Differential Equations Tutorial #1: answers.

Solutions

Exercise 1. Solve the following non-homogeneous linear differential equation:

$$f^{(3)} = 4f^{(2)} - 5f' + 2f + 4e^{3t}$$

with the initial conditions

$$f(0) = 7, \qquad f'(0) = 15, \qquad f''(0) = 40.$$

Solution. We call $(\mathcal{E})$ the equation. By the linear Cauchy-Lipschitz theorem, there will be a unique solution satisfying the initial condition; it will even be defined on all the interval (unspecified here, so we take it to be $\mathbb{R}$).
We first consider the associated homogeneous equation $(\mathcal{E}_H)$:

$$f^{(3)} = 4f^{(2)} - 5f' + 2f.$$

The linear Cauchy-Lipschitz theorem predicts that all solutions are defined on $\mathbb{R}$, and that the set S_H of solutions of $(\mathcal{E}_H)$ is a 3-dimensional vector space. A basis can be given computing the eigenvalues of the associated matrix:

$$A = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 2 & -5 & 4 \end{pmatrix}.$$

The characteristic polynomial of A is $\chi_A(\lambda) = -\lambda^3 + 4\lambda^2 - 5\lambda + 2$ (which also is the "characteristic equation" in the old-style method); it has 1 as an obvious root. Now $-\lambda^3 + 4\lambda^2 - 5\lambda + 2 = -(\lambda - 1)(\lambda^2 - 3\lambda + 2)$; here again 1 is an obvious root of the second factor, which yields $\chi_A(\lambda) = -(\lambda - 1)^2(\lambda - 2)$.
It follows that the roots are, with multiplicity, $1, 1, 2$; consequently, the functions e^t, te^t, e^{2t} form a basis of the space S_H. Now always by the Cauchy-Lipschitz theorem, the space S of solutions of $(\mathcal{E})$ is an affine space directed by S_H, i.e. $S = f_1 + S_H$ where f_1 is any solution of $(\mathcal{E})$.
It is reasonable to look for one admissible f_1 as λe^{3t}; such a function is a solution iff $3^3\lambda = 4 \cdot 3^2\lambda - 5 \cdot 3\lambda + 2\lambda + 4$, i.e. iff $4\lambda = 4$. So the map e^{3t} is a solution of $(\mathcal{E})$, and by the above, any solution of $(\mathcal{E})$ is of the form:

$$ae^t + bte^t + ce^{2t} + e^{3t}$$

It remains to find the only triple (a, b, c) satisfying the initial condition (existence and uniqueness have already been explained above). This gives rise to the linear system:

$$\begin{cases} a & & +c & = 7 - 1 \\ a & +b & +2c & = 15 - 3 \\ a & +2b & +4c & = 40 - 9 \end{cases}$$

In augmented matrix form, using Gauß elimination:

$$\left(\begin{array}{ccc|c} 1 & 0 & 1 & 6 \\ 1 & 1 & 2 & 12 \\ 1 & 2 & 4 & 31 \end{array}\right) \xrightarrow[\substack{L_2 \leftarrow L_2 - L_1 \\ L_3 \leftarrow L_3 - L_1}]{} \left(\begin{array}{ccc|c} 1 & 0 & 1 & 6 \\ 0 & 1 & 1 & 6 \\ 0 & 2 & 3 & 25 \end{array}\right) \xrightarrow[L_3 \leftarrow L_3 - 2L_2]{} \left(\begin{array}{ccc|c} 1 & 0 & 1 & 6 \\ 0 & 1 & 1 & 6 \\ 0 & 0 & 1 & 13 \end{array}\right)$$

Hence $c = 13$ and $a = b = -7$. As a conclusion, the only solution is:

$$-7(1 + t)e^t + 13e^{2t} + e^{3t}$$

Exercise 2. Solve the following ordinary differential equation:

$$(\mathcal{E}): \qquad F'(t) = A \cdot F(t) + B(t),$$

where

$$A = \begin{pmatrix} 2 & 3 \\ -1 & -2 \end{pmatrix}, \qquad \text{and} \qquad B(t) = \begin{pmatrix} 1 \\ b(t) \end{pmatrix}.$$

Hint: first solve the homogeneous equation, then try to find a special solution of $(\mathcal{E})$ of the form $F(t) = \exp(tA) \cdot \Lambda(t)$, with $\Lambda : \mathbb{R} \to M_2(\mathbb{R})$ a differentiable map.

