption Theorem there exists, for all $e \in E(\mathcal{G})$, a unitary $U_e \in C$ such that $Q \circ \pi_{r(e)}(r_e(b)) = U_e^* Q \circ \pi_{s(e)}(s_e(b)) U_e$ for all $b \in B_e$ and all $e \in E(\mathcal{G})$. Hence, P is not zero and we have a unique unital $*$-homomorphism $\pi : P \to C$ such that $\pi(u_e) = U_e$ and $\pi \circ \nu_p = Q \circ \pi_p$ for all $p \in V(\mathcal{G})$. In particular, the canonical map from A_p to P is faithful since $\pi_p(A_p) \cap \mathcal{K}(H) = \{0\}$ and π_p is faithful, which implies that $Q \circ \pi_p$ is faithful. Note that, when the C*-algebras A_p are not supposed to be separable and/or the graph

$\mathcal{G}$ is not countable anymore the result is still true by considering the universal representation, as in the proof of [Pe99, Theorem 4.2] (which was inspired by [Bl78]).

Remark 5.2. Let $\mathcal{A} \subset P$ be the $*$-algebra by the A_q for $q \in V(\mathcal{G})$ and the unitaries u_e, for $e \in E(\mathcal{G})$. Then $\mathcal{A}$ is a dense unital $*$-algebra of P. Moreover, for any fixed $p \in V(\mathcal{G})$, $\mathcal{A}$ is the linear span of A_p and elements of the form $a_0 u_{e_1} \ldots u_{e_n} a_n$ where $(e_1, \ldots, e_n)$ is a path in $\mathcal{G}$ from p to p, $a_0 \in A_p$ and $a_i \in A_{r(e_i)}$ for $1 \leqslant i \leqslant n$, see [FF13, Remark 3.7].

We now recall the construction of the reduced fundamental C*-algebra, when there is a family of conditional expectations $E_e^s : A_{s(e)} \to B_e^s$, for $e \in E(\mathcal{G})$. Set $E_e^r = E_{\bar{e}}^s : A_{r(e)} \to B_e^r$ and note that, in contrast with [FF13], we do not assume the conditional expectations E_e^s to be GNS-faithful. However, as it was already mentioned in [FF13], all the constructions can be easily carried out without the non-degeneracy assumption. Let us recall these constructions now. We shall omit the proofs which are exactly the same as the GNS-faithful case and concentrate only on the differences with the GNS-faithful case. The interested reader will observe that all the constructions are modeled on the vertex reduced amalgamated free product.

For every $e \in E(\mathcal{G})$ let $(K_e^s, \rho_e^s, \eta_e^s)$ be the GNS construction of the ucp map $s_e^{-1} \circ E_e^s : A_{s(e)} \to B_e$. This means that K_e^s is a right Hilbert B_e-module, $\rho_e^s : A_{s(e)} \to \mathcal{L}_{B_e}(K_e^s)$ and $\eta_e^s \in K_e^s$ are such that $K_e^s = \overline{\rho_e^s(A_{s(e)})\eta_e^s \cdot B_e}$ and $\langle \eta_e^s, \rho_e^s(a)\eta_e^s \cdot b \rangle = s_e^{-1} \circ E_e^s(a)b$. In particular, we have the formula $\rho_e^s(a)\eta_e^s \cdot b = \rho_e^s(as_e(b))\eta_e^s$. Let us notice that the submodule $\eta_e^s.B_e$ of K_e^s is orthogonally complemented. In fact, its orthogonal complement is the closure $(K_e^s)^\circ$ of $\{\rho_e^s(a)\eta_e^s : a \in A_{s(e)}, E_e^s(a) = 0\}$ which is easily seen to be a Hilbert B_e-submodule of K_e^s. Similarly, the orthogonal complement of $\eta_e^r.B_e$ in K_e^r will be denoted $(K_e^r)^\circ$.

Let $n \geqslant 1$ and $w = (e_1, \ldots, e_n)$ a path in $\mathcal{G}$. We define Hilbert C*-modules K_i for $0 \leqslant i \leqslant n$ by

- $K_0 = K_{e_1}^s$
- If $e_{i+1} \neq \bar{e}_i$, then $K_i = K_{e_{i+1}}^s$
- If $e_{i+1} = \bar{e}_i$, then $K_i = (K_{e_{i+1}}^s)^\circ$
- $K_n = A_{r(e_n)}$

For $0 \leqslant i \leqslant n-1$, K_i is a right Hilbert $B_{e_{i+1}}$-module and K_n will be seen as a right Hilbert $A_{r(e_n)}$-module. We define, for $1 \leqslant i \leqslant n-1$, the unital $*$-homomorphism

$$\rho_i = \rho_{e_{i+1}}^s \circ r_{e_i} : B_{e_i} \to \mathcal{L}_{B_{e_{i+1}}}(K_i),$$

and, $\rho_n = L_{A_{r(e_n)}} \circ r_{e_n} : B_{e_n} \to \mathcal{L}_{A_{r(e_n)}}(K_n)$. We can now define the right Hilbert $A_{r(e_n)}$-module

$$H_w = K_0 \underset{\rho_1}{\otimes} \ldots \underset{\rho_n}{\otimes} K_n$$

endowed with the left action of $A_{s(e_1)}$ given by the unital $*$-homomorphism defined by

$$\lambda_w = \rho_{e_1}^s \otimes \mathrm{id} : A_{s(e_1)} \to \mathcal{L}_{A_{r(e_n)}}(H_w).$$

For any two vertices $p, q \in V(\mathcal{G})$, we define the Hilbert A_p-module $H_{q,p} = \bigoplus_w H_w$, where the sum runs over all paths w in $\mathcal{G}$ from q to p. By convention, when $q = p$, the sum also runs over the empty path, where $H_\emptyset = A_p$ with its canonical Hilbert A_p-module structure. We equip this Hilbert C*-module with the left action of A_q which is given by $\lambda_{q,p} : A_q \to \mathcal{L}_{A_p}(H_{q,p})$ defined by $\lambda_{q,p} = \bigoplus_w \lambda_w$, where, when $q = p$ and $w = \emptyset$ is the empty path, $\lambda_\emptyset := L_{A_p}$.

For every $e \in E(\mathcal{G})$ and $p \in V(\mathcal{G})$, we define an operator $u_e^p : H_{r(e),p} \to H_{s(e),p}$ in the following way. Let w be a path in $\mathcal{G}$ from $r(e)$ to p and let $\xi \in \mathcal{H}_w$.

- If $p = r(e)$ and w is the empty path, then $u_e^p(\xi) = \eta_e^s \otimes \xi \in H_{(e)}$.
- If $n = 1$, $w = (e_1)$, $\xi = \rho_{e_1}^s(a)\eta_{e_1}^s \otimes \xi'$ with $a \in A_{s(e_1)}$ and $\xi' \in A_p$, then
  - If $e_1 \neq \overline{e}$, $u_e^p(\xi) = \eta_e^s \otimes \xi \in H_{(e,e_1)}$.
  - If $e_1 = \overline{e}$, $u_e^p(\xi) = \begin{cases} \eta_e^s \otimes \xi & \in H_{(e,e_1)} & \text{if} & E_{e_1}^s(a) = 0, \\ r_{e_1} \circ s_{e_1}^{-1}(a)\xi' & \in A_p & \text{if} & a \in B_{e_1}^s. \end{cases}$
- If $n \geqslant 2$, $w = (e_1, \ldots, e_n)$, $\xi = \rho_{e_1}^s(a)\eta_{e_1}^s \otimes \xi'$ with $a \in A_{s(e_1)}$ and $\xi' \in K_1 \underset{\rho_2}{\otimes} \ldots \underset{\rho_n}{\otimes} K_n$, then
  - If $e_1 \neq \overline{e}$, $u_e^p(\xi) = \eta_e^s \otimes \xi \in H_{(e,e_1,\ldots,e_n)}$.
  - If $e_1 = \overline{e}$, $u_e^p(\xi) = \begin{cases} \eta_e^s \otimes \xi & \in H_{(e,e_1,\ldots,e_n)} & \text{if} & E_{e_1}^s(a) = 0, \\ (\rho_1(s_{e_1}^{-1}(a)) \otimes \mathrm{id})\xi' & \in H_{(e_2,\ldots,e_n)} & \text{if} & a \in B_{e_1}^s. \end{cases}$

One easily checks that the operators u_e^p commute with the right actions of A_p on $H_{r(e),p}$ and $H_{s(e),p}$ and extend to unitary operators (still denoted u_e^p) in $\mathcal{L}_{A_p}(H_{r(e),p}, H_{s(e),p})$ such that $(u_e^p)^* = u_{\overline{e}}^p$. Moreover, for every $e \in \mathrm{E}(\mathcal{G})$ and every $b \in B_e$, the definition implies that

$$u_{\overline{e}}^p \lambda_{s(e),p}(s_e(b))u_e^p = \lambda_{r(e),p}(r_e(b)) \in \mathcal{L}_{A_p}(H_{r(e),p}).$$

Let $w = (e_1, \ldots, e_n)$ be a path in $\mathcal{G}$ and let $p \in \mathrm{V}(\mathcal{G})$, we set $u_w^p = u_{e_1}^p \ldots u_{e_n}^p \in \mathcal{L}_{A_p}(H_{r(e_n),p}, H_{s(e_1),p})$.

The *p-reduced fundamental C*-algebra* is the C*-algebra

$$P_p = \langle (u_z^p)^* \lambda_{q,p}(A_q)u_w^p | q \in \mathrm{V}(\mathcal{G}), w, z \text{ paths from } q \text{ to } p \rangle \subset \mathcal{L}_{A_p}(H_{p,p}).$$

We sometimes write $P_p^{\mathcal{G}} = P_p$. Let us now explain how one can canonically view P_p as a quotient of P. Let $\mathcal{T}$ be a maximal subtree in $\mathcal{G}$. Given a vertex $q \in \mathrm{V}(\mathcal{G})$, we denote by g_{qp} the unique geodesic path in $\mathcal{T}$ from q to p. For every $e \in \mathrm{E}(\mathcal{G})$, we define a unitary operator $w_e^p = (u_{g_{s(e)p}}^p)^* u_{(e,g_{r(e)p})}^p \in P_p$.
For every $q \in \mathrm{V}(\mathcal{G})$, we define a unital faithful $*$-homomorphism $\pi_{q,p} : A_q \to P_p$ by

$$\pi_{q,p}(a) = (u_{g_{qp}}^p)^* \lambda_{q,p}(a)u_{g_{qp}}^p \quad \text{for all } a \in A_q.$$

It is not difficult to check that the following relations hold:

- $w_{\overline{e}}^p = (w_e^p)^*$ for every $e \in \mathrm{E}(\mathcal{G})$,
- $w_e^p = 1$ for every $e \in \mathrm{E}(\mathcal{T})$,
- $w_{\overline{e}}^p \pi_{s(e),p}(s_e(b))w_e^p = \pi_{r(e),p}(r_e(b))$ for every $e \in \mathrm{E}(\mathcal{G})$, $b \in B_e$.

Thus, we can apply the universal property of the maximal fundamental C*-algebra P to get a unique surjective $*$-homomorphism $\lambda_p : P \to P_p$ such that $\lambda_p(u_e) = w_e^p$ for all $e \in \mathrm{E}(\mathcal{G})$ and $\lambda_p(a) = \pi_{q,p}(a)$ for all $a \in A_q$ and all $q \in \mathrm{V}(\mathcal{G})$. We sometimes write $\lambda_p^{\mathcal{G}} = \lambda_p$.

Let $p_0, p, q \in \mathrm{V}(\mathcal{G})$ and $a = \lambda_{p_0,p}(a_0)u_{e_1}^p \lambda_{s(e_2),p}(a_1)u_{e_2}^p \ldots u_{e_n}^p \lambda_{q,p}(a_n) \in \mathcal{L}_{A_p}(H_{q,p}, H_{p_0,p})$, where $w = (e_1, \ldots, e_n)$ is a (non-empty) path in $\mathcal{G}$ from p_0 to q, $a_0 \in A_{p_0}$ and, for $1 \leqslant i \leqslant n$, $a_i \in A_{r(e_i)}$. The operator a is said to be *reduced* (from p_0 to q) if for all $1 \leqslant i \leqslant n-1$ such that $e_{i+1} = \overline{e}_i$, we have $\mathbb{E}_{e_{i+1}}^s(a_i) = 0$.

Let $w = (e_1, \ldots, e_n)$ be a path from p to p. It is easy to check that any reduced operator of the form $a = \lambda_{p_0,p}(a_0)u_{e_1}^p \ldots u_{e_n}^p \lambda_{q,p}(a_n)$ is in P_p and that the linear span $\mathcal{A}_p$ of A_p and the reduced operators from p to p is a dense $*$-subalgebra of P_p.

Remark 5.3. The notion of reduced operator also makes sense in the maximal fundamental C*-algebra (if we assume the existence of conditional expectations) and, for any fixed $p \in V(\mathcal{G})$, the

linear span of A_p and the reduced operators from p to p is the $*$-algebra $\mathcal{A}$ introduced in Remark 5.2, which is dense in the maximal fundamental C*-algebra. Observe that, by definition, the morphism $\lambda_p : P \to P_p$ is the unique unital $*$-homomorphism which is formally equal to the identity on the reduced operators from p to p. More precisely, one has, for any reduced operator $a = a_0 u_{e_1} \ldots u_{e_n} a_n \in P$ from p to p, $\lambda_p(a) = \lambda_{p,p}(a_0) u_{e_1}^p \ldots u_{e_n}^p \lambda_{p,p}(a_n)$.

We will need the following purely combinatorial lemma which gives an explicit decomposition of the product of two reduced operators in P from p to p as a sum of reduced operators from p to p plus an element in A_p.

Lemma 5.4. [FF13, Lemma 3.17] Let $w = (e_n, \ldots, e_1)$ and $\mu = (f_1, \ldots, f_m)$ be paths from p to p. Set $n_0 = \max\{1 \leqslant k \leqslant \min(n,m) | e_i = \overline{f}_i, \forall i \leqslant k\}$. If the above set is empty, set $n_0 = 0$. Let $a = a_n u_{e_n} \ldots u_{e_1} a_0 \in P$ and $b = b_0 u_{f_1} \ldots u_{f_m} b_m \in P$ be reduced operators. Set $x_0 = a_0 b_0$ and, for $1 \leqslant k \leqslant n_0$, $x_k = a_k(s_{e_k} \circ r_{e_k}^{-1} \circ \mathbb{E}_{e_k}^r(x_{k-1})) b_k$ and $y_k = \mathcal{P}_{e_k}^r(x_{k-1})$. The following holds :

(1) If $n_0 = 0$, then $ab = a_n u_{e_n} \ldots u_{e_1} x_0 u_{f_1} \ldots u_{f_m} b_m$.
(2) If $n_0 = n = m$, then $ab = \sum_{k=1}^n a_n u_{e_n} \ldots u_{e_k} y_k u_{f_k} \ldots u_{f_n} b_n + x_n$.
(3) If $n_0 = n < m$, then $ab = \sum_{k=1}^n a_n u_{e_n} \ldots u_{e_k} y_k u_{f_k} \ldots u_{f_m} b_m + x_n u_{f_{n+1}} \ldots u_{f_m} b_m$.
(4) If $n_0 = m < n$, then $ab = \sum_{k=1}^m a_n u_{e_n} \ldots u_{e_k} y_k u_{f_k} \ldots u_{f_m} b_m + a_n u_{e_n} \ldots u_{e_{m+1}} x_m$.
(5) If $1 \leqslant n_0 < \min\{n, m\}$, then

$$ ab = \sum_{k=1}^n a_n u_{e_n} \ldots u_{e_k} y_k u_{f_k} \ldots u_{f_m} b_m + a_n u_{e_n} \ldots u_{e_{n_0+1}} x_{n_0} u_{f_{n_0+1}} \ldots u_{f_m} b_m. $$

Note that the preceding Lemma also holds in P_p, for all $p \in V(\mathcal{G})$, simply by applying the unital $*$-homomorphism λ_p which is formally the identity on the reduced operators from p to p, as explained in Remark 5.3.

In the following Proposition we completely characterize the p-reduced fundamental C*-algebra: it is the unique quotient of P for which there exists a GNS-faithful ucp map $P_p \to A_p$ which is zero on the reduced operators and "the identity on A_p". This proof of this result is almost contained in [FF13] in the GNS-faithful case but it is not explicitly stated like. Since all the arguments are given in [FF13] and since we already did the proof in great details in section 2 in the case of amalgamated free product, we will only sketch the proof of the next Proposition.

Proposition 5.5. For all $p \in V(\mathcal{G})$ the following holds.

