Université de Paris

Master Mathématiques fondamentales, 2e année, 2e semestre
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COHOMOLOGY OF COHERENT SHEAVES ON SCHEMES

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The exercises are independent one of another; you may solve them in any order. You may write your solution in English or in French. You may consult handwritten notes from the lectures.

EXERCICE 1

If A is a ring and M is an A-module, an element $a \in \mathrm{A}$ is said to be regular in M if the homothety $(a)_{\mathrm{M}}$ is injective. The support of M, its annihilator, its set of associated prime ideals are respectively denoted by $\mathrm{Supp}_{\mathrm{A}}(\mathrm{M})$, $\mathrm{Ann}_{\mathrm{A}}(\mathrm{M})$ and $\mathrm{Ass}_{\mathrm{A}}(\mathrm{M})$,

1 Let A be a ring and let M be an A-module. Let $a \in \mathrm{A}$ be an element which is not regular in M. Prove that there exists a prime ideal $\mathrm{P} \in \mathrm{Ass}_{\mathrm{A}}(\mathrm{M})$ such that $a \in \mathrm{P}$.

2 Let A be a ring, let $\mathrm{P}_1, \dots, \mathrm{P}_m$ be prime ideals of A, let $x_1, \dots, x_n \in \mathrm{M}$ and let $\mathrm{N} = \langle x_1, \dots, x_n \rangle$. We assume that for every $j \in \{1, \dots, m\}$, $\mathrm{N}_{\mathrm{P}_j} \not\subset \mathrm{P}_j \mathrm{M}_{\mathrm{P}_j}$. Prove that there exist $a_2, \dots, a_n \in \mathrm{A}$ such that for every $j \in \{1, \dots, m\}$, $x_1 + \sum_{i=2}^{n} a_i x_i \notin \mathrm{P}_j \mathrm{M}_{\mathrm{P}_j}$. (*Argue by induction on m.*)

In the rest of the exercise, we consider a noetherian ring A and finitely generated A-modules M, N.

3 Prove that $\mathrm{Ass}_{\mathrm{A}}(\mathrm{M}) \cap \mathrm{Supp}(\mathrm{N}) = \varnothing$ if and only if $\mathrm{Hom}_{\mathrm{A}}(\mathrm{N}, \mathrm{M}) = 0$. (*Let* P *be a prime ideal of* A *such that* $\mathrm{P} \in \mathrm{Ass}_{\mathrm{A}}(\mathrm{M}) \cap \mathrm{Supp}_{\mathrm{A}}(\mathrm{N})$*; find non trivial morphisms* $\mathrm{N}_{\mathrm{P}} \to \mathrm{A}_{\mathrm{P}}/\mathrm{PA}_{\mathrm{P}}$ *and* $\mathrm{A}_{\mathrm{P}}/\mathrm{PA}_{\mathrm{P}} \to \mathrm{M}_{\mathrm{P}}$.)

4 Let I be an ideal of A such that no element of I is regular in M. Prove that there exists a prime ideal $\mathrm{P} \in \mathrm{Ass}_{\mathrm{A}}(\mathrm{M})$ such that $\mathrm{I} \subset \mathrm{P}$.

5 Prove that $\mathrm{Hom}_{\mathrm{A}}(\mathrm{N}, \mathrm{M}) = 0$ if and only if $\mathrm{Ann}_{\mathrm{A}}(\mathrm{N})$ contains an element which is regular in M.

EXERCICE 2

Let X be an affine scheme and let U be a quasi-compact open subscheme of X. Let $\mathrm{A} = \Gamma(\mathrm{X}, \mathscr{O}_{\mathrm{X}})$ and let $\mathrm{B} = \Gamma(\mathrm{U}, \mathscr{O}_{\mathrm{X}})$.

We assume that $\mathrm{H}^p(\mathrm{U}, \mathscr{O}_{\mathrm{X}}) = 0$ *for all* $p > 0$.

