Differential Equations
Final test

Exercise 1.

1. State Peano's theorem.
2. State the Cauchy-Lipschitz theorem.

Exercise 2. Let $\alpha \in \mathbb{R}$. For $x \neq 0$, let $f_\alpha(x) = |x|^\alpha$; also let $f_\alpha(0) = 0$.

1. Sketch the graphs of $f_2, f_{\frac{1}{2}}, f_{-1}$.
2. Let $\alpha > 1$. Is f_α continuous at every point? locally Lipschitz around every point? globally Lipschitz? C^1?
3. Now let $\alpha \in (0,1)$. Same questions.
4. Finally suppose $\alpha < 0$. Same questions.
5. Now consider the *scalar* Cauchy problem:

$$\begin{cases} x'(t) & = & |x(t)|^\alpha, \\ x(0) & = & 0 \end{cases}$$

   Discuss existence and uniqueness of solutions depending on the value of α.

   Do not forget the cases $\alpha = 1$ and $\alpha = 0$.

This is an exercise on the theory: do not try to solve solveexplicitly.

Exercise 3.

1. Explain briefly Euler's method.
2. Notice that the map $x(t) = t^{3/2}$ is a solution to the following Cauchy problem:

$$x'(t) = \frac{3}{2}(x(t))^{1/3}, \qquad \text{with } x(0) = 0$$

   Take any integer n and step $h = \frac{1}{n}$.

   Show that the Euler method gives the null function.

3. Can you explain this?

Exercise 4. Solve the following coupled system:

$$\begin{cases} f'(t) & = & 2f(t) + g(t) + e^t \\ g'(t) & = & -f(t) + e^t \end{cases}$$

with initial condition:

$$f(0) = g(0) = 0$$

Be careful: computations can be long, you should treat this exercise last. If you have little time left, you may skip some computations and just explain your methods. And since I am a nice guy, you should find:

$$\exp(tA) = e^t \begin{pmatrix} 1+t & t \\ -t & 1-t \end{pmatrix}$$