NON-ARCHIMEDEAN INTEGRALS AS LIMITS OF COMPLEX INTEGRALS

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Abstract

We explain how non-Archimedean integrals considered by Chambert-Loir and Ducros naturally arise in asymptotics of families of complex integrals. To perform this analysis, we work over a nonstandard model of the field of complex numbers, which is endowed at the same time with an Archimedean and a non-Archimedean norm. Our main result states the existence of a natural morphism between bicomplexes of Archimedean and non-Archimedean forms which is compatible with integration.

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1. Introduction

1.1

Chambert-Loir and Ducros [6] recently developed a full-fledged theory of real-valued (p,q)-forms and currents on Berkovich spaces that is an analogue of the theory of differential forms on complex spaces. Their forms are constructed as pullbacks under tropicalization maps of the "superforms" introduced by Lagerberg [19]. They are able

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to integrate compactly supported (n,n)-forms for n the dimension of the ambient space (the output being a real number), and they obtain versions of the Poincaré–Lelong theorem and Stokes's theorem in this setting. Their work is guided throughout by an analogy with complex analytic geometry. The aim of the present work is to convert the analogy into a direct connection, showing how the non-Archimedean theory appears as an asymptotic limit of one-parameter families of complex (Archimedean) forms and integrals.

One way to view a family of complex varieties as degenerating to a non-Archimedean space is to consider the hybrid spaces first introduced by Berkovich [2] to provide a non-Archimedean interpretation of the weight-0 part of the mixed Hodge structure on the cohomology of a proper complex variety. For some other recent applications of hybrid spaces, see [4], [7], and [9].

The approach we follow in this paper is somewhat different. We work over an algebraically closed field C containing $\mathbf{C}$, which is a degree-2 extension of a real closed field R containing $\mathbf{R}$ and is endowed at the same time with a nonstandard Archimedean norm $|\cdot| : C \to R_+$ and a non-Archimedean norm $|\cdot|_\flat : C \to \mathbf{R}_+$ that essentially encapsulates the "order of magnitude" of $|\cdot|$ with respect to a given infinitesimal element which should be thought of as a "complex parameter tending to zero." This presents the advantage of working on spaces that have both Archimedean and non-Archimedean features and allows one to directly compare Archimedean constructions and their non-Archimedean counterparts. The fields R and C are constructed using ultrapowers. The field R was introduced by Robinson in [25], with the explicit hope that it will be useful for asymptotic analysis (see also [20]). It was brought to good use in [18] following the fundamental work of van den Dries and Wilkie [28], who have reformulated Gromov's theory of asymptotic cones of metric spaces in [10] using ultrapowers.

A long-term motivation for our work is the famous conjecture by Kontsevich and Soibelman [16], [17] relating large scale complex geometry and non-Archimedean geometry. Roughly speaking, the conjecture describes the Gromov–Hausdorff limit of a family of complex Calabi–Yau varieties with maximal degeneration in terms of non-Archimedean geometry. (We refer to [11]–[13], [23], [24], and [26] for some recent results in that direction.) Note that our results involve a renormalization in powers of $\log|t|$ which corresponds to what appears naturally when considering volume forms on Calabi–Yau varieties with maximal degeneration. From a model-theoretic perspective, this is related to considering measures on certain definable sets over the value group, in contrast to [1], where measures are reduced to the residue field.

1.2

Before going further, it may be useful to provide the flavor of our main results on a very elementary example. Let $\varphi : \mathbf{R} \to \mathbf{R}$ be a smooth function with compact support. Consider the complex $(1, 1)$-form

$$\omega_t = -\frac{1}{\log|t|}\varphi\Big(-\frac{\log|z(z-t)|}{\log|t|}\Big)\,\mathrm{d}\log|z| \wedge \frac{\mathrm{d}\arg z}{2\pi}$$

on $\mathbf{P}_1$, depending on the complex parameter t. Fix a real number $K > 1$. One may write

$$\int_{\mathbf{P}_1(\mathbf{C})} \omega_t = I_1 + I_2 + I_3$$

with

$$I_1 = \int_{|z|\le|t|/K} \omega_t, \qquad I_2 = \int_{|t|/K\le|z|\le K|t|} \omega_t, \qquad \text{and} \qquad I_3 = \int_{|z|\ge K|t|} \omega_t.$$

Using direct explicit computations, one may check that

$$\lim_{t\to 0} I_1 = \int_{x\le -1} \varphi(x-1)\,\mathrm{d}x, \qquad \lim_{t\to 0} I_2 = 0, \qquad \text{and}$$
$$\lim_{t\to 0} I_3 = \int_{x\ge -1} \varphi(2x)\,\mathrm{d}x,$$

from which one deduces the equality

$$\lim_{t\to 0}\int_{\mathbf{P}_1(\mathbf{C})} \omega_t = \int_{x\le -1} \varphi(x-1)\,\mathrm{d}x + \int_{x\ge -1} \varphi(2x)\,\mathrm{d}x.$$

Quite remarkably, the right-hand side of that equality admits a non-Archimedean interpretation. Indeed, consider the field of Laurent series $\mathbf{C}((t))$, fix $\tau \in (0, 1)$, and endow $\mathbf{C}((t))$ with the t-adic norm $|\,|_\flat$ normalized by $|t|_\flat = \tau$. On the Berkovich analytification $\mathbf{P}_1^{\mathrm{an}}$ of $\mathbf{P}_1$ over $\mathbf{C}((t))$ one can consider the $(1, 1)$-form

$$\omega_\flat = -\frac{1}{\log|t|_\flat}\varphi\Big(-\frac{\log|z(z-t)|_\flat}{\log|t|_\flat}\Big)\,\mathrm{d}'\log|z|_\flat \wedge \mathrm{d}''\log|z|_\flat$$

in the sense of Chambert-Loir and Ducros [6]. Furthermore, the integral in the sense of Chambert-Loir and Ducros of the form $\omega_\flat$ on $\mathbf{P}_1^{\mathrm{an}}$ is given by

$$\int_{\mathbf{P}_1^{\mathrm{an}}} \omega_\flat = \int_{x\le -1} \varphi(x-1)\,\mathrm{d}x + \int_{x\ge -1} \varphi(2x)\,\mathrm{d}x,$$

since the support of $\omega_\flat$ is contained in the standard skeleton $(0, \infty)$ of $\mathbf{G}_m^{\mathrm{an}}$, and the function z is of degree 1 at each point of this skeleton. Therefore, we finally deduce the equality

$$\lim_{t\to 0}\int_{\mathbf{P}_1(\mathbf{C})}\omega_t=\int_{\mathbf{P}_1^{\mathrm{an}}}\omega_\flat,$$

a very special case of our Corollary 8.4. We can already see here an instance of a general feature that will be exploited in our proof of the general case: asymptotically, the complex integrals we consider concentrate on the support of the corresponding non-Archimedean forms. This support is piecewise polyhedral, and only the faces of maximal dimension provide a nonzero contribution to the limit. In general, Chambert-Loir and Ducros integrals also involve degrees over these faces (see Section 9.1.11 for an explanation of how these relate to the number of sheets of a complex étale morphism).

1.3

Let us now sketch the construction of the nonstandard "asymptotic" field C. We fix a nonprincipal ultrafilter $\mathcal{U}$ on $\mathbf{C}$ containing all the neighborhoods of the origin (otherwise said, $\mathcal{U}$ converges to zero) and consider the ultrapowers ${}^*\mathbf{C}=\prod_{t\in\mathbf{C}^\times}\mathbf{C}/\mathcal{U}$ and ${}^*\mathbf{R}=\prod_{t\in\mathbf{C}^\times}\mathbf{R}/\mathcal{U}$. We say that an element (a_t) in ${}^*\mathbf{C}$ (resp., ${}^*\mathbf{R}$) is t-bounded if for some positive integer N, $|a_t|\le|t|^{-N}$ along $\mathcal{U}$ (i.e., the set of indices t for which this inequality holds belongs to $\mathcal{U}$). Similarly, it is said to be t-negligible if for every positive integer N, $|a_t|\le|t|^N$ along $\mathcal{U}$. The set of t-bounded elements in ${}^*\mathbf{C}$ (resp., ${}^*\mathbf{R}$) is a local ring which we denote by A (resp., A_{r}), with maximal ideal the subset of t-negligible elements which we denote by $\mathfrak{M}$ (resp., $\mathfrak{M}_{\mathrm{r}}$). We now set $C:=A/\mathfrak{M}$ and $R:=A_{\mathrm{r}}/\mathfrak{M}_{\mathrm{r}}$. The field R is a real closed field, and $C\simeq R(i)$ is algebraically closed. The norm $|\cdot|\colon {}^*\mathbf{C}\to{}^*\mathbf{R}_{\ge 0}$ induces an R-valued norm $|\cdot|\colon C\to R_{\ge 0}$.

1.4

Any usual smooth function $\varphi\colon U\to\mathbf{R}$ defined on some semialgebraic open subset U of $\mathbf{R}^n$ formally induces a map $U({}^*\mathbf{R})\to{}^*\mathbf{R}$ which is still denoted by φ. Allowing ourselves to compose these functions (which arise from *standard* smooth functions) with polynomial maps (which might have nonstandard coefficients), we define for every smooth, separated ${}^*\mathbf{R}$-scheme X of finite type a sheaf of so-called *smooth functions* for the (Grothendieck) semialgebraic topology on $X({}^*\mathbf{R})$, which we denote by $\mathscr{C}^\infty_X$. The natural inclusion map from $X({}^*\mathbf{R})$ into the (underlying set of) the scheme X underlies a morphism of locally ringed sites $\psi\colon (X({}^*\mathbf{R}),\mathscr{C}^\infty_X)\to(X,\mathscr{O}_X)$, and we can define the sheaf of smooth p-forms on $X({}^*\mathbf{R})$ by $\mathscr{A}^p_X:=\psi^*\Omega^p_{X/{}^*\mathbf{R}}$. One has for every p a natural differential $\mathrm{d}\colon\mathscr{A}^p_X\to\mathscr{A}^{p+1}_X$. We now assume that X is of pure dimension n and that $X({}^*\mathbf{R})$ is oriented (the notion of an orientation of a variety makes sense over an arbitrary real closed field; see Section 3.3). Let ω be a smooth n-form on some semialgebraic open subset U of $X({}^*\mathbf{R})$, and let E be a semialgebraic subset of U whose closure in U is definably compact. Choosing a description of (X, U, ω, E) through a "limited family" $(X_t, U_t, \omega_t, E_t)_t$, it is possible to define the integral $\int_E \omega$ as the class of the sequence $(\int_{E_t} \omega_t)_t$ in ${}^*\mathbf{R}$.

1.5

We now move from ${}^*\mathbf{R}$ to R, seeking to show that smooth functions, smooth forms, and their integrals remain well defined on R.

Let $\varphi\colon U \to \mathbf{R}$ be a usual smooth function defined on some semialgebraic open subset U of $\mathbf{R}^n$. Under some boundedness assumptions on φ (which are, for instance, automatically fulfilled if φ is compactly supported, or more generally if all its derivatives are polynomially bounded), the induced function $\varphi\colon U({}^*\mathbf{R}) \to {}^*\mathbf{R}$ in turn induces a map $U(R) \to R$, which we again denote by φ.

For instance, the map $\log|\cdot|$ from $\mathbf{C}^\times \simeq \mathbf{R}^2 \setminus \{(0,0)\}$ is smooth and satisfies the boundedness conditions alluded to above; it thus induces a map $\log|\cdot|\colon C^\times \to R$, which enables us to endow the field C with a real-valued non-Archimedean norm $|\cdot|_\flat$: $C \to \mathbf{R}_{\geq 0}$ as follows. For any z belonging to $C^\times$, one checks that the norm of $\frac{\log|z|}{\log|t|}$ is bounded by some positive real number in $\mathbf{R}$. One can thus consider its standard part $\alpha = \operatorname{std}(\frac{\log|z|}{\log|t|}) \in \mathbf{R}$. Fixing $\tau \in (0,1) \subset \mathbf{R}$, one sets $|z|_\flat := \tau^\alpha$ so that $|z|_\flat = |t|_\flat^\alpha$. With this non-Archimedean norm the field C is complete (even spherically complete; cf. [21]).

We repeat the procedure used in Section 1.4. Allowing ourselves to compose the functions defined at the beginning of Section 1.5 (which arise from *standard* smooth functions) with polynomial maps (which might have nonstandard coefficients), we define for every smooth, separated R-scheme X of finite type a sheaf of so-called *smooth functions* for the (Grothendieck) semialgebraic topology on $X(R)$, which we denote by $\mathscr{C}_X^\infty$. There is a natural morphism of locally ringed sites $\psi\colon (X(R), \mathscr{C}_X^\infty) \to (X, \mathscr{O}_X)$. One then sets $\mathscr{A}_X^p := \psi^*\Omega_{X/R}^p$, and one has for every p a natural differential $\mathrm{d}\colon \mathscr{A}_X^p \to \mathscr{A}_X^{p+1}$.

Assume now that X is of pure dimension n and oriented. A substantial part of Section 3 is devoted to the construction of an R-valued integration theory on $X(R)$.

1.6 PROPOSITION

Integration theory on $X(A_{\mathrm{r}})$ descends to $X(R)$.

Namely, to a semialgebraic subset K of $X(R)$, with definably compact definable closure, and a smooth n-form ω on a semialgebraic neighborhood of K in $X(R)$, we assign an integral $\int_K \omega$ which is an element of R. This is achieved in Section 5.10 by reducing to the case when X is liftable. Independence from the lifting follows from the fact, proved in Proposition 5.3, that the integrals obtained from two different liftings coincide up to a t-negligible element. A preliminary key statement in that direction is provided by Proposition 3.9, which states that if D is a semialgebraic subset of $({}^*\mathbf{R})^n$ contained in A^n_{r}, then the volume of D is t-negligible if and only if the image of D in R^n through the reduction map is of dimension at most $n-1$.

Assume that X is a smooth C-scheme of finite type and pure dimension n. One defines similarly the integral $\int_K \omega$ of a complex-valued (n,n)-form ω defined in a semialgebraic neighborhood of a semialgebraic subset K of $X(C)$, assuming that there exists a semialgebraic subset K' of K with definably compact closure such that ω vanishes on $K \setminus K'$.

1.7 Remark

Note that for an arbitrary real closed field S one cannot hope for a reasonable integration theory with values in S. Indeed, let, for instance, S be the algebraic closure of $\mathbf{Q}$ in $\mathbf{R}$. Then there is no such reasonable integration theory on S, otherwise $\pi = \int_{x^2+y^2\leq 1} \mathrm{d}x \wedge \mathrm{d}y$ would belong to S.

1.8

Fix a smooth C-scheme X of finite type and pure dimension n, and set $\lambda := -\log|t|$. In this text, we define two Dolbeault-like complexes $\mathsf{A}^{p,q}$ and $\mathsf{B}^{p,q}$. Informally, $\mathsf{A}^{p,q}$ and $\mathsf{B}^{p,q}$ should be thought of as living on $X(C)$ and X^{an}, respectively. But since we want to be able to compare them in some sense, we need them to be defined on the same site; this is the reason that we have chosen to define them as complexes of sheaves on the *Zariski* site of X.

1.8.1. The nonstandard Archimedean complex

Let us start with $\mathsf{A}^{p,q}$. We will explain what would be the most natural definition, why it is not convenient for our purpose, and what the actual definition is.

1.8.1.1. Basically, we would like a section of $\mathsf{A}^{p,q}$ on a given Zariski-open subset U of X to be a differential form on $U(C)$ which is locally for the semialgebraic topology on $X(C)$ of the form

$$\omega = \frac{1}{\lambda^p} \sum_{I,J} \varphi_{I,J}\Big(\frac{\log|f_1|}{\lambda}, \dots, \frac{\log|f_m|}{\lambda}\Big)\, \mathrm{d}\log|f_I| \wedge \mathrm{d}\,\mathrm{Arg}\, f_J,$$

where I (resp., J) runs through the set of subsets of $\{1,\dots,m\}$ of cardinality p (resp., q), where the f_i are regular invertible functions, $\mathrm{d}\log|f_I|$ stands for the wedge product $\mathrm{d}\log|f_{i_1}|\wedge\cdots\wedge\mathrm{d}\log|f_{i_p}|$ if $i_1 < i_2 < \cdots < i_p$ are the elements of I, and $\mathrm{d}\,\mathrm{Arg}\, f_J$ stands for the wedge product $\frac{\mathrm{d}\arg}{2\pi} f_{j_1} \wedge \cdots \wedge \frac{\mathrm{d}\arg}{2\pi} f_{j_q}$ if $j_1 < j_2 < \cdots < j_q$ are the elements of J.

1.8.1.2. But it would be difficult to use the definition suggested in Section 1.8.1.1, because the general forms described therein do not have non-Archimedean counterparts, since there is no natural way to turn the implicit semialgebraic covering of $U(C)$ in their definition into an open covering of U^{an}; hence we will take a slightly more restrictive definition, albeit flexible enough for our purpose.

We thus define a section of $\mathsf{A}^{p,q}$ on a Zariski-open subset U of X as a differential form on $U(C)$ that is locally *for the Zariski topology of* U of the form

$$\omega = \frac{1}{\lambda^p} \sum_{I,J} \varphi_{I,J}\Big(\frac{\log|f_1|}{\lambda}, \dots, \frac{\log|f_m|}{\lambda}\Big) \,\mathrm{d}\log|f_I| \wedge \mathrm{d}\,\mathrm{Arg}\, f_J,$$

where $(f_1, \dots, f_m)$ are regular functions (but they are not assumed to be invertible), where I (resp., J) is running through the set of subsets of $\{1, \dots, n\}$ of cardinality p (resp., q), and where each function $\varphi_{I,J}$ is defined on a suitable subset of $(\mathbf{R} \cup \{-\infty\})^m$ and satisfies some technical conditions which we explain now. Let $x = (x_1, \dots, x_m)$ be a point of $(\mathbf{R} \cup \{-\infty\})^m$, and let K denote the set of indices i such that $x_i = -\infty$. Then:

- around x the function $\varphi_{I,J}$ depends only on the x_i for $i \notin K$ and is smooth as a function of the latter;
- the function $\varphi_{I,J}$ even *vanishes* around x as soon as K intersects $I \cup J$.

It is clear that sections of $\mathsf{A}^{p,q}$ admit a local description as in Section 1.8.1.1. Note that if $K = \emptyset$, then the only requirement is for $\varphi_{I,J}$ to be smooth around x, and that our second condition ensures that $\mathrm{d}\log|f_i|$ or $\frac{\mathrm{d}\arg f_i}{2\pi}$ can actually appear only around points at which f_i is invertible (which is necessary for integrating such a form when $p = q = n$).

There exist natural differentials $d : \mathsf{A}^{p,q} \to \mathsf{A}^{p+1,q}$ and $\mathrm{d}^\sharp : \mathsf{A}^{p,q} \to \mathsf{A}^{p,q+1}$ mapping, respectively, a form

$$\frac{1}{\lambda^p}\varphi\Big(\frac{\log|f_1|}{\lambda}, \dots, \frac{\log|f_m|}{\lambda}\Big) \,\mathrm{d}\log|f_I| \wedge \mathrm{d}\,\mathrm{Arg}\, f_J$$

to

$$\frac{1}{\lambda^{p+1}} \sum_{1\le i\le m} \frac{\partial\varphi}{\partial x_i}\Big(\frac{\log|f_1|}{\lambda}, \dots, \frac{\log|f_m|}{\lambda}\Big) \,\mathrm{d}\log|f_i| \wedge \mathrm{d}\log|f_I| \wedge \mathrm{d}\,\mathrm{Arg}\, f_J$$

and to

$$\frac{1}{\lambda^{p}} \sum_{1\le i\le m} \frac{\partial\varphi}{\partial x_i}\Big(\frac{\log|f_1|}{\lambda}, \dots, \frac{\log|f_m|}{\lambda}\Big) \frac{\mathrm{d}\arg f_i}{2\pi} \wedge \mathrm{d}\log|f_I| \wedge \mathrm{d}\,\mathrm{Arg}\, f_J.$$

Here the map d is the usual differential, and $d^\sharp$ is designed to switch modulus and argument (see Section 4.2.2); it turns out to be analogous to the operator d^{c} of complex analytic geometry.

1.8.2. The non-Archimedean complex
We are now going to describe $\mathsf{B}^{p,q}$. Set $\lambda_\flat := -\log|t|_\flat$.

1.8.2.1. Basically, we would like a section of $\mathsf{B}^{p,q}$ on a given Zariski-open subset U of X to be a differential form on U^{an} in the sense of [6] which is locally on U^{an} of the form

$$\frac{1}{\lambda_\flat^p}\sum_{I,J}\varphi_{I,J}\Big(\frac{\log|f_1|_\flat}{\lambda_\flat},\dots,\frac{\log|f_m|_\flat}{\lambda_\flat}\Big)\,\mathrm{d}'\log|f_I|_\flat\wedge\mathrm{d}''\log|f_J|_\flat,$$

where I (resp., J) runs through the set of subsets of $\{1,\dots,m\}$ of cardinality p (resp., q), where the f_i are regular invertible functions, and where $\mathrm{d}'\log|f_I|_\flat$ standing for the wedge product $\mathrm{d}'\log|f_{i_1}|_\flat\wedge\cdots\wedge\mathrm{d}'\log|f_{i_p}|$ if $i_1<i_2<\cdots<i_p$ are the elements of I, and similarly for $\mathrm{d}''\log$.