(1) The morphism λ_p is faithful on A_p.
(2) There exists a unique ucp map $\mathbb{E}_p : P_p \to A_p$ such that $\mathbb{E}_p \circ \lambda_p(a) = a$ for all $a \in A_p$ and

$\mathbb{E}_p(\lambda_p(a_0 u_{e_1} \ldots u_{e_n} a_n)) = 0$ for all $a = a_0 u_{e_1} \ldots u_{e_n} a_n \in P$ a reduced operator from p to p.

Moreover, $\mathbb{E}_p$ is GNS-faithful.
(3) For any unital C*-algebra with a surjective unital $*$-homomorphism $\pi : P \to C$ and GNS-faithful ucp map $E : C \to A_p$ such that $E \circ \lambda(a) = a$ for all $a \in A_p$ and

$E(\pi(a_0 u_{e_1} \ldots u_{e_n} a_n)) = 0$ for all $a = a_0 u_{e_1} \ldots u_{e_n} a_n \in P$ a reduced operator from p to p

there exists a unique unital $*$-isomorphism $\nu : P_p \to C$ such that $\nu \circ \lambda_p = \pi$. Moreover, ν satisfies $E \circ \nu = \mathbb{E}_p$.

Proof. Assertion (1) follows from assertion (2), since $\mathbb{E}_p \circ \lambda_p(a) = a$ for all $a \in A_p$. Let us sketch the proof of assertion (2). Define $\xi_p = 1_{A_p} \in A_p \subset H_{p,p}$ and $\mathbb{E}_p(x) = \langle \xi_p, x\xi_p \rangle$ for all $x \in P_p$. Then

$\mathbb{E}_p : P_p \to A_p$ is a ucp map and, for all $a \in A_p$, $\mathbb{E}_p(\lambda_p(a)) = \langle 1_{A_p}, L_{A_p}(a)1_{A_p}\rangle = a$. Repeating the proof of [FF13, Proposition 3.18], we see that $\overline{P_p \xi_p \cdot A_p} = H_{p,p}$ and, for any reduced operator $a \in A_p$, one has $\langle \xi_p, a\xi_p \rangle = 0$. It follows that the triple $(H_{p,p}, \mathrm{id}, \xi_p)$ is a GNS-construction of $\mathbb{E}_p$ (in particular $\mathbb{E}_p$ is GNS-faithful) and $\mathbb{E}_p(\lambda_p(x)) = 0$ for any reduced operator $x \in P$ from p to p, since the map λ_p sends reduced operators in P from p to p to reduced operators in P_p.

The proof of (3) is a routine. Since E is GNS-faithful on C we may and will assume that $C \subset \mathcal{L}_{A_p}(K)$, where (K, id, η) is a GNS-construction of E. By the properties of E and $\mathbb{E}_p$, the operator $U : H_{p,p} \to K$ defined by $U(\lambda_p(x)\xi_p) = \pi(x)\eta$ for all $x \in P$ reduced operator from p to p or $x \in A_p \subset P$ extends to a unitary operator $U \in \mathcal{L}_{A_p}(H_{p,p}, K)$. By the definition of U, the map $\nu(x) = UxU^*$, for $x \in P_p$, does the job. The uniqueness is obvious. $\qquad\square$

Notation. We sometimes write $\mathbb{E}_p^{\mathcal{G}} = \mathbb{E}_p$.

For a connected subgraph $\mathcal{G}' \subset \mathcal{G}$ with a maximal subtree $\mathcal{T}' \subset \mathcal{G}'$ such that $\mathcal{T}' \subset \mathcal{T}$ we denote by $P_{\mathcal{G}'}$ the maximal fundamental C*-algebra of our graph of C*-algebras restricted to $\mathcal{G}'$ with respect to the maximal subtree $\mathcal{T}'$. By the universal property there exists a unique unital $*$-homomorphism $\pi_{\mathcal{G}'} : P_{\mathcal{G}'} \to P$ such that $\lambda_{\mathcal{G}'}(a) = a$ for all $a \in A_p$, $p \in V(\mathcal{G}')$ and $\pi_{\mathcal{G}'}(u_e) = u_e$ for all $e \in E(\mathcal{G}')$. The following Corollary says that we have a canonical identification of $P_p^{\mathcal{G}'}$ with the sub-C*-algebra of P_p generated by A_p and the reduced operators from p to p with associated path in $\mathcal{G}'$.

Proposition 5.6. *For all $p \in V(\mathcal{G}')$, there exists unique faithful $*$-homomorphism $\pi_p^{\mathcal{G}'} : P_p^{\mathcal{G}'} \to P_p$ such that $\pi_p^{\mathcal{G}'} \circ \lambda_p^{\mathcal{G}'} = \lambda_p \circ \pi_{\mathcal{G}'}$. The morphism $\pi_p^{\mathcal{G}'}$ satisfies $\mathbb{E}_p \circ \pi_p^{\mathcal{G}'} = \mathbb{E}_p^{\mathcal{G}'}$. Moreover, there exists a unique ucp map $\mathbb{E}_p^{\mathcal{G}'} : P_p \to P_p^{\mathcal{G}'}$ such that $\mathbb{E}_p^{\mathcal{G}'} \circ \pi_p^{\mathcal{G}'} = \mathrm{id}$ and $\mathbb{E}_p^{\mathcal{G}'}(\lambda_p(a)) = 0$ for all $a \in P$ a reduced operator from p to p with associated path containing at least one vertex which is not in $\mathcal{G}'$.*

Proof. The uniqueness of $\pi_p^{\mathcal{G}'}$ being obvious, let us show the existence. Define $P' = \pi_p^{\mathcal{G}'} \circ \lambda_p^{\mathcal{G}'}(P_{\mathcal{G}'}) \subset P_p$ and let $E : P' \to A_p$ be the ucp map defined by $E = \mathbb{E}_p|_{P'}$. By the universal property of Proposition 5.5, assertion 3, it suffices to check that E is GNS-faithul. Let $x \in P'$ such that $E(y^*x^*xy) = \mathbb{E}_p(y^*x^*xy) = 0$ for all $y \in P'$. In particular $\mathbb{E}_p(x^*x) = 0$ and we may and will assume that x^*x is the image under λ_p of a sum of reduced operators from p to p with associated vertices in $\mathcal{G}'$. Let us show that $x = 0$. Since $\mathbb{E}_p$ is GNS-faitful and since P' contains the image under λ_p of A_p and of the reduced operators from p to p in P whose associated path from is in $\mathcal{G}'$, it suffices to show that $\mathbb{E}_p(y^*x^*xy) = 0$ for all $y = \lambda_p(a)$, where $a \in P$ is a reduced operator from p to p whose associated path contains at least one vertex which is not in $\mathcal{G}'$. It follows easily from Lemma 5.4 since this Lemma implies that, for all $z \in P$ a reduced operator from p to p with all edges in $\mathcal{G}'$ or $z \in A_p$ and for all $a \in P$ a reduced operator from p to p with at least one vertex which is not in $\mathcal{G}'$, the product a^*za is equal to a sum of reduced operators from p to p with at least one vertex which is not in $\mathcal{G}'$. In particular, $\mathbb{E}_p(\lambda_p(a^*za)) = 0$ for all such a and z. Hence, $\mathbb{E}_p(yx^*xy) = 0$ for all $y \in P_p$. By construction, $\pi_p^{\mathcal{G}'}$ satisfies $\mathbb{E}_p \circ \pi_p^{\mathcal{G}'} = \mathbb{E}_p^{\mathcal{G}'}$. Finally, we construct the ucp map $\mathbb{E}_p^{\mathcal{G}'}$ (the uniqueness is obvious).

Let $H'_{p,p} = \bigoplus_{\omega \text{ a path in } \mathcal{G}' \text{ from } p \text{ to } p} H_\omega \subset H_{p,p}$. By convention the sum also contains the empty path for which $H_\emptyset = A_p$. Observe that $H'_{p,p}$ is a complemented Hilbert sub-A_p-module of

$H_{p,p}$. Let $Q \in \mathcal{L}_{A_p}(H_{p,p})$ be the orthogonal projection onto $H'_{p,p}$ and define the ucp map $\mathbb{E}_p^{\mathcal{G}'} : P_p \to \mathcal{L}_{A_p}(H'_{p,p})$ by $\mathbb{E}_p^{\mathcal{G}'}(x) = QxQ$.

Since $xH'_{p,p} \subset H'_{p,p}$ for all $x \in P_p^{\mathcal{G}'}$, the projection Q commutes with every $x \in P_p^{\mathcal{G}'}$. Hence, after the identification $P_p^{\mathcal{G}'} \subset P_p$, we have $\mathbb{E}_p^{\mathcal{G}'}(x) = x$ for all $x \in P_p^{\mathcal{G}'}$.

Let $a = a_0 u_{e_1} \ldots u_{e_n} a_n \in P$ be a reduced operator with $\omega = (e_1, \ldots e_n)$ a path in $\mathcal{G}$ from p to p such that $e_k \notin E(\mathcal{G}')$ for some $1 \le k \le n$. Observe that, by Lemma 5.4, for all $b \in P$ a reduced operator from p to p with associated path in $\mathcal{G}'$ or for $b \in A_p$ the product ab is a sum of reduced operators from p to p whose associated path has at least one edge from $\mathcal{G}'$. Hence, $\lambda_p(ab)\xi_p \in H_{p,p} \ominus H'_{p,p}$. It follows now easily from this observation that $Q\lambda_p(a)Q\lambda_p(b)\xi_p = 0$ for all $b \in P$ a reduced operator from p to p or $b \in A_p$. Hence, $Q\lambda_p(a)Q = 0$ and this concludes the proof. $\qquad\square$

The following definition is not contained in [FF13], it is the right definition of the reduced fundamental C*-algebra in the case of non GNS-faithful conditional expectations and it is the main contribution we are making in the present paper to the general theory developed in [FF13].

Definition 5.7. The *vertex-reduced fundamental C*-algebra* P_{vert} is the C*-algebra obtained by separation completion of P for the C*-semi-norm $\|x\|_v = \mathrm{Sup}\{\|\lambda_p(x)\| : p \in V(\mathcal{G})\}$ on P.

We sometimes write $P_{\mathrm{vert}}^{\mathcal{G}} = P_{\mathrm{vert}}$. We will denote by $\lambda : P \to P_{\mathrm{vert}}$ (or $\lambda_{\mathcal{G}}$) the canonical surjection. Note that, by construction of P_{vert}, for all $p \in V(\mathcal{G})$, there exists a unique unital (surjective) $*$-homomorphism $\lambda_{v,p} : P_{\mathrm{vert}} \to P_p$ such that $\lambda_{v,p} \circ \lambda = \lambda_p$. We sometimes write $\lambda_{v,p}^{\mathcal{G}} = \lambda_{v,p}$. We describe the fundamental properties of P_{vert} in the following Proposition.

Proposition 5.8. *The following holds.*
 (1) *The morphism λ is faithful on A_p for all $p \in V(\mathcal{G})$.*
 (2) *For all $p \in V(\mathcal{G})$, there exists a unique ucp map $\mathbb{E}_{A_p} : P_v \to A_p$ such that $\mathbb{E}_{A_p} \circ \lambda(a) = a$ for all $a \in A_p$ and all $p \in V(\mathcal{G})$ and,*

$\mathbb{E}_{A_p}(\lambda_v(a_0 u_{e_1} \ldots u_{e_n} a_n)) = 0$ *for all $a = a_0 u_{e_1} \ldots u_{e_n} a_n \in P$ a reduced operator from p to p.*

 Moreover, the family $\{\mathbb{E}_{A_p} : p \in V(\mathcal{G})\}$ is GNS-faithful.
 (3) *Suppose that C is a unital C*-algebra with a surjective unital $*$-homomorphism $\pi : P \to C$ and with ucp maps $E_{A_p} : C \to A_p$, for $p \in V(\mathcal{G})$, such that $E_{A_p} \circ \pi(a) = a$ for all $a \in A_p$, all $p \in V(\mathcal{G})$ and,*

$E_{A_p}(\pi(a_0 u_{e_1} \ldots u_{e_n} a_n)) = 0$ *for all $a = a_0 u_{e_1} \ldots u_{e_n} a_n \in P$ a reduced operator from p to p*

 and the family $\{E_{A_p} : p \in V(\mathcal{G})\}$ is GNS-faithful. Then, there exists a unique unital $*$-isomorphism $\nu : P_{\mathrm{vert}} \to C$ such that $\nu \circ \lambda = \pi$. Moreover, ν satisfies $E \circ \nu = \mathbb{E}_p$ for all $p \in V(\mathcal{G})$.

Proof. (1). It follows from (2) since $\mathbb{E}_{A_p} \circ \lambda(a) = a$ for all $a \in A_p$ and all $p \in V(\mathcal{G})$.

(2). By Proposition 5.5, the maps $\mathbb{E}_{A_p} = \mathbb{E}_p \circ \lambda_{v,p}$ satisfy the desired properties and it suffices to check that the family $\{\mathbb{E}_{A_p} : p \in V(\mathcal{G})\}$ is GNS-faithful. This is done exactly as in the proof of assertion (2) of Proposition 2.10.

(3). The proof follows the proof of assertion (3) of Proposition 2.10, by using the universal property stated in Proposition 5.5 and the definition of P_{vert}. $\qquad\square$

Notation. We sometimes write $\mathbb{E}^{\mathcal{G}}_{A_p} = \mathbb{E}_{A_p}$.

Proposition 5.9. *There exists a unique faithful $*$-homomorphism $\pi^{\mathcal{G}'}_{\mathrm{vert}} : P^{\mathcal{G}'}_{\mathrm{vert}} \to P_{\mathrm{vert}}$ such that $\pi^{\mathcal{G}'}_{\mathrm{vert}} \circ \lambda_{\mathcal{G}'} = \lambda \circ \pi_{\mathcal{G}'}$. The morphism $\pi^{\mathcal{G}'}_{\mathrm{vert}}$ satisfies $\mathbb{E}_p \circ \pi^{\mathcal{G}'}_p = \mathbb{E}^{\mathcal{G}'}_p$ for all $p \in V(\mathcal{G})$. Moreover, there exists a unique ucp map $\mathbb{E}_{\mathcal{G}'} : P_{\mathrm{vert}} \to P^{\mathcal{G}'}_{\mathrm{vert}}$ such that $\lambda^{\mathcal{G}'}_{v,p} \circ \mathbb{E}_{\mathcal{G}'} = \mathbb{E}^{\mathcal{G}'}_p \circ \lambda_{v,p}$ for all $p \in V(\mathcal{G}')$.*

Proof. Define $P' = \lambda \circ \pi_{\mathcal{G}'}(P_{\mathcal{G}'}) \subset P_{\mathrm{vert}}$ and consider, for $p \in V(\mathcal{G})$, the ucp map $E_{A_p} = \mathbb{E}_{A_p}|_{P'}$. Using the universal property of Proposition 5.8, assertion 3, it suffices to check that the family $\{E_{A_p} : p \in V(\mathcal{G})\}$ is GNS-faithful. Let $x \in P'$ such that $E_{A_p}(y^*x^*xy) = 0$ for all $y \in P'$ and all $p \in V(\mathcal{G})$. Arguing as in the proof of Proposition 5.6 we find that $\mathbb{E}_{A_p}(y^*x^*xy) = 0$ for all $y \in P_{\mathrm{vert}}$ and all $p \in V(\mathcal{G})$. Since the family $\{\mathbb{E}_{A_p} : p \in V(\mathcal{G})\}$ is GNS faithful, the family $\{E_{A_p} : p \in V(\mathcal{G})\}$ is also GNS-faithful. The construction of the canonical ucp map $\mathbb{E}_{\mathcal{G}'} : P_{\mathrm{vert}} \to P^{\mathcal{G}'}_{\mathrm{vert}}$ is similar to the construction made in the proof of Proposition 5.6. Indeed, let $A = \bigoplus_{p \in V(\mathcal{G})} A_p$ and consider the Hilbert A-module $\bigoplus_{p \in V(\mathcal{G})} H_{p,p}$ with the (faithful) left action of P_{vert} given by $\nu = \bigoplus_{p \in V(\mathcal{G})} \lambda_{v,p}$. As in the proof of Proposition 5.6, given any $p \in V(\mathcal{G}')$, we identify the Hilbert module of path in $\mathcal{G}'$ from p to p, with the canonical Hilbert A_p-submodule $H'_{p,p} \subset H_{p,p}$ and we also view $\bigoplus_{p \in V(\mathcal{G}')} H'_{p,p} \subset \bigoplus_{p \in V(\mathcal{G})} H_{p,p}$ as a Hilbert A-submodule. Note that the left action $\bigoplus_{p \in V(\mathcal{G}')} \lambda^{\mathcal{G}'}_{v,p}$ of $P^{\mathcal{G}'}_{\mathrm{vert}}$ on $\bigoplus_{p \in V(\mathcal{G}')} H'_{p,p}$ is faithful so that we may and will view $P^{\mathcal{G}'}_{\mathrm{vert}} \subset \mathcal{L}_A(\bigoplus_{p \in V(\mathcal{G}')} H'_{p,p})$. Let $Q \in \mathcal{L}_A(\bigoplus_{p \in V(\mathcal{G})} H_{p,p})$ be the orthogonal projection onto $\bigoplus_{p \in V(\mathcal{G}')} H'_{p,p}$. Then it is not difficult to check that the ucp map $x \mapsto Q\nu(x)Q$ has the desired properties. $\square$

