Differential Equations Assignment #1: answers.

Note. A mathematical text is written in human language; every formula should be part of a proper sentence. Needless (?) to say, a sentence begins with a capital letter and ends with a punctuation symbol. Otherwise the text is not meant to be read.
This elementary principle is often disregarded; however writing lines and lines of formulas is not acceptable. The idea is that your assignments should show your reasoning, not your computation; that someone with *less* knowledge than you should be able to see the key ideas, and the names of the methods.
By reading the solutions below, you will be impressed by the little amount of symbols. But high-level mathematics proceeds in sentences, not in formulas. *Do not focus on the computations, focus on the logic.*

Exercise 1. Consider the following equation:

$$x'(t) = 2tx(t) + \frac{1}{t}\exp(t^2)$$

defined on $\mathbb{R}_{>0}$, with initial condition $x(1) = 0$.

1. Write the so-called homogeneous equation (without any initial condition). Give the general solution (of the homogeneous equation).
2. Give one special solution of the original equation. Then give the general solution (of the original equation).
3. Now give the only solution satisfying the initial condition.

Solution.

1. The associated homogeneous equation is:

$$x'(t) = 2tx(t)$$

which as we know solves into $x(t) = \exp(\int 2tdt) = \lambda e^{t^2}$ for $\lambda \in \mathbb{R}$.

Notes.

(a) Many of you wrote "$\lambda(t) \cdot e^{t^2}$". This is simply wrong — λ must be a constant.

(b) Some of you have been using the so-called "separable variables method", as follows:

Write $\frac{x'}{x} = 2t$, which integrates as $\log(x) = e^{t^2} + C\ldots$

This is mathematically *not rigorous* for two reasons, and naive for one.

i. Division by x: but x could vanish. Indeed, the map $x(t) = 0$ is a solution.

ii. Use of the logarithm: there is no well-defined log function on $\mathbb{R}$.
The slightly more correct $\log|x| = e^{t^2}$ does not solve this issue: again, what about $x = 0$?

iii. How do you expect to generalise this division to the vector case?

In short, nothing beats quick use of the exponential. For desperate (non-linear) scalar situations, separating variables gives some intuition; its *rigorous* treatment is however much heavier than anything we discussed so far. Refrain from using something you cannot justify!

2. We use the method called “variation of the parameter”, and look for one special solution of the original equation in the form $x(t) = \lambda(t) \cdot e^{t^2}$.

   For this to be a solution, we need to have:

$$\lambda'(t) \cdot e^{t^2} = \frac{1}{t} e^{t^2}$$

   and therefore $\lambda'(t) = \frac{1}{t}$.

   This suggests to let $x(t) = \log(t) \cdot e^{t^2}$, which is well-defined on $I = \mathbb{R}_{>0}$. A quick computation shows that this is a solution of the original equation. Hence the general solution has the form:

$$(\lambda + \log(t)) \cdot e^{t^2}$$

   for some $\lambda \in \mathbb{R}$.

   Notes.

   (a) Do not forget to check that the x you define is a solution indeed. (A necessary condition is not always sufficient, so getting the condition on λ is a priori no serious guarantee.)

   (b) It is mathematically *and* grammatically incorrect to write “the one special solution”.

3. The linear case of the Cauchy-Lipschitz theorem predicts that there will be exactly one maximal solution to the equation with initial condition, and that it will be defined on $\mathbb{R}_{>0}$. For an arbitrary solution $x(t) = (\lambda + \log(t)) \cdot e^{t^2}$ to satisfy $x(1) = 0$, one needs $\lambda = 0$.

   One quickly checks that $x(t) = \log(t) \cdot e^{t^2}$ is a solution of the equation, and satisfies the initial condition; it is even unique as such by the Cauchy-Lipschitz theorem.

Exercise 2. We consider the following differential equation:

$$(\mathcal{E}_{\mathcal{H}}): \qquad x^{(4)} = x^{(3)} + 7x^{(2)} - 13x' + 6x.$$

1. Provide a basis of the space solution S_H.

2. Give the explicit solution with the initial conditions

$$x(0) = 2; \qquad x'(0) = 0; \qquad x''(0) = -2 \qquad \text{and} \qquad x^{(3)}(0) = -4$$

Solution.

