Differential Equations Assignment #2: answers.

Solutions

Exercise 1. Solve the equation on $\mathbb{R}$

$$F'(t) = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} F(t) + \begin{pmatrix} e^t \\ 0 \end{pmatrix}$$

with the initial condition

$$F(0) = \begin{pmatrix} 1 \\ 1 \end{pmatrix}.$$

Solution. Existence and uniqueness of a solution (with initial condition) are predicted by the linear Cauchy-Lipschitz theorem; we even know that the solution will be defined on $\mathbb{R}$.
For simplicity let $A = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$ and $B(t) = \begin{pmatrix} e^t \\ 0 \end{pmatrix}$. In order to solve the equation

$$(\mathcal{E}): \qquad F'(t) = A \cdot F(t) + B(t)$$

(with or without initial condition), we first handle the simpler equation:

$$(\mathcal{E}_H): \qquad F'(t) = A \cdot F(t).$$

The matrix tA is readily exponentiated:

$$\exp(tA) = \begin{pmatrix} e^t & te^t \\ 0 & e^t \end{pmatrix}$$

And as we know, the space of solutions of $(\mathcal{E}_H)$ is:

$$S_H = \{\exp(tA) \cdot X_1 : X_1 \in \mathbb{R}^2\}.$$

Now given one solution F_1 of $(\mathcal{E})$, the solution set of $(\mathcal{E})$ is exactly:

$$S = F_1 + S_H.$$

A possibility to find a special solution is of course to draw our inspiration from the very shape of B; let us look for it in the form:

$$F_1(t) = \begin{pmatrix} x(t)e^t \\ 0 \end{pmatrix}$$

For this to be a solution, one needs:

$$x'(t)e^t + x(t)e^t = x(t)e^t + 0 + e^t$$

which suggests to let:

$$F_1(t) = \begin{pmatrix} te^t \\ 0 \end{pmatrix}$$

One may check that F_1 is a solution of $(\mathcal{E})$.

As a conclusion,

$$S = \{\exp(tA) \cdot X_1 + F_1 : X_1 \in \mathbb{R}^2\}$$

Instead of endlessly rewriting it, let us see when a solution $F(t) = \exp(tA) \cdot X_1 + F_1$ satisfies the initial condition. One needs:

$$\begin{pmatrix} 1 \\ 1 \end{pmatrix} = F(0) = \exp(0) \cdot X_1 + F_1(0) = X_1 + \begin{pmatrix} 0 \\ 0 \end{pmatrix} = X_1$$

So the only candidate is:

$$\begin{aligned} F(t) &= \exp(tA) \cdot \begin{pmatrix} 1 \\ 1 \end{pmatrix} + \begin{pmatrix} te^t \\ 0 \end{pmatrix} \\ &= \begin{pmatrix} e^t & te^t \\ 0 & e^t \end{pmatrix} \cdot \begin{pmatrix} 1 \\ 1 \end{pmatrix} + \begin{pmatrix} te^t \\ 0 \end{pmatrix} \\ &= \begin{pmatrix} t + 2te^t \\ e^t \end{pmatrix} \end{aligned}$$

And as we know by existence of a solution, this only candidate must be the only solution.

> **Note.** If you do not believe in the theory, check that the above is a solution (with initial condition) indeed.

Exercise 2. We consider the following ordinary differential equation

$$(\mathcal{E}) \qquad \left\{ \begin{array}{rcl} u'(t) &=& \frac{5}{3} u^{2/5} \\ u(0) &=& 0 \end{array} \right. .$$

1. Prove that the solution set of $(\mathcal{E})$ is infinite. Provide a family of solutions.
2. Does this contradict the **Cauchy-Lipschitz theorem**? Justify!

Solution.

1. We adapt a construction seen in class. For $T \geq 0$ let:

$$u_T(t) = \left\{ \begin{array}{rcl} 0 & \text{if} & t \leq T \\ (t-T)^{5/3} & \text{if} & t \geq T \end{array} \right. .$$

We first observe that u_T is well-defined, since both definitions agree at T. It is clearly continuous for the same reason. Now u_T is differentiable (as a matter of fact, C^∞) both on $(-\infty, T)$ and $(T, +\infty)$; on the former, the derivative is 0; on the latter, it is:

$$\frac{5}{3}(t-T)^{2/3}$$

To conclude that u_T is differentiable at T one can either use the above and a theorem from real analysis ("limit of the derivative"), or return to the simple fact that for $h > 0$:

$$\frac{u_T(T+h) - u_T(T)}{h} = h^{2/3} \longrightarrow_{h \to 0} 0,$$

proving that u_T is right-semi-differentiable at T. Since the left-semi-derivative has the same value, we conclude that u_T is differentiable at T, and the formula $u_T'(t) = \frac{5}{3}(t-T)^{2/3}$ holds on $[T, \infty)$.

