Differential Equations Assignment #3

Exercise 1. Prove *Grönwall's Lemma*:

Let $f, g : I \to \mathbb{R}_{\geq 0}$ be continuous and $c \in \mathbb{R}$ be such that:

$$\forall t \geq t_0, \quad f(t) \leq c + \int_{t_0}^{t} f(s)g(s)ds$$

Then:

$$\forall t \geq t_0, \quad f(t) \leq c \exp\left(\int_{t_0}^{t} g(s)ds\right)$$

You may introduce the map:

$$h(t) = \frac{c + \int_{t_0}^{t} f(s)g(s)ds}{\exp\left(\int_{t_0}^{t} g(s)ds\right)}.$$

Exercise 2. Consider the following scalar Cauchy problem:

$$x'(t) = x(t) \text{ on } \mathbb{R} \text{ with initial condition } x(0) = 0$$

For $f : (-1, 1) \to \mathbb{R}$ let $T(f)$ be the map:

$$T(f)(t) = \int_{0}^{t} f(s)ds.$$

1. Let $f_0 = \exp$. Compute $f_1 = T(f_0)$, then $f_2 = T(f_1)$, then $f_3 = T(f_2)$.
2. Conjecture and prove something about f_n (defined by successive iterations).
3. How do you explain this in terms of differential equations?

Exercise 3. Consider the Cauchy problem:

$$x'(t) = 1 + x^2(t) \text{ for } t \in (-1, 2)$$

with $x(0) = 0$.

1. Prove that there is a unique solution in the neighborhood of 0.
2. Give an explicit formula (this is *not* a trick question: by chance, this non-linear equation can be solved using trigonometric functions).
3. Let $h > 0$ be any step. As in Euler's method, define $v_0 = 0$, then $v_{n+1} = v_n + h(1 + v_n^2)$. Prove that for all $n \in \mathbb{N}$, $v_n > 0$.
4. Hence Euler's method gives a strictly positive affine function. How do you reconcile this with the fact that $\lim_{\frac{\pi}{2}^+} \tan(t) = -\infty$?