Differential Equations
Final test: answers.

Exercise 1.

1. State Peano's theorem.

2. State the Cauchy-Lipschitz theorem.

Solution. See your lecture notes!

Exercise 2. Let $\alpha \in \mathbb{R}$. For $x \neq 0$, let $f_\alpha(x) = |x|^\alpha$; also let $f_\alpha(0) = 0$.

1. Sketch the graphs of $f_2, f_{\frac{1}{2}}, f_{-1}$.

2. Let $\alpha > 1$. Is f_α continuous at every point? locally Lipschitz around every point? globally Lipschitz? C^1?

3. Now let $\alpha \in (0, 1)$. Same questions.

4. Finally suppose $\alpha < 0$. Same questions.

5. Now consider the *scalar* Cauchy problem:

$$\begin{cases} x'(t) & = & |x(t)|^\alpha, \\ x(0) & = & 0 \end{cases}$$

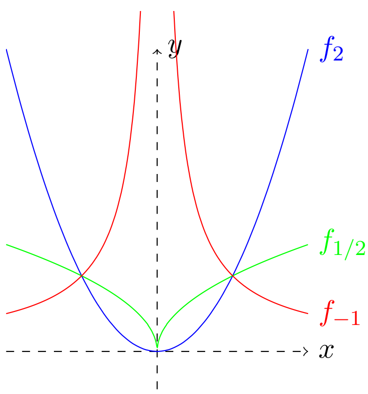
   Discuss existence and uniqueness of solutions depending on the value of α.

   Do not forget the cases $\alpha = 1$ and $\alpha = 0$.

This is an exercise on the theory: do not try to solve solveexplicitly.

Solution.

1.



2. Suppose $\alpha \geq 1$. On $(0, +\infty)$, the map f_α is given by:

$$f_\alpha(x) = x^\alpha = e^{\alpha \log x}.$$

   Since $\alpha \geq 0$, as $x \to 0^+$ we have $\log x \to -\infty$ and therefore:

$$\lim_{x \to 0^+} f_\alpha(x) = 0.$$

A similar analysis is valid on $(-\infty, 0)$, where f_α is given by $f_\alpha(x) = (-x)^\alpha = e^{\alpha \log(-x)}$. Since $f_\alpha(0) = 0$, we deduce continuity at 0..

On $(0, +\infty)$, the map f_α is even infinitely differentiable, with derivative:

$$f'_\alpha(x) = \alpha \frac{1}{x} e^{\alpha \log x} = \alpha x^{\alpha-1} = \alpha f_{\alpha-1}(x).$$

Moreover, for $h > 0$:

$$\frac{f_\alpha(h) - f_\alpha(0)}{h} = f_{\alpha-1}(h)$$

and since $\alpha - 1 \geq 0$, we know it has limit 0 as $h \to 0^+$. Since a similar analysis is possible on $(-\infty, 0)$, we find that f_α is differentiable at 0 with derivative $f'(0) = 0$.

But $\lim_{x\to 0} f'_\alpha(x) = 0$ too, so f_α is even C^1 on $\mathbb{R}$.

It is then of course locally Lipschitz around every point, but it is not globally Lipschitz: as one sees, the derivative goes to $+\infty$ at $+\infty$.

3. Suppose $\alpha \in (0, 1)$. The continuity argument remains valid; f_α is continuous on $\mathbb{R}$, and even C^∞ on both $(-\infty, 0) \cup (0, +\infty)$ (hence locally Lipschitz), but it is no longer differentiable at 0; as a matter of fact it is not even Lipschitz around 0.

   Suppose it is, say that it is k-Lipschitz on $(-\varepsilon, \varepsilon)$. Then for all $x, y \in [0, \varepsilon) \subseteq (-\varepsilon, \varepsilon)$, one has:

   $$|f_\alpha(x) - f_\alpha(y)| \leq k|x - y|$$

   By continuity one may let $y \to 0$ and find:

   $$\forall x \in [0, \varepsilon), \quad x^\alpha \leq kx$$

   This is however not the case. Indeed, as $\alpha < 1$, we know that $x^{\alpha-1} \to +\infty$ as $x \to 0^+$ (there is a proof in the next question anyway). So there exists $x \in [0, \varepsilon)$ such that $x^{\alpha-1} > k$, against the previous inequality.

