Differential Equations Assignment #3: answers.

Exercise 1. Prove *Grönwall's Lemma*:

Let $f, g : I \to \mathbb{R}_{\geq 0}$ be continuous and $c \in \mathbb{R}$ be such that:

$$\forall t \geq t_0, \quad f(t) \leq c + \int_{t_0}^{t} f(s)g(s)ds$$

Then:

$$\forall t \geq t_0, \quad f(t) \leq c \exp\left(\int_{t_0}^{t} g(s)ds\right)$$

You may introduce the map:

$$h(t) = \frac{c + \int_{t_0}^{t} f(s)g(s)ds}{\exp\left(\int_{t_0}^{t} g(s)ds\right)}.$$

Solution. The function h is continuous at t_0, differentiable on I, and for $t \geq t_0$ the derivative is:

$$h'(t) = \frac{f(t)g(t) - \left(c + \int_{t_0}^{t} f(s)g(s)ds\right) \cdot g(t)}{\exp\left(\int_{t_0}^{t} g(s)ds\right)} \leq 0$$

So, always for $t \geq t_0$, one has $h(t) \leq h(t_0)$. Therefore when $t \geq t_0$:

$$f(t) \leq c + \int_{t_0}^{t} f(s)g(s)ds = h(t) \cdot \exp\left(\int_{t_0}^{t} g(s)ds\right) \leq h(t_0) \cdot \exp\left(\int_{t_0}^{t} g(s)ds\right);$$

as $h(t_0) = c$ we are done.

Exercise 2. Consider the following scalar Cauchy problem:

$$x'(t) = x(t) \text{ on } \mathbb{R} \text{ with initial condition } x(0) = 0$$

For $f : (-1, 1) \to \mathbb{R}$ let $T(f)$ be the map:

$$T(f)(t) = \int_0^t f(s)ds.$$

1. Let $f_0 = \exp$. Compute $f_1 = T(f_0)$, then $f_2 = T(f_1)$, then $f_3 = T(f_2)$.
2. Conjecture and prove something about f_n (defined by successive iterations).
3. How do you explain this in terms of differential equations?

Solution.

1. By construction:

$$\begin{aligned} f_1(t) &= \int_0^t \exp(s)ds = e^t - 1; \\ f_2(t) &= \int_0^t (e^s - 1)ds = e^t - 1 - t; \\ f_3(t) &= \int_0^t (e^s - 1 - s)ds = e^t - 1 - t - \frac{t^2}{2}. \end{aligned}$$

2. We claim that for all $n \in \mathbb{N}$, one has:

$$f_n(t) = e^t - \sum_{k=0}^{n-1} \frac{1}{k!} t^k = \sum_{k=n}^{+\infty} \frac{1}{k!} t^k$$

Indeed, we just checked the claim for $n = 0, 1, 2, 3$. Suppose it holds at some fixed n; then by construction:

$$\begin{aligned} f_{n+1}(t) &= T(f_n)(t) \\ &= \int_0^t \left(e^s - \sum_{k=0}^{n-1} \frac{1}{k!} s^k \right) ds \\ &= e^t - 1 - \sum_{k=1}^{n} \frac{1}{k!} t^k \\ &= e^t - \sum_{k=0}^{n} \frac{1}{k!} t^k, \end{aligned}$$

which proves the claim by induction.

3. This is easily explained. The problem $x'(t) = x(t)$ with $x(0) = 0$ has a unique solution, the constant function 0.

   On the other hand, the equation $x' = x$ rewrites as $x'(t) = G(t, x(t))$ for $G(a, b) = b$. This map is clearly locally Lipschitz in its second variable; by the Picard proof of the Cauchy-Lipschitz theorem we know that the functional T as defined in the exercise:

   - has a unique fixed point,
   - which is the unique solution (here, 0),
   - and that any sequence of iterates will converge to it.

   So $(f_n) \to 0$ was to be expected.

Exercise 3. Consider the Cauchy problem:

$$x'(t) = 1 + x^2(t) \text{ for } t \in (-1, 2)$$

with $x(0) = 0$.

1. Prove that there is a unique solution in the neighborhood of 0.

2. Give an explicit formula (this is *not* a trick question: by chance, this non-linear equation can be solved using trigonometric functions).

3. Let $h > 0$ be any step. As in Euler's method, define $v_0 = 0$, then $v_{n+1} = v_n + h(1 + v_n^2)$.

   Prove that for all $n \in \mathbb{N}$, $v_n > 0$.

4. Hence Euler's method gives a strictly positive affine function.

   How do you reconcile this with the fact that $\lim_{\frac{\pi}{2}^+} \tan(t) = -\infty$?

Solution.

1. Since the map $x \mapsto 1 + x^2$ is C^∞ on $\mathbb{R}$, it is locally Lipschitz everywhere; by the Cauchy-Lipschitz theorem, around every initial condition there exists a unique solution.

2. The map tan satisfies the requirement. Notice that it is a non-global solution (as it goes to $+\infty$ at $\frac{\pi}{2}^- < 2$).

3. A trivial induction.

4. The local Cauchy-Lipschitz guarantees existence (and uniqueness) of a local solution; here, it is the restriction of tan to $(-1, \frac{\pi}{2})$. We also know that Euler's method will converge to this function on the relevant interval.

   What is pointed out is that this convergence does not hold on $(\frac{\pi}{2}, 3)$.

   But the Cauchy-Lipschitz theorem does not guarantee existence of a *global* solution. It would if $x \mapsto 1 + x^2$ were globally Lipschitz on all of $\mathbb{R}$, but it is not the case as the derivative goes to ∞ with x.

   Final note: no, tan on $(1, 3)$ is *not* a global solution: it is not defined at $\frac{\pi}{2}$.