1.8.2.2. But by analogy with $\mathsf{A}^{p,q}$, we shall rather define a section of $\mathsf{B}^{p,q}$ on a Zariski-open subset U of X as a differential form on U^{an} that is locally *for the Zariski topology of U* of the form

$$\frac{1}{\lambda_\flat^p}\sum_{I,J}\varphi_{I,J}\Big(\log\frac{\log|f_1|_\flat}{\lambda_\flat},\dots,\frac{\log|f_m|_\flat}{\lambda_\flat}\Big)\,\mathrm{d}'\log|f_I|_\flat\wedge\mathrm{d}''\log|f_J|_\flat,$$

where $(f_1,\dots,f_m)$ are regular functions, where I (resp., J) is running through the set of subsets of $\{1,\dots,n\}$ of cardinality p (resp., q) (with $\mathrm{d}'\log|f_I|_\flat$ standing for the wedge product $\mathrm{d}'\log|f_{i_1}|_\flat\wedge\cdots\wedge\mathrm{d}'\log|f_{i_p}|_\flat$ if $i_1<i_2<\cdots<i_p$ are the elements of I, and similarly for $\mathrm{d}''\log|f_J|_\flat$), and where each $\varphi_{I,J}$ satisfies the same conditions as those in the definition of $\mathsf{A}^{p,q}$.

It is clear that sections of $\mathsf{B}^{p,q}$ are locally of the form described in Section 1.8.2.1, and that $\mathsf{B}^{\bullet,\bullet}$ is stable under the two differential operators d' and d''.

1.9
Our main result, Theorem 8.1, states that the two sheaves of bigraded differential **R**-algebras $\mathsf{A}^{\bullet,\bullet}$ and $\mathsf{B}^{\bullet,\bullet}$ on the site X_{Zar}, consisting, respectively, of nonstandard Archimedean and non-Archimedean forms, are compatible in the following sense.

1.10 THEOREM
There exists a unique morphism of sheaves of bigraded differential **R**-*algebras* $\mathsf{A}^{\bullet,\bullet}\to\mathsf{B}^{\bullet,\bullet}$, *sending a nonstandard Archimedean form ω to the non-Archimedean form $\omega_\flat$, such that if ω is of the form*

$$\omega=\frac{1}{\lambda^{|I|}}\varphi\Big(\frac{\log|f_1|}{\lambda},\dots,\frac{\log|f_m|}{\lambda}\Big)\,\mathrm{d}\log|f_I|\wedge\mathrm{d}\,\mathrm{Arg}\,f_J,$$

with $f_1, \dots, f_m$ regular functions on a Zariski-open subset U of X, I, and J subsets of $\{1, \dots, m\}$, and φ a quasismooth function, then

$$\omega_\flat = \frac{1}{\lambda_\flat^{|I|}} \varphi\Big(\frac{\log|f_1|_\flat}{\lambda_\flat}, \dots, \frac{\log|f_m|_\flat}{\lambda_\flat}\Big)\, \mathrm{d}' \log|f_I|_\flat \wedge \mathrm{d}'' \log|f_J|_\flat.$$

Furthermore, we also prove in Theorem 8.1 that the mapping $\omega \mapsto \omega_\flat$ is compatible with integration. A special case of that compatibility can be stated as follows.

1.11 PROPOSITION
Assume that ω is an (n, n)-form defined on some Zariski-open subset U of X and that its support is contained in a definably compact semialgebraic subset of $U(C)$. Then the form $\omega_\flat$ on X^{an} is compactly supported, $\int_{U(C)} |\omega|$ is bounded by some positive real number in $\mathbf{R}$, and

$$\operatorname{std}\Big(\int_{U(C)} \omega\Big) = \int_{U^{\mathrm{an}}} \omega_\flat,$$

with std *standing for the standard part.*

Compatibility with integration is used in an essential way in proving that the mapping $\omega \mapsto \omega_\flat$ is well defined. Indeed, it allows us to use a result of Chambert-Loir and Ducros [6, Corollaire 4.3.7] stating that, in the boundaryless case, nonzero forms define nonzero currents. A key point in the proof of compatibility with integration is to show that the non-Archimedean degree involved in the construction of non-Archimedean integrals in [6] actually shows up in the asymptotics of the corresponding Archimedean integrals, which is done in Section 9.1.11.

This main result has very concrete consequences (see our Theorem 8.4, in which we express limits in the usual sense of complex integrals depending on a parameter in terms of non-Archimedean integrals).

2. General framework

2.1
We shall use in this paper basic facts and terminology from model theory, which can be found, for instance, in the books [22] and [27]. We shall make particular use of the theory DOAG of nontrivial divisible ordered abelian groups, the theory RCF of real closed fields, and the theory ACVF of algebraically closed nontrivially valued fields. Both DOAG and RCF are examples of o-minimal theories.

2.2
We fix a nonprincipal ultrafilter $\mathscr{U}$ on the set $\mathbf{C}$ of complex numbers; we assume that it converges to zero, which means that every neighborhood of the origin belongs to $\mathscr{U}$ (for our purpose, it would be sufficient to consider such an ultrafilter $\mathscr{U}$ on a sequence approaching zero). Note that since $\mathscr{U}$ is not principal, $\{0\} \notin \mathscr{U}$; as a consequence, every *punctured* neighborhood of zero also belongs to $\mathscr{U}$. In particular, there exists a family X_i, $i \in \mathbf{N}$, of elements of $\mathscr{U}$ such that $\bigcap_{i \in \mathbf{N}} X_i = \emptyset$; that is, the ultrafilter $\mathscr{U}$ is countably incomplete.

2.3 Convention
Unless otherwise stated, when we introduce a "sequence" $(a_t)_t$ the parameter t is always understood as running through some set belonging to $\mathscr{U}$ (e.g., a small punctured disk centered at the origin), which we shall usually not make explicit. We shall allow ourselves to shrink this set of parameters when necessary (without mentioning it), for instance if we work with finitely many sequences and need a common set of parameters.

If we work with some sequence $(M_t)_t$ of sets and then consider a sequence $(a_t)_t$ with $a_t \in M_t$ for every t, it will be understood that a_t is defined for t lying in *some* set belonging to $\mathscr{U}$ and on which $t \mapsto M_t$ does make sense; so we do *not* require that a_t be defined for every t such that M_t is.

We say that some specified property P is satisfied by a_t *along* $\mathscr{U}$ if the set of indices t such that a_t satisfies P belongs to $\mathscr{U}$; for example, $|a_t| < |t|$ along $\mathscr{U}$ means that the set of indices t such that $|a_t| < |t|$ belongs to $\mathscr{U}$.

2.4. Ultraproducts
Let $(M_t)_t$ be a sequence of sets. The *ultraproduct of the sets M_t along $\mathscr{U}$* is the quotient of the set of all sequences $(a_t)_t$ with $a_t \in M_t$ for all t by the equivalence relation for which $(a_t) \sim (b_t)$ if and only if $a_t = b_t$ along $\mathscr{U}$ (we remind the reader that according to Convention 2.3, a_t needs not to be defined for all t for which M_t exists, but only for a subset of such complex numbers t that belongs to $\mathscr{U}$). If all the sets M_t are groups (resp., rings, resp., ...) the ultraproduct of the sets M_t along $\mathscr{U}$ inherits a natural structure of group (resp., ring, resp., ...), which enjoys all the first-order properties that hold for M_t along $\mathscr{U}$; for example, if the group M_t is abelian along $\mathscr{U}$, then the ultraproduct of the groups M_t along $\mathscr{U}$ is abelian.

2.5 Remark
One can describe in a perhaps unusual way the ultraproduct of the sets M_t as $\operatorname{colim}_T M_T$ where T runs through the set of elements of $\mathscr{U}$ included in the domain

of $t \mapsto M_t$, where $M_T := \prod_{t \in T} M_t$, and where the transition maps are the obvious ones.

2.6. *The field* $^*\mathbf{C}$

We apply the above by taking M_t equal to the field $\mathbf{C}$ (resp., $\mathbf{R}$) for all t, and we denote by $^*\mathbf{C}$ (resp., $^*\mathbf{R}$) the corresponding ultraproduct. The field $^*\mathbf{R}$ is a real closed extension of $\mathbf{R}$; the field $^*\mathbf{C}$ is equal to $^*\mathbf{R}(i)$ and is an algebraically closed extension of $\mathbf{C}$. We still denote by $|\cdot|$ the "absolute value" on $^*\mathbf{C}$; this is the map from $^*\mathbf{C}$ to $^*\mathbf{R}_+$ that maps $a + bi$ to $\sqrt{a^2 + b^2}$. By (harmless) abuse, the image in $^*\mathbf{C}$ of the sequence $(t)_t$ will also be denoted by t; it should be thought of as a nonstandard complex number with infinitely small (but nonzero!) absolute value.

A sequence $(a_t)_t$ of complex numbers is called:

- *bounded* if there is some $N \in \mathbf{Z}_{\geq 0}$ such that $|a_t| \leq N$ along $\mathscr{U}$;
- *t-bounded* if there is some $N \in \mathbf{Z}_{\geq 0}$ such that $|a_t| \leq |t^{-N}|$ along $\mathscr{U}$;
- *negligible* if $|a_t| \leq \frac{1}{N}$ along $\mathscr{U}$ for all $N \in \mathbf{Z}_{>0}$;
- *t-negligible* if $|a_t| \leq |t^N|$ along $\mathscr{U}$ for all $N \in \mathbf{Z}_{\geq 0}$.

An element a of $^*\mathbf{C}$ is called *bounded* (resp., *t-bounded*; resp., *negligible*; resp., *t-negligible*) if it is the image of some bounded (resp., t-bounded; resp., negligible; resp., t-negligible) sequence. This amounts to requiring that $|a| \leq N$ for some integer $N \geq 0$ (resp., $|a| \leq |t^{-N}|$ for some integer $N \geq 0$; resp., $|a| \leq \frac{1}{N}$ for all integers $N > 0$; resp., $|a| \leq |t|^N$ for all integers $N \geq 0$). (Be aware that the above inequalities are understood in the huge real closed field $^*\mathbf{R}$.)

2.7. *The field* C

The set A of t-bounded elements of $^*\mathbf{C}$ is a subring of $^*\mathbf{C}$ which contains t. This is a local ring, whose maximal ideal $\mathfrak{m}$ is the set of t-negligible elements; the intersection $A_{\mathrm{r}} := A \cap {}^*\mathbf{R}$ is also a local ring, whose maximal ideal is $\mathfrak{m}_{\mathrm{r}} := \mathfrak{m} \cap {}^*\mathbf{R}$. We denote by C (resp., R) the residue field of A (resp., A_{r}), and we still denote by t the image of the element t of A in C. Note that $\mathfrak{m} \neq 0$: for instance, the sequence $(\exp(-1/|t|))_t$ is t-negligible and not equal to zero along $\mathscr{U}$, so it defines a nonzero element of $\mathfrak{m}$. One can directly describe C as the ring of t-bounded sequences modulo that of t-negligible sequences. The field R is a real closed extension of $\mathbf{R}$, we have $C = R(i)$, and C is an algebraically closed extension of $\mathbf{C}$. We still denote by $|\cdot|$ the "absolute value" on C; this is the map from C to R_+ that maps $a + bi$ to $\sqrt{a^2 + b^2}$. An element z of C is called *bounded* (resp., *negligible*) if it is the image of a bounded (resp., negligible) element of A. This amounts to requiring that z is the image of a bounded (resp., negligible) sequence or that $|z| \leq N$ for some $N \in \mathbf{Z}_{\geq 0}$ (resp., $|z| < \frac{1}{N}$ for all $N \in \mathbf{Z}_{>0}$).

If $z = a + bi$ is any bounded element of C, then the subset of $\mathbf{R}$ consisting of those real numbers that are at most a is nonempty and bounded above, and hence has a least upper bound $\alpha \in \mathbf{R}$; we define β analogously. By construction, $z - (\alpha + \beta i)$ is negligible, and $\alpha + \beta i$ is the only complex number having this property; it is called the *standard part* of z and it will be denoted by $\operatorname{std}(z)$. If $z \in R$, then $\operatorname{std}(z) \in \mathbf{R}$.

Any t-bounded complex-valued function f on an element of $\mathscr{U}$ (e.g., a small punctured disk centered at the origin) gives rise to an element of C, which we shall denote by f if no confusion arises, as we do for t. Let us give some examples.

- For every $\alpha \in \mathbf{R}$, the sequence $(|t|^\alpha)_t$ is t-bounded and is not t-negligible, so it gives rise to an element $|t|^\alpha$ of $C^\times$ (which actually belongs to $R_+^\times$). Note that if $\alpha \neq 0$, then $(|t|^\alpha - 1)_t$ is not t-negligible; hence $\alpha \mapsto |t|^\alpha$ is an injective order-reversing group homomorphism from $\mathbf{R}$ into $R_+^\times$.
- The field $\mathscr{M}$ of meromorphic functions around the origin has a natural embedding into C.
- If a is any nonzero element of C arising from a t-bounded and non-t-negligible sequence $(a_t)_t$, then the sequence $(\log|a_t|)_t$ is t-bounded, so it gives rise to an element of C. The latter depends only on a, and not on the specific sequence (a_t). To see it, we have to check that if $(\varepsilon_t)_t$ is a t-negligible sequence, then $(\log|a_t + \varepsilon_t| - \log|a_t|)$ is t-negligible as well. For that purpose, we first notice that if z is a standard complex number with $|z|$ small enough, then $\log|1 + z| \leq 2|z|$. Now our assumptions imply that the sequence $(\varepsilon_t a_t^{-1})_t$ is t-negligible and a fortiori negligible, so that

$$\log|a_t + \varepsilon_t| - \log|a_t| = \log\big|\big(1 + \varepsilon_t |a_t|^{-1}\big)\big| \leq 2|\varepsilon_t a_t^{-1}|$$

  holds along $\mathscr{U}$; using once again the fact that $(\varepsilon_t a_t^{-1})_t$ is t-negligible, we obtain the required result.

  The element of C defined by the sequence $(\log|a_t|)_t$ depending only on a, we denote it by $\log|a|$. The sequence $(\frac{\log|a_t|}{\log|t|})_t$ is bounded, so $\frac{\log|a|}{\log|t|}$ is bounded.

Set $\Lambda = \{r \in R_+^\times \,|\, |t|^{1/N} \leq r \leq |t|^{-1/N} \text{for all} N \in \mathbf{Z}_{>0}\}$; this is a convex subgroup of $R_+^\times$, and $R_+^\times/\Lambda$ thus inherits an ordering such that the quotient map is order-preserving. The composition

$$C^\times \xrightarrow{|\cdot|} R_+^\times \longrightarrow R_+^\times/\Lambda$$

is a valuation $|\cdot|_\flat$, and $|C^\times|_\flat = R_+^\times/\Lambda$. The valuation ring C° of $|\cdot|_\flat$ is the set of the elements $z \in C$ such that $|z| < |t|^{-1/N}$ for all integers $N > 0$, and the maximal ideal of C° is the set $C^{\circ\circ}$ of elements z of C such that $|z| < |t|^{1/N}$ for some integer $N > 0$ (note that C° contains the ring of bounded elements of C).

Let $z \in C^{\times}$, and set $\alpha = \mathrm{std}(\frac{\log|z|}{\log|t|})$. It follows immediately from the definitions that $|z| = |t|^{\alpha}$ modulo Λ and that $|t|^{\alpha}$ itself belongs to Λ if and only if $\alpha = 0$. Hence $\alpha \mapsto |t|^{\alpha} \mod \Lambda$ induces an order-reversing isomorphism between the ordered groups $\mathbf{R}$ and $|C^{\times}|_{\flat}$, which maps 1 to $|t|_{\flat} = |t| \mod \Lambda$.

We fix once and for all an order-preserving isomorphism between $|C^{\times}|_{\flat}$ and $\mathbf{R}_{+}^{\times}$, which amounts to choosing the image τ of $|t|_{\flat}$ in $(0,1)$. We will from now on use this isomorphism to see $|\cdot|_{\flat}$ as a real valuation (with value group the whole of $\mathbf{R}_{+}^{\times}$). If z is any element of $C^{\times}$, then we have

$$|z|_{\flat} = |t|_{\flat}^{\mathrm{std}(\frac{\log|z|}{\log|t|})} = \tau^{\mathrm{std}(\frac{\log|z|}{\log|t|})}.$$

The residue field $\widetilde{C} := C^{\circ}/C^{\circ\circ}$ is an algebraically closed extension of $\mathbf{C}$. Let us give an example of an element of $\widetilde{C}$ that is transcendent over $\mathbf{C}$. For every complex number λ and every integer $N > 0$, the (complex) inequalities $1 \leq |\log|t| - \lambda| \leq |t|^{-1/N}$ hold along $\mathscr{U}$; as a consequence, $1 \leq |\log|t| - \lambda| \leq |t|^{-1/N}$ in R for all integers $N > 0$, so $|\log|t| - \lambda|_{\flat} = 1$. Hence $|\log|t||_{\flat} = 1$ and if we denote by $\widetilde{\log|t|}$ the image of $\log|t|$ in $\widetilde{C}$, then $\widetilde{\log|t|} - \lambda \neq 0$ for all $\lambda \in \mathbf{C}$; as a consequence, $\widetilde{\log|t|}$ is transcendent over $\mathbf{C}$.

The non-Archimedean field C is complete, and even spherically complete (cf. [21]). Indeed, let $(B_n)_{n\in\mathbf{Z}_{\geq 0}}$ be a decreasing sequence of closed balls with positive radius in C. For every n, denote by r_n the radius of B_n and choose b_n in B_n; we want to prove that $\bigcap B_n$ is nonempty. For every $n \geq 1$, choose a preimage b_n of b_n in A, and a real number s_n with $r_{n-1} > s_n > r_n$, and denote by B_n the set of those $x \in {}^*\mathbf{C}$ such that $|x - \mathsf{b}_n| \leq |t|^{\log s_n/\log\tau}$. For each $n \geq 1$, the ball B_n contains the preimage of B_n in A and is contained in the preimage of B_{n-1}. The fact that every B_n contains the preimage of B_n in A implies that the intersection of finitely many of the sets B_n is nonempty. Since, as noted in Section 2.2, the ultrafilter $\mathscr{U}$ is countably incomplete, the ultraproduct ${}^*\mathbf{C}$ is $\aleph_1$-saturated by [15, Corollary 2.2], and thus the intersection of all the sets B_n is nonempty; however, this intersection is contained in the preimage of the intersection of all the sets B_n, so the latter is nonempty.

3. Smooth functions, smooth forms, and their integrals over ${}^*\mathbf{R}$ and ${}^*\mathbf{C}$

3.1. Semialgebraic topology

Let S be an arbitrary real closed field (we will use what follows for $S = {}^*\mathbf{R}$ and $S = R$). Let X be an algebraic variety over the field S, that is, X is a separated S-scheme of finite type. The set $X(S)$ is in a natural way a definable space of RCF. By quantifier elimination in RCF, the definable subsets of $X(S)$ are precisely its *semi-algebraic* subsets; that is, those subsets that can be defined locally for the Zariski topology of X by a Boolean combination of inequalities (strict or nonstrict) between regular functions.

3.1.1
The order topology on the field S induces a topology of $X(S)$, which is most of the time poorly behaved: except if $S = \mathbf{R}$, it is neither locally compact nor locally connected.

Let U be a semialgebraic subset of $X(S)$. We shall say that U is open (resp., closed) if it is open (resp., closed) for this topology. This amounts to requiring that U be defined—locally for the Zariski topology of X—by a positive Boolean combination of strict (resp., nonstrict) inequalities between regular functions (see [3, Théorème 2.7.1]). The topological closure of a semialgebraic subset U of $X(S)$ is semialgebraic (and so is its topological interior, by considering complements). Indeed, this can be checked on an affine chart, and hence we reduce to the case where $X = \mathbf{A}^n_S$; now since the topology on S^n has a basis consisting of products of open intervals, $\overline{U}$ is definable, so it is semialgebraic.

3.1.2
Since the interval $[0, 1]$ of S is not compact except if $S = \mathbf{R}$, naive topological compactness is not a relevant notion in our setting. We use *definable compactness* instead, which itself relies on the notion of a *definable type* (see, e.g., Section 2.3 and Chapter 4 of [14] for more information on these topics). Let us just recall here that a subset E of $X(S)$ is called *definably compact* if every definable type lying on E converges to a unique point of E. Since X is separated, any definably compact semialgebraic subset of $X(S)$ is closed. If E is a definably compact semialgebraic subset of $X(S)$, then a semialgebraic subset F of E is closed if and only if it is definably compact.

3.1.3
Assume that X is affine, and let $(f_1, \ldots, f_n)$ be a family of regular functions on X that generate the S-algebra $\mathscr{O}(X)$. If E is a semialgebraic subset of $X(S)$, then E is definably compact if and only if it is closed and *bounded*; that is, there exists $r > 0$ in S and such that $|f_i(x)| \leq r$ for all i and all $x \in E$.

3.2 LEMMA
Let X be a separated S-scheme of finite type, and let E be a definably compact semialgebraic subset of $X(S)$. Let $(U_i)_{i \in I}$ be a finite family of definable open subsets of $X(S)$ such that $E \subset \bigcup U_i$. There exists a family (E_i) with each E_i a definably compact semialgebraic subset of U_i and $E = \bigcup_i E_i$.

Proof

Up to refining the covering (U_i), we can assume that U_i is for every i contained in $X_i(S)$ for some open affine subscheme X_i of X. We argue by induction on $|I|$. The statement is clear if $|I| = 0$. Assume that $|I| > 0$ and that the statement is true in cardinality less than $|I|$. Choose an element i in I, and set $F = X(S) \setminus \bigcup_{j \in I, j \neq i} U_j$. By definition, F is a closed semialgebraic subset of $X(S)$ contained in U_i; thus $E \cap F$ is a definably compact semialgebraic subset of U_i.

Choose a semialgebraic open subset V of U_i that contains $E \cap F$ and whose closure $\overline{V}$ is definably compact and still contained in U_i (one can use a finite set of generators of the S-algebra $\mathscr{O}_X(X_i)$ to build semialgebraic continuous distance functions to $E \cap F$, to the boundary of U_i in $X_i(S)$, and to $(X \setminus X_i)(S)$, and then define V by a suitable positive Boolean combination of nonstrict inequalities involving these functions).