   As a conclusion, f_α is not Lipschitz around 0.

4. Suppose $\alpha < 0$. The map is no longer continuous at 0 since:

   $$\lim_{x\to 0^+} \alpha \log x = +\infty,$$

   implying $\lim_{x\to 0^+} f_\alpha(x) = +\infty \neq 0 = f_\alpha(0)$.

5. The problem is well-posed as $x'(t) = f(x(t))$ with $x(0) = 0$.

   If $\alpha > 1$ then f_α is locally Lipschitz around 0 (since it is C^1 there): by the Cauchy-Lipschitz theorem, we have existence *and* uniqueness of a local solution.

   When $\alpha = 1$, the map is globally Lipschitz! so we actually have existence and uniqueness of a global solution.

   If $\alpha \in (0, 1)$ the map is only continuous, so by Peano's theorem there always exists a solution but uniqueness is lost.

   The case $\alpha = 0$ is not extremely interesting: $f_\alpha(x) = 1$, a constant map, so certainly the Cauchy-Lipschitz theorem applies: existence and uniqueness of a global solution again.

   If $\alpha < 0$, the map is not continuous at 0; it seems hard to predict behaviour of the equation.

Exercise 3.

1. Explain briefly Euler's method.

2. Notice that the map $x(t) = t^{3/2}$ is a solution to the following Cauchy problem:

$$x'(t) = \frac{3}{2}(x(t))^{1/3}, \qquad \text{with } x(0) = 0$$

   Take any integer n and step $h = \frac{1}{n}$.

   Show that the Euler method gives the null function.

3. Can you explain this?

Solution.

1. See you lecture notes.

2. Let us discretise and consider $t_i = \frac{i}{n}$.

   We also let $x_0 = 0$, then $x_{i+1} = x_i + h\frac{3}{2}(x_i)^{1/3}$. A quick induction shows $x_i = 0$ for all i. In particular, the Euler method gives the zero function.

3. It so happens that the null function is another solution to the initial value problem.

   Because the evolution map is continuous not locally Lipschitz around the initial value $x_0 = 0$ (see the previous exercise for a proof), solutions do exist by Peano's theorem but may not be expected to be unique.

Exercise 4. Solve the following coupled system:

$$\begin{cases} f'(t) &= 2f(t) + g(t) + e^t \\ g'(t) &= -f(t) + e^t \end{cases}$$

with initial condition:

$$f(0) = g(0) = 0$$

Be careful: computations can be long, you should treat this exercise last. If you have little time left, you may skip some computations and just explain your methods. And since I am a nice guy, you should find:

$$\exp(tA) = e^t \begin{pmatrix} 1+t & t \\ -t & 1-t \end{pmatrix}$$

Solution. We first converted the coupled system into one vector equation. Let $X(t) = \begin{pmatrix} f(t) \\ g(t) \end{pmatrix}$; also let $A = \begin{pmatrix} 2 & 1 \\ -1 & 0 \end{pmatrix}$ and $B(t) = \begin{pmatrix} e^t \\ e^t \end{pmatrix}$.

Then the problem amounts to solving:

$$X'(t) = A \cdot X(t) + B(t)$$

with initial condition:

$$X(0) = \begin{pmatrix} 0 \\ 0 \end{pmatrix}.$$

By the Cauchy-Lipschitz theorem, there is a unique solution (which is even defined on $\mathbb{R}$). As usual, our strategy is to:

1. First solve the associated equation $(\mathcal{E}_H) : X'(t) = A \cdot X(t)$;

2. then solve $(\mathcal{E}) : X'(t) = A \cdot X(t) + B(t)$;

3. finally take the initial condition into account, to find the only solution.

Let us do it.