In particular, one sees that u_T is a solution of the equation; since $T \geq 0$, it satisfies the initial condition. As a conclusion, $\{u_T : T \geq 0\}$ is an infinite family of solutions.

Solutions

2. This does not contradict the Cauchy-Lipschitz theorem as it simply does not apply here. Indeed, the function $x \mapsto \frac{5}{3}x^{2/5}$ is not locally Lipschitz around 0. One argument can be to differentiate, and see that the derivative goes to ∞ as $t \to 0$. But why make things complicated? Let us write a short contradiction proof.

   Suppose that $f(x) = x^{2/5}$ is locally Lipschitz around 0, say k-Lipschitz for some $k \geq 0$. This means that there is $\varepsilon > 0$ such that for all $x, y \in [0, \varepsilon)$, $|f(x) - f(y)| \leq k|x - y|$. Letting y go to 0, by continuity, this implies $x^{2/5}|f(x)| \leq kx$ for $x \in [0, \varepsilon)$. However, if $x < \min(\varepsilon, k^{-5/3}$, then:

   $$\frac{x^{2/5}}{x} = x^{-3}5 = (x^{3/5})^{-1} > ((k^{-5/3})^{3/5})^{-1} = k$$

   which is a contradiction.

   Hence the Cauchy-Lipschitz is safe: its assumptions do not hold here.

Notes.

- Some of you decided to carelessly manipulate the equation and integrate — which was not asked.

  If you do this, after meaningless symbolic computations, you might have the impression that there exists a *unique* solution; some actually said so.

  This is silly for two reasons: first, the symbolic manipulations involve division by 0; you are too old for such a foolish thing. Second, you are supposed to show that there are *infinitely* many solutions; claiming something about "the unique", or even "the" solution is mathematically *wrong*.

- Do not write that it "contradicts the Cauchy-Lipschitz theorem": the theorem is proved, it is no longer possible to contradict it.

- To my great surprise, not all of you could state the Cauchy-Lipschitz theorem correctly.

Exercise 3.

1. Let $n \in \mathbb{Z} \setminus \{0, 1\}$; also let $a, b : \mathbb{R} \to \mathbb{R}$ be continuous maps, and consider the equation:

   $$x'(t) + a(t)x(t) + b(t)(x(t))^n = 0$$

   where $(x(t))^n$ is the n^{th} power of $x(t)$ (*not* the derivative).

   Suppose that x is a solution *that remains positive.* Let $y(t) = (x(t))^{1-n}$ and show that y satisfies a *linear* equation.

2. Solve

   $$tx'(t) + x(t) - t(x(t))^3 = 0$$

   on each interval where x keeps a constant sign.

Solution.

1. We let $y(t) = x(t)^{1-n}$, as suggested (assuming that x never vanishes). Then y is differentiable and one has:

   $$\begin{aligned} y'(t) &= (1-n)x'(t)x(t)^{-n} \\ &= (1-n)(-a(t)x(t) - b(t)x(t)^n)x(t)^{-n} \\ &= (1-n)(-a(t)y(t) - b(t)) \end{aligned}$$

   so $y'(t) = (n-1)(a(t)y(t) + b(t))$, a linear equation (really, they should say affine).

solutions

2. Suppose that $x(t)$ does not vanish on some interval I.

   Suppose that $0 \in I$. Then at $t = 0$ we find: $x(0) = 0$, so x vanishes at $0 \in I$: a contradiction. Hence $0 \notin I$.

   Then on I the equation rewrites as:

$$x'(t) + \frac{1}{t}x(t) - x^3(t) = 0$$

   As we know, letting $y(t) = x^{-2}(t)$ helps: it satisfies the equation:

$$y'(t) = \frac{2}{t}y - 2,$$

   which has (on I) global solutions by the linear Cauchy-Lipschitz theorem. One easily finds that $y(t) = \lambda t^2 + 2t$ for λ a real number.