Example 5.10. When the graph $\mathcal{G}$ has two vertices p_1 and p_2 and two edges e and $\bar{e}$ with $s(e) = p_1$ and $r(e) = p_2$ then we get the amalgamated free product, as explain in Section 2. When the graph has only one vertex p and two edges e and $\bar{e}$ with $s(e) = r(e) = p$ we obtain the HNN-extension. Let us recall the construction of HNN-extension. We have two unital C*-algebras A and B with two unital faithful $*$-homomorphisms $\pi_\epsilon : B \to A$, for $\epsilon \in \{-1, 1\}$. The full HNN-extension $P = \mathrm{HNN}(A, B, \pi_1, \pi_{-1})$ is the universal unital C*-algebra generated by A and a unitary u with the relations $u\pi_1(b)u^* = \pi_{-1}(b)$ for all $b \in B$. When there exists ucp maps $E_\epsilon : A \to B$ such that $E_\epsilon \circ \pi_\epsilon = \mathrm{id}_B$, for all $\epsilon \in \{-1, 1\}$, one can construct the vertex-reduced HNN-extension. We refer to [Fi13, Ue08] for the "edge-reduced" HNN-extension, when E_ϵ is GNS-faithful for all $\epsilon \in \{-1, 1\}$. An operator $x \in P$ is called reduced if $x = a_0 u^{\epsilon_1} \ldots u^{\epsilon_n} a_n$, with $n \geq 1$ and $a_k \in A$ such that, for all $1 \leq k \leq n-1$, we have $\epsilon_{k+1} = -\epsilon_k \implies E_{\epsilon_k}(a_k) = 0$. It is easy to see that the linear span of A and the reduced operator is a dense $*$-subalgebra $\mathcal{A} \subset P$. The vertex reduced HNN-extension $P_{\mathrm{vert}} = \mathrm{HNN}_{\mathrm{vert}}(A, B, \pi_1, \pi_{-1})$ is a quotient $\lambda : P \to P_{\mathrm{vert}}$ having a GNS-faithful ucp map $\mathbb{E} : P_{\mathrm{vert}} \to A$ such that $E(\lambda(a)) = a$ for all $a \in A$ and $E(\lambda(x)) = 0$ for all $x \in P$ reduced. Moreover, any quotient of P have such a GNS-faithful ucp map is isomorphic to P_{vert}. Let us now described the vertex-reduced HNN-extension in the extreme degenerated case i.e. when E_ϵ is an homomorphism for $\epsilon \in \{-1, 1\}$. Define the right Hilbert A-module $H = A \otimes l^2(\mathbb{Z})$ with the faithful unital $*$-homomorphism $\rho : A \to \mathcal{L}_A(H)$ defined by
$$\rho(a)(x \otimes e_n) = \begin{cases} ax \otimes e_n & \text{if} \quad n = 0 \\ (\pi_1 \circ E_{-1})^{\circ n}(a)x \otimes e_n & \text{if} \quad n > 0 \\ (\pi_{-1} \circ E_1)^{\circ -n}(a)x \otimes e_n & \text{if} \quad n < 0 \end{cases}$$
Define the unitary $u = 1 \otimes s \in \mathcal{L}_A(H)$, where $s \in \mathcal{L}(l^2(\mathbb{Z}))$ is the bilateral shift. It is easy to check that $u^*\rho(\pi_{-1}(b))u = \rho(\pi_1(b))$ for all $b \in B$. Let $\xi = 1_A \otimes e_0 \in \xi$. It can be easily

checked that $\overline{\langle \rho(A), u \rangle \xi \cdot A} = H$. Hence, the ucp map $\mathbb{E} : \langle \rho(A), u \rangle \to A$, $x \mapsto \langle \xi, x\xi \rangle$ is GNS-faithful. Moreover, it is obviously zero on the image of the reduced operator in P. By uniqueness, $\langle \rho(A), u \rangle \simeq \mathrm{HNN}_{\mathrm{vert}}(A, B, \pi_1, \pi_{-1})$ canonically.

We now described the *Serre's devissage* process for our vertex-reduced fundamental C*-algebras.

For $e \in E(\mathcal{G})$, let $\mathcal{G}_e$ be the graph obtained from $\mathcal{G}$ by removing the edges e and $\overline{e}$ i.e., $V(\mathcal{G}_e) = V(\mathcal{G})$ and $E(\mathcal{G}_e) = E(\mathcal{G}) \setminus \{e, \overline{e}\}$. The source range and inverse maps are the restrictions of the one for $\mathcal{G}$. The *Serre's devissage* shows that, when $\mathcal{G}_e$ is not connected, the vertex-reduced fundamental C*-algebra is a vertex-reduced amalgamated free product and, when $\mathcal{G}_e$ is connected, the vertex-reduced fundamental C*-algebra is a vertex-reduced HNN-extension.

Case 1: $\mathcal{G}_e$ is not connected. Let $\mathcal{G}_{s(e)}$ (respectively $\mathcal{G}_{r(e)}$) the connected component of $s(e)$ (resp. $r(e)$) in $\mathcal{G}_e$. Since $\mathcal{G}_e$ is not connected $e \in E(\mathcal{T})$ and the graphs $\mathcal{T}_{s(e)} := \mathcal{T} \cap \mathcal{G}_{s(e)}$ and $\mathcal{T}_{r(e)} := \mathcal{T} \cap \mathcal{G}_{r(e)}$ are maximal subtrees of $\mathcal{G}_{s(e)}$ and $\mathcal{G}_{r(e)}$ respectively. Let $P_{\mathcal{G}_{s(e)}}$ and $P_{\mathcal{G}_{r(e)}}$ be the maximal fundamental C*-algebras of our graph of C*-algebras restricted to $\mathcal{G}_{s(e)}$ and $\mathcal{G}_{r(e)}$ respectively and with respect to the maximal subtrees $\mathcal{T}_{s(e)}$ and $\mathcal{T}_{r(e)}$ respectively. Recall that we have canonical maps $\pi_{\mathcal{G}_{s(e)}} : P_{\mathcal{G}_{s(e)}} \to P$ and $\pi_{\mathcal{G}_{r(e)}} : P_{\mathcal{G}_{r(e)}} \to P$.

Let $P_{\mathcal{G}_{s(e)}} \underset{B_e}{*} P_{\mathcal{G}_{r(e)}}$ be the full free product of $P_{\mathcal{G}_{s(e)}}$ and $P_{\mathcal{G}_{r(e)}}$ amalgamated over B_e relative to the maps $s_e : B_e \to P_{\mathcal{G}_{s(e)}}$ and $r_e : B_e \to P_{\mathcal{G}_{r(e)}}$. Observe that, since $e \in E(\mathcal{T})$, we have $u_e = 1 \in P$. Hence, we have $s_e(b) = r_e(b)$ in P, for all $b \in B_e$. By the universal property of the full amalgamated free product there exists a unique unital $*$-homomorphism $\nu : P_{\mathcal{G}_{s(e)}} \underset{B_e}{*} P_{\mathcal{G}_{r(e)}} \to P$ such that $\nu|_{P_{\mathcal{G}_{s(e)}}} = \pi_{\mathcal{G}_{s(e)}}$ and $\nu|_{P_{\mathcal{G}_{r(e)}}} = \pi_{\mathcal{G}_{r(e)}}$. Moreover, by the universal property of P, there exists also a unital $*$-homomorphism $P \to P_{\mathcal{G}_{s(e)}} \underset{B_e}{*} P_{\mathcal{G}_{r(e)}}$ which is the inverse of ν. Hence, ν is a $*$-isomorphism. Actually, this is also true at the vertex-reduced level.

Note that we have injective unital $*$-homomorphisms $\iota_{s(e)} = \lambda_{\mathcal{G}_{s(e)}} \circ s_e : B_e \to P_{\mathrm{vert}}^{\mathcal{G}_{s(e)}}$ and $\iota_{s(e)} = \lambda_{\mathcal{G}_{r(e)}} \circ r_e : B_e \to P_{\mathrm{vert}}^{\mathcal{G}_{r(e)}}$ and conditional expectations $E_{s(e)} = \lambda_{\mathcal{G}_{s(e)}} \circ E_e^s \circ \mathbb{E}_{A_{s(e)}}^{\mathcal{G}_{s(e)}}$ from $P_{\mathrm{vert}}^{\mathcal{G}_{s(e)}}$ to $\iota_{s(e)}(B_e)$ and $E_{r(e)} = \lambda_{\mathcal{G}_{r(e)}} \circ E_e^r \circ \mathbb{E}_{A_{r(e)}}^{\mathcal{G}_{r(e)}}$ from $P_{\mathrm{vert}}^{\mathcal{G}_{r(e)}}$ to $\iota_{r(e)}(B_e)$ so that we can perform the vertex-reduced amalgamated free product. Following Section 2, we write

$$\pi : P_{\mathrm{vert}}^{\mathcal{G}_{s(e)}} \underset{B_e}{*} P_{\mathrm{vert}}^{\mathcal{G}_{r(e)}} \to P_{\mathrm{vert}}^{\mathcal{G}_{s(e)}} \underset{B_e}{\overset{v}{*}} P_{\mathrm{vert}}^{\mathcal{G}_{r(e)}}$$

the canonical surjection for the full amalgamated free product to the vertex-reduced amalgamated free product. and $\mathbb{E}_1$ (resp. $\mathbb{E}_2$) the canonical cup map from $P_{\mathrm{vert}}^{\mathcal{G}_{s(e)}} \underset{B_e}{\overset{v}{*}} P_{\mathrm{vert}}^{\mathcal{G}_{r(e)}}$ to $P_{\mathrm{vert}}^{\mathcal{G}_{s(e)}}$ (resp. to $P_{\mathrm{vert}}^{\mathcal{G}_{r(e)}}$).

Lemma 5.11. *There exists a unique $*$-isomorphism $\nu_e : P_{\mathrm{vert}}^{\mathcal{G}_{s(e)}} \underset{B_e}{\overset{v}{*}} P_{\mathrm{vert}}^{\mathcal{G}_{r(e)}} \to P_{\mathrm{vert}}$ such that:*

$$\nu_e \circ \pi \circ \lambda_{\mathcal{G}_{s(e)}} = \lambda \circ \pi_{\mathcal{G}_{s(e)}} \quad and \quad \nu_e \circ \pi \circ \lambda_{\mathcal{G}_{r(e)}} = \lambda \circ \pi_{\mathcal{G}_{r(e)}}.$$

Moreover we have $\mathbb{E}_{\mathcal{G}_{s(e)}} \circ \nu_e = \mathbb{E}_1$ and $\mathbb{E}_{\mathcal{G}_{r(e)}} \circ \nu_e = \mathbb{E}_2$.

Proof. The proof is the same as the proof of [FF13, Lemma 3.26], it suffices to prove that P_{vert} satisfies the universal property of $P_{\mathrm{vert}}^{\mathcal{G}_{s(e)}} \underset{B_e}{\overset{v}{*}} P_{\mathrm{vert}}^{\mathcal{G}_{r(e)}}$: the canonical ucp maps from P_{vert} to $P_{\mathrm{vert}}^{\mathcal{G}_{s(e)}}$

and $P_{\text{vert}}^{\mathcal{G}_{r(e)}}$ are the ones constructed in Proposition 5.9 i.e. $\mathbb{E}_{\mathcal{G}_{s(e)}}$ and $\mathbb{E}_{\mathcal{G}_{r(e)}}$. By the universal property, the result isomorphism ν_e intertwines the canonical ucp maps.   □

Case 2: $\mathcal{G}_e$ is connected.

Let $e \in E(\mathcal{G})$ and suppose that $\mathcal{G}_e$ is connected. Up to a canonical isomorphism of P we may and will assume that $\mathcal{T} \subset \mathcal{G}_e$. So that we have the canonical unital $*$-homomorphism $\pi_{\mathcal{G}_e} : P_{\mathcal{G}_e} \to P$. We consider the two unital faithful $*$-homomorphisms $s_e, r_e : B_e \to P_{\mathcal{G}_e}$. By definition, we have $u_e r_e(b) u_e^* = s_e(b)$ for all $b \in B_e$ and P is generated, as a C*-algebra, by $\pi_{\mathcal{G}_e}(P_{\mathcal{G}_e})$ and u_e. By the universal property of the maximal HNN-extension, there exists a unique unital (surjective) $*$-homomorphism $\nu : \text{HNN}(P_{\mathcal{G}_e}, B_e, s_e, r_e) \to P$ such that $\nu|_{P_{\mathcal{G}_e}} = \pi_{\mathcal{G}_e}$ and $\nu(u) = u_e$. Observe that, by the universal property of P, there exists a unital $*$-homomorphism $P \to \text{HNN}(P_{\mathcal{G}_e}, B_e, s_e, r_e)$ which is the inverse of ν. Hence ν is a $*$-isomorphism. Actually this is also true at the vertex-reduced level.

Define the faithful unital $*$-homomorphism $\pi_1, \pi_{-1} : B_e \to P_{\text{vert}}^{\mathcal{G}_e}$ by $\pi_{-1} = \lambda_{\mathcal{G}_e} \circ s_e$ and $\pi_1 = \lambda_{\mathcal{G}_e} \circ r_e$. Note that the ucp maps $E_\epsilon : P_{\text{vert}}^{\mathcal{G}_e} \to B_e$ defined by $E_1 = s_e^{-1} \circ E_e^s \circ \mathbb{E}_{s(e)}^{\mathcal{G}_e}$ and $E_{-1} = r_e^{-1} \circ E_e^r \circ \mathbb{E}_{r(e)}^{\mathcal{G}_e}$ satisfy $E_\epsilon \circ \pi_\epsilon = \text{id}_{B_e}$ for $\epsilon \in \{-1, 1\}$. Hence we may consider the vertex-reduced HNN-extension and the canonical surjection $\lambda_e : \text{HNN}(P_{\text{vert}}^{\mathcal{G}_e}, B_e, s_e, r_e) \to \text{HNN}_{\text{vert}}(P_{\text{vert}}^{\mathcal{G}_e}, B_e, \pi_1, \pi_{-1})$. Write $v = \lambda_e(u)$, where $u \in \text{HNN}(P_{\text{vert}}^{\mathcal{G}_e}, B_e, s_e, r_e)$ is the "stable letter". Recall that, by Proposition 5.9, we have the canonical faithful unital $*$-homomorphism $\pi_{\text{vert}}^{\mathcal{G}_e} : P_{\text{vert}}^{\mathcal{G}_e} \to P_{\text{vert}}$. Let $\mathbb{E} : \text{HNN}_{\text{vert}}(P_{\text{vert}}^{\mathcal{G}_e}, B_e, \pi_1, \pi_{-1}) \to P_{\text{vert}}^{\mathcal{G}_e}$ the canonical GNS-faithful ucp map.

Lemma 5.12. *There is a unique $*$-isomorphism $\nu_e : \text{HNN}_{\text{vert}}(P_{\text{vert}}^{\mathcal{G}_e}, B_e, \pi_1, \pi_{-1}) \to P_{\text{vert}}$ such that $\nu_e \circ \lambda_e|_{P_{\text{vert}}^{\mathcal{G}_e}} = \pi_{\text{vert}}^{\mathcal{G}_e}$ and $\nu_e(u) = u_e$. Moreover $\mathbb{E}_{\mathcal{G}_e} \circ \nu_e = \mathbb{E}$.*

Proof. Since we have $u_e \pi_{\text{vert}}^{\mathcal{G}_e}(\pi_{-1}(b)) u_e^* = \pi_{\text{vert}}^{\mathcal{G}_e}(\pi_1(b))$ for all $b \in B_e$, it suffices, by the universal property of the vertex-reduced HNN-extension explained in Example 5.10, to check that we have a GNS-faithful ucp map $P_{\text{vert}} \to P_{\text{vert}}^{\mathcal{G}_e}$ satisfying the conditions described in Example 5.10. This ucp map is the one constructed in Proposition 5.9: it is the map $\mathbb{E}_{\mathcal{G}_e}$ and the conditions can be checked as in the proof of [FF13, Lemma 3.27]. The fact that the resulting isomorphism ν_e intertwines the ucp maps follows from the universal property.   □

We end this preliminary section with an easy Lemma.