Set $G = X(S) \setminus V$. By definition, G is a closed semialgebraic subset of $X(S)$ and $G \cap E$ is thus definably compact.

We then have $E = (E \cap \overline{V}) \cup (G \cap E)$. Since $G \cap E$ avoids F, it is contained in $\bigcup_{j \neq i} U_j$. The conclusion follows by applying the induction hypothesis to the set $G \cap E$. □

3.3

Because of the bad properties of the order topology $X(S)$, we shall not use it except while speaking of closed or open semialgebraic subsets. Nevertheless, we shall use a closely related set-theoretic Grothendieck topology, namely the *semialgebraic* topology. The underlying category is that of semialgebraic open subsets of $X(S)$ with inclusion maps. A family $(U_i)_{i \in I}$ is a cover of U if there is a *finite* subset J of I such that $U = \bigcup_{i \in J} U_i$; this amounts to requiring that $(U_i \to U)$ induces a usual (open) cover at the level of type spaces.

If X is smooth, then $X(S)$ comes equipped with a *sheaf of orientations* (for the semialgebraic topology), defined mutatis mutandis as in the standard case. It is locally isomorphic to the constant sheaf associated with a two-element set; a global section of this sheaf is called an *orientation* on $X(S)$.

3.4. Smooth forms and integrals over the field $^*\mathbf{R}$

If U is an semialgebraic open subset of $\mathbf{R}^n$ for some n, then every smooth function (i.e., $\mathscr{C}^\infty$-function) $\varphi\colon U \to \mathbf{R}$ gives rise to a function $U(^*\mathbf{R}) \to {}^*\mathbf{R}$, which sends the class of a sequence $(a_t)_t$ with $a_t \in U$ along $\mathscr{U}$ to the class of $(\varphi(a_t))_t$; it will still be denoted by φ if no confusion arises.

3.4.1. Smooth functions and smooth forms on a variety
Let X be a smooth, separated ${}^*\mathbf{R}$-scheme of finite type. Let $\mathscr{F}$ be the assignment that sends a semialgebraic open subset U of $X({}^*\mathbf{R})$ to the set of functions from U to ${}^*\mathbf{R}$ of the form $\varphi \circ g$, where:

- g is a regular map from a Zariski-open subset of X containing U to $\mathbf{A}^m_{{}^*\mathbf{R}}$ for some m;
- φ is a smooth function from V to $\mathbf{R}$, where V is a semialgebraic open subset of $\mathbf{R}^m$ such that $g(U) \subset V({}^*\mathbf{R})$.

Then $\mathscr{F}$ is a presheaf; its associated sheaf (for the semialgebraic topology) is denoted by $\mathscr{C}^\infty$ or $\mathscr{C}^\infty_X$ and called the sheaf of *smooth functions* on $X({}^*\mathbf{R})$. It makes $X({}^*\mathbf{R})$ a locally ringed site.

The natural embedding of $X({}^*\mathbf{R})$ into (the underlying set of) the scheme X underlies a morphism of locally ringed sites $\psi\colon (X({}^*\mathbf{R}), \mathscr{C}^\infty_X) \to (X, \mathscr{O}_X)$; hence $\psi^*\Omega^p_{X/{}^*\mathbf{R}}$ is for every p a well-defined $\mathscr{C}^\infty_X$-module on $X({}^*\mathbf{R})$, which we denote by $\mathscr{A}^p$ or $\mathscr{A}^p_X$. The sheaf $\mathscr{A}^0_X$ is equal to $\mathscr{C}^\infty_X$, and the $\mathscr{C}^\infty_X$-module $\mathscr{A}^1_X$ is locally free (of rank n if X is of pure dimension n); for every p, we have $\mathscr{A}^p_X = \Lambda^p \mathscr{A}^1_X$. The sheaf $\mathscr{A}^p_X$ is called the sheaf of *smooth* p-forms on $X({}^*\mathbf{R})$. One has for every p a natural differential $\mathrm{d}\colon \mathscr{A}^p_X \to \mathscr{A}^{p+1}_X$. The sheaf ${}^*\mathbf{C} \otimes_{{}^*\mathbf{R}} \mathscr{A}^p_X$ is called the sheaf of *complex-valued* p-forms on $X({}^*\mathbf{R})$. Every complex-valued p-form ω defined on a semialgebraic open subset U of $X({}^*\mathbf{R})$ can be evaluated at any point u of U, giving rise to an element $\omega(u)$ of the ${}^*\mathbf{C}$-vector space ${}^*\mathbf{C} \otimes_{\mathscr{O}_{X,u}} \Omega^p_{X,u}$.

3.4.2. Integral of an n-form
We still denote by X a smooth, separated ${}^*\mathbf{R}$-scheme of finite type; we assume that it is of pure dimension n for some n, and that $X({}^*\mathbf{R})$ has been given an orientation. Let ω be a complex-valued smooth n-form on some semialgebraic open subset U of $X({}^*\mathbf{R})$, and let E be a semialgebraic subset of U whose closure in U is definably compact.

We now choose a description of (X, U, ω, E) through a "limited family" $(X_t, U_t, \omega_t, E_t)_t$, where X_t is for every t a smooth, separated $\mathbf{R}$-scheme of pure dimension n endowed with an orientation of $X_t(\mathbf{R})$, U_t is an open subset of $X_t(\mathbf{R})$, ω_t is a complex-valued smooth form on U_t, and E_t is a relatively compact semialgebraic subset of U_t. The expression "limited family" means that the sequence $(X_t, U_t, \omega_t, E_t)$ can be defined using finitely many smooth functions (defined on real intervals), a given set $T \in \mathscr{U}$, and finitely many polynomials with coefficients in $\mathbf{R}^T$.

For every t, the smooth manifold $X_t(\mathbf{R})$ is oriented; hence the integral $\int_{E_t} \omega_t$ is well defined. The sequence $(\int_{E_t} \omega_t)_t$ defines an element of ${}^*\mathbf{C}$ that depends only on (X, U, ω, E), and the chosen orientation on $X({}^*\mathbf{R})$. We denote it by $\int_E \omega$; if ω is real-valued, then $\int_E \omega$ is an element of ${}^*\mathbf{R}$.

3.4.3. The case of a nonstandard complex variety

Now let X be a smooth quasiprojective scheme over ${}^*\mathbf{C}$, and let Y be the Weil restriction $\mathrm{R}_{{}^*\mathbf{R}/{}^*\mathbf{C}}X$; this is a quasiprojective scheme over ${}^*\mathbf{R}$, equipped by definition with a canonical bijection $Y({}^{5*}\mathbf{R}) \simeq X({}^*\mathbf{C})$. This allows us to transfer to the set $X^*(\mathbf{C})$ all notions introduced above. Moreover, for every p, the sheaf $\mathscr{A}^p \otimes_{{}^*\mathbf{R}} {}^*\mathbf{C}$ of complex-valued smooth p-forms on $X^*(\mathbf{C})$ is equipped with a natural decomposition $\mathscr{A}^p \otimes_{{}^*\mathbf{R}} {}^*\mathbf{C} = \bigoplus_{i+j=p} \mathscr{A}^{i,j}$, where $\mathscr{A}^{i,j}$ is the sheaf of (i,j)-forms; that is, of complex-valued p-forms generated over $\mathscr{C}^\infty$ by forms of the type

$$\mathrm{d} f_1 \wedge \cdots \wedge \mathrm{d} f_i \wedge \mathrm{d}\overline{g_1} \wedge \cdots \wedge \mathrm{d}\overline{g_j}$$

for some regular functions $f_1, \dots, f_i, g_1, \dots, g_j$.

Assume that X is of pure dimension n for some n, let U be a semialgebraic open subset of $X({}^*\mathbf{C})$, and let ω be a smooth (n,n)-form on U. Let E be a semialgebraic subset of $X({}^*\mathbf{C})$ whose closure is definably compact. The (n,n)-form ω can then be integrated on E, using the canonical orientation of $X({}^*\mathbf{C})$. Indeed, choose a description of (X, U, ω, E) through a "limited family" $(X_t, U_t, \omega_t, E_t)_t$, where X_t is for every t a smooth, separated $\mathbf{C}$-scheme of pure dimension n, U_t is an open semialgebraic subset of $X_t(\mathbf{C})$, ω_t is a complex-valued smooth (n,n)-form on U_t, and E_t is a relatively compact semialgebraic subset of U_t; the integral $\int_E \omega$ is then given by the sequence $\int_{E_t} \omega_t$.

3.4.4

We have considered so far only differential forms with smooth coefficients. But by replacing the class of usual smooth functions (on open subsets of $\mathbf{R}^m$) by a broader class $\mathscr{C}$, we can define in the same way differential forms over ${}^*\mathbf{R}$ with coefficients in $\mathscr{C}$, and integrate those of maximal rank on relatively compact definable subsets (provided that $\mathscr{C}$ consists of locally integrable functions).

For instance, if we consider ω and E as in Sections 3.4.2 or 3.4.3, we can define $|\omega|$, which is a form with continuous piecewise smooth coefficients, and also define the integral $\int_E |\omega|$, which is a nonnegative element of ${}^*\mathbf{R}$.

3.5

We are thus able to integrate smooth forms on the field ${}^*\mathbf{R}$, but what we are actually seeking is a similar integration theory over R. Our basic strategy is very simple: it consists of lifting a differential form on the field ${}^*\mathbf{R}$, integrating it, and reducing the result modulo t-negligible elements. But of course, one has to check that it does not depend on our lifting. This requires a good understanding of the way our integrals interact with t-negligibility; this is the purpose of what follows.

3.6 Notation

Let S be a real closed field, and let $\mathfrak{D}$ be a nonempty, bounded above convex subset of S with no least upper bound in S. Then such a least upper bound nevertheless exists, but as a type on S; we denote it by d. We shall allow ourselves to say that a given definable subset I of S contains d (resp., that a given definable formula Φ is satisfied by d) if I (resp., the set of $x \in S$ satisfying Φ) contains $(\lambda, +\infty) \cap \mathfrak{D}$ for some $\lambda \in \mathfrak{D}$.

3.7 LEMMA

Let I be a definable interval of $S_{\geq 0}$ that contains d, and let f be a definable function from I to S. Assume that there exists $a \in I$ with $a < d$ such that $f(x) > d$ for all x with $a < x < d$; then there exists $x > d$ in I with $f(x) > d$.

Proof

Let J be the set of those $x \in I$ such that $f(x) > x$. This is a definable subset of S which contains all elements $y \in S$ with $a < y < d$. By o-minimality, J is a finite union of intervals with bounds in $S \cup \{-\infty, +\infty\}$; thus it contains some interval of the form (a, b) for some element $b \in S$ with $b > d$. Then for all $x \in S$ such that $d < x < b$, we have $f(x) > x > d$. $\square$

3.8

Let D be a definable subset of $({}^*\mathbf{R})^n$ with definably compact closure. The integral $\int_D \mathrm{d}x_1 \wedge \cdots \wedge \mathrm{d}x_n$ is called the *volume* of D and is denoted by $\mathrm{Vol}(D)$.

If D is a cube, that is, D is of the form $\prod_{1 \leq i \leq n} [a_i, b_i]$, then $\mathrm{Vol}(D) = \prod_i (b_i - a_i)$.

We remind the reader that A_{r} is the set of t-bounded elements of ${}^*\mathbf{R}$, and that $a \mapsto \overline{a}$ denote the reduction modulo the maximal ideal $\mathfrak{m}_{\mathrm{r}}$ of A_{r} (cf. Section 2.7).

3.9 PROPOSITION

Let D be a definable subset of $({}^*\mathbf{R})^n$ contained in A_{r}^n. The following are equivalent:

(i) *the volume of D is t-negligible;*

(ii) *for every n-form $\omega = \varphi \, \mathrm{d}x_1 \wedge \cdots \wedge \mathrm{d}x_n$ with φ a smooth function defined in a neighborhood of the closure of D and taking only t-bounded values on the latter, the integral $\int_D \omega$ is t-negligible;*

(iii) *every cube contained in D has t-negligible volume;*

(iv) *the image $\overline{D}$ of D in R^n through the reduction map is of dimension at most $n - 1$.*

3.10 Remark

It is known that $\overline{D}$ is a closed definable subset of R^n (no matter whether D is closed or not; see, e.g., [5]). Thus its dimension is well defined. But the reader could also rephrase (iv) by simply saying "$\overline{D}$ contains no n-cube with nonempty interior"; and this is indeed the rephrasing of (iv) that we shall actually use in the proof.

Proof of Proposition 3.9

We are going to prove (i) $\Rightarrow$ (ii) $\Rightarrow$ (iii) $\Rightarrow$ (i), and then (iv) $\Rightarrow$ (iii) and (i) $\Rightarrow$ (iv).

Assume that (i) is true, and let ω be as in (ii). By definable compactness of the closure of D, there exists a t-bounded positive element M such that $|\varphi| \leq M$ on D. Then $|\int_D \omega| \leq M \operatorname{Vol}(D)$; the volume of D being t-negligible, $\int_D \omega$ is t-negligible as well.

Now if (ii) is true, then in particular $\operatorname{Vol}(D)$ is t-negligible (take $\varphi = 1$); this implies that the volume of every definable subset of D, including any cube contained in D, is t-negligible.

Assume now that (iii) is true, and let us prove (i). We argue by induction on n. If $n = 0$, then there is nothing to prove. So assume that $n > 0$ and the result holds in dimension $n - 1$. Let $p\colon ({}^*\mathbf{R})^n \to ({}^*\mathbf{R})^{n-1}$ be the projection on the first $n - 1$ coordinates, and set $\Delta = p(D)$. If (D_i) is any finite covering of D by definable subsets, then it is sufficient to prove that (i) holds for every D_i (note that D_i obviously satisfies (iii)).

Hence using cellular decomposition we can assume that we are in one of the following two cases:

- there exists a continuous definable function f on Δ such that D is the graph of f;
- there exist two continuous definable functions f and g on Δ with $f < g$ such that $D = \{(x, y), f(x) < y < g(x)\}$.

In the first case, D is at most $(n-1)$-dimensional and its volume is zero. Let us assume from now on that we are in the second case. Since $D \subset A^n_{\mathrm{r}}$, there is a positive t-bounded element M such that $g - f < M$.

Let φ be the function that sends an element a of $[0, M]$ to the least upper bound of the $(n-1)$-volumes of all cubes contained in Δ over which $g - f > a$.

3.10.1

Let us prove by contradiction that there exists some t-negligible element a such that $\varphi(a)$ is t-negligible. We call *t-significant* an element which is not t-negligible, and we assume that $\varphi(a)$ is t-significant for all t-negligible a; we are going to exhibit a cube inside D with t-significant volume, which will contradict our assumptions.

By Lemma 3.7 (which we apply by taking for d the least upper bound of the set $\mathfrak{D}$ of t-negligible elements), there exists some t-significant a with $\varphi(a)$ also t-significant. Therefore, there exists some cube K inside Δ with t-significant $(n-1)$-volume over which $g - f > a$. For each family $\varepsilon = (\varepsilon_1, \ldots, \varepsilon_{2n-2})$ of elements of $\{-1, 1\}$, let K_ε be the subset of K on which $\partial_i g \in \varepsilon_i ({}^*\mathbf{R}_{\geq 0})$ and $\partial_i f \in \varepsilon_{n-1+i} ({}^*\mathbf{R}_{\geq 0})$ or all $1 \leq i \leq n-1$. Then K is the union of the sets K_ε, so one of the sets K_ε has a t-significant volume and hence contains a cube K' with t-significant volume (by the induction hypothesis). Replacing Δ by K', we assume from now on that Δ is a cube with t-significant volume on which each partial derivative of f and g has constant sign and on which $g - f > a$.

Write $\Delta = \prod [\alpha_i, \beta_i]$. Set $M = \sup_\Delta |f|$ and $K = 4M(\beta_1 - \alpha_1)^{-1}$. Since M is t-bounded and since $\beta_1 - \alpha_1$ is t-significant (because Δ has t-significant volume), K is t-bounded. Let $\Delta_K = \{x \in \Delta, |\partial_1 f(x)| \geq K\}$. We claim that $\mathrm{Vol}(\Delta_K) \leq \frac{\mathrm{Vol}(\Delta)}{2}$.

Indeed, fix $z = (z_2, \ldots, z_{n-1})$ in $\prod_{i \geq 2} [\alpha_i, \beta_i]$, and set $\Delta_{K,z} = \{y \in [\alpha_1, \beta_1], (y, z) \in \Delta_K\}$. By o-minimality, $\Delta_{K,z}$ is a finite union of closed intervals; let λ be the 1-dimensional volume (or, otherwise said, the total length) of $\Delta_{K,z}$. If γ and δ are two elements of $[\alpha_1, \beta_1]$ such that $\gamma \leq \delta$ and $[\gamma, \delta] \subset \Delta_{K,z}$, then by the mean value theorem one has $|f(\delta, z)) - f(\gamma, z)| \geq K(\delta - \gamma)$. By monotonicity of $f(\cdot, z)$, this implies that $|f(\beta_1, z) - f(\alpha_1, z)| \geq K\lambda$. Since $|f(\beta_1, z) - f(\alpha_1, z)| \leq 2M$ by the definition of M, we see that $\lambda \leq 2M/K = (\beta_1 - \alpha_1)/2$. Thus, by Fubini, $\mathrm{Vol}(\Delta_K) \leq \frac{\mathrm{Vol}(\Delta)}{2}$, as announced.

It follows that the complement of Δ_K in Δ has t-significant volume. By the induction hypothesis, it contains a cube with t-significant volume. Iterating this argument (which works for g as well as for f, and for the ith component as well as for the first one), we can furthermore assume that Δ is a cube with t-significant volume on which each partial derivative of f and g has an absolute value bounded above by some positive t-bounded constant N.

Let x be the point $(\frac{\alpha_i + \beta_i}{2})_i$ of Δ. Set $y = \frac{g(x) + f(x)}{2}$; the point (x, y) belongs to D. Set $r = (g(x) - f(x))/4$; since $g(x) - f(x) \geq a$, the number r is t-significant. Let N' be a t-bounded number such that $N' > \sqrt{n-1}N$ and $r/N' < \min_i(\beta_i - \alpha_i)/4$—such N' exists since $\beta_i - \alpha_i$ is t-significant for every i. Let Γ be the cube in $({}^*\mathbf{R})^n$ with center (x, y) and polyradius $(r/N', \ldots, r/N', r)$. If (ξ, η) belongs to Γ, then $\xi \in \Delta$. By the mean value theorem, $|f(\xi) - f(x)| \leq \frac{\sqrt{n-1} rN}{2N'} < \frac{r}{2}$ and similarly $|g(\xi) - g(x)| < \frac{r}{2}$. Thus $f(\xi) < \eta < g(\xi)$, and therefore D contains the cube Γ which has t-significant volume.

3.10.2

By the above, there exists some t-negligible element a such that $\varphi(a)$ is t-negligible. Let Δ' be the subset of Δ consisting of points over which $g - f > a$. By assumption, every cube contained in Δ' has t-negligible volume; by our induction hypothesis, the volume of Δ' is t-negligible. Since $g - f$ is uniformly t-bounded, it follows from Fubini's theorem that the volume of $p^{-1}(\Delta')$ is t-negligible. Let Δ'' be the complement of Δ' in Δ. The $(n-1)$-volume of Δ'' is t-bounded, and $g - f \leq a$ on Δ''. Applying Fubini's theorem again, we see that $p^{-1}(\Delta'')$ has t-negligible volume. Hence $D = p^{-1}(\Delta') \cup p^{-1}(\Delta'')$ has t-negligible volume. This ends the proof of (i) $\iff$ (ii) $\iff$ (iii).

3.10.3. Proof of (iv) ⇒ (iii) and (i) ⇒ (iv)

It is clear that (iv) $\Rightarrow$ (iii) since the reduction of every cube in A_{r}^n with t-significant volume is a cube with nonempty interior. We are going to prove (i) $\Rightarrow$ (iv) by contraposition. So assume that $\overline{D}$ is n-dimensional. Under this assumption, it contains a cube with nonempty interior; let us write it $\prod[\overline{a_i}, \overline{b_i}]$, where a_i and the b_i are t-bounded and $b_i - a_i$ is t-significant for all i. Let B be the definable set $\prod_i [a_i, b_i] \setminus D$.

We claim that every cube contained in the definable subset B has t-negligible volume. Indeed, let $\Delta = \prod[\alpha_i, \beta_i]$ be such a cube. If x is a point of A_{r}^n with $\overline{x} \in \prod(\overline{\alpha_i}, \overline{\beta_i})$, then $x \in \Delta$ (and hence $x \notin D$), so $\prod(\overline{\alpha_i}, \overline{\beta_i})$ does not intersect $\overline{D}$. On the other hand, since $\prod(\overline{\alpha_i}, \overline{\beta_i})$ is contained in $\prod[\overline{a_i}, \overline{b_i}]$ (because $\Delta \subset \prod_i [a_i, b_i]$), and $\overline{D}$ contains $\prod_i [\overline{a_i}, \overline{b_i}]$, the open cube $\prod(\overline{\alpha_i}, \overline{\beta_i})$ is contained in $\overline{D}$. Thus $\prod(\overline{\alpha_i}, \overline{\beta_i})$ is empty, and there is at least one index i such that $\overline{\beta_i} - \overline{\alpha_i} = 0$, which means that $\beta_i - \alpha_i$ is t-negligible; a fortiori, the volume of Δ is t-negligible.

