An algorithm computing genus-one invariants of the Landau–Ginzburg model of the quintic Calabi–Yau three-fold

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(joint work with Yongbin Ruan and Dimitri Zvonkine)

The computation of the Gromov–Witten theory of compact Calabi–Yau varieties (in all genera) has been a central problem in mathematics and physics for the last twenty years. The genus-zero theory has been known since the middle 90's by the celebrated mirror theorems of Givental and Lian–Liu–Yau. On the other hand, the computation of the higher genus theory for compact Calabi–Yau manifolds is unfortunately a hard problem for both mathematicians and physicists. The genus-one theory was computed by Zinger almost ten years ago after a great deal of hard work. So far, the computations in the genus $g \geq 2$ are out of mathematicians' reach.

Almost twenty years ago, a far-reaching correspondence was proposed to connect two areas of physics, the Landau–Ginzburg (LG) model and Calabi–Yau (CY) geometry [21] [22]. During the last ten years, the LG/CY correspondence has been investigated extensively in mathematics. For example, a mathematical LG/CY correspondence conjecture was formulated. It asserts that Gromov–Witten theory of a Calabi-Yau hypersurface within a weighted projective space is equivalent (via an analytic continuation) to its Landau–Ginzburg counterpart: the theory of Fan, Jarvis and Ruan of the isolated singularity of the corresponding affine cone. We mention several different recent approaches to this enumerative geometry of the Landau–Ginzburg model, see for instance Polishchuk–Vaintrob [19] and Chang–Li–Li [1]; the main result presented here can be phrased in equivalent terms in each of these setups.

It is generally believed that the Landau–Ginzburg side is easier to compute. In this perspective, the LG/CY correspondence provides a possible strategy to compute higher genus Gromov–Witten invariants of Calabi–Yau hypersurface. The genus zero FJRW theory has been computed and its LG/CY correspondence has been verified in many examples [5, 6, 8, 10, 13, 15, 17]. Now, the attention is shifted to higher genus case. We can divide the above strategy into two problems: (1) computing higher genus FJRW invariants and (2) establishing the LG/CY correspondence.

In this text, issued from a collaboration with Y. Ruan and D. Zvonkine, we illustrate an effective procedure to take a step forward in the first direction: the computation of genus-one FJRW invariants. For simplicity, we state our formula only for the invariants of the isolated singularity $W = 0$ attached to a degree-5 homogeneous Fermat polynomial in five variables. The approach extends automatically to homogeneous polynomials of any degree; even under the Fermat condition some modifications seem to be needed for more general weighted homogeneous polynomials; furthermore, the Fermat condition plays a key in the approach presented here. We consider the Landau–Ginzburg model $W\colon [\mathbb{C}^5/\boldsymbol{\mu}_5] \to \mathbb{C}$ where $\zeta \in \boldsymbol{\mu}_5$ acts on $\mathbb{C}^5$ linearly as $\zeta\mathbb{I}_5$; this model, under the LG/CY correspondence, is the LG counterpart of the quintic CY three-fold in $\mathbb{P}^4$.

The relevant quantum invariants are intersection numbers $\langle \phi_{i_1}, \dots, \phi_{i_n} \rangle_{g,n}$ attached to n cohomology classes ϕ_{i_j} in the relative orbifold Chen–Ruan cohomology of the pair $([\mathbb{C}^5/\boldsymbol{\mu}_5], M)$, where M is the Milnor fibre of W over a point of $\mathbb{C}^\times$. By [6], since W defines a Calabi–Yau three-fold $X \subset \mathbb{P}^4$ we have a degree-preserving identification with the cohomology of X. Here, we restrict to the much simpler narrow sector corresponding to the ambient cohomology of X, the subring spanned by the restriction of the hyperplane class H from $\mathbb{P}^4$. This is a 4-dimensional subspace whose entries are labelled by the four primitive 5th roots of unity. As in [5] we can restrict to the case where all the entries are given by the class ϕ_2 corresponding to $\exp(2\pi\mathtt{i}\frac{2}{5})$, the counterpart of the hyperplane class H in X.

In genus $g = 1$, we are led to compute $\langle \phi_2, \dots, \phi_2, \phi_2 \rangle_{1,n}$ where n, the number of markings, is a multiple of 5: $n = 5k$. The relevant invariants boil down to the 5-fold intersection of a virtual cycle defined on the moduli space $\mathrm{Spin} := \mathrm{Spin}^5_{1,5k}(2, \dots, 2)$ of 5-spin, genus-1, $5k$-pointed curves of type $(2, \dots, 2)$: the moduli space compactifying the space of smooth curves $[C, x_1, \dots, x_{5k}] \in \mathcal{M}_{1,5k}$ equipped with a 5-spin structure L

$$L^{\otimes 5} \xrightarrow{\cong} \omega_{C,\log}(-\textstyle\sum_{i=1}^{5k} 2[x_i]) = \omega_C(-\sum_{i=1}^{5k}[x_i]).$$

On the open part parametrising smooth curves, the virtual cycle c_{vir} is simply the top Chern class of the rank-k vector bundle $R^1\pi_*\mathcal{L}$ where $\pi\colon \mathcal{U} \to \mathrm{Spin}$ is the universal curve with the universal

5-spin structure $\mathcal{L}$. Over the whole moduli space Spin there are several compatible definitions of c_{vir} which satisfy the composition axioms of cohomological field theory, see [1,2,11,18]. Then, the relevant intersection numbers in genus one are the rational numbers

$$\langle \phi_2, \dots, \phi_2, \phi_2 \rangle_{1,5k} = \int_{[\mathrm{Spin}^5_{1,5k}(2,\dots,2)]} (c_{\mathrm{vir}})^5 \in \mathbb{Q}$$

for any positive integer k (by an easy dimension count $5k = \dim \overline{\mathcal{M}}_{1,5k} = \dim \mathrm{Spin}$ equals $5\deg(c_{\mathrm{vir}})$ because $\deg(c_{\mathrm{vir}}) = -\chi(L) = -\deg(L) = k$).