Solution. We first attack the homogeneous equation $(\mathcal{E}_H): \quad F'(t) = A \cdot F(t)$. As we know, solutions are of the form $\exp(tA) \cdot X_0$, where the initial condition X_0 ranges over $\mathbb{R}^2$.
We then look for one solution of $(\mathcal{E})$ in the recommended form, namely as $F_1(t) = \exp(tA) \cdot \Lambda(t)$. For this to be a solution, one needs:

$$(\exp(tA) \cdot \Lambda(t))' = A \exp(tA)\Lambda(t) + B(t),$$

which simplifies into $\exp(tA) \cdot \Lambda'(t) = B(t)$, or equivalently $\Lambda'(t) = \exp(-tA) \cdot B(t)$, which we now determine.
To compute the matrix exponential we determine the eigenvalues: since $\mathrm{tr}(A) = 0$ and $\det(A) = -1$, they are ± 1. Let us determine the corresponding eigenspaces:

$$E_1(A) = \ker(A - I_2) = \ker \begin{pmatrix} 1 & 3 \\ -1 & -3 \end{pmatrix} = \mathbb{R} \begin{pmatrix} 3 \\ -1 \end{pmatrix};$$

and

$$E_{-1}(A) = \ker(A + I_2) = \ker \begin{pmatrix} 3 & 3 \\ -1 & -1 \end{pmatrix} = \mathbb{R} \begin{pmatrix} 1 \\ -1 \end{pmatrix}.$$

So let us introduce the coordinate change matrix:

$$P = \begin{pmatrix} 3 & 1 \\ -1 & -1 \end{pmatrix}$$

which has inverse

$$P^{-1} = -\frac{1}{2} \begin{pmatrix} -1 & -1 \\ 1 & 3 \end{pmatrix} = \frac{1}{2} \begin{pmatrix} 1 & 1 \\ -1 & -3 \end{pmatrix}.$$

Now PAP^{-1} and $P(tA)P^{-1}$ are diagonal, with:

$$P(tA)P^{-1} = \begin{pmatrix} t & \\ & -t \end{pmatrix},$$

the exponential of which is easy to guess, so that:

$$\begin{aligned} \exp(tA) &= P^{-1} \begin{pmatrix} e^t & \\ & e^{-t} \end{pmatrix} P \\ &= \frac{1}{2} \begin{pmatrix} 1 & 1 \\ -1 & -3 \end{pmatrix} \begin{pmatrix} e^t & \\ & e^{-t} \end{pmatrix} \begin{pmatrix} 3 & 1 \\ -1 & -1 \end{pmatrix} \\ &= \frac{1}{2} \begin{pmatrix} 1 & 1 \\ -1 & -3 \end{pmatrix} \begin{pmatrix} 3e^t & e^t \\ -e^{-t} & -e^{-t} \end{pmatrix} \\ &= \frac{1}{2} \begin{pmatrix} 3e^t - e^{-t} & e^t - e^{-t} \\ 3(e^{-t} - e^t) & 3e^{-t} - e^t \end{pmatrix} \end{aligned}$$

solutions

We then get (mind the $-t$, which is what we want):

$$\exp(-tA) \cdot B(t) = \frac{1}{2} \begin{pmatrix} -e^t + 3e^{-t} & -e^t + e^{-t} \\ 3(e^t - e^{-t}) & 3e^t - e^{-t} \end{pmatrix} \cdot \begin{pmatrix} 1 \\ b(t) \end{pmatrix}$$

Integrating formally one then gets $\Lambda(t)$ and $F_1(t) = \exp(tA) \cdot \Lambda(t)$ explicitly. One can check, with patience, that this is a solution.
Finally, the general solution of $(\mathcal{E})$ has the form $F_1(t) + \exp(tA) \cdot F_0$ for $F_0 \in \mathbb{R}^2$.

Exercise 3. Determine the exponential of the following matrix:

$$A = \begin{pmatrix} 2 & 1 & -1 \\ -1 & 0 & 1 \\ 0 & 0 & 2 \end{pmatrix}$$

Solution. We first determine the spectre by computing the characteristic polynomial, which we obtain by expanding the first column:

$$\begin{aligned} \chi_A(\lambda) = |A - \lambda I_3| &= (2-\lambda) \cdot \begin{vmatrix} -\lambda & 1 \\ 0 & 2-\lambda \end{vmatrix} + \begin{vmatrix} 1 & -1 \\ 0 & 2-\lambda \end{vmatrix} \\ &= -\lambda \cdot (2-\lambda)^2 + (2-\lambda) = (2-\lambda)(1 - 2\lambda + \lambda^2) \\ &= -(\lambda - 2)(\lambda - 1)^2 \end{aligned}$$