1. We consider the characteristic polynomial associated to the problem:

$$P(\lambda) = \lambda^4 - \lambda^3 - 7\lambda^2 + 13\lambda - 6.$$

   The value 1 is an obvious root; dividing (there is no need for Euclidean division here, you can do it mentally) one finds $P(\lambda) = (\lambda - 1)(\lambda^3 - 7\lambda + 6)$. Here again 1 is a root of the second factor, yielding $P(\lambda) = (\lambda - 1)^2(\lambda^2 - \lambda - 6)$. The latter term has discriminant 25, and solutions $\frac{1 \pm 5}{2}$, namely -2 and 3.

   Hence the characteristic polynomials has roots (counted with multiplicity): $-2, 1, 1, 3$. So the functions $e^t, te^t, e^{-2t}, e^{3t}$ are solutions of the differential equation $(\mathcal{E}_{\mathcal{H}})$.

   Because it is linear homogeneous of order 4, the Cauchy-Lipschitz theorem predicts that the space S_H of solutions has dimension 4; since the solutions we gave are clearly linearly independent, they form a basis of S_H.

   As a conclusion, *a* basis of S_H is $\{e^t, te^t, e^{-2t}, e^{3t}\}$.

Notes.

(a) One should not need to write down a polynomial division here; the factorisation was too simple.

(b) Never introduce the discriminant Δ without saying the word "discriminant" (this is a typical example of unclear mathematical exposition; as a rule, never use an unexplained notation, however universal you think it is).

(c) A vector space almost never has a unique basis. It is therefore mathematically incorrect to write "the basis is".

2. We now take the initial condition into account. The general solution has the form:

$$x(t) = ae^t + bte^t + ce^{-2t} + de^{3t}.$$

We can then compute and evaluate the successive derivatives:

$$\begin{aligned} x(0) &= a + c + d; \\ x'(0) &= a + b - 2c + 3d; \\ x''(0) &= a + 2b + 4c + 9d; \\ x'''(0) &= a + 3b - 8c + 27d. \end{aligned}$$

Hence the initial condition gives rise to the following system in matrix form, where we let Λ be the vector of coordinates in the given basis:

$$\begin{pmatrix} 1 & 0 & 1 & 1 \\ 1 & 1 & -2 & 3 \\ 1 & 2 & 4 & 9 \\ 1 & 3 & -8 & 27 \end{pmatrix} \cdot \Lambda = \begin{pmatrix} 2 \\ 0 \\ -2 \\ -4 \end{pmatrix}.$$

This system is quickly handled by introducing an augmented matrix and using Gauß elimination, as follows:

$$\left(\begin{array}{cccc|c} 1 & 0 & 1 & 1 & 2 \\ 1 & 1 & -2 & 3 & 0 \\ 1 & 2 & 4 & 9 & -2 \\ 1 & 3 & -8 & 27 & -4 \end{array}\right) \xrightarrow[\substack{L_2 \leftarrow L_2 - L_1 \\ L_3 \leftarrow L_3 - L_1 \\ L_4 \leftarrow L_4 - L_1}]{} \left(\begin{array}{cccc|c} 1 & 0 & 1 & 1 & 2 \\ 0 & 1 & -3 & 2 & -2 \\ 0 & 2 & 3 & 8 & -4 \\ 0 & 3 & -9 & 26 & -6 \end{array}\right)$$

$$\xrightarrow[\substack{L_3 \leftarrow L_3 - 2L_2 \\ L_4 \leftarrow L_4 - 3L_2}]{} \left(\begin{array}{cccc|c} 1 & 0 & 1 & 1 & 2 \\ 0 & 1 & -3 & 2 & -2 \\ 0 & 0 & 9 & 4 & 0 \\ 0 & 0 & 0 & 20 & 0 \end{array}\right).$$

At this stage one clearly has $c = d = 0$; finally $a = 2$ and $b = -2$. So the only solution with initial condition is $x(t) = 2e^t - 2te^t = 2(1-t)e^t$.