Now by what we have already proved, this implies that $\int_{\prod[a_i,b_i]\setminus D} \mathrm{d}x_1 \wedge \cdots \wedge \mathrm{d}x_n$ is t-negligible. As a consequence,

$$\int_{\prod[a_i,b_i]\cap D} \mathrm{d}x_1 \wedge \cdots \wedge \mathrm{d}x_n = \int_{\prod[a_i,b_i]} \mathrm{d}x_1 \wedge \cdots \wedge \mathrm{d}x_n$$

modulo a t-negligible element, but $\int_{\prod[a_i,b_i]} \mathrm{d}x_1 \wedge \cdots \wedge \mathrm{d}x_n = \prod(b_i - a_i)$, which is t-significant. Thus $\int_{\prod[a_i,b_i]\cap D} \mathrm{d}x_1 \wedge \cdots \wedge \mathrm{d}x_n$ is t-significant as well, and so is $\int_D \mathrm{d}x_1 \wedge \cdots \wedge \mathrm{d}x_n$. □

3.11

A definable subset D of $({}^*\mathbf{R})^n$ is called *t-bounded* if it is contained in A_{r}^n; it is called *t-negligible* if it is t-bounded and satisfies the equivalent properties of Proposition 3.9. We shall say that two t-bounded definable subsets D and D' of $({}^*\mathbf{R})^n$ almost coincide (resp., are almost disjoint) if their symmetric difference (resp., their intersection) is t-negligible. If D is a t-bounded definable subset of $({}^*\mathbf{R})^n$, then a finite family (D_i) of t-bounded definable subsets of $({}^*\mathbf{R})^n$ will be called an *almost partition* of D if $\bigcup D_i$ is almost equal to D and the subsets D_i are pairwise almost disjoint.

A definable subset D of R^n is called *negligible* if it is of dimension at most $n-1$. We shall say that two definable subsets D and D' of R^n almost coincide (resp., are almost disjoint) if their symmetric difference (resp., their intersection) is negligible. If D is a definable subset of R^n, then a finite family (D_i) of definable subsets of R^n will be called an almost partition of D if $\bigcup D_i$ is almost equal to D and the subsets D_i are pairwise almost disjoint.

3.12 LEMMA
Let D and Δ be two t-bounded definable subsets of $({}^*\mathbf{R})^n$. Then D and Δ are almost disjoint if and only if $\overline{D}$ and $\overline{\Delta}$ are almost disjoint.

Proof
If $\overline{D}$ and $\overline{\Delta}$ are almost disjoint, then $\overline{D \cap \Delta} \subset \overline{D} \cap \overline{\Delta}$ is negligible, so $D \cap \Delta$ is t-negligible by Proposition 3.9. Conversely, assume that $D \cap \Delta$ is t-negligible, and let us prove that $\overline{D}$ and $\overline{\Delta}$ are almost disjoint. We argue by contradiction, so we assume that there exist elements $a_1, \dots, a_n, b_1, \dots, b_n$ in A_{r} with $b_i - a_i > 0$ and t-significant for all i such that $\prod[\overline{a_i}, \overline{b_i}] \subset \overline{D} \cap \overline{\Delta}$. Set $P = \prod[a_i, b_i] \subset A_{\mathrm{r}}^n$. The volume of the cube P is t-significant and the volume of $P \cap D \cap \Delta$ is t-negligible, so the volume of $P \setminus (D \cap \Delta) = (P \setminus D) \cup (P \setminus \Delta)$ is t-significant. So at least one of the two definable sets $P \setminus D$ and $P \setminus \Delta$ has t-significant volume. Assume without loss of generality that $P \setminus D$ has t-significant volume. By Proposition 3.9, there exists $c_1, \dots, c_n, d_1, \dots, d_n$ in A_{r} with $d_i - c_i > 0$ and t-significant for all i such that $\prod[c_i, d_i] \subset P \setminus D$. Set $x = (\frac{c_1+d_1}{2}, \dots, \frac{c_n+d_n}{2})$. Then x is a point of P whose distance to D is t-significant. As a consequence, $\overline{x} \notin \overline{D}$. But since $x \in P$, its reduction $\overline{x}$ belongs to $\prod[\overline{a_i}, \overline{b_i}] \subset \overline{D} \cap \overline{\Delta}$, which is a contradiction. □

3.13 PROPOSITION
Let D and Δ be two t-bounded definable subsets of $({}^*\mathbf{R})^n$.
(1) *The set D is almost equal to Δ if and only if $\overline{D}$ is almost equal to $\overline{\Delta}$.*
(2) *The set $\overline{D \cap \Delta}$ is almost equal to $\overline{D} \cap \overline{\Delta}$.*

Proof
Set $P = D \setminus \Delta$ and $Q = \Delta \setminus D$. By Lemma 3.12 above, $\overline{Q}$ and $\overline{D \cap \Delta}$ are almost disjoint, and so are $\overline{P}$ and $\overline{D \cap \Delta}$ as well as $\overline{P}$ and $\overline{Q}$. Moreover, we have

$$\overline{D} = \overline{P} \cup \overline{D \cap \Delta} \quad \text{and} \quad \overline{\Delta} = \overline{Q} \cup \overline{D \cap \Delta}.$$

Hence $\overline{D}$ is almost equal to $\overline{\Delta}$ if and only if $\overline{P}$ and $\overline{Q}$ are negligible, which amounts to requiring that P and Q be t-negligible (see Proposition 3.9), that is to say, that D and Δ almost coincide, whence (1). Moreover, $\overline{D} \cap \overline{\Delta} = \overline{D \cap \Delta} \cup (\overline{P} \cap \overline{Q})$, and in view of the negligibility of $\overline{P} \cap \overline{Q}$ this implies (2). □

3.14 COROLLARY
Let K be a definably compact definable subset of R^n. There exists a definable, definably compact and t-bounded subset E of $({}^*\mathbf{R})^n$ such that $\overline{E}$ almost coincides with K.

Proof
Choose $a_1, \dots, a_n$ and $b_1, \dots, b_n$ in A_{r} such that $b_i > a_i$ for all i and $K \subset \prod[\overline{a_i}, \overline{b_i}]$. By using the description of definably closed subsets of R^n provided by Théorème 2.7.1 of [3], we can assume that there exist finitely many polynomials $f_1, \dots, f_m$ in $R[T_1, \dots, T_n]$ such that K is the intersection of $\prod[\overline{a_i}, \overline{b_i}]$ with the set of points x such that $f_j(x) \geq 0$ for all j. By Proposition 3.13 above, we may assume that $m = 1$ and write f instead of f_1. If f is constant, then the set K is either empty or the whole of $\prod[\overline{a_i}, \overline{b_i}]$ and the statement is obvious. If f is nonconstant, let g be a polynomial with t-bounded coefficients that lifts f. Let E be the intersection of $\prod[a_i, b_i]$ and the nonnegative locus of g; it suffices to prove that $\overline{E}$ is almost equal to K. By definition, $\overline{E} \subset K$. Now let x be a point on K at which f is positive, and let ξ be any preimage of x on $\prod[a_i, b_i]$. Since $f(x) > 0$, we have $g(\xi) > 0$, and hence $\xi \in E$ and $x \in \overline{E}$. Thus the difference $K \setminus E$ is contained in the zero locus of f, which is at most $(n-1)$-dimensional since f is nonconstant. □

4. Smooth functions and smooth forms over R and C

4.1. Smooth functions and smooth forms over the field R
Recall that A denotes the ring of t-bounded elements of ${}^*\mathbf{C}$, $\mathfrak{m}$ denotes its maximal ideal (i.e., the set of t-negligible elements), and A_{r} and $\mathfrak{m}_r$ denote the intersections of A and $\mathfrak{m}$ with ${}^*\mathbf{R}$. The reduction modulo $\mathfrak{m}$ will be denoted by $a \mapsto \overline{a}$.

4.1.1
Let U be a semialgebraic open subset of $\mathbf{R}^m$ for some m.

4.1.1.1. If x is a point of R^m lying on $U(R)$ and if ξ is any point of A_{r}^m lifting x, then ξ lies on $U({}^*\mathbf{R})$: this comes from the fact that U can be defined by a positive Boolean combination of strict inequalities (which follows from Théorème 2.7.1 of [3]). For short, we shall call such a ξ a *lifting* of x in $U({}^*\mathbf{R})$.

4.1.1.2. Let φ be a smooth function from U to $\mathbf{R}$. Let $x \in U(R)$. We shall say that φ is *tame* at x if it satisfies the following condition: for every lifting ξ of x in $U({}^*\mathbf{R})$ and every multi-index I, the element $\partial^I \varphi(\xi)$ of ${}^*\mathbf{R}$ is t-bounded.

If this is the case, then for every ξ and every I as above, the element $\overline{\partial^I \varphi(\xi)}$ of R does not depend on ξ (since $\partial^I \varphi$ is Lipshitz with t-bounded constant around ξ).

4.1.1.3. If φ is tame at x, then so are all of its partial derivatives; the sum and the product of two smooth functions on U that are tame at x are themselves tame at x.

4.1.1.4. If φ is tame at x, then we shall denote by $\varphi(x)$ the element $\overline{\varphi(\xi)}$ for ξ any lifting of x in $U({}^*\mathbf{R})$ (it is well defined in view of Section 4.1.1.2).

4.1.2. Examples

In each of the following examples, the function φ is tame at every point of $U(R)$:

- $U = \mathbf{C}^\times$ (viewed as a semialgebraic subset of $\mathbf{C} \simeq \mathbf{R}^2$) and $\varphi = |\cdot|$;
- $U = \mathbf{R}^\times$ and $\varphi = z \mapsto z^n$ for some $n \in \mathbf{Z}$;
- $U = \mathbf{R}_{>0}$ and $\varphi = \log$;
- $U = \mathbf{R}$ and φ is any trigonometric polynomial.

The function $x \mapsto \exp(1/x)$ (defined on $\mathbf{R}^\times$) is not tame at the element t of ${}^*\mathbf{R}^\times$; indeed, $\exp(1/t)$ of ${}^*\mathbf{R}$ is not t-bounded.

4.1.3. Composition of tame functions

Let U be a semialgebraic open subset of $\mathbf{R}^m$, and let V be a semialgebraic open subset of $\mathbf{R}^n$. Let $\varphi = (\varphi_1, \dots, \varphi_n)$ be smooth functions from U to $\mathbf{R}^n$, and assume that $\varphi(U) \subset V$. Let ψ be a smooth function on V.

Let x be a point of U such that every φ_i is tame at x and such that ψ is tame at $\varphi(x)$. It follows straightforwardly from the definition that $\psi \circ \varphi$ is tame at x.

Using this together with Examples 4.1.2, we see that

$$\mathbf{C}^\times \to \mathbf{R}, \qquad z \mapsto \log|z|$$

is tame at every point of $C^\times$ and that

$$\mathbf{C}^\times \setminus \{z, |z| = 1\} \to \mathbf{R}, \qquad z \mapsto 1/\log|z|$$

is tame at every point of $C^\times \setminus \{z \in C^\times, |z| = 1\}$.

4.1.4. Smooth functions and smooth forms on a variety

Let X be a smooth, separated R-scheme of finite type.

Let U be a semialgebraic open subset of $X(R)$, and let g be a regular map from a Zariski-open subset of X containing U to $\mathbf{A}^m_R$ for some m. A (U,g)-*tame* smooth function is a smooth function φ defined on some semialgebraic open subset V of $\mathbf{R}^m$ with $g(U) \subset V(R)$ such that φ is tame at $g(x)$ for every $x \in U$.

Let $\mathscr{F}$ be the assignment that sends a semialgebraic open subset U of $X(R)$ to the set of functions from U to R of the form $\varphi \circ g$, where g is a regular map from a Zariski-open subset of X containing U to $\mathbf{A}^m_R$ for some m and where φ is a (U,g)-tame smooth function.

Then $\mathscr{F}$ is a presheaf; its associated sheaf for the semialgebraic topology is denoted by $\mathscr{C}^\infty$ or $\mathscr{C}^\infty_X$ and called the sheaf of *smooth functions* on $X(R)$. It makes $X(R)$ a locally ringed site.

The natural embedding of $X(R)$ into the scheme X underlies a morphism of locally ringed sites $\psi\colon (X(R), \mathscr{C}^\infty_X) \to (X, \mathscr{O}_X)$; hence $\psi^*\Omega^p_{X/R}$ is for every p a well-defined $\mathscr{C}^\infty_X$-module on $X(R)$, which we denote by $\mathscr{A}^p$ or $\mathscr{A}^p_X$. The sheaf $\mathscr{A}^0_X$ is equal to $\mathscr{C}^\infty_X$, and the $\mathscr{C}^\infty_X$-module $\mathscr{A}^1_X$ is locally free (of rank n if X is of pure dimension n); for every p, we have $\mathscr{A}^p_X = \Lambda^p \mathscr{A}^1_X$. The sheaf $\mathscr{A}^p_X$ is called the sheaf of *smooth* p-forms on $X(R)$. One has for every p a natural differential $\mathrm{d}\colon \mathscr{A}^p_X \to \mathscr{A}^{p+1}_X$. The sheaf $C \otimes_R \mathscr{A}^p_X$ is called the sheaf of *complex-valued* p-forms on $X(R)$.

4.2. *The case of a variety over* C

By considering the Weil restriction, we can apply the above to smooth schemes of finite type over the field C. For such a scheme X and every m, we get a sheaf $\mathscr{A}^m_X$ of R-vector spaces on $X(C)$ (equipped with the semialgebraic topology). This sheaf comes with a natural decomposition

$$C \otimes_R \mathscr{A}^m_X = \bigoplus_{p+q=m} \mathscr{A}^{p,q}_X,$$

where $\mathscr{A}^{p,q}$ is the sheaf of (i, j)-forms, that is, of C-valued p-forms generated over $\mathscr{C}^\infty$ by forms of the type

$$\mathrm{d}f_1 \wedge \cdots \wedge \mathrm{d}f_p \wedge \mathrm{d}\overline{g_1} \wedge \cdots \wedge \mathrm{d}\overline{g_q}$$

for some regular functions $f_1, \dots, f_p, g_1, \dots, g_q$ (this is analogous to Section 3.4.3).

4.2.1. *Polar coordinates*

The usual real functions cos and sin are tame at every point of R; hence $\theta \mapsto \cos\theta + i \sin\theta$ is a well-defined smooth C-valued function on R, which we denote by $\theta \mapsto e^{i\theta}$. The map $\theta \mapsto e^{i\theta}$ is a surjective homomorphism from R to $\{z \in C^\times, |z| = 1\}$. The map $\theta \mapsto e^{i\theta}$ is not injective; its kernel consists of elements of the form $2\pi n$ where n is a (possibly) nonstandard integer; that is, it can be written as the (class of the) limit of a t-bounded sequence of integers. For every $a \in R$, the restriction of $\theta \mapsto e^{i\theta}$ to $[a, a + 2\pi)$ and $(a, a + 2\pi]$ is injective.

Every element z of $C^\times$ can be written as $re^{i\theta}$ with $r \in R_{>0}$ and $\theta \in R$. The element r is unique (it is equal to $|z|$), but θ is not—we say that θ is an argument of z.

Making z vary, we obtain two "functions" r and θ on $C^\times = \mathbf{G}_{\mathrm{m}}(C)$. More precisely, r is an actual function which is tame at every point and takes its values in $R_{>0}$, and $\mathrm{d}r$ and $\mathrm{d}\log r = \frac{\mathrm{d}r}{r}$ are well-defined differential forms on $C^\times$. But θ is only a multivalued function; nevertheless, the differential form $\mathrm{d}\theta$ is also well defined. Let us quickly explain how. Let $z_0 \in C^\times$, and let a be any element of R such that z_0 has an argument θ_0 in $(a-\pi, a+\pi)$ (this always holds for $a=0$ or $a=\pi$). Then on a suitable semialgebraic neighborhood U of z_0 in $C^\times$ we have a single-valued smooth argument function θ with values in $(a-\pi, a+\pi)$ (and $\theta(z_0)=\theta_0$). The smooth form $\mathrm{d}\theta$ is well defined on U. From the equality $z = re^{i\theta}$ we get

$$\mathrm{d}z = e^{i\theta}\,\mathrm{d}r + rie^{i\theta}\,\mathrm{d}\theta,$$

and then

$$\mathrm{d}\theta = -\frac{i}{r}e^{-i\theta}\,\mathrm{d}z - i\,\frac{\mathrm{d}r}{r}.$$

This last formula does not involve the choice of z_0, a, and θ_0 anymore, and we use it to define $\mathrm{d}\theta$ on the whole of $C^\times$.

If we see z as an invertible function on $C^\times$, we shall write $\mathrm{d}\log|z|$ instead of $\frac{\mathrm{d}r}{r}$ and $\mathrm{d}\arg z$ instead of $\mathrm{d}\theta$.

From the equality $z\overline{z} = r^2$ we get

$$\mathrm{d}\log|z| = \frac{1}{2}\cdot\frac{2\,\mathrm{d}r}{r} = \frac{1}{2}\Big(\frac{\mathrm{d}z}{z} + \frac{\mathrm{d}\overline{z}}{\overline{z}}\Big).$$

From the equality $\frac{z}{\overline{z}} = e^{2i\theta}$ we get

$$\mathrm{d}\arg z = \frac{1}{2}\cdot 2\,\mathrm{d}\theta = \frac{1}{2i}\cdot\frac{\mathrm{d}(e^{2i\theta})}{e^{2i\theta}} = \frac{1}{2i}\Big(\frac{\mathrm{d}z}{z} - \frac{\mathrm{d}\overline{z}}{\overline{z}}\Big).$$

4.2.2. The definition of $\mathrm{d}^\sharp$

Let X be a smooth scheme of finite type over C. Our purpose is to define an operator $\mathrm{d}^\sharp$ on complex-valued smooth forms on $X(C)$ (which is a nonstandard avatar of d^{c} up to a constant).

Let us denote for short by $\mathscr{C}^\infty_{X,C}$ (resp., $\mathscr{A}^p_{X,C}$) the sheaf $C \otimes_R \mathscr{C}^\infty_X$ (resp., $C \otimes_R \mathscr{A}^p_X$). The sheaf $\mathscr{A}^1_{X,C}$ of complex-valued smooth 1-forms on $X(C)$ admits a canonical decomposition $\mathscr{A}^1_{X,C} = \mathscr{A}^{1,0} \oplus \mathscr{A}^{0,1}$. The formula $(\omega,\omega') \mapsto (-i\omega, i\omega')$ defines an order-4 automorphism J of the $\mathscr{C}^\infty_{X,C}$-module $\mathscr{A}^1_{X,C}$; we still denote by J the induced automorphism of $\mathscr{A}^p_{X,C}$. We remark that $\mathscr{A}^{2n}_{X,C} \simeq \mathscr{A}^{n,0} \otimes_{\mathscr{C}^\infty_{X,C}} \mathscr{A}^{0,n}$, so the operator J on $\mathscr{A}^{2n}_{X,C}C$ is nothing but $(-i)^n i^n \mathrm{Id} = \mathrm{Id}$.

We then define the derivation $\mathrm{d}^\sharp\colon \mathscr{C}^\infty_{X,C} \to \mathscr{A}^1_{X,C}$ as being equal to $(\mathrm{J}\circ \mathrm{d})/2\pi$ (this is an avatar of the classical operator d^{c}); it extends to a compatible system of exterior derivations

$$\mathrm{d}^\sharp := \frac{1}{2\pi}\mathrm{J}\circ d\circ \mathrm{J}^{-1}\colon \mathscr{A}^p_{X,C} \to \mathscr{A}^{p+1}_{X,C}.$$

Let us see how it acts on polar coordinates. We have

$$\begin{aligned}
\mathrm{d}^\sharp(\log r) &= \frac{1}{2\pi}\mathrm{J}(\mathrm{d}\log r)\\
&= \frac{1}{2\pi}\mathrm{J}\Big(\frac{1}{2}\Big(\frac{\mathrm{d}z}{z}+\frac{\mathrm{d}\overline{z}}{\overline{z}}\Big)\Big)\\
&= \frac{1}{2\pi}\Big(\frac{1}{2}\Big(-i\,\frac{\mathrm{d}z}{z}+i\,\frac{\mathrm{d}\overline{z}}{\overline{z}}\Big)\Big)\\
&= \frac{1}{2\pi}\Big(\frac{1}{2i}\Big(\frac{\mathrm{d}z}{z}-\frac{\mathrm{d}\overline{z}}{\overline{z}}\Big)\Big)\\
&= \frac{\mathrm{d}\theta}{2\pi}
\end{aligned}$$

and

$$\begin{aligned}
\mathrm{d}^\sharp(\theta) &= \frac{1}{2\pi}\mathrm{J}(\mathrm{d}\theta)\\
&= \frac{1}{2\pi}\mathrm{J}\Big(\frac{1}{2i}\Big(\frac{\mathrm{d}z}{z}-\frac{\mathrm{d}\overline{z}}{\overline{z}}\Big)\Big)\\
&= \frac{1}{2\pi}\Big(\frac{1}{2i}\Big(-i\,\frac{\mathrm{d}z}{z}-i\,\frac{\mathrm{d}\overline{z}}{\overline{z}}\Big)\Big)\\
&= \frac{1}{2\pi}\Big(\frac{1}{2}\Big(-\frac{\mathrm{d}z}{z}-\frac{\mathrm{d}\overline{z}}{\overline{z}}\Big)\Big)\\
&= -\frac{\mathrm{d}\log r}{2\pi}.
\end{aligned}$$

Note that since $(\mathrm{d}^\sharp)^2=0$, this implies that $\mathrm{d}^\sharp(\mathrm{d}\log r)=0$ and $\mathrm{d}^\sharp(\mathrm{d}\theta)=0$.