1 Let R be a ring, let $0 \to \mathrm{M}' \to \mathrm{M} \to \mathrm{M}'' \to 0$ be an exact sequence of R-modules and let N be an R-module. If M″ is flat, prove that the induced complex

$$0 \to \mathrm{M}' \otimes_{\mathrm{R}} \mathrm{N} \to \mathrm{M} \otimes_{\mathrm{R}} \mathrm{N} \to \mathrm{M}'' \otimes_{\mathrm{R}} \mathrm{N} \to 0$$

is exact.

2 Let $\mathscr{F}$ be a quasi-coherent sheaf on U and let $f \in \mathrm{A}$ be such that $\mathrm{D}(f) \subset \mathrm{U}$. Prove that the restriction map induces an isomorphism $\Gamma(\mathrm{U}, \mathscr{F})_f \simeq \Gamma(\mathrm{D}(f), \mathscr{F})$.

3 Let $f \in \mathrm{A}$ be such that $\mathrm{D}(f) \subset \mathrm{U}$. Prove that A_f is a flat B-module.

4 Prove that B is a flat A-module.

5 Prove that there exist $n \in \mathbf{N}$ and $f_1, \dots, f_n \in \Gamma(X, \mathscr{O}_X)$ such that $U = \bigcup_{i=1}^n D(f_i)$.

6 Denoting by $\mathscr{C}_{\mathscr{U}}(\mathscr{O}_X)$ the Čech complex of $\mathscr{O}_X$ associated with the open covering $\mathscr{U} = (D(f_i))_{1 \leqslant i \leqslant n}$ of U, prove that one has an exact sequence

$$0 \to B \to \mathscr{C}^0_{\mathscr{U}}(\mathscr{O}_X) \to \mathscr{C}^1_{\mathscr{U}}(\mathscr{O}_X) \to \dots$$

whose terms are flat B-modules.

7 Prove that this exact sequence stays exact after tensoring with any B-module M.

8 Let $\mathscr{F}$ be a quasi-coherent sheaf on U. Prove that $H^p(U, \mathscr{F}) = 0$ for all $p > 0$.

9 Prove that U is affine.

10 Give an example of a quasi-compact open subset U of a scheme X such that $H^p(U, \mathscr{O}_X) = 0$ for all $p > 0$ but such that U is not affine.

EXERCICE 3

Let A be a ring. For $a \in A$ and an A-module M, one denotes by $M[a]$ the submodule of elements $m \in M$ such that $am = 0$; one says that a is *regular* in M if $M[a] = 0$. One says that a is regular if $A[a] = 0$.

1 Let M be a flat A-module. Prove that $M[a] = 0$ for any regular element $a \in A$. Give an example of an A-module M where a is regular but $M[a] \neq 0$.

2 Let X be a scheme over Spec(A) and let $\mathscr{F}$ be a quasi-coherent sheaf on X which is flat over A. For any nonzero divisor $a \in A$, construct an exact sequence $0 \to \mathscr{F} \xrightarrow{a} \mathscr{F} \to \mathscr{F}/a\mathscr{F} \to 0$, where the arrow labeled a is induced by multiplication by a on sections.

3 Under the preceding hypotheses, construct exact sequences

$$0 \to H^p(X, \mathscr{F})/aH^p(X, \mathscr{F}) \to H^p(X, \mathscr{F}/a\mathscr{F}) \to H^{p+1}(X, \mathscr{F})[a] \to 0,$$

for all integers p.

We now assume that A is a discrete valuation ring (which is not a field). Let s and η be the points of Spec(A) corresponding respectively to the maximal ideal and the zero ideal of A; let a be a generator of the maximal ideal of A, let $k = \kappa(s) = A/(a)$ be the residue field of A and let $K = \kappa(\eta) = A_a$ be its fraction field.

4 Let M be a finitely generated A-module. Prove that

$$\dim_k(M \otimes_A k) - \dim_k(M[a]) = \dim_K(M \otimes_A K).$$

(*Use that* M *is of the form* L/L′, *where* L *is a free finitely generated* A*-module and* L′ *is a submodule of* L.)

5 Assume that X is projective over A. Prove (without using the results discussed in class) that

$$\dim_{\kappa(s)} H^p(X_s, \mathscr{F}_s) \geqslant \dim_{\kappa(\eta)} H^p(X_\eta, \mathscr{F}_\eta).$$