Note. More than ever in this question, what matters is not the mathematical formulas, but the sentences linking them.

Exercise 3. Consider the following matrix:

$$A = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$$

Prove that:

$$\exp(A) = \begin{pmatrix} \cos(1) & -\sin(1) \\ \sin(1) & \cos(1) \end{pmatrix}$$

Solution. We treat A as a complex-valued matrix. Its characteristic polynomial is λ^2+1, which has roots $\pm i$. The 2×2 matrix A has two distinct eigenvalues, so it is diagonalisable.
Let us compute the eigenspaces:

$$\ker(A - iI) = \ker\begin{pmatrix} -i & -1 \\ 1 & -i \end{pmatrix} = \ker\begin{pmatrix} 1 & -i \end{pmatrix} = \mathbb{C}\begin{pmatrix} i \\ 1 \end{pmatrix}$$

and likewise,

$$\ker(A + iI) = \ker\begin{pmatrix} i & -1 \\ 1 & i \end{pmatrix} = \ker\begin{pmatrix} 1 & i \end{pmatrix} = \mathbb{C}\begin{pmatrix} 1 \\ i \end{pmatrix}.$$

Hence $\left(\begin{pmatrix} i \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ i \end{pmatrix}\right)$ is an eigenbasis, which suggests to indtroduce the following coordinate change matrix:

$$P = \begin{pmatrix} i & 1 \\ 1 & i \end{pmatrix}.$$

Its inverse is easily computed:

$$P^{-1} = -\frac{1}{2}\begin{pmatrix} i & -1 \\ -1 & i \end{pmatrix};$$

by construction (no computation is required here),

$$P^{-1}AP = \begin{pmatrix} i & \\ & -i \end{pmatrix}.$$

Now $\exp(PAP^{-1})$ is easily computed, which yields:

$$\begin{aligned}
\exp A &= P\begin{pmatrix} e^i & \\ & e^{-i} \end{pmatrix}P^{-1} &&= \begin{pmatrix} i & 1 \\ 1 & i \end{pmatrix}\cdot\begin{pmatrix} e^i & \\ & e^{-i} \end{pmatrix}\cdot\begin{pmatrix} \frac{-i}{2} & \frac{1}{2} \\ \frac{1}{2} & \frac{-i}{2} \end{pmatrix} \\
&= \begin{pmatrix} ie^i & e^{-i} \\ e^i & ie^{-i} \end{pmatrix}\cdot\begin{pmatrix} \frac{-i}{2} & \frac{1}{2} \\ \frac{1}{2} & \frac{-i}{2} \end{pmatrix} &&= \frac{1}{2}\begin{pmatrix} e^i + e^{-i} & i(e^i - e^{-i}) \\ -i(e^i - e^{-i}) & e^i + e^{-i} \end{pmatrix} \\
&= \begin{pmatrix} \cos(1) & -\sin(1) \\ \sin(1) & \cos(1) \end{pmatrix}.
\end{aligned}$$

Notes.

1. The function $\sqrt{\cdot}$ is defined on $\mathbb{R}_{\geq 0}$. Writing $\sqrt{-1}$ is simply meaningless.
2. *An* eigenvector, *the* eigenspace. It is not correct to write "the eigenvector" (since any multiple will be another such).

   By the way, the notation $E_\lambda(M)$ refers to the eigen*space*; it is therefore incorrect to write $E_\lambda(M) = v$, one should write $E_\lambda(v) = \mathbb{C}v$.
3. When you first introduce P, you *must* write "Let $P =$". Otherwise the reader may believe (s)he has missed an earlier definition.
4. Last but not least, a technical remark. Because the matrix here was not too hard, it was possible to "brute-force" the exponential by actually computing the series. Now be honest and ask yourself:

   (a) if you would have found the correct answer had it not been given to you;

   (b) what the odds of success of this method might be in dimension 4.

   The general method proceeds by conjugating the matrix to one with simpler exponential; in practice, and although it is the definition, computing the series never works.