More generally, if f is an invertible regular function defined on some Zariski-open subset U of X, we can define $\mathrm{d}\log|f|$ and $\mathrm{d}\arg f$. Those are smooth forms on $U(C)$, and we have the following equalities:

$$\begin{aligned}
\mathrm{d}\log|f| &= \frac{1}{2}\Big(\frac{\mathrm{d}f}{f}+\frac{\mathrm{d}\overline{f}}{\overline{f}}\Big),\\
\mathrm{d}\arg f &= \frac{1}{2i}\Big(\frac{\mathrm{d}f}{f}-\frac{\mathrm{d}\overline{f}}{\overline{f}}\Big),\\
\mathrm{d}^\sharp\big(\log|f|\big) &= \frac{\mathrm{d}\arg f}{2\pi},\\
\mathrm{d}^\sharp(\arg f) &= -\frac{\mathrm{d}\log|f|}{2\pi}.
\end{aligned}$$

4.3

Now we introduce a particular class of smooth functions and forms on smooth schemes over C that will play a crucial role in our work. Roughly speaking, these are the functions and forms that have a natural counterpart in the Berkovich setting—we will make this rather vague formulation more precise later.

4.4 Definition

Let V be an open subset of $(\mathbf{R} \cup \{-\infty\})^m$ which can be defined by a Boolean combination of $\mathbf{Q}$-linear inequalities, and let φ be a function from V to $\mathbf{C}$. We shall say that φ is a *reasonably smooth* function if there exists:

- a finite open cover $(V_i)_i$ of V, where each V_i is also defined by $\mathbf{Q}$-linear inequalities;
- for every i, a subset J_i of $\{1, \dots, m\}$ with $\Omega_i := p_{J_i}(V_i) \subset \mathbf{R}^{J_i}$, where p_{J_i} is the projection onto the coordinates belonging to J_i;
- for every i, a smooth function φ_i on Ω_i such that $\varphi|_{V_i} = \varphi_i \circ p_{J_i}|_{V_i}$.

The data $(V_i, J_i, \Omega_i, \varphi_i)_i$ will be called a *nice description* of φ.

If J is some subset of $\{1, \dots, m\}$, then we shall say that φ is J*-vanishing* if there exists an open subset V' of V satisfying the following:

- V' can be defined by $\mathbf{Q}$-linear inequalities;
- $\varphi|_{V'} = 0$;
- for every $x = (x_1, \dots, x_m) \in V \setminus V'$ and every $i \in J$, the coordinate x_i is not equal to $(-\infty)$.

Note that φ is automatically $\emptyset$-vanishing; indeed, if $J = \emptyset$, then the above conditions are fulfilled by $V' = \emptyset$.

For instance, a reasonably smooth function φ on $\mathbf{R} \cup \{-\infty\}$ is nothing but a smooth function φ on $\mathbf{R}$ such that there exists $\lambda \in \mathbf{R}$ with $\varphi(x) = \lambda$ for $x \ll 0$ (and the value of φ at $-\infty$ is then set equal to λ); it is 1-vanishing if and only if $\lambda = 0$.

4.5

Let V be an open subset of $(\mathbf{R} \cup \{-\infty\})^m$ which can be defined by a Boolean combination of $\mathbf{Q}$-linear inequalities. The following facts follow straightforwardly from the definition.

4.5.1

If $\varphi \colon V \to \mathbf{R}$ is a reasonably smooth function, then it is continuous, and $\varphi|_{V \cap \mathbf{R}^m}$ is smooth.

4.5.2
For $V \subset \mathbf{R}^m$, a function from V to $\mathbf{R}$ is reasonably smooth if and only if it is smooth.

4.5.3
The set of reasonably smooth functions on V is a subalgebra of the algebra of $\mathbf{R}$-valued functions on V. It is endowed with partial derivation operators defined in the obvious way.

Let φ be a reasonably smooth function on V that is J-vanishing for some subset J of $\{1, \ldots, m\}$. Let $j \in J$. Let us show that $\partial_j \varphi$ is $(J \cup \{j\})$-vanishing.

Let $(V_i, J_i, \Omega_i, \varphi_i)_i$ be a nice description of φ, and let V' be an open subset of V that witnesses the fact that φ is J-vanishing. Let V'' be the union of V' and of all the open sets V_i such that $j \notin J_i$. We claim that V'' witnesses the fact that $\partial_j \varphi$ is $(J \cup \{j\})$-vanishing. Indeed, $\partial_j \varphi$ is zero on V' since so is φ, and if i is such that $j \notin J_i$, then $\varphi|_{V_i}$ does not depend on the jth coordinate, so $\partial_j \varphi$ is zero on V_i; thus $\partial_j \varphi$ is zero on V''.

Let $x \in V \setminus V''$; choose i such that $x = (x_1, \ldots, x_m) \in V_i$. By definition of V'', the set J_i contains j. Hence $x_j \neq (-\infty)$, whence our claim.

4.6. Smooth functions and smooth forms on a C-scheme: A fundamental example

Let V be an open subset of $(\mathbf{R} \cup \{-\infty\})^m$ which can be defined by a Boolean combination of $\mathbf{Q}$-linear inequalities, and let φ be a reasonably smooth function from V to $\mathbf{C}$.

Let W be the semialgebraic open subset of $\mathbf{C}^{m+1}$ consisting of points $(a_1, \ldots, a_m, b)$ such that $0 < |b| < 1$ and $(-\log|a_i|/\log|b|)_i \in V$. By construction,

$$\Phi\colon (a_1, \ldots, a_m, b) \mapsto \varphi\bigl(-\log|a_1|/\log|b|, \ldots, -\log|a_m|/\log|b|\bigr)$$

is a well-defined $\mathscr{C}^\infty$-map from W to $\mathbf{C}$.

Let X be a smooth C-scheme of finite type, and let U be a semialgebraic open subset of $X(C)$. Let $g = (g_1, \ldots, g_m)$ be a regular map from a Zariski-open subset of X containing U to $\mathbf{A}^m_C$, and assume that $(g_1, \ldots, g_m, t)(U) \subset W(C)$ (here the element t of C is viewed as a constant regular function).

4.6.1. The smooth function Φ on W is $(U, (g_1, \ldots, g_m, t))$-tame

To see it, fix a nice description $(V_i, J_i, \Omega_i, \varphi_i)_i$ of φ. For every i, denote by W_i the preimage of V_i in W under the map $(a_1, \ldots, a_m, b) \mapsto (-\log|a_j|/\log|b|)$, and let U_i denote the preimage of W_i in U under the map $(g_1, \ldots, g_m, t)$.

We fix i, and we are going to show that Φ is $(U_i, (g_1, \ldots, g_m, t))$-tame, which will imply our claim. In view of Sections 4.1.1.3 and 4.1.3, it suffices to prove that for every $x \in U_i$, the map φ_i is tame at the point $y := (-\log|g_j(x)|/\log|t|)_{j \in J_i}$ of

$\Omega_i(R)$. But the coordinates of y are bounded (as is $\log r/\log|t|$ for every $r \in R_{>0}$), so the coordinates of η are bounded for every lifting η of y, which implies that all partial derivatives of φ_i are bounded, and a fortiori t-bounded, at η; thus φ_i is tame at y.

4.6.2

We can thus compose Φ and $(g_1, \ldots, g_m, t)$ to get a smooth map on U, which we can safely write as

$$x \mapsto \varphi\big(-\log|g_1|/\log|t|, \ldots, -\log|g_m|/\log|t|\big);$$

its restriction to every U_i can be written as

$$x \mapsto \varphi_i\big(-\log|g_j|/\log|t|\big)_{j \in J_i}.$$

4.6.3

Let I and J be two subsets of $\{1, \ldots, m\}$ of respective cardinalities p and q such that φ is $(I \cup J)$-vanishing.

Let U' be the pre-image of V' in U under $(-\log|g_1|/\log|t|, \ldots, -\log|g_m|/\log|t|)$, and let U'' be the subset of U consisting of points at which all the functions g_i with $i \in I \cup J$ are invertible. Let ω be the (p,q)-form on U'' equal to

$$\Big(\frac{-1}{\log|t|}\Big)^p \varphi\big(-\log|g_1|/\log|t|, \ldots, -\log|g_m|/\log|t|\big)\, \mathrm{d}\log|g_I| \wedge \mathrm{d}\arg g_J$$

(where $\mathrm{d}\log|g_I| = \mathrm{d}\log|g_{i_1}| \wedge \cdots \wedge \mathrm{d}\log|g_{i_p}|$ if $i_1 < i_2 < \cdots < i_p$ are the elements of I, and similarly for $\mathrm{d}\arg|g_J|$). Since φ is $(I \cup J)$-vanishing, the restriction of ω to $U' \cap U''$ is zero, so that ω and the zero form on U' glue to a (p,q)-form on U which (obviously) does not depend on V'; we shall allow ourselves to denote it by

$$\Big(\frac{-1}{\log|t|}\Big)^p \varphi\big(-\log|g_1|/\log|t|, \ldots, -\log|g_m|/\log|t|\big)\, \mathrm{d}\log|g_I| \wedge \mathrm{d}\arg g_J.$$

5. Integrals of smooth forms over R and C

5.1

The purpose of this section is to integrate forms on a smooth scheme defined over the field R. The rough idea is quite natural (and unsurprising): lift the situation over A_{r}, compute the integral over $^*\mathbf{R}$ like in Section 3, and then take its class modulo the ideal $\mathfrak{m}_{\mathrm{r}}$ of t-negligible elements.

First of all, we shall assume that we are given two different liftings of a very specific form, and we show that the integrals over $^*\mathbf{R}$ to which they give rise coincide modulo $\mathfrak{m}_{\mathrm{r}}$ (Proposition 5.3 below); the proof rests in a crucial way on our former study of cubes with t-negligible volume and uses the notion of "almost equality" over $^*\mathbf{R}$ as well as over R (see Proposition 3.9, Section 3.11, and Proposition 3.13), together with Hensel's lemma.

Then we shall handle the general case, the point being that a form on a smooth R-scheme always admits locally for the Zariski topology a lifting of the kind dealt with by Proposition 5.3, so this part is somewhat tedious but rather formal once Proposition 5.3 is taken for granted.

5.2

If $\mathscr{X}$ is an affine A_{r}-scheme of finite type, a definable subset E of $\mathscr{X}(^*\mathbf{R})$ will be called *t-bounded* if it is contained in $\mathscr{X}(A_{\mathrm{r}})$. We remark that E is t-bounded if and only if its topological closure is t-bounded, and if this is the case, then the latter is even definably compact. Indeed, by embedding $\mathscr{X}$ in an affine space and arguing componentwise we reduce to the case where $\mathscr{X} = \mathbf{A}^1_{A_{\mathrm{r}}}$, for which our statement follows from o-minimality.

5.3 PROPOSITION

Let Z be a smooth R-scheme of finite type and pure dimension n, and let $h = (h_1, \dots, h_n)$ be an étale map $Z \to \mathbf{A}^n_R$ factorizing through an immersion $(h, h_{n+1})\colon Z \hookrightarrow \mathbf{A}^{n+1}_R$. Let $\mathscr{X}$ and $\mathscr{Y}$ be two smooth $\mathrm{A_r}$-schemes of finite type and of pure relative dimension n, equipped with identifications $\mathscr{X}_R \simeq Z$ and $\mathscr{Y}_R \simeq Z$. Let $f = (f_1, \dots, f_n)\colon \mathscr{X} \to \mathbf{A}^n_{A_{\mathrm{r}}}$ and $g = (g_1, \dots, g_n)\colon \mathscr{Y} \to \mathbf{A}^n_{A_{\mathrm{r}}}$ be two étale maps, factorizing respectively through a closed immersion $(f, f_{n+1})\colon \mathscr{X} \hookrightarrow \mathbf{A}^{n+1}_{A_{\mathrm{r}}}$ and $(g, g_{n+1})\colon \mathscr{Y} \hookrightarrow \mathbf{A}^{n+1}_{A_{\mathrm{r}}}$; assume that for all i one has $f_i|_Z = g_i|_Z = h_i$.

Let E (resp., F) be a t-bounded semialgebraic subset of $\mathscr{X}(^*\mathbf{R})$ (resp., $\mathscr{Y}(^*\mathbf{R})$); assume that the subsets $\overline{E}$ and $\overline{F}$ of $Z(R)$ almost coincide.

Let φ be a smooth function defined on a neighborhood of E in $\mathscr{X}(A_{\mathrm{r}})$, of the form $\varphi_0 \circ \lambda$ with φ_0 a $\mathscr{C}^\infty$-function and λ a tuple of regular functions on $\mathscr{X}$; let ψ be a smooth function defined on a neighborhood of F in $\mathscr{Y}(A_{\mathrm{r}})$, of the form $\psi_0 \circ \mu$ with ψ_0 a $\mathscr{C}^\infty$ function and μ a tuple of regular functions on $\mathscr{Y}$. Assume that there exists a semialgebraic open subset O of $Z(R)$ containing $\overline{E}$ and $\overline{F}$ such that φ_0 is $(O, \lambda|_Z)$-tame, ψ_0 is $(O, \mu|_Z)$-tame, and the smooth functions $\varphi_0 \circ (\lambda|_O)$ and $\psi_0 \circ (\mu|_O)$ coincide on some semialgebraic subset of O almost equal to $\overline{E}$ and $\overline{F}$.

Then $\int_E \varphi \, \mathrm{d}f_1 \wedge \cdots \wedge \mathrm{d}f_n$ and $\int_F \psi \, \mathrm{d}g_1 \wedge \cdots \wedge \mathrm{d}g_n$ are t-bounded and coincide up to a t-negligible element.

Proof

We begin with noting that our tameness assumptions on φ_0 (resp., ψ_0) imply that φ (resp., ψ) takes only t-bounded values on E (resp., F); this in turn implies that it is *uniformly* t-bounded on E (resp., F). The t-boundedness of the integrals involved in our statement follows immediately.

Throughout the proof, we will use the map f_{n+1} (resp., g_{n+1}; resp., h_{n+1}) to see any fiber of f (resp., g; resp., h) as a subset of the affine line over its ground field, and we will repeatedly use the following fact, which is a consequence of the Henselian property of the local ring A_{r}: if w is a point of A_{r}^n with image $\overline{w}$ in R^n, then for every $z \in Z(R)$ lying above $\overline{w}$ there exists a unique preimage ζ of w in $\mathscr{X}(A_{\mathrm{r}})$ (resp., $\mathscr{Y}(A_{\mathrm{r}})$) with $\overline{\zeta} = z$.

The subsets $\overline{E}$ and $\overline{F}$ of $Z(R)$ are definable, closed, and bounded (because E and F are bounded), so they are definably compact. The sets $h(\overline{E})$ and $h(\overline{F})$ are definably compact, and they almost coincide since $\overline{E}$ and $\overline{F}$ almost coincide, so they have the same n-dimensional locus Θ; and the set $h(\overline{E} \vartriangle \overline{F})$ is negligible. It follows that there exists an almost partition (Θ_i) of Θ (and thus of $h(\overline{E})$ as well as of $h(\overline{F})$) by definably compact definable subsets satisfying the following: for every i, there exists an integer n_i such that the subset Θ'_i of Θ_i consisting of points having exactly n_i preimages in $\overline{E} \cap \overline{F}$ and no preimage in $\overline{E} \vartriangle \overline{F}$ is almost equal to Θ_i.

Now for every i, there exists a t-bounded definable subset Ω_i of $({}^*\mathbf{R})^n$ such that $\overline{\Omega_i}$ is almost equal to Θ_i (see Corollary 3.14). By Proposition 3.13, the family (Ω_i) is an almost partition of $f(E)$ as well as of $g(F)$. For every i, let Ω'_i be the subset of Ω_i consisting of points having exactly n_i preimages in E under f and exactly n_i preimages in F under g.

5.3.1

Let us fix i, and prove that Ω'_i is almost equal to Ω_i. It is sufficient (since f and g play exactly the same role) to prove that the set H of points of Ω_i having exactly n_i preimages in E under f is almost equal to Ω_i.

We argue by contradiction, so we assume that the set H consisting of points $x \in \Omega_i$ such that $f^{-1}(x) \cap E$ has cardinality different from n_i has t-significant volume. Then its image $\overline{H}$ is a nonnegligible subset of Θ_i, which implies that $\overline{H} \cap \Theta'_i$ has dimension n. Let us choose a cube (with nonzero volume) $\mathcal{C}$ in $\overline{H} \cap \Theta'_i$ having the following property: there exist an integer N, a subset I of $\{1, \ldots, N\}$ of cardinality n_i, and a t-bounded element $A > 1$ in ${}^*\mathbf{R}$ such that each fiber of h over $\mathcal{C}$ consists of exactly N points $z_1 < z_2 < \cdots < z_N$ all contained in $[1 - \overline{A}, \overline{A} - 1]$ and such that $z_j \in \overline{E}$ if and only if $j \in I$.

Let us choose a cube $\mathcal{D} \subset A_{\mathrm{r}}^n$ lifting $\mathcal{C}$. Since $\mathcal{C} \subset \overline{H}$, the intersection $\mathcal{D} \cap H$ is not t-negligible (see Lemma 3.12), and hence contains a cube $\mathcal{D}'$ with t-significant volume. Every point of $\mathcal{D}'$ has exactly N t-bounded preimages, all contained in $[-A, A]$; let $\sigma_1 < \cdots < \sigma_N$ denote the corresponding continuous sections of the étale map f above $\mathcal{D}'$. If $x \in \mathcal{D}'$ and if $j \in \{1, \ldots, N\} \setminus I$, then $\sigma_j(x) \notin E$, because

$\overline{\sigma_j(x)} \notin \overline{E}$ by the very definition of I. For each $j \in I$, set $\mathcal{D}'_j = \sigma_j^{-1}(E)$. Let $\xi \in \overline{\mathcal{D}'}$, and let $j \in I$. The point ξ belongs to $\mathcal{C}$, so its jth preimage ζ under h belongs to $\overline{E}$, so there exists a point $z \in E$ such that $\overline{z} = \zeta$, which implies that $\overline{f(z)} = \xi$; thus $z = \sigma_j(f(z))$ and $f(z)$ belongs to $\mathcal{D}'_j$. As a consequence, $\overline{\mathcal{D}'_j} = \overline{\mathcal{D}'}$. In view of Proposition 3.13, it follows that $\overline{\bigcap_{j\in I} \mathcal{D}'_j}$ is almost equal to $\overline{\mathcal{D}'}$. In particular, $\bigcap_{j\in I} \mathcal{D}'_j$ is nonempty, but for every $x \in \bigcap_{j\in I} \mathcal{D}'_j$ the intersection $f^{-1}(x) \cap E$ has exactly n_i elements, which is a contradiction.

5.3.2

Now we remark that if $\mathcal{N}$ is a t-negligible t-bounded definable subset of $({}^*\mathbf{R})^n$, then

$$\int_{E\cap f^{-1}(\mathcal{N})} \varphi \,\mathrm{d} f_1 \wedge \cdots \wedge \mathrm{d} f_n \qquad \text{and} \qquad \int_{F\cap g^{-1}(\mathcal{N})} \psi \,\mathrm{d}g_1 \wedge \cdots \wedge \mathrm{d}g_n$$

are t-negligible. Indeed, let N be an integer such that the fibers of $f|_E$ and of $g|_F$ all have cardinality at most N, and let M be a t-bounded positive element such that $|\varphi|$ and $|\psi|$ are bounded by M on E and F, respectively.

Then

$$\Big|\int_{E\cap f^{-1}(\mathcal{N})} \varphi \,\mathrm{d} f_1 \wedge \cdots \wedge \mathrm{d} f_n\Big| \leq NM \int_{\mathcal{N}} \mathrm{d}T_1 \wedge \cdots \wedge \mathrm{d}T_n$$

and

$$\Big|\int_{F\cap g^{-1}(\mathcal{N})} \psi \,\mathrm{d}g_1 \wedge \cdots \wedge \mathrm{d}g_n\Big| \leq NM \int_{\mathcal{N}} \mathrm{d}T_1 \wedge \cdots \wedge \mathrm{d}T_n,$$

whence our claim.

5.3.3. Conclusion

In view of Sections 5.3.1 and 5.3.2, it is sufficient to prove that for all i the integrals

$$\int_{E\cap f^{-1}(\Omega'_i)} \varphi \,\mathrm{d} f_1 \wedge \cdots \wedge \mathrm{d} f_n \qquad \text{and} \qquad \int_{F\cap g^{-1}(\Omega'_i)} \psi \,\mathrm{d}g_1 \wedge \cdots \wedge \mathrm{d}g_n$$

agree up to a t-negligible element. So let us fix i. We denote by $\sigma_1 < \sigma_2 < \cdots < \sigma_{n_i}$ the continuous sections of $f|_E$ over Ω'_i and by $\tau_1 < \tau_2 < \cdots < \tau_{n_i}$ the continuous sections of $g|_F$ over Ω'_i. For all $x \in \Omega'_i$ and all j between 1 and n_i, the elements $\overline{\sigma_j(x)}$ and $\overline{\tau_j(x)}$ coincide: both are the jth preimage of $\overline{x}$ in $\overline{E} \cap \overline{F}$. We have by construction

$$\int_{E\cap f^{-1}(\Omega'_i)} \varphi \,\mathrm{d} f_1 \wedge \cdots \wedge \mathrm{d} f_n = \sum_j \int_{\Omega'_i} (\varphi \circ \sigma_j) \,\mathrm{d}T_1 \wedge \ldots \mathrm{d}T_n$$

and

$$\int_{F\cap g^{-1}(\Omega_i')} \psi \,\mathrm{d}g_1 \wedge \cdots \wedge \mathrm{d}g_n = \sum_j \int_{\Omega_i'} (\psi \circ \tau_j)\,\mathrm{d}T_1 \wedge \ldots \mathrm{d}T_n.$$

The difference

$$\int_{E\cap f^{-1}(\Omega_i')} \varphi \,\mathrm{d}f_1 \wedge \cdots \wedge \mathrm{d}f_n - \int_{F\cap g^{-1}(\Omega_i')} \psi \,\mathrm{d}g_1 \wedge \cdots \wedge \mathrm{d}g_n$$

is thus equal to

$$\sum_j \int_{\Omega_i'} (\varphi \circ \sigma_j - \psi \circ \tau_j)\,\mathrm{d}T_1 \wedge \cdots \wedge \mathrm{d}T_n.$$

By our assumptions on ψ and ψ, the difference $|\varphi \circ \sigma_j - \psi \circ \tau_j|$ is t-negligible for every j at every point of Ω_i'. Therefore, there exists a positive t-negligible element ε such that $|\varphi \circ \sigma_j - \psi \circ \tau_j)| \leq \varepsilon$ for all j at every point of Ω_i'. As a consequence,

$$\begin{aligned}\Big|\int_{E\cap f^{-1}(\Omega_i')} \varphi \,\mathrm{d}f_1 \wedge \cdots \wedge \mathrm{d}f_n - \int_{F\cap g^{-1}(\Omega_i')} \psi \,\mathrm{d}g_1 \wedge \cdots \wedge \mathrm{d}g_n\Big| \\ \leq n_i \varepsilon \int_{\Omega_i'} \mathrm{d}T_1 \wedge \cdots \wedge \mathrm{d}T_n,\end{aligned}$$

which ends the proof. □

5.4 COROLLARY

Let $\mathscr{X}$ be a smooth A_{r}-scheme of finite type and pure relative dimension n. Let $f = (f_1, \ldots, f_n)\colon \mathscr{X} \to \mathbf{A}^n_{A_{\mathrm{r}}}$ be an étale map factorizing through an immersion $(f, f_{n+1})\colon \mathscr{X} \hookrightarrow \mathbf{A}^{n+1}_{A_{\mathrm{r}}}$. Let E be a t-bounded semialgebraic subset of $\mathscr{X}({}^*\mathbf{R})$; we remind the reader that $\overline{E}$ denotes the image of E under the reduction map (and not its topological closure). The following are equivalent.