Lemma 5.13. *If $x = a_0 u_{e_1} \ldots u_{e_n} a_n \in P$ is a reduced operator from p to p and $a_n \in B_{e_n}^r$ then*

$$\mathbb{E}_p(\lambda_p(x^* x)) = \mathbb{E}_{e_n}^r \circ \mathbb{E}_p(\lambda_p(x^* x)).$$

Proof. Define $x_0 = a_0^* a_0$ and for $1 \leq k \leq n$, $x_k = a_k^*(r_{e_k} \circ s_{e_k}^{-1} \circ \mathbb{E}_{e_k}^s(x_{k-1})) a_k$. We apply Lemma 5.4 to the pair $a = x^*$ and $b = x$ in case (2). It follows that $x^* x = y + x_n$, where y is a sum of reduced operators from p to p. Hence $\mathbb{E}_p(\lambda_p(y)) = 0$ and, since $a_n \in B_{e_n}^r$, we have $x_n = a_n^*(r_{e_n} \circ s_{e_n}^{-1} \circ \mathbb{E}_{e_n}^s(x_{n-1})) a_n \in B_{e_n}^r$.   □

5.2. **Boundary maps.** Define the completely positive map $\mathbb{E}_e = E_e^r \circ \mathbb{E}_{A_{r(e)}} : P_{\text{vert}} \to B_e^r$. Note that the GNS construction of $\mathbb{E}_e$ is given by $(H_{r(e),r(e)} \underset{E_e^r}{\otimes} B_e^r, \lambda_{v,r(e)} \otimes 1, \xi_{r(e)} \otimes 1)$. To simplify

the notations, we will denote by (K_e, ρ_e, η_e) the GNS construction of $\mathbb{E}_e$. We define $\mathcal{R}_e \subset K_e$ has the Hilbert B_e^r-submodule of K_e of the "words ending with e". More precisely,

$$\mathcal{R}_e := \overline{\mathrm{Span}}\{\rho_e(\lambda(x))\eta_e \mid x = a_0 u_{e_1} \ldots u_{e_n} a_n \in P \text{ reduced from } r(e) \text{ to } r(e)$$

$$\text{with } e_n = e \text{ and } a_n \in B_e^r\} \subset K_e.$$

It is easy to see from the definition that $\mathcal{R}_e$ is a Hilbert B_e^r-submodule of K_e. Moreover, it is complemented in K_e with orthogonal complement given by:

$$\mathcal{L}_e := \overline{\mathrm{Span}}\{\rho_e(\lambda(x))\eta_e \mid x \in A_{r(e)} \text{ or } x = a_0 u_{e_1} \ldots u_{e_n} a_n \in P \text{ reduced from } r(e) \text{ to } r(e) \text{ with}$$

$$e_n \neq e \text{ or } e_n = e \text{ and } a_n \in A_{r(e)} \ominus B_e^r\}.$$

Let $Q_e \in \mathcal{L}_{B_e^r}(K_e)$ be the orthogonal projection onto $\mathcal{R}_e$ and define

$$X_e = \{x = a_0 u_{e_1} \ldots u_{e_n} a_n \in P \text{ reduced from } r(e) \text{ to } r(e) \text{ with } e_k \notin \{\bar{e}, e\} \text{ for all } 1 \leq k \leq n\},$$

Lemma 5.14. *The following holds.*

(1) *For all reduced operator* $a = a_n u_{e_n} \ldots u_{e_1} a_0 \in P$ *from* $r(e)$ *to* $r(e)$ *we have*

$$\mathrm{Im}(Q_e \rho_e(\lambda(a)) - \rho_e(\lambda(a)) Q_e) \subset X_a, \text{ where:}$$

$$X_a = \begin{cases} Y_a := \rho_e(\lambda(a_n))\eta_e \cdot B_e^r \oplus \left(\displaystyle\bigoplus_{k \in \{1,\ldots,n\}, e_k = e} \rho_e(\lambda(a_n u_{e_n} \ldots u_{e_k}))\eta_e \cdot B_e^r \right) & \text{if } e \text{ is not a loop,} \\[2em] Y_a \oplus \left(\displaystyle\bigoplus_{k \in \{1,\ldots,n\}, e_k = \bar{e}} \rho_e(\lambda(a_n u_{e_n} \ldots u_{e_k} a_{k-1}))\eta_e \cdot B_e^r \right) & \text{if } e \text{ is a loop.} \end{cases}$$

(2) Q_e *commutes with* $\rho_e(\lambda(a))$ *for all* $a \in \overline{\mathrm{Span}}\left(A_{r(e)} \cup X_e\right)$.

Proof. It is obvious that $\rho_e(\lambda(a))$ commutes with Q_e for all $a \in A_{r(e)}$. Hence, (2) follows from (1). Let us prove (1). During the proof we will use the notation $\mathcal{P}_e^r(x) = x - E_e^r(x)$ for $x \in A_{r(e)}$.

Let $n \geq 1$ and $a = a_n u_{e_n} \ldots u_{e_1} a_0 \in P$ a reduced operator from $r(e)$ to $r(e)$. Suppose that $b \in A_{r(e)}$. We have $Q_e \rho_e(\lambda(b))\eta_e = 0$ and $ab = a_n u_{e_n} \ldots u_{e_1} a_0 b \in P$ is reduced. Hence, if $e_1 \neq e$, we have $Q_e \rho_e(\lambda(ab))\eta_e = 0$ and, if $e_1 = e$, we have

$$ab = a_n u_{e_n} \ldots u_e E_e^r(x_0) + a_n \ldots u_e \mathcal{P}_e^r(x_0) \quad \text{where} \quad x_0 = a_0 b.$$

It follows that $Q_e \rho_e(\lambda(ab))\eta_e = \rho_e(\lambda(a_n u_{e_n} \ldots u_e E_e^r(x_0)))\eta_e$. To conclude we have, $\forall b \in A_{r(e)}$,

$$(Q_e \rho_e(\lambda(a)) - \rho_e(\lambda(a)) Q_e)\rho_e(\lambda(b))\eta_e = \begin{cases} 0 \in X_a & \text{if } e_1 \neq e, \\ \rho_e(\lambda(a_n u_{e_n} \ldots u_{e_1}))\eta_e \cdot E_e^r(a_0 b) \in X_a & \text{if } e_1 = e. \end{cases}$$

Suppose that $b = b_0 u_{f_1} \ldots u_{f_m} b_m \in P$ is a reduced operator from $r(e)$ to $r(e)$. Let $0 \leq n_0 \leq \min\{n, m\}$ be the integer associated to the couple (a, b) in Lemma 5.4. This Lemma implies that, when $n_0 = 0$ or $n_0 = n < m$ or $1 \leq n_0 < \min\{n, m\}$, ab is reduced word or a sum of reduced words that end by $u_{f_m} b_m$. Hence, in this cases, we have $\rho_e(\lambda(b))\eta_e \in \mathcal{R}_e \implies \rho_e(\lambda(ab))\eta_e \in \mathcal{R}_e$ and $\rho_e(\lambda(b))\eta_e \in \mathcal{L}_e \ominus A_{r(e)} \implies \rho_e(\lambda(ab))\eta_e \in \mathcal{L}_e \ominus A_{r(e)}$. It follows that $(Q_e \rho_e(\lambda(a)) - \rho_e(\lambda(a)) Q_e)\rho_e(\lambda(b))\eta_e = 0 \in X_a$.

Suppose now that $n_0 = m < n$. Lemma 5.4 implies that $ab = y + z$ where y is a sum of reduced words that end by $u_{f_m} b_m$ and $z = a_n u_{e_n} \ldots u_{e_{m+1}} x_m$. Hence we have $\rho_e(\lambda(b))\eta_e \in \mathcal{R}_e \implies \rho_e(\lambda(y))\eta_e \in \mathcal{R}_e$ and $\rho_e(\lambda(b))\eta_e \in \mathcal{L}_e \ominus A_{r(e)} \implies \rho_e(\lambda(y))\eta_e \in \mathcal{L}_e \ominus A_{r(e)}$. It follows that

$$(Q_e \rho_e(\lambda(a)) - \rho_e(\lambda(a))Q_e)\rho_e(\lambda(b))\eta_e = \begin{cases} Q_e \rho_e(\lambda(z))\eta_e & \text{if } \rho_e(\lambda(b))\eta_e \in \mathcal{L}_e, \\ Q_e \rho_e(\lambda(z))\eta_e - \rho_e(\lambda(z))\eta_e & \text{if } \rho_e(\lambda(b))\eta_e \in \mathcal{R}_e. \end{cases}$$

We have $\rho_e(\lambda(z))\eta_e = \rho_e(\lambda(a_n u_{e_n} \ldots u_{e_{m+1}} x_m))\eta_e$ hence,

$$Q_e \rho_e(\lambda(z))\eta_e = \begin{cases} 0 \in X_a & \text{if } e_{m+1} \neq e \text{ or } e_{m+1} = e \text{ and } x_m \in A_{r(e)} \ominus B_e^r, \\ \rho_e(\lambda(a_n u_{e_n} \ldots u_{e_{m+1}}))\eta_e.x_m \in X_a & \text{if } e_{m+1} = e \text{ and } x_m \in B_e^r. \end{cases}$$

Hence $(Q_e \rho_e(\lambda(a)) - \rho_e(\lambda(a))Q_e)\rho_e(\lambda(b))\eta_e \in X_a$ if $\rho_e(\lambda(b))\eta_e \in \mathcal{L}_e$ and, if $\rho_e(\lambda(b))\eta_e \in \mathcal{R}_e$, we have $f_m = e$ and $b_m \in B_e^r$. Since $n_0 = m$ we have $e_m = \overline{f_m} = \overline{e}$ and $x_m = a_m(r_e \circ s_e^{-1} \circ E_e^s(x_{m-1}))b_m$. Note that, since $r(f_m) = r(e)$ and $f_m = \overline{e}$ we find that $s(e) = r(f_m) = r(e)$. Hence e must be a loop. Moreover, $\rho_e(\lambda(z))\eta_e = \rho_e(\lambda(a_n u_{e_n} \ldots u_{e_{m+1}} a_m))\eta_e \cdot x_m' \in X_a$, where $x_m' = (r_e \circ s_e^{-1} \circ E_e^s(x_{m-1}))b_m \in B_e^r$. It follows that $(Q_e \rho_e(\lambda(a)) - \rho_e(\lambda(a))Q_e)\rho_e(\lambda(b))\eta_e \in X_a$ also when $\rho_e(\lambda(b))\eta_e \in \mathcal{R}_e$.

Suppose that $n_0 = n = m$. Lemma 5.4 implies that $ab = y + x_m$ where y is a sum of reduced words that end by $u_{f_m} b_m$. As before, we deduce that:

$$(Q_e \rho_e(\lambda(a)) - \rho_e(\lambda(a))Q_e)\rho_e(\lambda(b))\eta_e = \begin{cases} Q_e \rho_e(\lambda(x_m))\eta_e = 0 & \text{if } \rho_e(\lambda(b))\eta_e \in \mathcal{L}_e, \\ Q_e \rho_e(\lambda(x_m))\eta_e - \rho_e(\lambda(x_m))\eta_e & \text{if } \rho_e(\lambda(b))\eta_e \in \mathcal{R}_e. \end{cases}$$

And, if $\rho_e(\lambda(b))\eta_e \in \mathcal{R}_e$ then $f_m = e$ and $b_m \in B_e^r$. Since $n_0 = m = n$, we deduce that $e_n = \overline{f_m} = \overline{e}$ (hence e is a loop) and $x_m = a_n(r_e \circ s_e^{-1} \circ E_e^s(x_{n-1}))b_n \in a_n B_e^r$. Hence,

$$Q_e \rho_e(\lambda(x_m))\eta_e - \rho_e(\lambda(x_m))\eta_e = -\rho_e(\lambda(x_m))\eta_e = -\rho_e(\lambda(a_n))\eta_e \cdot x_n' \in X_a,$$

where $x_n' = (r_e \circ s_e^{-1} \circ E_e^s(x_{n-1}))b_n \in B_e^r$. This concludes the proof of the Lemma. $\qquad\square$

Define $V_e = 2Q_e - 1 \in \mathcal{L}_{B_e^r}(K_e)$. We have $V_e^2 = 1$, $V_e = V_e^*$ and, for all $x \in P_{\text{vert}}$, Lemma 5.14 implies that $V_e \rho_e(x) - \rho_e(x)V_e \in \mathcal{K}_{B_e^r}(K_e)$. Hence we get an element $y_e^{\mathcal{G}} \in KK^1(P_{\text{vert}}, B_e^r)$. Define $x_e^{\mathcal{G}} = y_e^{\mathcal{G}} \underset{B_e^r}{\otimes} [r_e^{-1}] \in KK^1(P_{\text{vert}}, B_e)$.

Remark 5.15. Note that we also have an element $z_e^{\mathcal{G}} = [\lambda] \underset{P_{\text{vert}}}{\otimes} x_e^{\mathcal{G}} \in KK^1(P, B_e)$.

Recall that for a subgraph $\mathcal{G}' \subset \mathcal{G}$ with a maximal subtree $\mathcal{T}' \subset \mathcal{G}'$ such that $\mathcal{T}' \subset \mathcal{T}$ we have the canonical unital faithful $*$-homomorphism $\pi_{\text{vert}}^{\mathcal{G}'} : P_{\text{vert}}^{\mathcal{G}'} \to P_{\text{vert}}$ defined in Proposition 5.9.

Proposition 5.16. *For all connected subgraph $\mathcal{G}' \subset \mathcal{G}$ with maximal subtree $\mathcal{T}' \subset \mathcal{T}$, we have*

(1) *if $e \in E(\mathcal{G}')$ then $[\pi_{\text{vert}}^{\mathcal{G}'}] \underset{P_{\text{vert}}}{\otimes} x_e^{\mathcal{G}} = x_e^{\mathcal{G}'} \in KK^1(P_{\text{vert}}^{\mathcal{G}'}, B_e)$,*

(2) *if $e \notin E(\mathcal{G}')$ then $[\pi_{\text{vert}}^{\mathcal{G}'}] \underset{P_{\text{vert}}}{\otimes} x_e^{\mathcal{G}} = 0 \in KK^1(P_{\text{vert}}^{\mathcal{G}'}, B_e)$.*

(3) *$\sum_{r(e)=p} x_e^{\mathcal{G}} \underset{B_e}{\otimes} [r_e] = 0 \in KK^1(P_{\text{vert}}, A_p)$ for all $p \in V(\mathcal{G})$.*

(4) *For all $e \in E(\mathcal{G})$ we have $x_{\overline{e}}^{\mathcal{G}} = -x_e^{\mathcal{G}}$.*

Proof. Let $\mathcal{G}' \subset \mathcal{G}$ be a connected subgraph with maximal subtree $\mathcal{T}' \subset \mathcal{T}$ and $e \in E(\mathcal{G})$.

(1). Suppose that $e \in E(\mathcal{G}')$ (hence $\bar{e} \in E(\mathcal{G}')$). Recall that we have the canonical ucp map $\mathbb{E}_{\mathcal{G}'} : P_{\text{vert}} \to P_{\text{vert}}^{\mathcal{G}'}$ from Proposition 5.9. Moreover, by definition of $\pi_{\text{vert}}^{\mathcal{G}'}$ of we have $\mathbb{E}_e^{\mathcal{G}'} = \mathbb{E}_e \circ \pi_{\text{vert}}^{\mathcal{G}'}$, where $\mathbb{E}_e^{\mathcal{G}'} = E_e^r \circ \mathbb{E}_{A_{r(e)}}^{\mathcal{G}'}$.