It seems natural to attempt to define c_{vir} by means of the natural kth Chern class $c_k^5 := c_k(-R\pi_*\mathcal{L})$. However the identification $c_{\mathrm{vir}} = c_k$ does not extend to the entire compactified space Spin, although it holds on the locus parametrising smooth curves. Indeed the class c_k fails to satisfy the cohomological field theory properties. A close look to the definition of c_{vir} shows that c_k differs from c_{vir} precisely on the closed locus Z of *effective* 5-spin structures: $(C, x_1, \dots, x_n, L)$ for which $h^0(L) > 0$. These are all the curves arising from specialisations of the star-shaped nodal curve where all the markings lie on rational tails each one bearing a number of markings which is divisible by 5. All tails are joined by a separating node to a genus-1 subcurve lying in the middle. The locus Z is closed but not irreducible unless $k = 1$.

A slight modification of such definition coincides with c_{vir} on Z, but fails to match c_{vir} on $\Omega = \mathrm{Spin} \setminus Z$. Indeed, we can set

$$\widetilde{c}_{\mathrm{vir}} := \left[c\left(-R\pi_* L\right) \left(-4 - 4 \sum_{i>0} \left(\frac{4}{5} \lambda \right)^i \right) \right]_k ,$$

where $[\cdots]_k$ denotes the degree-k part in the Chow ring and λ is the Hodge class $c_1(\pi_*\omega)$; we get $\widetilde{c}_{\mathrm{vir}}|_Z = c_{\mathrm{vir}}|_Z$. This happens essentially by the composition axiom of the cohomological field theory of 5-spin curves which allows to express virtual cycles of family of spin curves joined by separating (narrow) nodes in terms of products of virtual cycles coming from each component of the normalisation of the separating nodes. On Z, the higher direct image of L splits into two parts, one coming from the tails of the form $[0 \to B]$ and one of the form $[L_0 \to L_1]$, coming from the middle genus-one subcurve. Here L_0 and L_1 are line bundles related to the Hodge bundle. The above term $-4\left(1 + 1\sum_{i>0}\left(\frac{4}{5}\lambda\right)^i\right)$ is the genus-1 virtual cycle in absence of markings, *i.e.* -4 times the fundamental class, multiplied by the inverse of the total Chern class of $-[L_0 \to L_1]$ (it is not difficult to express this term by means of the first Chern class of the Hodge bundle). In this way, on Z, we finally get the virtual class, *i.e.* $-4c_{\mathrm{top}}(B)$.

We now write

$$(c_{\mathrm{vir}})^5 = (c_{\mathrm{vir}})^5 - (c_k)^5 + (c_k)^5 = \left(\sum_{i=0}^4 (c_{\mathrm{vir}})^i (c_k)^{4-i} \right) (c_{\mathrm{vir}} - c_k) + (c_k)^5$$

and we notice that, since $c_{\mathrm{vir}} - c_k$ is supported on Z, we can replace $\sum_{i=0}^4 (c_{\mathrm{vir}})^i (c_k)^{4-i}$ by $\sum_{i=0}^4 (\widetilde{c}_{\mathrm{vir}})^i (c_k)^{4-i}$. Then, we get the following formula.

Theorem 1. *We have*

$$\int_{[\mathrm{Spin}^5_{1,5k}(\underline{2})]} (c_{\mathrm{vir}})^5 = \int_{[\mathrm{Spin}^5_{1,5k}(\underline{2})]^{\mathrm{vir}}} \sum_{i=0}^4 (\widetilde{c}_{\mathrm{vir}})^i (c_k)^{4-i} - \int_{[\mathrm{Spin}^5_{1,5k}(\underline{2})]} \sum_{i=1}^4 (\widetilde{c}_{\mathrm{vir}})^i (c_k)^{5-i},$$

where the second summand is an integration of tautological classes determined by Grothendieck–Riemann–Roch (see [4]*) and the first summand is a polynomial in the same tautological classes over the cycle* $[\mathrm{Spin}^5_{1,5k}(\underline{2})]^{\mathrm{vir}} := c_{\mathrm{vir}} \cap [\mathrm{Spin}^5_{1,5k}(\underline{2})]$ *and can be computed in terms of* 5*-spin invariants via* [12,16].

The classes c_k and $\widetilde{c}_{\mathrm{vir}}$ can be expressed in Givental's formalism

$$c_k = \left[\exp\left\{ \sum_{i>0} s_i \mathrm{ch}_i(R\pi_*\mathcal{L}) \right\} \right]_k ,$$

and

$$\widetilde{c}_{\text{vir}} = -4 \left[\exp \left\{ \sum_{i>0} s_i \text{ch}_i(R\pi_* \mathcal{L}) + \widetilde{s}_i \text{ch}_i(R\pi_* \omega) \right\} \right]_k ,$$

for $s_i = (-1)^i (i-1)!$ and $\widetilde{s}_i = s_i \frac{4^i - 1}{5^i}$. By [16], the virtual cycle $[\text{Spin}^5_{1,5k}(\underline{2})]^{\text{vir}}$ also admits an expression in Givental formalism. The possibility of using Givental's formalism is an important advantage on any other approaches to the quintic threefold in higher genus. It should be also noticed that the above formula holds both in genus zero and genus one since λ vanishes in genus zero.

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