(i) *The image $f(E)$ is t-negligible.*

(ii) *The image $f(\overline{E})$ is of dimension less than n.*

(iii) *The reduction $\overline{E}$ is of dimension less than n.*

(iv) *For every smooth function φ of the form $\varphi_0 \circ \lambda$ with φ_0 a $\mathscr{C}^\infty$-function and λ a tuple of regular functions on $\mathscr{X}$ such that φ_0 is $(O, \lambda|_{\mathscr{X}_R})$-tame on some semialegbraic open subset O of $\mathscr{X}(R)$ containing $\overline{E}$, the integral $\int_E \varphi \,\mathrm{d}f_1 \wedge \cdots \wedge \mathrm{d}f_n$ is t-negligible.*

(v) *The integral $\int_E \mathrm{d}f_1 \wedge \cdots \wedge \mathrm{d}f_n$ is t-negligible.*

Proof

Implication (i) $\Rightarrow$ (ii) comes from the fact that $f(\overline{E}) = \overline{f(E)}$. Implication (ii) $\Rightarrow$ (iii) comes from étaleness of f. Implication (iii) $\Rightarrow$ (iv) follows from Proposition 5.3

(apply it with $\mathscr{Y} = \mathscr{X}$, $g = g$, $g_{n+1} = f_{n+1}$, and $F = \emptyset$). Implication (iv) $\Rightarrow$ (v) is obvious. Assume that (v) holds. For every i, let D_i denote the subset of $f(E)$ consisting of points having exactly i preimages on E, and let N be such that $D_i = \emptyset$ for $i > N$. We then have

$$\int_E \mathrm{d} f_1 \wedge \cdots \wedge \mathrm{d} f_n = \sum_{i=1}^{N} \int_{D_i} \mathrm{d}T_1 \wedge \cdots \wedge \mathrm{d}T_n.$$

As a consequence, $\int_{D_i} \mathrm{d}T_1 \wedge \cdots \wedge \mathrm{d}T_n$ is t-negligible for every i, so $\int_{f(E)} \mathrm{d}T_1 \wedge \cdots \wedge \mathrm{d}T_n$ is t-negligible, whence (i). $\square$

5.5

Let us keep the notation of Corollary 5.4 above. We shall say that E is t-negligible if it satisfies the equivalent conditions (i)–(v) (note that condition (iii) does not involve the functions f_i, so the notion of t-negligibility does not depend on the choice of the functions f_i). We shall say that two t-bounded definable subsets of $\mathscr{X}({}^*\mathbf{R})$ almost coincide if their symmetric difference is t-negligible, and that two definable subsets of $\mathscr{X}(R)$ almost coincide if their symmetric difference is of dimension less than n.

5.6 Lemma

Let $\mathscr{X}$ be a smooth A_{r}-scheme of finite type and pure relative dimension n. Assume that there exists an étale map $f = (f_1, \dots, f_n)\colon \mathscr{X} \to \mathbf{A}^n_{A_{\mathrm{r}}}$ factorizing through an immersion $(f, f_{n+1})\colon \mathscr{X} \hookrightarrow \mathbf{A}^{n+1}_{A_{\mathrm{r}}}$. Let E and F be two t-bounded definable subsets of $\mathscr{X}({}^*\mathbf{R})$. Then E and F are almost disjoint if and only if $\overline{E}$ and $\overline{F}$ are almost disjoint.

Proof

If $\dim(\overline{E} \cap \overline{F}) < n$, then $\dim \overline{E \cap F} < n$ because $\overline{E \cap F} \subset \overline{E} \cap \overline{F}$; thus if $\overline{E}$ and $\overline{F}$ are almost disjoint, so are E and F. Assume now that E and F are almost disjoint. Set $G = f(E \cup F)$. For every triple (i, j, k), denote by $\overline{G}_{i,j,k}$ the subset of points of $\overline{G}$ having i preimages in $\overline{E}$, j in $\overline{F}$, and k in $\overline{E} \cup \overline{F}$. By Corollary 3.14, there exists for every (i, j, k) a t-bounded definably compact definable subset $\Gamma_{i,j,k}$ of $({}^*\mathbf{R})^n$ such that $\overline{\Gamma_{i,j,k}}$ is almost equal to the definable closure of $\overline{G}_{i,j,k}$, hence is also almost equal to $\overline{G}_{i,j,k}$. By the same reasoning as in Section 5.3.1, the subset of points of $\Gamma_{i,j,k}$ having exactly i preimages in E (resp., j preimages in F; resp., k preimages in $E \cup F$) is almost equal to $\Gamma_{i,j,k}$; hence so is the intersection $\Gamma'_{i,j,k}$ of these three subsets. The family $(\Gamma'_{i,j,k})$ is an almost partition of G.

Let (i, j, k) be a triple with $k < i + j$. Since $\overline{E} \cap \overline{F}$ has dimension less than n, the set $\overline{G_{i,j,k}}$ is negligible; as a consequence, $\Gamma_{i,j,k}$ and $\Gamma'_{i,j,k}$ are t-negligible. This implies that $f(E \cap F)$ is t-negligible, whence the t-negligibility of $E \cap F$ itself. $\square$

5.7 PROPOSITION

Let $\mathscr{X}$ be as in Lemma 5.6 above, and let E and F be two t-bounded definable subsets of $\mathscr{X}(^*\mathbf{R})$.

(1) *The set E is almost equal to F if and only if $\overline{E}$ is almost equal to $\overline{F}$.*

(2) *The set $\overline{E \cap F}$ is almost equal to $\overline{E} \cap \overline{F}$.*

Proof

The proof is the same as that of Proposition 3.13, except that one uses Lemma 5.6 instead of Lemma 3.12. □

5.8 COROLLARY

Let $\mathscr{X}$ be as in Lemma 5.6, and let K be a definably compact definable subset of $\mathscr{X}(R)$. There exists a definable, definably compact and t-bounded subset E of $\mathscr{X}(^*\mathbf{R})$ such that $\overline{E}$ almost coincides with K.

Proof

By writing K as the union of its intersections with the Zariski-connected components of $\mathscr{X}_R$, we can assume that it lies on such a component X. By boundedness of K and the Henselian property of A_{r} (which ensures that any R-point of $\mathscr{X}$ can be lifted to an A_{r}-point), we can choose a t-bounded, definably compact definable subset F of $\mathscr{X}(^*\mathbf{R})$ such that $K \subset \overline{F} \subset X(R)$. By Théorème 2.7.1 of [3], we can assume that there exist finitely many regular functions $f_1, \ldots, f_m$ on $\mathscr{X}_R$ such that K is the intersection of $\overline{F}$ with the set of points x such that $f_j(x) \geq 0$ for all j. By Proposition 5.7 above, we may assume that $m = 1$, and write f instead of f_1. If f is constant on X, then the set K is either empty or the whole of $\overline{F}$ and the statement is obvious. If f is nonconstant on X, let g be a regular function on $\mathscr{X}$ that lifts f. Let E be the intersection of F and the nonnegative locus of g; it suffices to prove that $\overline{E}$ is almost equal to K. By definition, $\overline{E} \subset K$. Now let x be a point on K at which f is positive, and let ξ be any preimage of x on F. Since $f(x) > 0$, we have $g(\xi) > 0$, hence $\xi \in E$ and $x \in \overline{E}$. Thus the difference $K \setminus E$ is contained in the zero locus of f in $X(R)$ which is at most $(n-1)$-dimensional since $f|_X$ is nonconstant. □

5.9 Definition

Let X be a smooth R-scheme of finite type and pure dimension n. We shall say for short that X is *liftable* if there exists a smooth affine A_{r}-scheme $\mathscr{X}$, an isomorphism $\mathscr{X}_R \simeq X$, and $n+1$ regular functions $f_1, \ldots, f_{n+1}$ on $\mathscr{X}$ such that $(f_1, \ldots, f_{n+1})$ defines an immersion $\mathscr{X} \hookrightarrow \mathbf{A}^{n+1}_{A_{\mathrm{r}}}$ and $(f_1, \ldots, f_n)\colon \mathscr{X} \to \mathbf{A}^n_{A_{\mathrm{r}}}$ is étale.

5.10. Integral of a smooth n-form

Let X be a smooth R-scheme of finite type and pure dimension n, let K be a definable subset of $X(R)$ with definably compact definable closure, and let ω be a smooth n-form on a semialgebraic open neighborhood O of K in $X(R)$. The purpose of what follows is to define $\int_K \omega$.

5.10.1

We first assume that X is liftable and ω is of the form $\varphi(u_1, \dots, u_m)\omega_0$ almost everywhere on K, with u_i regular functions, φ an $(O, (u_1, \dots, u_m))$-tame smooth function, and where ω_0 is an algebraic n-form. Choose $\mathscr{X}$ and $f_1, \dots, f_{n+1}$ as in Definition 5.9. The sheaf $\Omega_{X/R}$ is then free with basis $(\mathrm{d} f_i|_X)_{1 \le i \le n}$; therefore, up to multiplying φ with a regular function, we might assume that $\omega_0 = (\mathrm{d} f_1 \wedge \dots \wedge \mathrm{d} f_n)|_X$.

Choose a t-bounded definable subset E of $\mathscr{X}({}^*\mathbf{R})$ such that $\overline{E}$ is almost equal to the definable closure of K (see Corollary 5.8) and for every i, choose a regular function v_i on $\mathscr{X}$ lifting u_i.

By Proposition 5.3, the integral $\int_E \varphi(v_1, \dots, v_m)\, \mathrm{d} f_1 \wedge \dots \mathrm{d} f_n$ does not depend on our various choices up to a t-negligible element. We can thus set

$$\int_K \omega = \overline{\int_E \varphi(v_1, \dots, v_m)\, \mathrm{d} f_1 \wedge \dots \mathrm{d} f_n};$$

this is an element of R. Note that if K' is any definable subset almost equal to K, then $\int_{K'} \omega = \int_K \omega$ (since the same E can be used for both computations).

The assignment $K \mapsto \int_K \omega$ is finitely additive. Indeed, if K is a finite union $\bigcup_{j \in J} K_j$ of definable subsets, we can choose for every j an almost lifting E_j of K_j; now for every subset I of J the sets $\overline{\bigcap_{j \in I} E_j}$ and $\bigcap_{j \in I} K_j$ almost coincide by Proposition 5.7, and additivity follows from additivity of integrals over the field ${}^*\mathbf{R}$.

5.10.2

We still assume that X is liftable, but we no longer assume that ω is of the form $\varphi(u_1, \dots, u_m)\omega_0$ on K. By the very definition of an n-form, there exist finitely many definably open subsets $U_1, \dots, U_r$ of $X(R)$ that cover K and such that $\omega|_{U_i}$ has the required form. By Lemma 3.2, we can write the definable closure of K as a finite union $\bigcup_{j \in J} K_j$ with each K_j definably compact and contained in U_j. By additivity, $\sum_{\emptyset \neq I \subset J} (-1)^{|I|+1} \int_{\bigcap_{j \in I} K_j} \omega$ does not depend on the choice of the sets U_j and K_j, and we can use this formula as a definition for $\int_K \omega$. The assignment $K \mapsto \int_K \omega$ remains additive in this more general setting, and $\int_K \omega$ depends only on the class of K modulo almost equality.

5.10.3

We still assume that X is liftable. Let s be an algebraic function on X, set $X' = D(s)$ (the invertibility locus of s), and assume that the definable closure of K is contained in $X'(R)$. We then have a priori two different definitions for $\int_K \omega$, the one using X and the other one using the principal open subset X', which is also (obviously) liftable. Let us show that both integrals coincide. By replacing K by its closure (to which it is almost equal), we can assume that it is definably compact.

By cutting K into finitely many sufficiently small pieces (see Lemma 3.2) and using additivity, we can assume that ω is of the form $\varphi(u_1, \ldots, u_m)\omega_0$ almost everywhere on K, with u_i regular functions on X, φ an $(O, (u_1, \ldots, u_m))$-tame smooth function, and ω_0 a section of $\Omega^n_{X/R}$ (this can be achieved since $\Omega_{X/R}$ is free because X is liftable). Lift every u_i to a regular function v_i on $\mathscr{X}$, and lift ω_0 to a relative n-form ω' on $\mathscr{X}$.

Let us choose data $(\mathscr{X}, f_1, \ldots, f_{n+1})$ that witness the liftability of X. Lift every u_i to a regular function v_i on $\mathscr{X}$, lift ω_0 to a relative n-form ω' on $\mathscr{X}$, and lift s to a regular function σ on $\mathscr{X}$. Set $\mathscr{X}' = D(\sigma)$. Then $(\mathscr{X}', f_1, \ldots, f_n, f_{n+1})$ witnesses the liftability of $\mathscr{X}'$. Now choose a t-bounded definable subset of $\mathscr{X}'(^*\mathbf{R})$ that almost lifts K. Then it is definable, t-bounded and an almost lifting of K as a subset of $\mathscr{X}(^*\mathbf{R})$ as well. Therefore, the X and the X' version of $\int_K \omega$ both are equal to the class of $\int_E \varphi(v_1, \ldots, v_m)\omega'$ modulo the t-negligible elements.

5.10.4

The scheme X is no longer assumed to be liftable, but we assume that there exist two liftable affine open subsets X' and X'' of X such that the definable closure of K is contained in $X'(R) \cap X''(R)$. We then have a priori two different definitions for $\int_K \omega$, the one using X' and the other one using X''. We want to prove that they coincide. By replacing K by its closure (to which it is almost equal), we can assume that it is definably compact.

Let us first note the following. Let x be a point of $X' \cap X''$. Choose an affine neighborhood Y of x in $X' \cap X''$ equal to $D(s)$ as a subset of X', for some regular function s on X'. Now choose an affine neighborhood Z of x in Y equal to $D(w)$ as a subset of X'', for some regular function w on X''. The restriction of w to Y is equal to a/s^ℓ for some $\ell \geq 0$ and some regular function a on X'; as a consequence, $Z = D(as)$ as a subset of X''.

Hence we can cover $X' \cap X''$ by finitely many open subschemes $Y_1, \ldots, Y_r$, each of which is principal in both X' and X''. Now write $K = \bigcup K_i$ with every K_i definable, definably compact and contained in Y_i (see Lemma 3.2). For every nonempty subset I of $\{1, \ldots, r\}$, it follows from Section 5.10.3 that $\int_{\bigcap_{i \in I} K_i} \omega$ does not depend on whether one is working with X' or X'' (because it can be computed working with

Y_j, where j is any element of I). By additivity, it follows that $\int_K \omega$ also does not depend on whether one is working with X' or X''.

5.10.5

Now let us explain how to define $\int_K \omega$ in general. Let K' be the closure of K, which is definably compact. We choose a finite cover $(X_i)_{i\in I}$ of X by liftable open subschemes (which is possible since X is smooth). We then write K' as a finite union $\bigcup K_i$, where every K_i is a definably compact semialgebraic subset of $X_i(R)$ (see Lemma 3.2).

We then set

$$\int_K \omega = \sum_{\emptyset\neq J\subset I} (-1)^{|J|+1} \int_{\bigcap_{i\in J} K_i} \omega,$$

which makes sense because, as it follows straightforwardly from the above, it does not depend on (X_i) nor on (K_i).

5.11

Let X be a smooth R-scheme of finite type and pure dimension n. It follows from our construction that

$$(K,\omega) \mapsto \int_K \omega$$

(where K is a semialgebraic subset of $X(R)$ with definably compact closure and ω is an n-form defined on a definable neighborhood of K) is $\mathbf{R}$-linear in ω, additive in K, and that it depends on K only up to almost equality.

We can extend this definition to forms with coefficients in a reasonable class of functions (like piecewise smooth) by requiring everywhere in the above that φ belongs to the involved class (instead of being smooth) and satisfies some tameness condition. For instance, $\int_K |\omega|$ makes sense (and is nonnegative; see Section 3.4.4). It follows from the definition that $\int_K \omega$ depends only on $\omega|_K$; in particular, it is zero if ω vanishes almost everywhere on K. We can thus extend the definition of $\int_K \omega$ when we only assume that there exists a definable subset K' of K with definably compact closure such that ω vanishes on $K \setminus K'$.

And of course, we can also define by linearity the integrals of complex-valued forms (see Section 4.1.4).

5.12. The complex case

We now consider a smooth C-scheme of finite type X and pure dimension n, and a complex-valued (n,n)-form ω with coefficients belonging to a reasonable class of functions defined in a semialgebraic open neighborhood of a semialgebraic subset K

of $X(C)$. Assume that there exists a semialgebraic subset K' of K with definably compact closure such that ω vanishes on $K \setminus K'$.

Then $\int_K \omega$ is well defined. Its computation requires (among other things) to lift locally $\mathrm{R}_{C/R} X$ to a smooth A_{r}-scheme and ω to a $(2n)$-form on this scheme, which can be achieved by lifting locally X to a smooth A-scheme and ω to an (n,n)-form on this scheme.

6. Archimedean and non-Archimedean complexes of forms

6.1

We denote by λ the element $-\log|t|$ of $R_{>0}$, and by Log the normalized logarithm function $a \mapsto \log a/\lambda$ from $R_{>0}$ to R.

We recall that C is equipped with a non-Archimedean absolute value $|\cdot|_\flat$, which sends a nonzero element z to $\tau^{\mathrm{std}(\frac{\log|z|}{\log|t|})}$ where τ is an element of $(0,1)$ which has been fixed once and for all, and where $\mathrm{std}(\cdot)$ denotes the standard part (see Section 2.7). We set $\lambda_\flat = -\log\tau = -\log|t|_\flat \in \mathbf{R}_{>0}$, and we denote by $\mathrm{Log}_\flat$ the normalized logarithm function $a \mapsto \log a/\lambda_\flat$ from $\mathbf{R}_{>0}$ to $\mathbf{R}$.

If a is any element of $C^\times$, then it follows from the definitions that

$$\mathrm{Log}_\flat |a|_\flat = \mathrm{std}\big(\mathrm{Log}\,|a|\big).$$

6.2. Analytification of C-schemes

The field C is a complete non-Archimedean field, so Berkovich geometry makes sense over it.

Let X be a C-scheme of finite type, and let X^{an} denote its Berkovich analytification. Let x be a point of X^{an}. In the proof of our main theorem, we shall use the fact that x has a basis of open (resp., affinoid) neighborhoods V in X^{an} satisfying the following: there exists an affine open subscheme Ω of X such that V is an open subset (resp., a Weierstrass domain) of Ω^{an} that admits a description by a system of inequalities of the form

$$|\varphi_1|_\flat < R_1, \dots, |\varphi_n|_\flat < R_n, \quad \text{resp., } |\varphi_1|_\flat \le R_1, \dots, |\varphi_n|_\flat \le R_n,$$

where the functions φ_i belong to $\mathscr{O}(\Omega)$, and with R_i positive real numbers.

Let us prove it. We first chose an open affine subscheme U of X with $x \in U^{\mathrm{an}}$, a family $(f_1, \dots, f_n)$ of regular functions on U that generate $\mathscr{O}(U)$ over C, and let R be a positive real number such that $|f_i(x)|_\flat < R$ for all i; let W be the Weierstrass domain of U^{an} defined by the inequalities $|f_i|_\flat \le R$. Now it follows from the general theory of Berkovich spaces that x has a basis of open (resp., affinoid) neighborhoods described by a system of inequalities of the form

$$
\begin{aligned}
&|f_1|_\flat < R, \dots, |f_n|_\flat < R, \qquad |g_1|_\flat < r_1, \dots, |g_m|_\flat < r_m,\\
&|h_1|_\flat > s_1, \dots, |h_\ell|_\flat > s_\ell,\\
\text{resp.,}\quad &|f_1|_\flat \le R, \dots, |f_n|_\flat \le R, \qquad |g_1|_\flat \le r_1, \dots, |g_m|_\flat \le r_m,\\
&|h_1|_\flat \ge s_1, \dots, |h_\ell|_\flat \ge s_\ell
\end{aligned}
$$

with g_i and h_i analytic functions on W, and s_i and r_i positive real numbers. But $\mathscr{O}(U)$ is dense in $\mathscr{O}(W)$, so we can assume by approximation that the functions g_i and h_i belong to $\mathscr{O}(U)$. Then the domain described by the above system of inequalities can also be described as the locus of validity of

$$
\begin{aligned}
&|f_1|_\flat < R, \dots, |f_n|_\flat < R, \qquad |g_1|_\flat < r_1, \dots, |g_m|_\flat < r_m,\\
&|h_1^{-1}|_\flat < s_1^{-1}, \dots, |h_\ell^{-1}|_\flat < s_\ell^{-1},\\
\text{resp.,}\quad &|f_1|_\flat \le R, \dots, |f_n|_\flat \le R, \qquad |g_1|_\flat \le r_1, \dots, |g_m|_\flat \le r_m,\\
&|h_1^{-1}|_\flat \le s_1^{-1}, \dots, |h_\ell^{-1}|_\flat \le s_\ell^{-1}
\end{aligned}
$$

on $D(h_1 h_2 \dots h_\ell)^{\mathrm{an}}$, whence our claim.