Let (K_e, ρ_e, η_e) be the GNS construction of $\mathbb{E}_e$ and define $K_e' = \overline{\rho_e \circ \pi_{\text{vert}}^{\mathcal{G}'}(P_{\text{vert}}^{\mathcal{G}'})\eta_e \cdot B_e^r}$. Observe that K_e' is complemented. Indeed, we have $K_e' \oplus L_e = K_e$, where

$$L_e = \overline{\text{Span}}\{\rho_e(x)\eta_e \cdot b : b \in B_e^r \text{ and } x \in P_{\text{vert}} \text{ such that } \mathbb{E}_{\mathcal{G}'}(x) = 0\}.$$

Let $R_e \in \mathcal{L}_{B_e^r}(K_e)$ be the orthogonal projection onto K_e'. Since $\rho_e \circ \pi_{\text{vert}}^{\mathcal{G}'}(x)K_e' \subset K_e'$ for all $x \in P_{\text{vert}}^{\mathcal{G}'}$, R_e commutes with $\rho_e \circ \pi_{\text{vert}}^{\mathcal{G}'}(x)$ for all $x \in P_{\text{vert}}^{\mathcal{G}'}$. It is also easy to check that R_e commutes with Q_e hence with V_e.

Since $\mathbb{E}_e^{\mathcal{G}'} = \mathbb{E}_e \circ \pi_{\text{vert}}^{\mathcal{G}'}$ the triple (K_e', ρ_e', η_e'), where $\rho_e'(x) = \rho_e \circ \pi_{\text{vert}}^{\mathcal{G}'}(x)R_e$ for $x \in P_{\text{vert}}^{\mathcal{G}'}$ and $\eta_e' = \eta_e$, is a GNS construction of $\mathbb{E}_e^{\mathcal{G}'}$. Let $Q_e' \in \mathcal{L}_{B_e^r}(K_e')$ be the associated operator such that $x_e^{\mathcal{G}'} = [(K_e', \rho_e', V_e')]$, with $V_e' = 2Q_e' - 1$. By definition we have $Q_e' = Q_e R_e$ hence, $V_e' = V_e R_e$. It follows that $[\pi_{\text{vert}}^{\mathcal{G}'}] \underset{P_{\text{vert}}}{\otimes} x_e^{\mathcal{G}} = x_e^{\mathcal{G}'} \oplus y$, where $y \in KK^1(P_{\text{vert}}^{\mathcal{G}'}, B_e)$ is represented by the triple $(L_e, \pi_e, V_e(1 - R_e))$, where $\pi_e = \rho_e \circ \pi_{\text{vert}}^{\mathcal{G}'}(\cdot)(1 - R_e)$. To conclude the proof of (1) it suffices to check that this triple is degenerated. Since V_e and $(1 - R_e)$ commute, $V_e(1 - R_e)$ is self-adjoint and $(V_e(1 - R_e))^2 = 1 - R_e = \text{id}_{L_e}$. Hence, it suffices to check that, for all $a \in P_{\text{vert}}^{\mathcal{G}'}$,

$$(Q_e\rho_e \circ \pi_{\text{vert}}^{\mathcal{G}'}(a) - \rho_e \circ \pi_{\text{vert}}^{\mathcal{G}'}(a)Q_e)(1 - R_e) = 0.$$

We already know from assertion (2) of Lemma 5.14 that $Q_e\rho_e(\lambda(a)) = \rho_e(\lambda(a))Q_e$ for all $a \in A_{r(e)}$ (and all $a \in X_e$). Let $a = a_n u_{e_n} \ldots u_{e_1} a_0 \in P_{\mathcal{G}'}$ and $b = b_0 u_{f_1} \ldots u_{f_m} b_m \in P$ be reduced operators from $r(e)$ to $r(e)$ and suppose that $\mathbb{E}_{\mathcal{G}'}(\lambda(b)) = 0$. Hence, there exists $k \in \{1, \ldots m\}$ such that $f_k \notin E(\mathcal{G}')$ and it follows that the integer n_0 associated to the pair $(\pi_{\mathcal{G}'}(a), b)$ in Lemma 5.4 satisfies $n_0 < k$ since $e_l \in E(\mathcal{G}')$ for all $l \in \{1, \ldots n\}$. Applying Lemma 5.4 in case (5), we see that $\pi_{\mathcal{G}'}(a)b$ is a sum of reduced operators that end with $u_{f_m}b_m$. Hence, $\rho_e(\lambda(b))\eta_e \in \mathcal{R}_e \implies \rho_e(\lambda(\pi_{\mathcal{G}'}(a)b))\eta_e \in \mathcal{R}_e$ and $\rho_e(\lambda(b))\eta_e \in \mathcal{L}_e \implies \rho_e(\lambda(\pi_{\mathcal{G}'}(a)b))\eta_e \in \mathcal{L}_e$. It follows that

$$[Q_e\rho_e(\pi_{\text{vert}}^{\mathcal{G}'}(\lambda_{\mathcal{G}'}(a))) - \rho_e(\pi_{\text{vert}}^{\mathcal{G}'}(\lambda_{\mathcal{G}'}(a)))Q_e]\rho_e(\lambda(b))\eta_e$$
$$= [Q_e\rho_e(\lambda(\pi_{\mathcal{G}'}(a))) - \rho_e(\lambda(\pi_{\mathcal{G}'}(a)))Q_e]\rho_e(\lambda(b))\eta_e = 0.$$

This concludes the proof of (1).

(2). Suppose that $e \notin E(\mathcal{G}')$ (hence $\bar{e} \notin E(\mathcal{G}')$). The element $[\pi_{\text{vert}}^{\mathcal{G}'}] \underset{P_{\text{vert}}}{\otimes} x_e^{\mathcal{G}}$ is represented by the triple (K_e, π_e, V_e), where $\pi_e = \rho_e \circ \pi_{\text{vert}}^{\mathcal{G}'}$. Since $V_e^2 = 1$ and $V_e^* = V_e$, it suffices to show that Q_e commutes with $\rho_e(\pi_{\text{vert}}^{\mathcal{G}'}(x))$ for all $x \in P_{\text{vert}}^{\mathcal{G}'}$. It follows from assertion (2) of Lemma 5.14 since $e, \bar{e} \notin E(\mathcal{G}')$ implies $\pi_{\text{vert}}^{\mathcal{G}'}(P_{\text{vert}}^{\mathcal{G}'}) \subset \overline{\text{Span}}\left(\lambda(A_{r(e)}) \cup \lambda(X_e)\right)$.

(3). For $p \in V(\mathcal{G})$ we use the notation $(H_p, \pi_p, \xi_p) := (H_{p,p}, \lambda_{v,p}, \xi_p)$ for the GNS construction of the canonical ucp map $\mathbb{E}_{A_p} : P_{\text{vert}} \to A_p$. Observe that $\xi_p \cdot A_p$ is orthogonally complemented in H_p and set $H_p^\circ = H_p \ominus \xi_p \cdot A_p$. Define $K_p = \bigoplus_{e \in E(\mathcal{G}), r(e)=p} K_e \underset{B_e^r}{\otimes} A_p$ and observe that, by Lemma 5.13, we have an isometry $F_p \in \mathcal{L}_{A_p}(H_p^\circ, K_p)$ defined by

$$F_p(\pi_p(\lambda(a_0 u_{e_1} \ldots u_{e_n} a_n))\xi_p) = \rho_{e_n}(\lambda(a_0 u_{e_1} \ldots u_{e_n}))\eta_{e_n} \otimes a_n,$$

for all $a_0 u_{e_1} \ldots u_{e_n} a_n \in P$ reduced operator from p to p. We extend F_p to partial isometry, still denoted $F_p \in \mathcal{L}_{A_p}(H_p, K_p)$ by $F_p|_{\xi_p \cdot A_p} = 0$. Then $F_p^* F_p = 1 - Q_{\xi_p}$, where $Q_{\xi_p} \in \mathcal{L}_{A_p}(H_p)$ is the orthogonal projection onto $\xi_p \cdot A_p$. Moreover, $F_p F_p^* = \bigoplus_{e \in E(\mathcal{G}), r(e)=p} Q_e \otimes 1$.

Define $\rho_p = \bigoplus_{e \in E(\mathcal{G}), r(e)=p} \rho_e \otimes 1 : P_{\text{vert}} \to \mathcal{L}_{A_p}(K_p)$.

Lemma 5.17. *For any $a = a_n u_{e_n} \ldots u_{e_1} a_0 \in P$ reduced operator from p to p we have:*

$$\text{Im}(F_p \pi_p(\lambda(a)) - \rho_p(\lambda(a)) F_p) \subset Z_a \text{ with,}$$

$$Z_a = \bigoplus_{1 \le k \le n, \, r(e_k)=p} (\rho_{e_k}(\lambda(a_n u_{e_n} \ldots u_{e_k})) \eta_{e_k} \otimes 1) \cdot A_p \bigoplus$$

$$\bigoplus_{1 \le k \le n-1, \, s(e_k)=p} (\rho_{\bar{e}_k}(\lambda(a_n u_{e_n} \ldots u_{e_{k+1}} a_k)) \eta_{\bar{e}_k} \otimes 1) \cdot A_p \oplus (\rho_{\bar{e}_n}(\lambda(a_n)) \eta_{\bar{e}_n} \otimes 1) \cdot A_p$$

Proof. If $b \in A_p$ then $F_p \pi_p(\lambda(b)) \xi_p = 0$ and $ab = a_n u_{e_n} \ldots u_{e_1} a_0 b \in P$ is reduced from p to p. Hence, $F_p \pi_p(\lambda(ab)) \xi_p = \rho_{e_1}(\lambda(a_n u_{e_n} \ldots u_{e_1})) \eta_{e_1} \otimes a_0 b$ and we have

$$(F_p \pi_p(\lambda(a)) - \rho_p(\lambda(a)) F_p) \pi_p(\lambda(b)) \xi_p = (\rho_{e_1}(\lambda(a_n u_{e_n} \ldots u_{e_1})) \eta_{e_1} \otimes 1) \cdot a_0 b \in Z_a.$$

Suppose that $b = b_0 u_{f_1} \ldots u_{f_m} b_m \in P$ is a reduced operator from p to p and write $b = b' b_m$, where $b' = b_0 u_{f_1} \ldots u_{f_m}$. Let $0 \le n_0 \le \min\{n, m\}$ and, for $1 \le k \le n_0$, $x_k \in A_{s(e_k)}$ be the data associated to the couple (a, b') in Lemma 5.4. By Lemma 5.4 we can write $ab' = y + z$, where y is either reduced and ends with u_{f_m} or is a sum of reduced operators that end with u_{f_m} and:

$$z = \begin{cases} a_n u_{e_n} \ldots u_{e_{m+1}} x_m & \text{if } n_0 = m < n \\ x_n & \text{if } n_0 = n = m \\ 0 & \text{if } n_0 = 0 \text{ or } n_0 = n < m \text{ or } 1 \le n_0 < \min\{n, m\} \end{cases}$$

Since y is a sum of reduced operators ending with u_{f_m} we have $F_p \pi_p(\lambda(y)) \xi_p = \rho_{f_m}(\lambda(y)) \eta_{f_m} \otimes 1$ and,

$$(F_p \pi_p(\lambda(a)) - \rho_p(\lambda(a)) F_p) \pi_p(\lambda(b)) \xi_p = F_p \pi_p(\lambda(ab')) \xi_p \cdot b_m - \rho_{f_m}(\lambda(ab')) \eta_{f_m} \otimes b_m$$

$$= F_p \pi_p(\lambda(y)) \xi_p \cdot b_m - \rho_{f_m}(\lambda(y)) \eta_{f_m} \otimes b_m + F_p \pi_p(\lambda(z)) \xi_p \cdot b_m - \rho_{f_m}(\lambda(z)) \eta_{f_m} \otimes b_m$$

$$= F_p \pi_p(\lambda(z)) \xi_p \cdot b_m - \rho_{f_m}(\lambda(z)) \eta_{f_m} \otimes b_m.$$

Hence, if $n_0 = 0$, $n_0 = n < m$ or $1 \le n_0 < \min\{n, m\}$ then

$$(F_p \pi_p(\lambda(a)) - \rho_p(\lambda(a)) F_p) \pi_p(\lambda(b)) \xi_p = 0 \in Z_a.$$

If $n_0 = m < n$ then $z = a_n u_{e_n} \ldots u_{e_{m+1}} x_m$ and $\bar{f}_m = e_m$ which implies that $r(e_{m+1}) = s(e_m) = r(f_m) = p$ and, since $x_m = a_m s_{e_m} \circ r_{e_m}^{-1} \circ E_{e_m}^r(x_{m_1}) \in a_m B_{\bar{e}_m}^r$, we have

$$\rho_{f_m}(\lambda(z)) \eta_{f_m} \otimes b_m = \rho_{\bar{e}_m}(\lambda(a_n u_{e_n} \ldots u_{e_{m+1}} x_m)) \eta_{\bar{e}_m} \otimes b_m$$

$$\in (\rho_{\bar{e}_m}(\lambda(a_n u_{e_n} \ldots u_{e_{m+1}} a_m)) \eta_{\bar{e}_m} \otimes 1) \cdot A_p \subset Z_a. \text{ Also,}$$

$$F_p \pi_p(\lambda(z)) \xi_p \cdot b_m = F_p \pi_p(\lambda(a_n u_{e_n} \ldots u_{e_{m+1}} x_m)) \xi_p \cdot b_m$$

$$= \rho_{e_{m+1}}(\lambda(a_n u_{e_n} \ldots u_{e_{m+1}})) \eta_{e_{m+1}} \otimes x_m b_m \in Z_a$$

Finally, if $n_0 = n = m$ then $z = x_n = a_n s_{e_n} \circ r_{e_n}^{-1} \circ E_{e_n}^r(n-1) \in a_n B_{\bar{e}_n}^r$ and, since $f_m = f_n = \bar{e}_n$, we have $\rho_{f_m}(\lambda(z)) \eta_{f_m} \otimes b_m = \rho_{\bar{e}_n}(\lambda(x_n)) \eta_{\bar{e}_n} \otimes b_n \in (\rho_{\bar{e}_n}(\lambda(a_n)) \eta_{\bar{e}_n} \otimes 1) \cdot A_p \subset Z_a$ and $F_p \pi_p(\lambda(z)) \xi_p \cdot b_m = F_p \pi_p(\lambda(x_n)) \xi_p \cdot b_m = 0 \in Z_a$. It concludes the proof. $\square$

It follows from Lemma 5.17 that $F_p\pi_p(x) - \rho_p(x)F_p \in \mathcal{K}_{A_p}(H_p, K_p)$ for all $x \in P_{\text{vert}}$ and, since F_p is a partial isometry satisfying $F_pF_p^* - 1 = -Q_{\xi_p} \in \mathcal{K}_{A_p}(H_p)$, we can apply Lemma 2.2 to conclude that $[(K_p, \rho_p, V_p)] = 0 \in KK^1(P_{\text{vert}}, A_p)$, where $V_p = 2F_pF_p^* - 1 = \bigoplus_{e \in E(\mathcal{G}), r(e)=p} V_e \otimes 1$, where V_e has been defined previously by $V_e = 2Q_e - 1$. It follows from the definitions that (K_p, ρ_p, V_p) is a triple representing the element $\sum_{r(e)=p} x_e^{\mathcal{G}} \underset{B_e}{\otimes} [r_e]$. This concludes the proof of (3).

(4). Note that, for all $e \in E(\mathcal{G})$ and all $x \in P$, we have $\mathbb{E}_{\overline{e}}(\lambda(x)) = \lambda(u_e)\mathbb{E}_e(\lambda(u_e^*xu_e))\lambda(u_e^*)$. It follows from this formula that the operator $W_e : K_{\overline{e}} \underset{s_e^{-1}}{\otimes} B_e \to K_e \underset{r_e^{-1}}{\otimes} B_e$ defined by $W_e(\rho_{\overline{e}}(\lambda(x))\eta_{\overline{e}} \otimes b) = \rho_e(\lambda(xu_e))\eta_e \otimes b$, for $x \in P$ and $b \in B_e$, is a unitary operator in $\mathcal{L}_{B_e}(K_{\overline{e}} \underset{s_e^{-1}}{\otimes} B_e, K_e \underset{r_e^{-1}}{\otimes} B_e)$. Moreover, it is clear that W_e intertwines the representations $\rho_e(\cdot) \otimes 1$ and $\rho_{\overline{e}}(\cdot) \otimes 1$ and we have $W_e^*(Q_e \otimes 1)W_e = 1 \otimes 1 - Q_{\overline{e}} \otimes 1$. The proof of (4) follows. $\qquad\square$

Remark 5.18. Assertions (2) and (3) of the preceding Proposition obviously hold for the elements $z_e^{\mathcal{G}} = [\lambda] \underset{P_{\text{vert}}}{\otimes} x_e^{\mathcal{G}} \in KK^1(P, B_e)$ and also assertions (1) and (2) with $\pi_{\mathcal{G}'}$ instead of $\pi_{\text{vert}}^{\mathcal{G}'}$ since we have $\pi_{\text{vert}}^{\mathcal{G}'} \circ \lambda_{\mathcal{G}'} = \lambda \circ \pi_{\mathcal{G}'}$ for any connected subgraph $\mathcal{G}' \subset \mathcal{G}$, with maximal subtree $\mathcal{T}' \subset \mathcal{T}$.