1. Such a question begs for matrix exponentials. Since $\det A = 1$ and $\mathrm{Tr}(A) = 2$, it should be clear that A has as double eigenvalue 1. It is however not the identity matrix, so it is not diagonalisable. Let us bring it into triangular form;

$$\ker(A - I) = \ker \begin{pmatrix} 1 & 1 \\ -1 & -1 \end{pmatrix} = \mathbb{R} \begin{pmatrix} 1 \\ -1 \end{pmatrix}$$

   So $\begin{pmatrix} 1 \\ -1 \end{pmatrix}$ is an eigenvector; of course $(A - I)^2 = 0$ so completing this into a suitable coordinate change matrix is a trivial matter.

   Let $P = \begin{pmatrix} 1 & 0 \\ -1 & 1 \end{pmatrix}$; we do this because we are happy with triangular matrices. The inverse is obviously $P^{-1} = \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}$; if necessary one may check that:

$$P^{-1}AP = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$$

   It is then obvious that:

$$\begin{aligned}
\exp(tA) &= P \exp \begin{pmatrix} t & t \\ 0 & t \end{pmatrix} P^{-1} \\
&= \begin{pmatrix} 1 & 0 \\ -1 & 1 \end{pmatrix} \cdot \begin{pmatrix} e^t & te^t \\ 0 & e^t \end{pmatrix} \cdot \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix} \\
&= \begin{pmatrix} e^t & te^t \\ -e^t & (1-t)e^t \end{pmatrix} \cdot \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix} \\
&= \begin{pmatrix} (1+t)e^t & te^t \\ -te^t & (1-t)e^t \end{pmatrix} \\
&= e^t \begin{pmatrix} 1+t & t \\ -t & 1-t \end{pmatrix}
\end{aligned}$$

   All solutions of $(\mathcal{E}_H)$ have the form $\exp(tA) \cdot X_0$ for $X_0 \in \mathbb{R}^2$.

2. We now turn to equation $(\mathcal{E})$. As we know, the solution set is an affine space; to determine it one element is enough. Let us look for it in the form $\exp(tA) \cdot X_1(t)$, where $X_1(t)$ is a differentiable map.

   For this to fulfil the equation, one needs:

$$A \exp(tA) X_1(t) + \exp(tA) X_1'(t) = A \exp(tA) X_1(t) + B(t),$$

   or equivalently $X_1'(t) = \exp(-tA) \cdot B(t)$.

   It it time to compute a bit. One therefore has:

$$\begin{aligned}
X_1'(t) &= \exp(-tA) \cdot B(t) \\
&= e^{-t} \begin{pmatrix} 1-t & -t \\ t & 1+t \end{pmatrix} \cdot \begin{pmatrix} e^t \\ e^t \end{pmatrix} \\
&= \begin{pmatrix} 1-t & -t \\ t & 1+t \end{pmatrix} \cdot \begin{pmatrix} 1 \\ 1 \end{pmatrix} \\
&= \begin{pmatrix} 1-2t \\ 1+2t \end{pmatrix}
\end{aligned}$$

Integrating, we can let:

$$X_1(t) = \begin{pmatrix} t - t^2 \\ t + t^2 \end{pmatrix}$$

so that, again:

$$\begin{aligned} X(t) &= \exp(tA) \cdot X_1 \\ &= e^t \begin{pmatrix} 1+t & t \\ -t & 1-t \end{pmatrix} \cdot \begin{pmatrix} t - t^2 \\ t + t^2 \end{pmatrix} \\ &= e^t \begin{pmatrix} t + t^2 \\ t - t^2 \end{pmatrix} \end{aligned}$$

One may check that this is a solution.

As a conclusion, all solutions of $(\mathcal{E})$ have the form:

$$e^t \begin{pmatrix} 1+t & t \\ -t & 1-t \end{pmatrix} \cdot X_0 + e^t \begin{pmatrix} t + t^2 \\ t - t^2 \end{pmatrix}$$

3. For this to satisfy $X(0) = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$, we must have:

$$\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \cdot X_0 + \begin{pmatrix} 0 \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix},$$

i.e. $X_0 = 0$.

As a conclusion, the only solution to the equation with initial condition is:

$$X(t) = e^t \begin{pmatrix} t + t^2 \\ t - t^2 \end{pmatrix}$$