At this stage it is unclear whether the matrix will be diagonalisable, but this does not look favorable. And a computation confirms this:

$$E_1(A) = \ker \begin{pmatrix} 1 & 1 & -1 \\ -1 & -1 & 1 \\ 0 & 0 & 1 \end{pmatrix} = \ker \begin{pmatrix} 1 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} = \mathbb{R} \begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix}$$

which is 1-dimensional, whereas the algebraic multiplicity of the eigenvalue 1 is equal to 2. Still, we let $v_1 = \begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix}$.
We have to dig deeper and investigate:

$$\ker(A - I_3)^2 = \ker \begin{pmatrix} 0 & 0 & -1 \\ 0 & 0 & 1 \\ 0 & 0 & 1 \end{pmatrix} = \mathrm{Vect} \left(\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} \right)$$

So the vector $v_2 = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$ lies in the second kernel, but not in the first (in symbols, $v_2 \in \ker(A - I_3)^2 \setminus \ker(A - I_3)$). Notice that there are many possible choices for v_2.
We finally turn to:

$$E_2(A) = \ker \begin{pmatrix} 0 & 1 & -1 \\ -1 & -2 & 1 \\ 0 & 0 & 0 \end{pmatrix} = \mathbb{R} \begin{pmatrix} 1 \\ -1 \\ -1 \end{pmatrix}$$

and we let $v_3 = \begin{pmatrix} 1 \\ -1 \\ -1 \end{pmatrix}$.
All the above suggests to introduce the coordinate change matrix from the standard basis to basis (v_1, v_2, v_3):

$$P = \begin{pmatrix} 1 & 1 & 1 \\ -1 & 0 & -1 \\ 0 & 0 & -1 \end{pmatrix}$$

Solutions

which has inverse:

$$P^{-1} = -\begin{pmatrix} 0 & 1 & -1 \\ -1 & -1 & 0 \\ 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} 0 & -1 & 1 \\ 1 & 1 & 0 \\ 0 & 0 & -1 \end{pmatrix}$$

It is a fact only discretely suggested in class that $P^{-1}AP$ has a decent form; let us check it here:

$$\begin{aligned} P^{-1}AP &= \begin{pmatrix} 0 & -1 & 1 \\ 1 & 1 & 0 \\ 0 & 0 & -1 \end{pmatrix} \begin{pmatrix} 2 & 1 & -1 \\ -1 & 0 & 1 \\ 0 & 0 & 2 \end{pmatrix} \begin{pmatrix} 1 & 1 & 1 \\ -1 & 0 & -1 \\ 0 & 0 & -1 \end{pmatrix} \\ &= \begin{pmatrix} 0 & -1 & 1 \\ 1 & 1 & 0 \\ 0 & 0 & -1 \end{pmatrix} \begin{pmatrix} 1 & 2 & 2 \\ -1 & -1 & -2 \\ 0 & 0 & -2 \end{pmatrix} \\ &= \begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 2 \end{pmatrix} \end{aligned}$$

The latter rewrites as $D + N$, where:

$$D = \begin{pmatrix} 1 & & \\ & 1 & \\ & & 2 \end{pmatrix} \quad \text{and} \quad N = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

are diagonal, resp. nilpotent ($N^2 = 0$), and commute.
From this we obtain:

$$\exp(P^{-1}AP) = \exp(D) \cdot \exp(N) = \begin{pmatrix} e & & \\ & e & \\ & & e^2 \end{pmatrix} \cdot \begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} e & e & 0 \\ 0 & e & 0 \\ 0 & 0 & e^2 \end{pmatrix}$$

It remains to compute:

$$\begin{aligned} \exp(A) = P\exp(P^{-1}AP)P^{-1} &= \begin{pmatrix} 1 & 1 & 1 \\ -1 & 0 & -1 \\ 0 & 0 & -1 \end{pmatrix} \begin{pmatrix} e & e & 0 \\ 0 & e & 0 \\ 0 & 0 & e^2 \end{pmatrix} \begin{pmatrix} 0 & -1 & 1 \\ 1 & 1 & 0 \\ 0 & 0 & -1 \end{pmatrix} \\ &= \begin{pmatrix} 1 & 1 & 1 \\ -1 & 0 & -1 \\ 0 & 0 & -1 \end{pmatrix} \begin{pmatrix} e & 0 & e \\ e & e & 0 \\ 0 & 0 & -e^2 \end{pmatrix} \\ &= \begin{pmatrix} 2e & e & e - e^2 \\ -e & 0 & e^2 - e \\ 0 & 0 & e^2 \end{pmatrix} \end{aligned}$$

solutions