We study now in details the behavior of our elements $x_e^{\mathcal{G}}$ under the *Serre's devissage* process.

The case of an amalgamated free product. Let A_1, A_2 and B be C*-algebras with unital faithful $*$-homomorphisms $\iota_k : B \to A_k$ and conditional expectations $E_k : A_k \to \pi_k(B)$ for $k = 1, 2$. Let $A_v = A_1 \overset{v}{\underset{B}{*}} A_2$ be the associated vertex-reduced amalgamated free product, $A_f = A_1 \underset{B}{*} A_2$ the full amalgamated free product and $\pi : A_f \to A_v$ the canonical surjection. Let (K, ρ, η) be the GNS construction of the canonical ucp map $E : A \to B$ (which is the composition of the canonical surjection from A to the edge-reduced amalgamated free product with the canonical ucp map from the edge-reduced amalgamated free product to B) and K_i, for $i = 1, 2$, be the closed subspace of K generated by $\{\rho(\pi(x))\eta : x = a_1 \ldots a_n \in A_f$ reduced and ends with $A_i \ominus B\}$. Observe that K_i is a complemented Hilbert submodule of K. Actually we have $K = K_1 \oplus K_2 \oplus \eta \cdot B$. Let $Q_i \in \mathcal{L}_B(K)$ be the orthogonal projection onto K_i. The following Proposition is easy to check and left to the reader.

Proposition 5.19. (K, ρ, V), where $V = 2Q_1 - 1$ defines an element $x_A = [(K, \rho, V)] \in KK^1(A_v, B)$.

Let $e \in E(\mathcal{G})$ and suppose that $\mathcal{G}_e$ is not connected. We keep the same notations as the one used in the Serre's devissage process explained in the previous Section. In particular we have the $*$-isomorphism $\nu_e : A_{\mathcal{G}_e} := P_{\text{vert}}^{\mathcal{G}_{s(e)}} \overset{v}{\underset{B_e}{*}} P_{\text{vert}}^{\mathcal{G}_{r(e)}} \to P_{\text{vert}}$ from Lemma 5.11. We now have two canonical elements in $KK^1(P_{\text{vert}}, B_e)$: $x_e^{\mathcal{G}}$ and $x_{\mathcal{G}_e} := [\nu_e^{-1}] \underset{A_{\mathcal{G}_e}}{\otimes} y_{\mathcal{G}_e}$, where $y_{\mathcal{G}_e}$ is the element associated to the vertex-reduced amalgamated free product $A_{\mathcal{G}_e}$ constructed in Proposition 5.19. These two elements are actually equal.

Lemma 5.20. *We have* $x_{\mathcal{G}_e} = x_e^{\mathcal{G}} \in KK^1(P_{\text{vert}}, B_e)$.

Proof. The proof is a simple identification: there is not a single homotopy to write, only an isomorphism of Kasparov's triples. The key of the proof is to realize that the two ucp maps

$P_{\text{vert}} \to B_e$ defined by $\varphi = r_e^{-1} \circ \mathbb{E}_e$ and $\psi = E \circ \nu_e^{-1}$ are equal, where $E : A_{\mathcal{G}_e} \to B_e$ is the canonical ucp map and it directly follows from the fact that ν_e intertwines the canonical ucp maps. Having this observation in mind, one construct an isomorphism of Kasparov's triples. Recall that (K_e, ρ_e, η_e) denotes the GNS construction of the ucp map $\mathbb{E}_e : P_{\text{vert}} \to B_e^r$ and (K, ρ, η) denotes the GNS of the ucp map $E : A_{\mathcal{G}_e} \to B_e$.

Since $K = \overline{\rho \circ \nu_e^{-1}(P_{\text{vert}})\eta \cdot B_e}$, $K_e \underset{r_e^{-1}}{\otimes} B_e = \overline{\rho_e(P_{\text{vert}})\eta_e \otimes 1 \cdot B_e}$ and

$$\langle \eta, \rho \circ \nu_e^{-1}(x)\eta \rangle_K = \psi(x) = \varphi(x) = \langle \eta_e \otimes 1, \rho_e(x)\eta_e \otimes 1 \rangle_{K_e \underset{r_e^{-1}}{\otimes} B_e} \quad \text{for all } x \in P_{\text{vert}},$$

it follows that the map $U : K \to K_e \underset{r_e^{-1}}{\otimes} B_e$, $U(\rho \circ \nu_e^{-1}(x))\eta \cdot b) = \rho_e(x)\eta_e \otimes 1 \cdot b$ for $x \in P_{\text{vert}}$ and $b \in B_e$, defines a unitary $U \in \mathcal{L}_{B_e}(K, K_e \underset{r_e^{-1}}{\otimes} B_e)$. Moreover, U intertwines the representations $\rho \circ \nu_e^{-1}$ and $\rho_e(\cdot) \otimes 1$. Observe that $x_{\mathcal{G}_e}$ is represented by the triple $(K, \rho \circ \nu_e^{-1}, V)$, where $V = 2Q - 1$ and Q is the orthogonal projection on the closed linear span of the $\rho(\pi(x_1 \ldots x_n))$, where $x_1 \ldots x_n \in P_{\text{vert}}^{\mathcal{G}_{s(e)}} \underset{B_e}{*} P_{\text{vert}}^{\mathcal{G}_{r(e)}}$ is a reduced operator in the free product sense and $x_n \in P_{\text{vert}}^{\mathcal{G}_{s(e)}}$. Moreover, $x_e^{\mathcal{G}}$ is represented by the triple $(K_e \underset{r_e^{-1}}{\otimes} B_e, \rho_e(\cdot) \otimes 1, V_e)$, where $V_e = Q_e \otimes 1$ and Q_e is the orthogonal projection onto the closed linear span of the $\rho_e(\lambda(a_0 u_{e_1} \ldots u_{e_n} a_n))\eta_e$, where $a_0 u_{e_1} \ldots u_{e_n} a_n \in P$ is reduced from $r(e)$ to $r(e)$ with $e_n = e$ and $a_n \in B_e^r$.

To conclude the proof, it suffices to observe that $UVU^* = V_e$. $\square$

We study now the case of an HNN-extension.

The case of an HNN extension. For $\epsilon \in \{-1, 1\}$, let $\pi_\epsilon : B \to A$ be a unital faithful $*$-homomorphism $E_\epsilon : A \to B$ be a ucp map such that $E_\epsilon \circ \pi_\epsilon = \text{id}_B$. Let C_f be the fulld HNN-extension with stable letter $u \in \mathcal{U}(C)$, C_v the vertex-reduced HNN-extension and $\pi : C_f \to C_v$ the canonical surjection. Let (K, ρ, η) be the GNS construction of the ucp map $E = E_1 \circ E_A : C_v \to B$, where $E_A : C_v \to A$ is the canonical GNS-faithful ucp map. Define the sub B-module

$K_+ = \overline{\text{Span}}\{\rho(\pi(x))\eta : x = a_0 u^{\epsilon_1} \ldots u^{\epsilon_n} a_n \in C_f$ is a reduced operator with $\epsilon_n = 1$ and $a_n \in \pi_1(B)\}$.

Observe that K_+ is complemented and let $Q_+ \in \mathcal{L}_B(K)$ be the orthogonal projection onto K_+. The following proposition is easy to check.

Proposition 5.21. (K, ρ, V), where $V = 2Q_+ - 1$, defines an element $x_C \in KK^1(C_v, B)$.

Let $e \in E(\mathcal{G})$ and suppose that $\mathcal{G}_e$ is connected. Up to a canonical isomorphism of P we may and will assume that $\mathcal{T} \subset \mathcal{G}_e$. Recall that we have a canonical $*$-isomorphism $\nu_e : C_{\mathcal{G}_e} := \text{HNN}_{\text{vert}}(P_{\text{vert}}^{\mathcal{G}_e}, B_e, \pi_1, \pi_{-1}) \to P_{\text{vert}}$ defined in Lemma 5.12. As before, we get two canonical elements in $KK^1(P_{\text{vert}}, B_e)$: $x_e^{\mathcal{G}}$ and $x_{\mathcal{G}_e} := [\nu_e^{-1}] \underset{C_{\mathcal{G}_e}}{\otimes} y_{\mathcal{G}_e}$, where $y_{\mathcal{G}_e} \in KK^1(C_{\mathcal{G}_e}, B_e)$ is the element associated to the vertex-reduced HNN-extension $C_{\mathcal{G}_e}$ constructed in Proposition 5.21. As before, these two elements are actually equal.

Lemma 5.22. We have $x_{\mathcal{G}_e} = x_e^{\mathcal{G}} \in KK^1(P_{\text{vert}}, B_e)$.

Proof. Recall that (K, ρ, η) denotes the GNS construction of the canonical ucp map $E : C_{\mathcal{G}_e} \to B_e$. The proof is similar to the proof of Lemma 5.20 and is just a simple identification. Since ν_e

intertwines the canonical ucp maps, the two ucp maps $\varphi, \psi : P_{\text{vert}} \to B_e$ defined by $\varphi = \mathbb{E}_e$ and $\psi = E \circ \nu_E^{-1}$ are equal. As before, one can deduce easily from this equality an isomorphism of Kasparov's triples. Since the arguments are the same, we leave the details to the reader. $\qquad \square$

Remark 5.23. The analogue of Lemmas 5.20, 5.22 are obviously still valid for the elements $z_e^{\mathcal{G}} \in KK^1(P, B_e)$ defined in Remark 5.15.

5.3. **The exact sequence.** For any separable C*-algebra C, let $F^*(-)$ be $KK^*(C, -)$. It is a $\mathbb{Z}_2$-graded covariant functor. If f is a morphism of C*-algebras, we will denote by f^* the induced morphism.

In the sequel $P_{\mathcal{G}}$ or simply P denotes either the full or the vertex reduced fundamental C*-algebra in the context of GNS faithful conditional expectations. We define the boundary maps $\gamma_e^{\mathcal{G}}$ from $F^*(P_{\mathcal{G}}) = KK^*(D, P_{\mathcal{G}})$ to $KK^{*+1}(D, B_e) = F^{*+1}(B_e)$ by $\gamma_e^{\mathcal{G}}(y) = y \otimes_P x_e$ when P is the full fundamental C*-algebra or $\gamma_e^{\mathcal{G}}(y) = y \otimes_P z_e$ when P is the vertex reduced one.

If $\mathcal{G}$ is a graph, then E^+ is the set of positive edges, V the set of vertices and for any $v \in V$, the map from A_v to $P_{\mathcal{G}}$ is π_v or sometimes $\pi_v^{\mathcal{G}}$ if it is necessary to indicate which graph algebras we consider. If one removes an edge e_0 (and its opposite) to $\mathcal{G}$, the new graph is called $\mathcal{G}_0$, P_0 is the algebra associated to it and π_v^0 is the embedding of A_v in P_0. We also have for $\mathcal{G}_1 \subset \mathcal{G}$ a morphism $\pi_{\mathcal{G}_1}$ from $P_{\mathcal{G}_1}$ to $P_{\mathcal{G}}$.

Theorem 5.24. *In the presence of conditional expectations (not necessarily GNS -faithful), we have, for P the full or vertex reduced fundamental C*-algebra, a long exact sequence*

$$\longrightarrow \bigoplus_{e \in E^+} F^*(B_e) \xrightarrow{\sum_e s_e^* - r_e^*} \bigoplus_{v \in V} F^*(A_v) \xrightarrow{\sum_v \pi_v^*} F^*(P) \xrightarrow{\oplus_e \gamma_e^{\mathcal{G}}} \bigoplus_{e \in E^+} F^{*+1}(B_e) \longrightarrow$$

Proof. First note that it is indeed a chain complex. Because s_e and r_e are conjugated in P, we only have to check that $\gamma_e \circ \pi_v^* = 0$ (which is point 2 of prop 5.16) and $\sum_{e \in E^+} x_e \otimes [r_e] - x_e \otimes [s_e] = 0$. As $x_e = -x_{\bar{e}}$ (point 4 of 5.16) and $s_{\bar{e}} = r_e$, this is the same as point 3 of 5.16.

Also if the graph contains only one geometric edge (i.e. two opposite oriented edges), we are in the case of the amalgamated free product or the HNN extension and the complex is known to be exact because of the result of section 4 as noted by several authors ([Ge97], [Th03], [Ue08]). For convenience we will briefly recall why and also we will identify the boundary map, freely using the notations of that section. Let's do the full amalgamated free product A_f first. Recall that D fits into a short exact sequence :

$$0 \to A_1 \otimes S \oplus A_2 \otimes S \xrightarrow{\bar{\kappa}_1 \oplus \bar{\kappa}_2} D \xrightarrow{ev_0} B \to 0.$$

Therefore there is a long exact sequence for our functor F^* :

$$F^*(A_1 \otimes S \oplus A_2 \otimes S) \to F^*(D) \to F^*(B) \to F^{*+1}(A_1 \otimes S \oplus A_2 \otimes S).$$

But $F^*(A_k \otimes S)$ identifies with $F^{*+1}(A_k)$ and $F^*(D)$ with $F^{*+1}(A_f)$. Via these identifications, the map from $F^*(B)$ to $F^*(A_k)$ becomes i_k^* or its opposite (this is seen using the mapping cone exact sequence) and the map from $F^*(A_k)$ to $F^*(A_f)$ is j_k^*. The only thing left is the identification of the boundary map from $F^*(A_f)$ to $F^{*+1}(B)$. It is obviously the Kasparov product by $x \otimes [ev_0]$ where x is the element in $KK^1(A_f, D)$ that implements the K-equivalence. It has been described in 4.11 and is exactly the element of 5.19. Therefore the boundary map is exactly given by the corresponding $\gamma_e^{\mathcal{G}}$ for the graph of the free product. Note also that x, as defined at the end of section 4.1, actually factorizes as $[\lambda] \otimes_A z$ where A is the vertex reduced

free product, λ the canonical homomorphism from the full free product onto the vertex reduced one and $z \in KK^1(A, D)$. Therefore the same identifications and the same exact sequence hold for the vertex reduced free product A and theorem 5.24 is true for free products.

Now let's tackle the HNN extension case. Let's call C_m the full HNN extension of (A, B, θ) and E and E_θ the conditional expectations from A to B and $\theta(B)$. An explicit isomorphism is known to exist between C_m and $e_{11}M_2(A) \underset{B \oplus B}{*} M_2(B)e_{11}$ where $B \oplus B$ imbeds diagonally in $M_2(A)$ via the canonical inclusion and θ, e_{11} is the matrix unit $\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$ and the conditional expectations are $E_1 \begin{pmatrix} a_1 & a_2 \\ a_3 & a_4 \end{pmatrix} = E(a_1) \oplus E_\theta(a_4)$ from $M_2(A)$ to $B \oplus B$ and $E_2 \begin{pmatrix} b_1 & b_2 \\ b_3 & b_4 \end{pmatrix} = b_1 \oplus b_4$ from $M_2(B)$ to $B \oplus B$. The exact sequence for the HNN extension is then deduced from this isomorphism of C*-algebras (cf. [Ue08] for example).

If we call j_A and j_B the inclusions of $M_2(A)$ respectively $M_2(B)$ in the free product then the unitary u in C_m that implements θ is mapped to $j_A \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} j_B \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$.

It is then clear that a reduced word in C_m that ends with u times b with b in B is mapped into a reduced word in the free product that ends with $j_B \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} b & 0 \\ 0 & 0 \end{pmatrix} = j_B \begin{pmatrix} 0 & b' \\ b & 0 \end{pmatrix} e_{11}$ i.e. that ends in $j_B(M_2(B)) \ominus (B \oplus B)$. Therefore in this situation the element described in 5.21 is the same as the element described in 5.19 and we have identified the correct boundary map.

Let's have a look now at the vertex reduced situation. Recall that, given two unital C*-algebras D_1, D_2, in the situation of a unital inclusion of $M_2(D_1) \subset M_2(D_2)$ a GNS faithful conditional expectation from $M_2(D_2)$ onto $M_2(D_1)$ comes from a GNS faithful conditional expectation from D_2 onto D_1. Hence with the notations of section 2 and using the universal properties, $M_2(A) \overset{2}{\underset{B \oplus B}{*}} M_2(B)$ is isomorphic to $M_2(A) \overset{e}{\underset{B \oplus B}{*}} M_2(B)$ and as a consequence $M_2(A) \overset{1}{\underset{B \oplus B}{*}} M_2(B)$ is isomorphic to $M_2(A) \overset{v}{\underset{B \oplus B}{*}} M_2(B)$. Again using the universal properties it is obvious that the vertex reduced HNN extension of (A, B, θ) is $e_{11}M_2(A) \overset{1}{\underset{B \oplus B}{*}} M_2(B)e_{11}$. Therefore the identification described earlier for the full free product and HNN extension is again true for the vertex reduced free product and corresponding vertex reduced HNN extension. Hence theorem 5.24 is again valid for HNN extensions.

