Differential Equations
Assignment #2

Exercise 1. Solve the equation on $\mathbb{R}$

$$F'(t) = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} F(t) + \begin{pmatrix} e^t \\ 0 \end{pmatrix}$$

with the initial condition

$$F(0) = \begin{pmatrix} 1 \\ 1 \end{pmatrix}.$$

Exercise 2. We consider the following ordinary differential equation

$$(\mathcal{E}) \qquad \begin{cases} u'(t) &= \frac{5}{3}u^{2/5} \\ u(0) &= 0 \end{cases}.$$

1. Prove that the solution set of $(\mathcal{E})$ is infinite. Provide a family of solutions.
2. Does this contradict the **Cauchy-Lipschitz theorem**? Justify!

Exercise 3.

1. Let $n \in \mathbb{Z} \setminus \{0, 1\}$; also let $a, b : \mathbb{R} \to \mathbb{R}$ be continuous maps, and consider the equation:

   $$x'(t) + a(t)x(t) + b(t)(x(t))^n = 0$$

   where $(x(t))^n$ is the n^{th} power of $x(t)$ (*not* the derivative).

   Suppose that x is a solution *that remains positive*. Let $y(t) = (x(t))^{1-n}$ and show that y satisfies a *linear* equation.

2. Solve

   $$tx'(t) + x(t) - t(x(t))^3 = 0$$

   on each interval where x keeps a constant sign.

Exercise 4 (some theory)**.** Let $A : I \to M_d(\mathbb{R})$ have the property:

$$\forall (t_1, t_2) \in I^2 \quad A(t_1) \cdot A(t_2) = A(t_2) \cdot A(t_1)$$

Prove that the unique solution of the differential equation on I with initial condition:

$$X'(t) = A(t) \cdot X(t), \qquad \text{with } X(t_0) = X_0$$

has the form:

$$X(t) = \exp\left(\int_{t_0}^{t} A(s)ds\right) \cdot X_0$$

Hint: prove that $\int_{t_0}^{t} A(s)ds$ and $\int_{t}^{t+h} A(s)ds$ commute.