   On I this should remain positive (notice that we thus find again that $0 \notin I$); this yields constraints on I depending on λ but these were not explicitly asked. Since $y(t) = x^{-2}(t)$, we find:

$$x(t) = \frac{\delta}{\sqrt{\lambda t^2 + 2t}}$$

   where $\delta = \pm 1$.

Notes.

1. No, $a^2 = b > 0$ does *not* imply $a = \sqrt{b}$.
2. You are too old to divide by 0.
3. The Cauchy-Lipschitz theorem can be invoked for y, not for x; the mathematical difficulty is that one needs x to behave well (here, not to vanish) in order to define y.

Exercise 4 (some theory)**.** Let $A : I \to M_d(\mathbb{R})$ have the property:

$$\forall (t_1, t_2) \in I^2 \quad A(t_1) \cdot A(t_2) = A(t_2) \cdot A(t_1)$$

Prove that the unique solution of the differential equation on I with initial condition:

$$X'(t) = A(t) \cdot X(t), \qquad \text{with } X(t_0) = X_0$$

has the form:

$$X(t) = \exp\left(\int_{t_0}^{t} A(s)ds\right) \cdot X_0$$

Hint: prove that $\int_{t_0}^{t} A(s)ds$ and $\int_{t}^{t+h} A(s)ds$ commute.

Solution. The linear Cauchy-Lipschitz theorem predicts existence and uniqueness; it is then safe to let

$$X(t) = \exp\left(\int_{t_0}^{t} A(s)ds\right) \cdot X_0,$$

a well-defined and differentiable map, and check that $X(t)$ satisfies the equation and initial condition. The latter is trivial since:

$$X(t_0) = \exp\left(\int_{t_0}^{t_0} A(s)ds\right) \cdot X_0 = \exp(0) \cdot X_0 = I \cdot X_0 = X_0,$$

so we turn to the equation itself. It suffices to check that $X'(t) = A(t) \cdot X(t)$.

solutions

Our goal is obviously to compute the derivative of $X(t)$. Since we have not computed the differential of exp, we return to the basic definition as a limit of difference quotients:

$$X'(t_1) = \lim_{h\to 0} \frac{1}{h}(X(t_1+h) - X(t_1)).$$

Fix $t_1 \in I$ and let $h \in \mathbb{R}$ be small. Then:

$$\begin{aligned} X(t_1+h) &= \exp\left(\int_{t_0}^{t_1+h} A(s)ds\right) \cdot X_0 \\ &= \exp\left(\int_{t_0}^{t_1} A(s)ds + \int_{t_1}^{t+h} A(s)ds\right) \cdot X_0 \end{aligned}$$

We claim that the matrix integrals inside the exponential commute. Indeed let $s_1 \in (t_0, t_1)$ and $s_2 \in (t_1, t_1+h)$. Then $A(s_1)A(s_2) = A(s_2)A(s_1)$, so integrating over s_1:

$$\left(\int_{t_0}^{t_1} A(s_1)ds_1\right) \cdot A(s_2) = A(s_2) \cdot \left(\int_{t_0}^{t_1} A(s_1)ds_1\right).$$

Now integrating over s_2, we find:

$$\left(\int_{t_0}^{t_1} A(s_1)ds_1\right) \cdot \left(\int_{t_1}^{t_1+h} A(s_2)ds_2\right) = \left(\int_{t_1}^{t_1+h} A(s_2)ds_2\right) \cdot \left(\int_{t_0}^{t_1} A(s_1)ds_1\right),$$

as desired.
Returning to $X(t_1+h)$, we find:

$$\begin{aligned} X(t_1+h) &= \exp\left(\int_{t_0}^{t_1} A(s)ds + \int_{t_1}^{t_1+h} A(s)ds\right) \cdot X_0 \\ &= \exp\left(\int_{t_0}^{t_1} A(s)ds\right) \cdot \exp\left(\int_{t_1}^{t_1+h} A(s)ds\right) \cdot X_0 \\ &= \exp\left(\int_{t_1}^{t_1+h} A(s)ds\right) \cdot \exp\left(\int_{t_0}^{t_1} A(s)ds\right) \cdot X_0 \\ &= \exp\left(\int_{t_1}^{t_1+h} A(s)ds\right) \cdot X(t_1) \end{aligned}$$