We now prove exactness at each place by induction on the cardinal of edges and "devissage". Note that 5.20 and 5.22 allow us to decompose our fundamental algebra in HNN or free product while using the same boundary maps γ_e.

Lemma 5.25. *We have the exactness of*

$$\bigoplus_{e \in E^+} F^*(B_e) \overset{\sum_e s_e^* - r_e^*}{\longrightarrow} \bigoplus_{v \in V} F^*(A_v) \overset{\sum_v \pi_v^*}{\longrightarrow} F^*(P)$$

Choose a positive edge e_0. Then without this edge (and its opposite), the graph $\mathcal{G}_0$ is either connected (Case I) or has two connected components $\mathcal{G}_1$ and $\mathcal{G}_2$ (Case II).

For Case I, P is the HNN extension of $P_{\mathcal{G}_0}$ and B_{e_0}.

Note $v_0 = s(e_0) = r(e_0)$ and that the set of vertices of $\mathcal{G}$ is the same as the set of vertices of $\mathcal{G}_0$.

Let $x = \oplus x_v$ be in $\oplus_{v \in V} F^*(A_v)$ such that $\sum_v \pi_v^*(x_v) = 0$. If $y = \sum_v \pi_v^{0*}(x_v)$, then clearly $\pi_{\mathcal{G}_0}(y) = 0$. Then long exact sequence for P seen as an HNN extension implies then that there exists $y_0 \in F^*(B_{e_0})$ such that $(\pi_{v_0} \circ s_{e_0})^*(y_0) - (\pi_{v_0} \circ r_{e_0})^*(y_0) = y = \sum_v \pi_v^{0*}(x_v)$

So $\sum_v \pi_v^{0*}(\oplus_{v \neq v_0} x_v \oplus (x_{v_0} - s_{e_0}^*(y_0) + r_{e_0}^*(y_0)) = 0$. Using the exactness for P_0 as $\mathcal{G}_0$ has one less edge, we get that there exists for any $e \neq e_0$ a y_e such that $\sum_{e \neq e_0} s_e^*(y_e) - r_e^*(y_e) = \oplus_{v \neq v_0} x_v \oplus (x_{v_0} - s_{e_0}^*(y_0) + r_{e_0}^*(y_0))$. Thus $\sum_{e \neq e_0} s_e^*(y_e) - r_e^*(y_e) + s_{e_0}^*(y_0) - r_{e_0}^*(y_0) = x$.

For case II, P is the amalgamated free product of $P_1 = P_{\mathcal{G}_1}$ and $P_2 = P_{\mathcal{G}_2}$ over B_{e_0}.

For $i = 1, 2$, note V_i the vertices of $\mathcal{G}_i$. We know that V is the disjoint union of V_1 and V_2. The map π_v^i will be the embedding of A_v in P_i. Note also $v_1 = s(e_0)$ and $v_2 = r(e_0)$.

Let $x = \oplus x_v$ be in $\oplus_{v \in V} F^*(A_v)$ such that $\sum_v \pi_v^*(x_v) = 0$. Let $x_i = \oplus_{v \in V_i} \pi_v^{i*}(x_v)$. Clearly $\pi_{\mathcal{G}_1}^*(x_1) + \pi_{\mathcal{G}_2}^*(x_2) = 0$. Then the long exact sequence for P seen as an amalgamated free product gives a $y_0 \in F^*(B_{e_0})$ such that $(\pi_{v_1}^1 \circ s_{e_0})^*(y_0) - (\pi_{v_2}^2 \circ r_{e_0})^*(y_0) = x_1 \oplus x_2$. Set $\bar{x}_1 = \oplus_{v \in V_1} x_v - s_{e_0}^*(y_0)$ and $\bar{x}_2 = \oplus_{v \in V_2} x_v + r_{e_0}^*(y_0)$. So we have that $\sum_{v \in V_i} \pi_v^{i*}(\bar{x}_i) = 0$ for $i = 1, 2$. Therefore by induction as $\mathcal{G}_i$ has strictly less edges than $\mathcal{G}$, there exists for any $e \neq e_0$ a $y_e \in F^*(B_e)$ such that $\bar{x}_1 \oplus \bar{x}_2 = \sum_{e \neq e_0} s_e^*(y_e) - r_e^*(y_e)$. Hence $x = \sum_{e \neq e_0} s_e^*(y_e) - r_e^*(y_e) + s_{v_0}^*(y_0) - r_{v_0}^*(y_0)$ and we are done. $\qquad\square$

Lemma 5.26. *The following chain complex is exact in the middle*

$$\bigoplus_{v \in V} F^*(A_v) \xrightarrow{\sum_v \pi_v^*} F^*(P) \xrightarrow{\oplus_e \gamma_e^{\mathcal{G}}} \bigoplus_{e \in E^+} F^{*+1}(B_e)$$

Proof. For case I.

Let x be in $F^*(P)$ such that for any e, $\gamma_e^{\mathcal{G}}(x) = 0$. In particular for the edge e_0. Using the long exact sequence for P seen as an HHN extension, and since $\gamma_{e_0}^{\mathcal{G}}(x) = 0$ we get that there exists x_0 in $F^*(P_0)$ such that $\pi_{\mathcal{G}_0}^*(x_0) = x$. For any edges $e \neq e_0$, one has $\gamma_e^{\mathcal{G}_0}(x_0) = \gamma_e^{\mathcal{G}}(\pi_{\mathcal{G}_0}^*(x_0)) = 0$. Hence by induction there exists for any $v \in V(\mathcal{G}_0) = V(\mathcal{G})$ a $y_v \in F^*(A_v)$ such that $\sum_v \pi_v^{0*}(y_v) = x_0$. Hence $x = \sum_v (\pi_{\mathcal{G}_0} \circ \pi_v^0)^*(y_v) = \sum_v \pi_v^*(y_v)$.

For case II.

Using that P is the free product of P_1 and P_2, we get an $x_i \in F^*(P_i)$ for $i = 1, 2$ such that $x = \pi_{\mathcal{G}_1}^*(x_1) + \pi_{\mathcal{G}_2}^*(x_2)$. Now for $i = 1, 2$, and for any edge e of $\mathcal{G}_i$, $\gamma_e^{\mathcal{G}_i}(x_i) = \gamma_e^{\mathcal{G}}(\pi_{\mathcal{G}_i}^*(x_i)) = \gamma_e^{\mathcal{G}}(x) - \gamma_e^{\mathcal{G}}(\pi_{\mathcal{G}_j}^*(x_j))$ with $j \neq i$. But e is not an edge of $\mathcal{G}_j$, so $\gamma_e^{\mathcal{G}} \circ \pi_{\mathcal{G}_j}^* = 0$. Hence $\gamma_e^{\mathcal{G}_i}(x_i) = 0$. By induction we get for any vertex of $V_1 \cup V_2 = V(\mathcal{G})$ a $y_v \in F^*(A_v)$ such that $x_i = \sum_{v \in V_i} \pi_v^{i*}(y_v)$ for $i = 1, 2$. Therefore $x = \sum_v \pi_v^*(y_v)$. $\qquad\square$

Lemma 5.27. *The following chain complex is exact in the middle*

$$F^{*-1}(P) \xrightarrow{\oplus_e \gamma_e^{\mathcal{G}}} \bigoplus_{e \in E^+} F^*(B_e) \xrightarrow{\sum_e s_e^* - r_e^*} \bigoplus_{v \in V} F^*(A_v)$$

Proof. Let's turn first to case I.

Let $x = \oplus_{e \in E^+} x_e$ such that $\sum_e s_e^*(x_e) - r_e^*(x_e) = 0$. Then for the distinguished vertex v_0, one has

$$\pi_{v_0}^0{}^*(s_{e_0}^*(x_{e_0})) - \pi_{v_0}^0{}^*(r_{e_0}^*(x_{e_0})) = -\sum_{e \neq e_0} \pi_{v_0}^0{}^*(s_e^*(x_{e_0})) - \pi_{v_0}^0{}^*(r_e^*(x_{e_0}))$$

But as e is an edge of $\mathcal{G}_0$, s_e and r_e are conjugated by a unitary of P_0. Therefore there difference are 0 in any KK-groups. Thus $\pi_{v_0}^0{}^*(s_{e_0}^*(x_{e_0})) - \pi_{v_0}^0{}^*(r_{e_0}^*(x_{e_0})) = 0$. Using the long exact sequence for P as an HHN extension, we get a y_0 in $F^{*-1}(P)$ such that $\gamma_{e_0}^{\mathcal{G}}(y_0) = x_{e_0}$. Set now $\bar{x}_e = x_e - \gamma_e^{\mathcal{G}}(y_0)$ for any $e \neq e_0$ and compute

$$\begin{aligned}
\sum_{e \neq e_0} s_e^*(\bar{x}_e) - r_e^*(\bar{x}_e) &= \sum_{e \neq e_0} s_e^*(x_e) - r_e^*(x_e) - \sum_e s_e^*(\gamma_e^{\mathcal{G}}(y_0)) - r_e^*(\gamma_e^{\mathcal{G}}(y_0)) \\
&\quad + s_{e_0}^*(\gamma_{e_0}^{\mathcal{G}}(y_0)) - r_{e_0}^*(\gamma_{e_0}^{\mathcal{G}}(y_0)) \\
&= \sum_e s_e^*(x_e) - r_e^*(x_e)
\end{aligned}$$

by the third property of γ_e. Hence $\sum_{e \neq e_0} s_e^*(\bar{x}_e) - r_e^*(\bar{x}_e) = 0$. By induction there exists $\bar{y}_1$ in $F^{*-1}(P_0)$ such that for all $e \neq e_0$, $\gamma_e^{\mathcal{G}_0}(y_1) = \bar{x}_e$. Set at last $y_1 = \pi_{\mathcal{G}_0}^*(\bar{y}_1)$ which is an element of $F^{*-1}(P)$. Now $\gamma_{e_0}^{\mathcal{G}}(y_0 + y_1) = x_0 + \gamma_{e_0}^{\mathcal{G}} \circ \pi_{\mathcal{G}_0}^*(\bar{y}_1)$. But e_0 is not an edge of $\mathcal{G}_0$ so $\gamma_{e_0}^{\mathcal{G}} \circ \pi_{\mathcal{G}_0}^* = 0$. Hence $\gamma_{e_0}^{\mathcal{G}}(y_0 + y_1) = x_0$.

On the other end, for $e \neq e_0$, $\gamma_e^{\mathcal{G}}(y_0 + y_1) = \gamma_e^{\mathcal{G}}(y_0) + \bar{x}_e$ as $\gamma_e^{\mathcal{G}_0} = \gamma_e^{\mathcal{G}} \circ \pi_{\mathcal{G}_0}^*$. So $\gamma_e^{\mathcal{G}}(y_0 + y_1) = x_e$.

And let's finish with case II. Call E_i the edges of $\mathcal{G}_i$ for $i = 1, 2$. Note that for any positive edge e, if $s(e) \in V_1$ then either $e \in E_1$ or $e = e_0$ and if $r(e) \in V_2$ then $e \in E_2$.

Let $x = \oplus_{e \in E^+} x_e$ such that $\sum_e s_e^*(x_e) - r_e^*(x_e) = 0$. The equality can be rewritten as $\sum_{e \in E_1^+} s_e^*(x_e) - r_e^*(x_e) + s_{e_0}^*(x_{e_0}) = 0$ in $\oplus_{v \in V_1} F^*(A_v)$ and $\sum_{e \in E_2^+} s_e^*(x_e) - r_e^*(x_e) - r_{e_0}^*(x_{e_0}) = 0$ in $\oplus_{v \in V_2} F^*(A_v)$.

Let's compute now $\pi_{v_1}^1(x_{e_0})$. It is $-\sum_{e \in E_1^+} (\pi_{s(e)}^1 \circ s_e)^*(x_e) - (\pi_r(e)^1 \circ r_e)^*(x_e)$ by the preceding remark. But as s_e and r_e are conjugated in P_1 because e is an edge of $\mathcal{G}_1$, this is 0. In the same way $\pi_{v_2}^2(x_{e_0}) = 0$. Therefore using the long exact sequence for P as a free product of P_1 and P_2, there is a y_0 in $F^{*-1}(P)$ such that $\gamma_{e_0}^{\mathcal{G}}(y_0) = x_{e_0}$.

For all $e \neq e_0$ set $\bar{x}_e = x_e - \gamma_e^{\mathcal{G}}(y_0)$. Then $\sum_{e \in E_1^+} s_e^*(\bar{x}_e) - r_e^*(\bar{x}_e) = \sum_{e \in E_1^+} s_e^*(x_e) - r_e^*(x_e) - \left(\sum_{e \in E_1^+} s_e^* \circ \gamma_e^{\mathcal{G}}(y_0) - r_e^* \circ \gamma_e^{\mathcal{G}}(y_0) \right)$.

But the third property of the $\gamma_e^{\mathcal{G}}$ implies that $0 = \sum_{e \in E_1^+} s_e^* \circ \gamma_e^{\mathcal{G}} + s_{e_0}^* \circ \gamma_{e_0}^{\mathcal{G}} - \sum_{e \in E_1^+} r_e^* \circ \gamma_e^{\mathcal{G}}$ using the remark made at the begining of this proof. Hence $\sum_{e \in E_1^+} s_e^*(\bar{x}_e) - r_e^*(\bar{x}_e) = \sum_{e \in E_1^+} s_e^*(x_e) - r_e^*(x_e) + s_{e_0}^*(x_{e_0}) = 0$. Similarly $\sum_{e \in E_2^+} s_e^*(\bar{x}_e) - r_e^*(\bar{x}_e) = 0$. Therefore by induction, there exists for $i = 1, 2$, an element y_i in $F^{*-1}(P_i)$ such that for all e in E_i^+, $\gamma_e^{\mathcal{G}_i}(y_i) = \bar{x}_e$. Set now $y = y_0 + \pi_{\mathcal{G}_1}(y_1) + \pi_{\mathcal{G}_2}(y_2)$ in $F^{*-1}(P)$. Then $\gamma_{e_0}^{\mathcal{G}}(y) = x_{e_0} + \gamma_{e_0}^{\mathcal{G}} \circ \pi_{\mathcal{G}_1}^*(y_1) + \gamma_{e_0}^{\mathcal{G}} \circ \pi_{\mathcal{G}_2}^*(y_2) = x_{e_0}$ as $\gamma_{e_0}^{\mathcal{G}} \circ \pi_{\mathcal{G}_i} = 0$ since e_0 is not an edge of $\mathcal{G}_1$ nor $\mathcal{G}_2$. On the other end, for $e \in E_1$, $\gamma_e^{\mathcal{G}}(y) = \gamma_e^{\mathcal{G}}(y_0) + \gamma_e^{\mathcal{G}_1}(y_1) + 0$ as e is not an edge of $\mathcal{G}_2$. Hence $\gamma_e^{\mathcal{G}}(y) = \gamma_e^{\mathcal{G}}(y_0) + \bar{x}_e = x_e$. The same is of course true for an edge in E_2. So we are done. $\qquad \square$

Let's treat now the case $F^*(-) = KK(-, C)$. Again if f is a morphism of C*-algebras we will adopt the same notation f^* for the induced morphism. Now the map $\gamma_e^{\mathcal{G}}$ from $F(B_e)$ to $F(P)$ is defined as $\gamma_e^{\mathcal{G}}(z) = x_e^{\mathcal{G}} \otimes_{B_e} z$.

Theorem 5.28. *When the conditional expectations are GNS faithful, we have, for P the full or reduced fundamental C*-algebra, a long exact sequence*

$$\longleftarrow \bigoplus_{e \in E^+} F^*(B_e) \xleftarrow{\sum_e s_e^* - r_e^*} \bigoplus_{v \in V} F^*(A_v) \xleftarrow{\sum_v \pi_v^*} F^*(P) \xleftarrow{\oplus_e \gamma_e^{\mathcal{G}}} \bigoplus_{e \in E^+} F^{*+1}(B_e) \longleftarrow$$

Proof. As before this is a chain complex and the same identifications proves it for free products and HNN extension. We will now show exactness with the three following lemmas.