6.3. Two complexes of differential forms

We fix a smooth C-scheme of finite type X and pure dimension n.

6.3.1

Let us begin with some notation. Let U be an open subscheme of X, and let $f = (f_1, \dots, f_m)$ be a family of regular functions on U. Let I and J be two subsets of $\{1, \dots, m\}$. We shall denote by $\mathscr{S}^{I,J,(f_i)}$ the set of pairs (V, φ) where:

(a) V is an open subset of $(\mathbf{R} \cup \{-\infty\})^m$, defined by $\mathbf{Q}$-linear inequalities such that V_R contains $(\operatorname{Log}|f_1|, \dots, \operatorname{Log}|f_m|)(U(C))$;

(b) φ is a reasonably smooth function on V which is $(I \cup J)$-vanishing.

We identify two pairs (V, φ) and (V', φ') if φ and φ' agree on $V \cap V'$; therefore, we shall most of the time omit mentioning V, and elements of $\mathscr{S}^{I,J,(f_i)}$ will be called $(I \cup J)$-*vanishing reasonably smooth functions*.

We denote by $\mathscr{S}_\flat^{I,J,(f_i)}$ the set of pairs (V, φ) satisfying condition

($a_\flat$) V is an open subset of $(\mathbf{R} \cup \{-\infty\})^m$, defined by $\mathbf{Q}$-linear inequalities and containing $(\operatorname{Log}_\flat|f_1|_\flat, \dots, \operatorname{Log}_\flat|f_m|_\flat)(U^{\mathrm{an}})$

and condition (b) above. Here also, we identify two pairs (V, φ) and (V', φ') if φ and φ' agree on $V \cap V'$ and elements of $\mathscr{S}_\flat^{I,J,(f_i)}$ will be called $(I, J)_\flat$-*vanishing smooth functions*.

Note that $\mathscr{S}^{I,J,(f_i)} \subset \mathscr{S}_\flat^{I,J,(f_i)}$.

6.3.2. The nonstandard Archimedean complex
Let U be a Zariski open subset of $X(C)$. Let us denote by $\mathsf{A}^{p,q}_{\mathrm{presh}}(U)$ the set of those $(p+q)$-smooth forms ω on $U(C)$ for which there exist:

- a finite family $(f_1,\ldots,f_m)$ of regular functions on U;
- for every pair (I,J) with I and J two subsets of $\{1,\ldots,m\}$ of respective cardinality p and q, an $(I\cup J)$-vanishing reasonably smooth function $\varphi_{I,J}\in\mathscr{S}^{I,J,(f_i)}$

such that

$$\omega=\sum_{I,J}\varphi_{I,J}\bigl(\mathrm{Log}\,|f_1|,\ldots,\mathrm{Log}\,|f_m|\bigr)\,\mathrm{d}\,\mathrm{Log}\,|f_I|\wedge\mathrm{d}\,\mathrm{Arg}\,f_J,$$

where $\mathrm{d}\,\mathrm{Log}\,|f|_I$ stands for $\mathrm{d}\,\mathrm{Log}\,|f_{i_1}|\wedge\cdots\wedge\mathrm{d}\,\mathrm{Log}\,|f_{i_p}|$ if $i_1<i_2<\cdots<i_p$ are the elements of I, and $\mathrm{d}\,\mathrm{Arg}\,f_J$ stands for $\frac{\mathrm{d}\arg f_{j_1}}{2\pi}\wedge\cdots\wedge\frac{\mathrm{d}\arg f_{j_q}}{2\pi}$ if $j_1<j_2<\cdots<j_q$ are the elements of J.

We denote by $\mathsf{A}^{p,q}$ the sheaf on X^{Zar} associated to the presheaf $\mathsf{A}^{p,q}_{\mathrm{presh}}$, and by $\mathsf{A}^{\bullet,\bullet}$ the direct sum $\bigoplus_{p,q}\mathsf{A}^{p,q}$.

We set for short $\mathsf{A}^0=\mathsf{A}^{0,0}$. By construction, $\mathsf{A}^0(X)$ is the subsheaf (of C-algebras) of the pushforward of $C\otimes_R\mathscr{C}^\infty_X$ to X^{Zar}, whose sections are the smooth functions that are locally on X^{Zar} of the form $\varphi(\mathrm{Log}\,|f_1|,\ldots,\mathrm{Log}\,|f_m|)$ for some finite family $(f_1,\ldots,f_m)$ of regular functions and some reasonably smooth function φ on a suitable open subset of $(\mathbf{R}\cup\{-\infty\})^m$.

The sheaf $\mathsf{A}^{\bullet,\bullet}$ has a natural structure of bigraded A^0-algebra; it follows from Section 4.5.3 that the differentials d and $\mathrm{d}^\sharp$ induce two differentials on $\mathsf{A}^{\bullet,\bullet}$, which are still denoted by d and $\mathrm{d}^\sharp$. The differential d is of bidegree $(1,0)$ and maps a form

$$\varphi\bigl(\mathrm{Log}\,|f_1|,\ldots,\mathrm{Log}\,|f_m|\bigr)\,\mathrm{d}\,\mathrm{Log}\,|f_I|\wedge\mathrm{d}\,\mathrm{Arg}\,f_J$$

to

$$\sum_{1\le i\le m}\frac{\partial\varphi}{\partial x_i}\bigl(\mathrm{Log}\,|f_1|,\ldots,\mathrm{Log}\,|f_m|\bigr)\,\mathrm{d}\,\mathrm{Log}\,|f_i|\wedge\mathrm{d}\,\mathrm{Log}\,|f_I|\wedge\mathrm{d}\,\mathrm{Arg}\,f_J.$$

The differential $\mathrm{d}^\sharp$ is of bidegree $(0,1)$ and maps a form

$$\varphi\bigl(\mathrm{Log}\,|f_1|,\ldots,\mathrm{Log}\,|f_m|\bigr)\,\mathrm{d}\,\mathrm{Log}\,|f_I|\wedge\mathrm{d}\,\mathrm{Arg}\,f_J$$

to

$$\sum_{1\le i\le m}\frac{\partial\varphi}{\partial x_i}\bigl(\mathrm{Log}\,|f_1|,\ldots,\mathrm{Log}\,|f_m|\bigr)\frac{\mathrm{d}\arg}{2\pi}f_i\wedge\mathrm{d}\,\mathrm{Log}\,|f_I|\wedge\mathrm{d}\,\mathrm{Arg}\,f_J.$$

The operator J also acts on $\mathsf{A}^{\bullet,\bullet}$; it maps a form

$$\varphi\big(\operatorname{Log}|f_1|,\dots,\operatorname{Log}|f_m|\big)\,\mathrm{d}\operatorname{Log}|f_I|\wedge \mathrm{d}\operatorname{Arg} f_J$$

to

$$(-1)^q(2\pi)^{p-q}\varphi\big(\operatorname{Log}|f_1|,\dots,\operatorname{Log}|f_m|\big)\,\mathrm{d}\operatorname{Arg} f_I\wedge \mathrm{d}\operatorname{Log}|f_J|$$

and acts trivially on $\mathsf{A}^{n,n}$.

6.3.3. The non-Archimedean complex

Let U be a Zariski-open subset of X. Let us denote by $\mathsf{B}^{p,q}_{\text{presh}}(U)$ the set of those (p,q)-smooth forms ω on U^{an} in the sense of [6] for which there exist:

- a finite family $(f_1,\dots,f_m)$ of regular functions on U;
- for every pair (I,J) with I and J two subsets of $\{1,\dots,m\}$ of respective cardinality p and q, an $(I,J)_\flat$-vanishing reasonably smooth function $\varphi_{I,J}\in\mathscr{S}_\flat^{I,J,(f_i)}$

such that

$$\omega=\sum_{I,J}\varphi_{I,J}\big(\operatorname{Log}_\flat|f_1|_\flat,\dots,\operatorname{Log}_\flat|f_m|_\flat\big)\,\mathrm{d}'\operatorname{Log}_\flat|f_I|_\flat\wedge \mathrm{d}''\log|f_J|_\flat,$$

where $\mathrm{d}'\operatorname{Log}_\flat|f_I|_\flat$ stands for $\mathrm{d}'\operatorname{Log}_\flat|f_{i_1}|_\flat\wedge\dots\wedge \mathrm{d}'\operatorname{Log}_\flat|f_{i_p}|_\flat$ if $i_1<i_2<\dots<i_p$ are the elements of I, and similarly for $\mathrm{d}''\log|f_J|_\flat$.

We denote by $\mathsf{B}^{p,q}$ the sheaf on X^{Zar} associated to the presheaf $\mathsf{B}^{p,q}_{\text{presh}}$. We denote by $\mathsf{B}^{\bullet,\bullet}$ the direct sum $\bigoplus_{p,q}\mathsf{B}^{p,q}$. We set for short $\mathsf{B}^0=\mathsf{B}^{0,0}$. By construction, B^0 is the subsheaf (of C-algebras) of the pushforward of $C\otimes_R A^0_{X^{\text{an}}}$ to X^{Zar}, whose sections are the smooth functions that are locally on X^{Zar} of the form $\varphi(\operatorname{Log}_\flat|f_1|_\flat,\dots,\operatorname{Log}_\flat|f_m|_\flat)$ for some finite family $(f_1,\dots,f_m)$ of regular functions and some reasonably smooth function φ on a suitable open subset of $(\mathbf{R}\cup\{-\infty\})^m$.

The sheaf $\mathsf{B}^{\bullet,\bullet}$ is a bigraded B^0-algebra which is stable under d' and d''.

6.4 Remark

Every (p,q)-form in the sense of [6] can be written locally *for the Berkovich topology* as a sum

$$\sum\psi_{I,J}\big(\log|g_i|_\flat,\dots,\log|g_m|_\flat\big)\,\mathrm{d}'\log|g_I|_\flat\wedge \mathrm{d}''\log|g_J|_\flat,$$

where the $\psi_{I,J}$ are smooth and with g_i *invertible* analytic functions.

By the very definition of an $(I\cup J)$-vanishing reasonably smooth function, a section

$$\omega=\sum_{I,J}\varphi_{I,J}\big(\operatorname{Log}_\flat|f_1|_\flat,\dots,\operatorname{Log}_\flat|f_m|_\flat\big)\,\mathrm{d}'\operatorname{Log}_\flat|f_I|_\flat\wedge \mathrm{d}''\log|f_J|_\flat$$

of $\mathsf{B}^{p,q}_{\mathrm{presh}}$ fulfills this condition, because locally for the Berkovich topology, every nonzero term of the sum can be rewritten by involving only the functions f_i which are invertible. But the reader should be aware that ω cannot in general be written locally *for the Zariski topology* as a sum

$$\sum \psi_{I,J}\big(\log|g_i|_\flat,\dots,\log|g_m|_\flat\big)\,\mathrm{d}'\log|g_I|_\flat\wedge\mathrm{d}''\log|g_J|_\flat$$

with g_i invertible algebraic functions.

(Consider for example a non-zero smooth function φ on $\mathbf{R}$ that vanishes on $(-\infty, A)$ for some A, and the section $\varphi(\mathrm{Log}_\flat|T|_\flat)\,\mathrm{d}'\,\mathrm{Log}_\flat|T|_\flat\wedge\mathrm{d}''\log|T|_\flat$ of $\mathsf{B}^{1,1}$ on $\mathbf{A}^{1,\mathrm{an}}_C$.)

7. Pseudopolyhedra

The purpose of this section is to describe the domains on which we shall integrate our forms, in both the Archimedean and non-Archimedean settings. These domains will be the preimages under functions of the form $\mathrm{Log}|f|$ (resp., $\mathrm{Log}_\flat|f|_\flat$) of some specific subsets of $(R\cup\{-\infty\})^n$ (resp., $(\mathbf{R}\cup\{-\infty\})^n$) that we call *pseudopolyhedra*.

7.1 Definition

Let S be a nontrivial divisible ordered abelian group with additive notation (in practice we shall consider only cases where S underlies a real closed field). A subset of $(S\cup\{-\infty\})^m$ is called a *pseudopolyhedron* if it is a finite union of sets of the form

$$\Big\{(x',x'')\in\prod_{i\in I}[-\infty,b_i]\times\prod_{i\in J}[a_i,b_i]\ \text{s.t.}\ \varphi_1(x'')\leq 0,\dots,\varphi_r(x'')\leq 0\Big\}$$

where

- I and J are subsets of $\{1,\dots,m\}$ that partition it;
- for $1\leq i\leq m$, a_i and b_i are elements of S;
- for $1\leq j\leq r$, φ_j is an affine form whose linear part has coefficients in $\mathbf{Q}$.

A subset of S^m is a *polyhedron* if this is a pseudopolyhedron of $(S\cup\{-\infty\})^m$. This amounts to requiring that it be a finite union of sets of the form

$$\Big\{x\in\prod_{i\in\{1,\dots,n\}}[a_i,b_i]\ \text{s.t.}\ \varphi_1(x)\leq 0,\dots,\varphi_r(x)\leq 0\Big\},$$

with a_i, b_i, and φ_i as above.

7.1.1

Let X be an analytic space over C, and let $f_1,\dots,f_m$ be analytic functions on X. Let P be a pseudopolyhedron of $(\mathbf{R}\cup\{-\infty\})^m$. The set

$$\big(\mathrm{Log}_\flat|f_1|_\flat,\dots,\mathrm{Log}_\flat|f_m|_\flat\big)^{-1}(P)$$

is a closed analytic domain of X.

7.1.2

Let P be a pseudopolyhedron of $(R \cup \{-\infty\})^m$. The subset

$$|t|^{-P} := \left\{|t|^{-x}, x \in P\right\}$$

(with the convention that $|t|^{\infty} = 0$) is an RCF-definable subset of $R^m_{\geq 0}$; indeed, it is defined by monomial inequalities. One sees easily that if P depends DOAG-definably on some set of parameters $a_1, \dots, a_\ell \in R$, then $|t|^{-P}$ depends RCF-definably on $|t|^{a_1}, \dots, |t|^{a_\ell}$.

7.1.3

In practice, we shall encounter pseudopolyhedra over the real closed fields $\mathbf{R}$ and R.

7.1.3.1. Let $P \subset (\mathbf{R} \cup \{-\infty\})^m$ be a pseudopolyhedron over $\mathbf{R}$. It gives rise by base change to a pseudopolyhedron over $P_R \subset (R \cup \{-\infty\})^m$ over the field R which has the following properties: it can be written as a finite union of subsets of $(R \cup \{-\infty\})^m$ admitting a description like in Definition 7.1 with the additional requirement that all the elements a_i and b_i are bounded; we shall say for short that such a pseudopolyhedron is bounded.

7.1.3.2. Let Π be a bounded pseudopolyhedron in $(R \cup \{-\infty\})^m$. For every x in $R \cup \{-\infty\}$ which is either negative unbounded or equal to $-\infty$, we set $\operatorname{std}(x) = -\infty$; with this convention, the definition

$$\operatorname{std}(\Pi) := \left\{\left(\operatorname{std}(x_1), \dots, \operatorname{std}(x_m)\right)\right\}_{(x_1,\dots,x_m)\in\Pi}$$

makes sense, and $\operatorname{std}(\Pi)$ is a pseudopolyhedron of $(\mathbf{R} \cup \{-\infty\})^m$.

To see this, we can assume that Π is of the form

$$\left\{(x', x'') \in \prod_{i\in I}[-\infty, b_i] \times \prod_{i\in J}[a_i, b_i] \text{ s.t. } \varphi_1(x'') \leq 0, \dots, \varphi_r(x'') \leq 0\right\},$$

where the notation is as in Definition 7.1 and where the elements a_i and b_i are all bounded. Set

$$\Theta = \left\{x \in \prod_{i\in J}[a_i, b_i] \text{ s.t. } \varphi_1(x) \leq 0, \dots, \varphi_r(x) \leq 0\right\}.$$

This is a bounded polyhedron of R^J and one has

$$\operatorname{std}(\Pi) = \left(\prod_{i\in I}\left[-\infty, \operatorname{std}(b_i)\right]\right) \times \operatorname{std}(\Theta).$$

So it suffices to prove that $\mathrm{std}(\Theta)$ is a polyhedron. Otherwise said, we can assume that $I = \emptyset$ and $J = \{1, \ldots, m\}$ and it suffices to show that $\mathrm{std}(\Pi)$ is a polyhedron.

In fact we shall prove more generally that $\mathrm{std}(\Pi)$ is a polyhedron when Π is any bounded DOAG-definable subset of R^m. We use induction on m; there is nothing to prove if $m = 0$. Assume now that $m > 0$ and that the result holds for integers less than m. By cell decomposition for an o-minimal theory, we can assume that Π is an open cell. So there exists an open DOAG-definable subset Δ of R^{m-1} and two DOAG-definable functions λ and μ from Δ to R such that $\lambda < \mu$ on Δ and Π is equal to the set of those m-tuples $(x_1, \ldots, x_m) \in R^m$ such that

$$(x_1, \ldots, x_{m-1}) \in \Delta \qquad \text{and} \qquad \lambda(x_1, \ldots, x_{m-1}) < x_m < \mu(x_1, \ldots, x_{m-1})\}.$$

Up to refining the original cell decomposition, we can even assume that λ and μ are affine with their linear parts having coefficients in $\mathbf{Q}$.

Since the cell Π is bounded, its projection Δ onto R^{m-1} is bounded as well, and the constant terms of both λ and μ are bounded too, thus the standard parts $\mathrm{std}(\lambda)$ and $\mathrm{std}(\mu)$ make sense as affine functions from $\mathbf{R}^{m-1} \to \mathbf{R}$, with linear parts having coefficients in $\mathbf{Q}$.

Now a direct computation shows that $\mathrm{std}(\Pi)$ is equal to the set of those m-tuples $(x_1, \ldots, x_m) \in \mathbf{R}^m$ such that

$$(x_1, \ldots, x_{m-1}) \in \mathrm{std}(\Delta) \qquad \text{and}$$
$$\mathrm{std}(\lambda)(x_1, \ldots, x_{m-1}) \leq x_m \leq \mathrm{std}(\mu)(x_1, \ldots, x_{m-1})\}.$$

Since $\mathrm{std}(\Delta)$ is a polyhedron of $\mathbf{R}^{m-1}$ by our induction hypothesis, we are done.

7.2

Let U be a Zariski-open subset of X, let $g_1, \ldots, g_\ell$ be regular functions on U, and let P be a pseudopolyhedron of $(\mathbf{R} \cup \{-\infty\})^\ell$. Let Q be the closed analytic domain $(\mathrm{Log}_\flat |g|_\flat)^{-1}(P)$ of U^{an} (with $g = (g_1, \ldots, g_\ell)$). A point x of $U(C)$ belongs to Q if and only if $\mathrm{Log}_\flat |g(x)|_\flat \in P$, which is equivalent to

$$\frac{-\log(\tau^{\mathrm{std}(\frac{\log|g(x)|}{\log|t|})})}{\log \tau} \in P,$$

which we may rewrite as

$$-\mathrm{std}\Big(\frac{\log|g(x)|}{\log|t|}\Big) \in P$$

or equivalently as

$$\mathrm{Log}\big|g(x)\big| \in P_R + \mathfrak{n}^m,$$

where we denote by $\mathfrak{n}$ the set of negligible elements of R.

7.3 Notation

If Π is a pseudopolyhedron of $(R \cup \{-\infty\})^\ell$ for some ℓ and if a is a nonnegative element of R, we shall denote by Π_a the pseudopolyhedron $\Pi + [-a, a]^\ell$. If Π and a are bounded, then Π_a is bounded as well.

7.4 LEMMA

Let X be a C-scheme of finite type, let $g\colon X \to \mathbf{A}^\ell_C$ be a morphism, and let Π be a bounded pseudopolyhedron of $(R \cup \{-\infty\})^\ell$. The following are equivalent:

(i) *the analytic domain* $(\mathrm{Log}_\flat |g|_\flat)^{-1}(\mathrm{std}(\Pi))$ *of X^{an} is compact;*

(ii) *there exists a positive* standard *number ε such that the semialgebraic subset* $(\mathrm{Log}\,|g|)^{-1}(\Pi_\varepsilon)$ *of $X(C)$ is definably compact.*

Proof

Choose a finite affine open cover (X_i) of X and for each i, a finite family (f_{ij}) of regular functions on X_i that generate $\mathscr{O}_X(X_i)$ as a C-algebra. For every i and every positive bounded element M of R (resp., every positive real number M), denote by K_i^M (resp., $K_{i\flat}^M$) the subset of $X_i(C)$ consisting of points at which $\mathrm{Log}\,|f_{ij}| \le M$ for all j (resp., the subset of X_i^{an} consisting of points at which $\mathrm{Log}_\flat |f_{ij}|_\flat \le M$ for all j). For every positive real number M and every positive real number ε, we have the inclusions

$$K_i^M \subset K_{i\flat}^M(C) \subset K_i^{M+\varepsilon}.$$

Assume that (i) holds. As $(\mathrm{Log}_\flat |g|_\flat)^{-1}(\mathrm{std}(\Pi))$ is compact, it is contained in $\bigcup_i K_{i\flat}^M$ for some positive real number M.

