Differential Equations Tutorial #3: answers.

Exercise 1. Redo all quizzes.

Exercise 2. Consider the *pendulum equation*:

$$\theta''(t) = \sin(\theta(t))$$

1. Write this as a (non-linear) first-order, vector equation.
2. Is the function G you wrote Lipschitz? globally Lipschitz?
3. Deduce that for all (θ_0, v_1), there is a unique solution such that $\theta(0) = \theta_0$ and $\theta'(0) = v_1$.

Solution.

1. Let us introduce the function:

$$\begin{array}{rccc} G: & \mathbb{R}^3 & \to & \mathbb{R}^2 \\ & (a,b,c) & \mapsto & (c, \sin(b)) \end{array}$$

   For $\theta : \mathbb{R} \to \mathbb{R}$ any differentiable map, let $\Theta(t) = \begin{pmatrix} \theta(t) \\ \theta'(t) \end{pmatrix}$.

   Then clearly, $\theta''(t) = \sin\theta(t)$ iff $\Theta'(t) = \begin{pmatrix} \theta'(t) \\ \sin\theta(t) \end{pmatrix}$ iff $\Theta'(t) = G(t, \Theta(t))$.

2. We contend that G is globally 1-Lipschitz in the space variables (b,c), with respect to the usual norm. Indeed, for any time a and any (b_1, c_1) and (b_2, c_2) in $\mathbb{R}^2$, one has:

$$\|G(a, b_1, c_1) - G(a, b_2, c_2)\| = \|(c_1 - c_2, \sin b_1 - \sin b_2)\|$$

   But sin itself is globally 1-Lipschitz as it is differentiable with derivative everywhere bounded by 1, so:

$$\begin{aligned} \|G(a, b_1, c_1) - G(a, b_2, c_2)\| &= \|(c_1 - c_2, \sin b_1 - \sin b_2)\| \\ &= \sqrt{(c_1 - c_2)^2 + (\sin b_1 - \sin b_2)^2} \\ &\leq \sqrt{(c_1 - c_2)^2 + (b_1 - b_2)^2} \\ &= \|(b_1, c_1) - (b_2, c_2)\| \end{aligned}$$

   Note. The optimal Lipschitz constant depends on the choice of the norm. But in a finite-dimensional space, since all norms are equivalent, the property of being Lipschitz does not depend on the norm.

3. It is then a clear consequence of the Cauchy-Lipschitz theorem that for all $\Theta_0 = \begin{pmatrix} \theta_0 \\ v_1 \end{pmatrix}$, there is a unique solution Θ with $\Theta(0) = \Theta_0$. Which exactly says that there is a unique solution θ with:

$$\begin{pmatrix} \theta(0) \\ \theta'(0) \end{pmatrix} = \begin{pmatrix} \theta_0 \\ v_1 \end{pmatrix}.$$

Exercise 3. Consider the function $f : \mathbb{R} \to \mathbb{R}$ defined as follows:

$$f(x) = \begin{cases} -1 & \text{if} \quad x < -1 \\ \sqrt{1-x^2} & \text{if} \quad x \in [-1, 1) \\ 0 & \text{if} \quad x \geq 1 \end{cases}$$

We consider the Cauchy problem $x'(t) = f(x(t))$ with initial condition $x(0) = a$. Discuss existence and uniqueness depending on a.

Solution.

- The map is constant on $(-\infty, -1)$, hence C^∞, hence locally Lipschitz; as a consequence of the Cauchy-Lipschitz theorem (here the linear version would do!), for $a \in (-\infty, -1)$, there is a unique solution (which could be easily computed but it is not our purpose).
- At -1 the function is not continuous: it is hard to predict what happens when $a = -1$.
- Again, on $(-1, 1)$, the map is C^∞: by the Cauchy-Lipschitz theorem, for a there, one has a unique solution.
- At 1 things are different: the function is continuous but not locally Lipschitz. So there exist solutions by Peano's theorem, but uniqueness is not to be expected a priori.
- On $(1, +\infty)$, the function is constant again: all is fine.

Exercise 4. Consider equation $x'(t) = x^2(t)$ on $[0, 1]$ with $x(0) = 1$. Draw a $\frac{1}{4}$-approximate solution.

Solution. Drawn in class.