Lemma 5.29. *We have the exactness of*

$$\bigoplus_{e \in E^+} F^*(B_e) \xleftarrow{\sum_e s_e^* - r_e^*} \bigoplus_{v \in V} F^*(A_v) \xleftarrow{\sum_v \pi_v^*} F^*(P)$$

Proof. Let $x = \oplus x_v \in \oplus_v F(A_v)$ such that $\sum_e s_e^*(\oplus x_v) - r_e^*(\oplus x_v) = 0$.

For case I. Then $\sum_{e \neq e_0} s_e^*(\oplus x_v) - r_e^*(\oplus x_v) = 0$. Hence there is a y_0 in $F(P_0)$ such that for all v, $\pi_v^{0*}(y_0) = x_v$. But $s_{e_0}^* \circ \pi_{v_0}^{0*}(y_0) = s_{e_0}^*(x_{v_0}) = r_{e_0}^*(x_{v_0}) = r_{e_0}^* \circ \pi_{v_0}^{0*}(y_0)$. Using the exact sequence for P as an HNN of P_0 and the two copies of B_{e_0}, we get that there is $y \in F(P)$ such that $\pi_{\mathcal{G}_0}^*(y) = y_0$. Now for all v, $\pi_v^*(y) = \pi_v^{0*}(y_0) = x_v$.

For case II. Then $\sum_{e \in E_k^+} s_e^*(\oplus x_v) - r_e^*(\oplus x_v) = 0$ for $k = 1, 2$. Hence there is $y_k \in F(P_k)$ such that $\pi_v^{k*}(y_k) = x_v$ for any $v \in V_k$. As $s_{e_0}^* \circ \pi_{v_1}^{1*}(y_1) = s_{e_0}^*(x_{v_1}) = r_{e_0}^*(x_{v_2}) = r_{e_0}^* \circ \pi_{v_2}^{2*}(y_2)$, using the exact sequence for P as a free product, we have a $y \in F(P)$ such that $\pi_{\mathcal{G}_k}^*(y) = y_k$ for $k = 1, 2$. Then for $k = 1, 2$ and all $v \in V_k$, $\pi_v^*(y) = \pi_v^{k*}(y_k) = x_v$. $\square$

Lemma 5.30. *The following chain complex is exact in the middle*

$$\bigoplus_{v \in V} F^*(A_v) \xleftarrow{\sum_v \pi_v^*} F^*(P) \xleftarrow{\oplus_e \gamma_e^{\mathcal{G}}} \bigoplus_{e \in E^+} F^{*+1}(B_e)$$

Proof. Let y be in $F(P)$ such that $\pi_v^*(y) = 0$ for all v.

Case I. Let $y_0 = \pi_{\mathcal{G}_0}^*(y)$. Then for all v, $\pi_v^{0*}(y_0) = \pi_v^*(y) = 0$. Therefor there exists $x = \sum_{e \neq e_0} x_e$ such that $\sum_{e \neq e_0} \gamma_e^{\mathcal{G}_0}(x_e) = y_0$. Put $z = y - \sum_{e \neq e_0} \gamma_e^{\mathcal{G}*}(x_e)$. Then $\pi_{\mathcal{G}_0}^*(z) = y_0 - \sum_{e \neq e_0} \gamma_e^{\mathcal{G}_0}(x_e) = 0$. Hence there is a $x_{e_0} \in F(B_{e_0})$ such that $\gamma_{e_0}(x_{e_0}) = z$ and $y = \sum_{e \neq e_0} \gamma_e^{\mathcal{G}}(x_e) + \gamma_{e_0}(x_{e_0})$.

Case II. Let $y_k = \pi_{\mathcal{G}_k}^*(y)$ for $k = 1, 2$. For all $v \in V_k$, $\pi_v^{k*}(y_k) = \pi_v^*(y) = 0$, hence there exists $x_k = \oplus_{e \in E_k^+} x_e$ such that $\sum_{e \in E_k^+} \gamma_e^{\mathcal{G}_k}(x_e) = y_k$. Let $z = y - \sum_{e \neq e_0} \gamma_e^{\mathcal{G}*}(x_e)$. Then for $k = 1, 2$, $\pi_{\mathcal{G}_k}^*(z) = y_k - \sum_{e \in E_k^+} \gamma_e^{\mathcal{G}_k}(x_e) = 0$ as $\pi_{\mathcal{G}_2}^* \circ \gamma_e^{\mathcal{G}_1} = 0$ because of 5.16. Hence $z = \gamma_{e_0}(x_{e_0})$ for some x_{e_0} in $F(B_{e_0})$ and we are done. $\square$

Lemma 5.31. *The following chain complex is exact in the middle*

$$F^{*-1}(P) \xleftarrow{\oplus_e \gamma_e^{\mathcal{G}}} \bigoplus_{e \in E^+} F^*(B_e) \xleftarrow{\sum_e s_e^* - r_e^*} \bigoplus_{v \in V} F^*(A_v)$$

Proof. Let $x = \oplus x_e$ in $F(\oplus_e B_e)$ such that $\sum_{e \in E^+} \gamma_e^{\mathcal{G}}(x_e) = 0$.

Case I. Then $0 = \pi_{\mathcal{G}_0}{}^*(\sum_{e \in E^+} \gamma_e^{\mathcal{G}}(x_e)) = \sum_{e \neq e_0} \gamma_e^{\mathcal{G}_0}(x_e)$ as $\pi_{\mathcal{G}_0}{}^* \circ \gamma_{e_0} = 0$. Hence by induction, there is a $z = \oplus z_v$ in $\oplus_v F(A_v)$ such that for all $e \neq e_0$, $x_e = s_e^*(z_{s(e)}) - r_e^*(z_{r(e)})$. Put $x_0 = x_{e_0} - s_{e_0}^*(z_{v_0}) - r_{e_0}^*(z_{v_0})$. As by 5.5 $\sum_{e \in E^+} \gamma_e \circ s_e^* - \gamma_e \circ r_e^* = 0$, we deduce that $\gamma_{e_0} \circ (-s_{e_0}^*(z_{v_0}) + r_{e_0}^*(z_{v_0})) = \sum_{e \neq e_0} \gamma_e(s_e^*(\oplus z_v)) - \gamma_e(r_e^*(\oplus z_v)) = \sum_{e \neq e_0} \gamma_e(x_e)$. Hence $\gamma_{e_0}(x_0) = \gamma_{e_0}(x_{e_0}) + \sum_{e \neq e_0} \gamma_e(x_e) = 0$. Using the long exact sequence for P as an HNN of P_0 and B_{e_0}, we get a $z_0 \in F(P_0)$ such that $x_0 = s_{e_0}^*(\pi_{v_0}^0{}^*(z_0)) - r_{e_0}^*(\pi_{v_0}^0{}^*(z_0))$. So $x_{e_0} = s_{e_0}^*(z_{v_0} + \pi_{v_0}^0{}^*(z_0)) - r_{e_0}^*(z_{v_0} + \pi_{v_0}^0{}^*(z_0))$ and we are done.

Case II. $0 = \pi_{\mathcal{G}_k}{}^*(\sum_{e \in E^+} \gamma_e^{\mathcal{G}}(x_e)) = \sum_{e \neq E_k^+} \gamma_e^{\mathcal{G}_k}(x_e)$ for $k = 1, 2$. Hence there is a $z = \oplus z_v$ such that for all $e \in E_k^+$, $x_e = s_e^*(z_{s(e)}) - r_e^*(z_{r(e)})$. Write $x_0 = x_{e_0} - s_{e_0}^*(z_{v_1}) - r_{e_0}^*(z_{v_2})$. As before we have that $\gamma_{e_0}(x_0) = 0$ and by exactness for the free product of P_1 and P_2 there is $z_1 \in F(P_1)$ and $z_2 \in F(P_2)$ such that $x_0 = s_{e_0}^*(\pi_{v_1}^1{}^*(z_1)) - r_{e_0}^*(\pi_{v_2}^2{}^*(z_2))$. Finally $x_{e_0} = s_{e_0}^*(z_{v_1} + \pi_{v_1}^1{}^*(z_1)) - r_{e_0}^*(z_{v_2} + \pi_{v_2}^2{}^*(z_2))$ and this concludes the proof.

$\square$

$\square$

6. Applications

In this section we collect some applications of our results to K-equivalence, K-amenability of quantum groups and computation of KK-theory. We also explain how our results unify and simplify many other known results and allow to recover them as corollaries.

Our first application is entirely new. We will deduce many important results out of it, in particular all the known results about K-amenability from Cuntz [Cu82], Julg-Valette [JV84], Pimsner [Pi86], Vergnioux [Ve04], Fima [Fi13] and Fima-Freslon [FF13] will become obvious corollaries of our result and so their proofs are greatly simplified and unified. We will also deduce much more in the context of quantum groups. The fact that we do not need to assume that the conditional expectations are GNS-faithful will be crucial for the applications.

Let A_1, A_2 be unital C*-algebras with a common unital subalgebra B and C_1, C_2 be unital C*-algebras with a common unital subalgebra B' and conditional expectations $E_k : A_k \to B$ and $E'_k : C_k \to B'$. Suppose that we have unital $*$-homomorphisms $\lambda_k : A_k \to C_k$ such that $\lambda_1(b) = \lambda_2(b) \in B'$ for all $b \in B$ and define $\lambda_0 : B \to B'$ by $\lambda_0(b) := \lambda_1(b) = \lambda_2(b)$. By the universal property of the full amalgamated free product, there exists a unique unital $*$-homomorphism $\lambda : A_1 \underset{B}{*} A_2 \to C_1 \underset{B}{*} C_2$ such that

$$\lambda(a) = \begin{cases} \lambda_1(a) & \text{if} \quad a \in A_1, \\ \lambda_2(a) & \text{if} \quad a \in A_2. \end{cases}$$

Theorem 6.1. *If, for $k = 1, 2$, $E'_k \circ \lambda_k = \lambda_0 \circ E_k$ and λ_k is a K-equivalence for all $k \in \{0, 1, 2\}$ then λ is a K-equivalence.*

Proof. Consider the following diagram with exact rows

$$KK(D,B) \rightarrow KK(D,A_1) \oplus KK(D,A_2) \rightarrow KK(D, A_1 \underset{B}{*} A_2) \rightarrow KK^1(D,B) \rightarrow KK^1(D,A_1) \oplus KK^1(D,A_2)$$

$$\downarrow \cdot \underset{B}{\otimes} [\lambda_0] \quad \downarrow \left(\cdot \underset{A_1}{\otimes} [\lambda_1] \right) \oplus \left(\cdot \underset{A_2}{\otimes} [\lambda_2] \right) \quad \downarrow \cdot \underset{A_1 \underset{B}{*} A_2}{\otimes} [\lambda] \quad \downarrow \cdot \underset{B}{\otimes} [\lambda_0] \quad \downarrow \left(\cdot \underset{A_1}{\otimes} [\lambda_1] \right) \oplus \left(\cdot \underset{A_2}{\otimes} [\lambda_2] \right)$$

$$KK(D,B') \rightarrow KK(D,C_1) \oplus KK(D,C_2) \rightarrow KK(D, C_1 \underset{B'}{*} C_2) \rightarrow KK^1(D,B') \rightarrow KK^1(D,C_1) \oplus KK^1(D,C_2)$$

By the Five Lemma and the hypothesis, it suffices to check that very square of the diagram is commutative. For a unital inclusion $X \subset Y$ of unital C*-algebras, we write $\iota_{X \subset Y}$ the inclusion map. We also write $A = A_1 \underset{B}{*} A_2$ and $C = C_1 \underset{B'}{*} C_2$ the full amalgamated free products. The first square on the left and the last square on the right of the diagram are obviously commutative since, by definition of λ_0, $\lambda_k \circ \iota_{B \subset A_k} = \iota_{B' \subset C_k} \circ \lambda_0$ for all $k \in \{1, 2\}$. The second square on the left is commutative since, by definition of λ, we have $\lambda \circ \iota_{A_k \subset A} = \iota_{C_k \subset C} \circ \lambda_k$ for all $k \in \{1, 2\}$. Hence, it suffices to check that the third square, starting from the left, is commutative. Note that the commutativity of this square is equivalent to the equality $[x_A] \underset{B}{\otimes} [\lambda_0] = [\lambda] \underset{B}{\otimes} [x_C] \in KK^1(A, B')$,

where x_A and x_C are the KK^1 elements constructed in Proposition 5.19 associated with the amalgamated free products A and C respectively. Indeed if we call K_1 the space appearing in the definition of x_A in 5.19 and K_1' the same space for x_C. Then it follows easily from the assumption $E_k' \circ \lambda_k = \lambda_0 \circ E_k$ for $k = 1, 2$ that $K_1 \otimes_{\lambda_0} B'$ is isomorphic to K_1' and that the induced left action of A on K_1' is λ □

We now formulate the same result in the context of HNN-extensions. Let $B \subset A$, $B' \subset C$ be unital inclusions of unital C*-algebras and $\theta : B \rightarrow A$, $\theta' : B' \rightarrow C$ be unital and faithful *-homomorphisms. Suppose that we have conditional expectations $E_\epsilon : A \rightarrow B_\epsilon$ and $E_\epsilon' : C \rightarrow B_\epsilon'$ for $\epsilon \in \{-1, 1\}$, where B_ϵ, B_ϵ' are defined as usual. Let $\lambda_1 : A \rightarrow C$ be a unital *-homomorphism such that $\lambda_1(b) \in B'$ for all $b \in B$ and define $\lambda_0 : B \rightarrow B'$ by $\lambda_0 = \lambda_1|_B$. Suppose moreover that $\lambda_1 \circ \theta = \theta' \circ \lambda_0$. By the universal property of full HNN-extensions, there exists unique unital *-homomorphism $\lambda : \text{HNN}(A, B, \theta) \rightarrow \text{HNN}(C, B', \theta')$ such that $\lambda|_A = \lambda_1$ and $\lambda(u) = u'$, where u and u' are the "stable letters" in $\text{HNN}(A, B, \theta)$ and $\text{HNN}(C, B', \theta')$ respectively.

Corollary 6.2. *If, for $\epsilon \in \{-1, 1\}$, $E_\epsilon' \circ \lambda_1 = \lambda_0 \circ E_\epsilon$ and λ_k is a K-equivalence for all $k \in \{0, 1\}$ then λ is a K-equivalence.*

Proof. It follows from Theorem 6.1 and the relation between amalgamated free products and HNN-extensions discovered in [Ue08]. □

Let $(A_p, B_e, \mathcal{G})$ be a graph of C*-algebra. Fix a maximal subtree $\mathcal{T} \subset \mathcal{G}$ and let P we the maximal fundamental C*-algebra relative to $\mathcal{T}$. Suppose that we have a compatible family of conditional expectations $\mathbb{E}_e^s : A_{s(e)} \rightarrow B_e^s$. We write P_{vert} the vertex reduced fundamental C*-algebra and $\lambda : P \rightarrow P_{\text{vert}}$ the canonical surjective unital *-homomorphism.

Corollary 6.3. *Suppose that $\mathcal{G}$ is a finite graph then the class of the canonical surjection $[\lambda] \in KK(P, P_{\text{vert}})$ is invertible.*

Proof. Using the *Serre's devissage* machinery developed in the beginning of section 5, the results follows by induction and Theorem 6.1 and Corollary 6.2. □

Remark 6.4. The previous result is actually true without assuming the graph $\mathcal{G}$ finite. Indeed the inverse of $[\lambda]$ and the homotopy showing that it is an inverse can be constructed directly, without using induction.

Remark 6.5. Corollary 6.3 implies the result of Pimsner [Pi86, Corollary 19] saying that a countable discrete group Γ acting without inversion on a tree $\mathcal{T}$ is K-amenable if and only if all the vertex stabilizers are K-amenable (whenever the quotient graph $\mathcal{T}/\Gamma$ is finite). Indeed, by induction and Bass-Serre's theory, it suffices to prove the result for amalgamated free products and HNN-extensions and these two cases follow from Theorem 6.1 and Corollary 6.2. More generally, the following also holds.

Corollary 6.6. *The following holds.*

(1) *If G be the fundamental compact quantum group of a graph of compact quantum groups $(G_p, G_e, \mathcal{G})$ then $\widehat{G}$ is K-amenable if and only if $\widehat{G}_p$ is K-amenable for all $p \in V(\mathcal{G})$.*

(2) *If G is the compact quantum group obtained from the graph product of the family of compact quantum groups G_p, $p \in V(\mathcal{G})$ (see [CF14]) then $\widehat{G}$ is K-amenable if and only if $\widehat{G}_p$ is K-amenable for all $p \in V(\mathcal{G})$.*

Remark 6.7. The first assertion of the previous Corollary strengthens the results of [FF13, Fi13, Ve04].

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