Therefore:

$$X(t_1+h) - X(t_1) = \left(\exp\left(\int_{t_1}^{t_1+h} A(s)ds\right) - I\right) \cdot X(t_1)$$

We keep in mind that we want to find $X'(t) = A(t)X(t)$; that is, we now wish to prove:

$$\lim_{h\to 0} \frac{1}{h}\left(\exp\left(\int_{t_1}^{t_1+h} A(s)ds\right) - I\right) = A(t_1)$$

If we prove this we are done. Intuitively, on the short segment $[t_1, t_1+h]$ the function $A(s)$ is almost constant; so the integral is more or less $hA(t_1)$, which is small (since h is); hence the exponential is approximately $I + hA(t_1)$; and finally the derivative should be $A(t_1)$. Let us do this properly.
Fix $\varepsilon \in (0, 1)$. By continuity of A at t_1 (I realise that the assumption was not stated explicitly in the assignment; however, we always considered continuous coefficients), there is η such that:

$$\forall h \in (-\eta, \eta), \quad \|A(t_1+h) - A(t_1)\| < \varepsilon$$

solutions

Of course since we are trying to compute a limit, we may freely suppose that $|h| < \eta$.
It follows that:

$$\int_{t_1}^{t_1+h} A(s)ds = \int_{t_1}^{t_1+h} A(t_1)ds + \int_{t_1}^{t_1+h} (A(s) - A(t_1))ds = hA(t_1) + R(h),$$

where the error term satisfies:

$$\|R(h)\| = \left\| \int_{t_1}^{t_1+h} (A(s) - A(t_1))ds \right\| \leq \int_{t_1}^{t_1+h} \|A(s) - A(t_1)\| \, ds \leq \varepsilon |h|$$

Here again, $hA(t_1)$ and $R(h)$ commute, so:

$$\exp\left(\int_{t_1}^{t_1+h} A(s)ds \right) = \exp(hA(t_1)) \cdot \exp(R(h)).$$

Now as a function of h (bear in mind that t_1 is fixed), the map $\exp(hA(t_1))$ is differentiable at 0 with derivative $A(t_1)$, as proved in class; it means that it has Taylor expansion $I + hA(t_1) + o(h)$.
On the other hand, one has:

$$\exp R(h) = I + R(h) + \frac{1}{2!} R(h)^2 + \dots.$$

But since $\|R(h)\| \leq \varepsilon h$ with $\varepsilon < 1$, the term of order $n \geq 2$ has norm $\leq \frac{1}{n!} \varepsilon^n |h|^n \leq \frac{1}{n!} \varepsilon |h|^2$; it follows that:

$$\| \exp(R(h)) - I - R(h) \| = \left\| \sum_{k=2}^{\infty} \frac{1}{k!} (R(h))^k \right\| \leq |h^2| \sum_{k=2}^{\infty} \frac{1}{k!} \leq \varepsilon |h|^2 e$$

Hence $\exp(R(h)) = I + R(h) + o(h)$.
Put together:

$$\begin{aligned} \exp\left(\int_{t_1}^{t_1+h} A(s)ds \right) &= \exp(hA(t_1)) \cdot \exp(R(h)) \\ &= (I + hA(t_1) + o(h)) \cdot (I + R(h) + o(h)) \\ &= I + hA(t_1) + R(h) + o(h) \end{aligned}$$

so

$$\begin{aligned} Q(h) := \frac{1}{h} \left(\exp\left(\int_{t_1}^{t_1+h} A(s)ds \right) - I \right) &= \frac{1}{h} \left(I + hA(t_1) + R(h) + o(h) - I \right) \\ &= A(t_1) + \frac{1}{h} R(h) + o(1) \end{aligned}$$

But since $\|R(h)\| \leq \varepsilon |h|$, this means that $Q(h)$ can be made arbitrarily close to $A(t_1)$. Which is our claim.

> **Note.** There is no such thing as "vector division". This is a *serious mathematical mistake* against which I had warned you several times.
> In class I made a point of never dividing, even in the scalar case; apparently you preferred to rely on unsatisfactory recipes you learnt in the past. Unfortunately, in this exercise, "not quite rigorous" became "completely out".

solutions