Let a be a positive infinitesimal element of R. The subset $(\mathrm{Log}\,|g|)^{-1}(\Pi_a)$ of $X(C)$ is contained in $(\mathrm{Log}_\flat |g|_\flat)^{-1}(\mathrm{std}(\Pi_a)) = (\mathrm{Log}_\flat |g|_\flat)^{-1}(\mathrm{std}(\Pi))$; it is thus contained in the definably compact semialgebraic subset $\bigcup_i K_i^{M+1}$.

Let I be the set of positive elements a of R such that $(\mathrm{Log}\,|g|)^{-1}(\Pi_a) \subset \bigcup_i K_i^{M+1}$. This is a definable subset of $R_{>0}$ which contains by the above every positive infinitesimal element; thus it contains some standard positive element ε. The semialgebraic subset $(\mathrm{Log}\,|g|)^{-1}(\Pi_\varepsilon)$ of $X(C)$ is closed by its very definition, and is contained in the definably compact semialgebraic subset $\bigcup_i K_i^{M+1}$ by the choice of ε, so it is definably compact; thus (ii) holds.

Conversely, assume that (ii) holds. Then there exists a positive real number M such that $(\mathrm{Log}\,|g|)^{-1}(\Pi_\varepsilon) \subset \bigcup_i K_i^M$.

The set of C-points of $(\mathrm{Log}_\flat |g|_\flat)^{-1}(\Pi)$ is contained in $(\mathrm{Log}\,|g|)^{-1}(\Pi_\varepsilon)$, hence in $\bigcup_i K_i^M$. The latter is itself contained in the set of C-points of $\bigcup_i K_{i\flat}^M$; thus $(\mathrm{Log}_\flat |g|_\flat)^{-1}(\Pi) \subset \bigcup_i K_{i\flat}^M$, which implies that $(\mathrm{Log}_\flat |g|_\flat)^{-1}(\Pi)$ is compact. $\square$

7.5 Notation

Let X and g be as in Lemma 7.4 above. The set of bounded pseudopolyhedra Π of $(R \cup \{-\infty\})^\ell$ such that the equivalent assertions (i) and (ii) of Lemma 7.4 hold will be denoted by $\Theta(g)$. For any $\Pi \in \Theta(g)$, we will denote by $\Lambda(g, \Pi)$ the set of positive real numbers ε as in (ii).

7.6 Remark

Let X and g be as in Lemma 7.4 above, and let $\Pi \in \Theta(g)$. The set $\Lambda(g, \Pi)$ is nonempty by definition; choose ε therein. If η is any real number in $(0, \varepsilon)$, then it is clear that $\Pi_\eta \in \Theta(g)$ and that $(\varepsilon - \eta) \in \Lambda(g, \Pi_\eta)$.

8. Main theorem: Statement and consequences

8.1 THEOREM

Let X be a smooth scheme over C of pure dimension n. There exists a unique morphism of sheaves of bigraded differential $\mathbf{R}$*-algebras on* X^{Zar}

$$\mathsf{A}^{\bullet,\bullet} \longrightarrow \mathsf{B}^{\bullet,\bullet}$$

$$\omega \longmapsto \omega_\flat$$

such that for every Zariski-open subset U of X, every finite family $(f_1, \ldots, f_m)$ of regular functions on U, every pair (I, J) of subsets of $\{1, \ldots, m\}$, and every $(I \cup J)$-vanishing reasonably smooth function φ in $\mathscr{S}^{I,J,(f_i)}$, one has

$$\begin{aligned}&\big[\varphi\big(\mathrm{Log}\,|f_1|, \ldots, \mathrm{Log}\,|f_m|\big)\,\mathrm{d}\,\mathrm{Log}\,|f_I| \wedge \mathrm{d}\,\mathrm{Arg}\, f_J\big]_\flat \\ &\qquad = \varphi\big(\mathrm{Log}_\flat\,|f_1|_\flat, \ldots, \mathrm{Log}_\flat\,|f_m|_\flat\big)\,\mathrm{d}'\,\mathrm{Log}_\flat\,|f_I|_\flat \wedge \mathrm{d}''\log|f_J|_\flat.\end{aligned}$$

Moreover, this morphism enjoys the following properties; let U be a Zariski-open subset of X, and let $\omega \in \mathsf{A}^{p,q}(U)$.

(1) *Assume that the support of ω is contained in some definably compact semialgebraic subset of $U(C)$. Then $\omega_\flat$ is compactly supported.*
We assume from now on that $p = q = n$.

(2) *Let $g\colon U \to \mathbf{A}^\ell_C$ be a morphism, and let Π be an element of $\Theta(g)$. The integral $\int_{(\mathrm{Log}\,|g|)^{-1}(\Pi)} |\omega|$ is bounded, which implies that $\int_{(\mathrm{Log}\,|g|)^{-1}(\Pi)} \omega$ is bounded too.*

(3) *Let (V_i) be a finite family of Zariski-open subsets of U; for every i, let g_i be a morphism from $V_i \to \mathbf{A}^{\ell_i}_C$, and let Π_i be an element of $\Theta(g_i)$. Then*

$$\mathrm{std}\Big(\int_{\bigcup_i (\mathrm{Log}\,|g_i|)^{-1}(\Pi_{i,\varepsilon})} \omega\Big) \longrightarrow \int_{\bigcup_i (\mathrm{Log}_\flat\,|g_i|_\flat)^{-1}(\mathrm{std}(\Pi_i))} \omega_\flat \qquad \text{(a)}$$

and

$$\operatorname{std}\Big(\int_{\bigcup_i (\operatorname{Log}|g_i|)^{-1}(\Pi_{i,\varepsilon})} |\omega|\Big) \longrightarrow \int_{\bigcup_i (\operatorname{Log}_\flat |g_i|_\flat)^{-1}(\operatorname{std}(\Pi_i))} |\omega_\flat|_\flat \tag{b}$$

when the positive standard *number* ε *belongs to* $\bigcap_i \Lambda(g_i, \Pi_i)$ *and tends to zero.*

Moreover, there exists a positive negligible element $\alpha \in R$ *such that*

$$\operatorname{std}\Big(\int_{\bigcup_i (\operatorname{Log}|g_i|)^{-1}(\Pi_{i,\varepsilon})\setminus\bigcup_i (\operatorname{Log}|g_i|)^{-1}(\Pi_{i,\alpha})} |\omega|\Big) \longrightarrow 0 \tag{c}$$

when the positive standard *number* ε *belongs to* $\bigcap_i \Lambda(g_i, \Pi_i)$ *and tends to zero.*

(4) *Assume that the support of* ω *is contained in a definably compact semialgebraic subset of* $U(C)$*, which implies by (1) that* $\omega_\flat$ *is compactly supported. Then* $\int_{U(C)} |\omega|$ *is bounded and*

$$\operatorname{std}\Big(\int_{U(C)} \omega\Big) = \int_{U^{\mathrm{an}}} \omega_\flat \tag{d}$$

and

$$\operatorname{std}\Big(\int_{U(C)} |\omega|\Big) = \int_{U^{\mathrm{an}}} |\omega_\flat|_\flat. \tag{e}$$

8.2 Remark

Statement (3c) has the following consequence. Assume that we are given for every small enough positive standard ε in $\bigcap_i \Lambda(g_i, \Pi_i)$ a semialgebraic subset D_ε of $U(C)$ satisfying

$$\bigcup_i \big(\operatorname{Log}|g_i|\big)^{-1}(\Pi_{i,\alpha}) \subset D_\varepsilon \subset \bigcup_i \big(\operatorname{Log}|g_i|\big)^{-1}(\Pi_{i,\varepsilon}).$$

Then

$$\operatorname{std}\Big(\int_{D_\varepsilon} \omega\Big) \longrightarrow \int_{\bigcup_i (\operatorname{Log}_\flat |g_i|_\flat)^{-1}(\operatorname{std}(\Pi_i))} \omega_\flat \tag{f}$$

and

$$\operatorname{std}\Big(\int_{D_\varepsilon} |\omega|\Big) \longrightarrow \int_{\bigcup_i (\operatorname{Log}_\flat |g_i|_\flat)^{-1}(\operatorname{std}(\Pi_i))} |\omega_\flat|_\flat \tag{g}$$

when the positive *standard* number ε tends to zero.

8.3. A statement about ordinary limits of complex integrals

Our purpose now is to state a corollary of our main theorem in a more classical language, namely, in terms of limits of usual complex integrals, without using any ultrafilter nor any nonstandard model of **R** or **C**.

Let us recall that $\mathscr{M}$ denotes the field of meromorphic functions around the origin of **C**. Let X be a smooth $\mathscr{M}$-scheme of finite type and pure dimension n, and let (U_i) be a finite Zariski-open cover of X. For every i, let $(f_{ij})_{1\leq j\leq n_i}$ be a finite family of regular functions on U_i; for every subset I and J of $\{1,\ldots,n_j\}$ of cardinality n, let $\varphi_{i,I,J}$ be a reasonably smooth and $(I\cup J)$-vanishing complex-valued function defined on some suitable open subset of $(\mathbf{R}\cup\{-\infty\})^{n_i}$.

Since $\mathscr{M}$ is the field of meromorphic functions around the origin, X gives rise to a complex analytic space, relatively algebraic, over a small enough punctured disk D^*, which we still denote by X. Up to shrinking D^*, we can assume that every U_i is a relative Zariski-open subset of the analytic space X, and that the functions f_{ij} are relatively algebraic holomorphic functions on U_i.

Assume that there exists a relative (n,n)-form ω on X whose support is proper over D^* and such that

$$\omega|_{U_i}=\Big(\frac{-1}{\log|t|}\Big)^n\sum_{I,J}\varphi_{i,I,J}\Big(-\frac{\log|f_{i1}|}{\log|t|},\ldots,-\frac{\log|f_{in_i}|}{\log|t|}\Big)\,\mathrm{d}\log|f_{i,I}|\wedge\mathrm{d}\,\mathrm{Arg}\,f_{i,J}$$

for every i (otherwise said, the forms locally defined by the above formulas coincide on overlaps, and the global form obtained by gluing them is relatively compactly supported).

The t-adic completion of $\mathscr{M}$ is the field $\mathbf{C}((t))$ of Laurent series. Fix $\tau\in(0,1)$, and endow $\mathbf{C}((t))$ with the t-adic absolute value $|\cdot|_\flat$ that maps t to τ; let us denote by X^{an} the Berkovich analytification of $X\times_{\mathscr{M}}\mathbf{C}((t))$.

Then the existence of our morphism of sheaves of bigraded differential **R**-algebras implies the existence of a compactly supported (n,n)-form $\omega_\flat$ on X^{an} (in the sense of [6]) such that

$$\begin{aligned}\omega_\flat|_{U_i^{\mathrm{an}}}&\\ &=\Big(\frac{-1}{\log\tau}\Big)^n\sum_{I,J}\varphi_{i,I,J}\Big(-\frac{\log|f_{i1}|_\flat}{\log\tau},\ldots,-\frac{\log|f_{in_i}|_\flat}{\log\tau}\Big)\,\mathrm{d}'\log|f_{i,I}|_\flat\\ &\quad\wedge\mathrm{d}''\log|f_{i,J}|_\flat\end{aligned}$$

for every i.

Now assertion (4) has the following consequence.

8.4 THEOREM
We have

$$\lim_{t\to 0}\int_{X_t}\omega|_{X_t}=\int_{X^{\mathrm{an}}}\omega_\flat.$$

Proof
Let (z_n) be a zero sequence of nonzero complex numbers such that $\int_{X_{z_n}}\omega|_{X_{z_n}}$ has a limit in $\mathbf{R}\cup\{-\infty,+\infty\}$ when n tends to infinity, and let $\mathscr{U}$ be any ultrafilter on $\mathbf{C}$ containing all cofinite subsets of $\{z_n\}_n$. Then applying our general construction with this specific $\mathscr{U}$ (recall that $\mathscr{M}$ has a natural embedding into our field C of nonstandard complex numbers), we see that

$$\int_{X_{z_n}}\omega|_{X_{z_n}}\longrightarrow\int_{X^{\mathrm{an}}}\omega_\flat$$

when n tends to infinity. As this holds for an arbitrary sequence (z_n) as above, we are done. □

9. Proof of the main theorem

9.1. Compatibility with integration
We shall in some sense establish the good behavior with respect to integration *before* showing the existence of the morphism $\omega\mapsto\omega_\flat$. Let us make this more precise.

9.1.1. Our setting
We assume that ω can be written as

$$\sum_{I,J}\varphi_{I,J}\big(\mathrm{Log}\,|f_1|,\dots,\mathrm{Log}\,|f_m|\big)\,\mathrm{d}\,\mathrm{Log}\,|f_I|\wedge\mathrm{d}\,\mathrm{Arg}\,f_J,$$

where I and J run through the set of subsets of $\{1,\dots,m\}$ of cardinality n, where $(f_i)_{1\le i\le m}$ is a family of regular invertible functions on U, and where $\varphi_{I,J}$ is an $(I\cup J)$-vanishing reasonably smooth function in $\mathscr{S}^{I,J,(f_i)}$ for each (I,J). We denote by $\omega_\flat$ the form

$$\sum_{I,J}\varphi_{I,J}\big(\mathrm{Log}_\flat\,|f_1|_\flat,\dots,\mathrm{Log}_\flat\,|f_m|_\flat\big)\,\mathrm{d}'\,\mathrm{Log}_\flat\,|f_I|_\flat\wedge\mathrm{d}''\log|f_J|_\flat$$

(we insist that our morphism has not yet been defined, so $\omega_\flat$ is currently just a notation for the form above).

We also assume that the open covering (V_i) is the trivial covering consisting of one open subset $V_1=U$ and we write g instead of g_1, Π instead of Π_1, and ℓ instead of ℓ_1.

Section 9.1 will be devoted to the proofs of (2) and (3) in this setting.

9.1.2. Proof of (2)

We shall in fact prove that $\int_K |\omega|$ is bounded for any t-bounded definably compact semialgebraic subset K of $U(C)$; so, let us fix such a subset K. Since K is definably compact and since $\mathrm{Log}\,|f_i|$ only takes bounded values on the invertible locus of f_i, there exists a positive standard real number A such that $\mathrm{Log}\,|f_i| \leq A$ on K for all i; thus there exists a positive standard real number N such that $|\varphi_{I,J}(\mathrm{Log}\,|f_1|, \ldots, \mathrm{Log}\,|f_m|)| \leq N$ on K for all (I, J).

Fix I and J. By the very definition of $(I \cup J)$-vanishing reasonably smooth functions, there exist two open subsets $V'_{I,J} \subset V_{I,J}$ of $(\mathbf{R} \cup \{-\infty\})^m$, defined by $\mathbf{Q}$-linear inequalities, and such that the following holds:

- $\varphi_{I,J}$ is defined on $V_{I,J}$ and $(\mathrm{Log}\,|f_1|, \ldots, \mathrm{Log}\,|f_m|)(U(C)) \subset V_{I,J}(R)$;
- $\varphi_{I,J}|_{V'_{I,J}} = 0$, and for every $i \in I \cup J$, the ith coordinate function does not take the value $-\infty$ on $V_{I,J} \setminus V'_{I,J}$.

Let $K_{I,J}$ be the preimage of $V_{I,J} \setminus V'_{I,J}$ in K under $(\mathrm{Log}\,|f_1|, \ldots, \mathrm{Log}\,|f_m|)$. This is a definably compact semialgebraic subset of K on which $|f_i|$ does not vanish as soon as $i \in I \cup J$; by construction, $\varphi_{I,J}(\mathrm{Log}\,|f_1|, \ldots, \mathrm{Log}\,|f_m|)$ vanishes on $K \setminus K_{I,J}$.

By enlarging A, we may assume that for all I, J and all $i \in I \cup J$ one has the minoration $\log |f_i| \geq -A$ on $K_{I,J}$.

For every subset L of $\{1, \ldots, m\}$, denote by D_L the subset of $U(C)$ consisting of points at which every f_i with $i \in L$ is invertible. Let $i \in \{1, \ldots, m\}$; on $\mathrm{D}_{\{i\}}$ we set $f_i = r_i e^{2i\pi\alpha_i}$ for every i (where $r_i = |f_i|$ and α_i is a multivalued function, which we will use only through the well-defined differential form $\mathrm{d}\alpha_i$). Let I and J be two subsets of $\{1, \ldots, m\}$ of cardinality n. Let $i_1 < \cdots < i_n$ be the elements of I, and let $j_1 < \cdots < j_n$ be those of J; on $\mathrm{D}_{I \cup J}$, we set $\frac{\mathrm{d}r_I}{r_I} = \frac{\mathrm{d}r_{i_1}}{r_{i_1}} \wedge \cdots \wedge \frac{\mathrm{d}r_{i_n}}{r_{i_n}}$ and $\mathrm{d}\alpha_J = \mathrm{d}\alpha_{j_1} \wedge \cdots \wedge \mathrm{d}\alpha_{j_n}$. Let S^1_R denote the "unit circle" $\{z \in C, |z| = 1\}$. Let $u_{I,J}$ be the map from $\mathrm{D}_{I,J}$ to $(R_{>0})^n \times (S^1_R)^n$ that maps a point x to $(|f_{i_1}(x)|, \ldots, |f_{i_n}(x)|, \frac{f_{j_1}(x)}{|f_{j_1}(x)|}, \ldots, \frac{f_{j_n}(x)}{|f_{j_n}(x)|})$.

We denote by ρ_j the coordinate function on $(R_{>0})^n \times (S^1_R)^n$ corresponding to the jth factor $R_{>0}$, and by ϖ_j the multivalued argument function corresponding to the jth factor S^1_R. The form $\mathrm{d}\varpi_j$ is well defined (we can describe it alternatively as the pullback under the projection to the jth factor $S^1_R \simeq \{(x, y) \in R^2, x^2 + y^2 = 1\}$ of the form $x\,\mathrm{d}y - y\,\mathrm{d}x$). Let $E_{I,J}$ denote the étale locus of $u_{I,J}$; by definability and o-minimality, there exists an integer d such that the fibers of $u_{I,J}|_{E_{I,J} \cap K}$ are all of cardinality at most d for all I and J.

We then have (recall that $\lambda = -\log |t|$)

$$\int_K |\omega| \leq \frac{N}{\lambda^n} \sum_{I,J} \int_{K_{I,J}} \left| \frac{\mathrm{d}r_I}{r_I} \wedge \mathrm{d}\alpha_J \right| \tag{h}$$

$$= \frac{N}{\lambda^n} \sum_{I,J} \int_{K_{I,J} \cap E_{I,J}} \left| \frac{\mathrm{d}r_I}{r_I} \wedge \mathrm{d}\alpha_J \right| \tag{i}$$

$$\leq \frac{Nd}{\lambda^n} \sum_{I,J} \int_{u_{I,J}(K_{I,J} \cap E_{I,J})} \left| \frac{\mathrm{d}\rho_1}{\rho_1} \wedge \cdots \wedge \frac{\mathrm{d}\rho_n}{\rho_n} \wedge \frac{d\varpi_1}{2\pi} \wedge \cdots \wedge \frac{\mathrm{d}\varpi_n}{2\pi} \right| \tag{j}$$

$$\leq \frac{Nd}{\lambda^n} \sum_{I,J} \int_{|f_I|(K_{I,J} \cap E_{I,J})} \frac{\mathrm{d}\rho_1}{\rho_1} \wedge \cdots \wedge \frac{\mathrm{d}\rho_n}{\rho_n} \tag{k}$$

$$\leq \binom{m}{n}^2 \frac{Nd}{\lambda^n} \int_{[|t|^A, |t|^{-A}]^n} \frac{\mathrm{d}\rho_1}{\rho_1} \wedge \cdots \wedge \frac{\mathrm{d}\rho_n}{\rho_n} \tag{l}$$

$$\leq \binom{m}{n}^2 \frac{Nd}{\lambda^n} \left(-2A \log |t|\right)^n \tag{m}$$

$$= \binom{m}{n}^2 Nd(2A)^n. \tag{n}$$

Hence $\int_K |\omega|$ is bounded, as announced.

9.1.3. Proof of (3)(c)
The proofs of (a) and (b) will rest on several steps allowing ourselves to reduce to a simpler case, in which it will be possible to perform some explicit computations that are the core of our proof. But to achieve this reduction we shall need (c); hence we start by proving it.

Let $\Pi \in \Theta(g)$. Choose a positive standard real number a in $\Lambda(g, \Pi)$ (such an a exists in view of Remark 7.6). For every nonnegative standard real number ε, we set $P_\varepsilon = \mathrm{std}(\Pi) + [-\varepsilon, \varepsilon]^\ell \subset \mathbf{R}^\ell$ (so $P_0 = \mathrm{std}(\Pi)$). Let us introduce some notation:

- $V_\varepsilon = (\mathrm{Log}_\flat |g|_\flat)^{-1}(P_\varepsilon) \subset U^{\mathrm{an}}$, for ε a standard element of $[0, a]$;
- $V_{\varepsilon,\eta} = (\mathrm{Log}_\flat |g|_\flat)^{-1}(\overline{P_\varepsilon \setminus P_\eta}) \subset U^{\mathrm{an}}$, for ε a standard element of $[0, a]$ and η a standard element of $(0, \varepsilon)$;
- $K_\varepsilon = (\mathrm{Log}\,|g|)^{-1}(\Pi_\varepsilon) \subset U(C)$, for ε any element of R lying on $[0, a]$;
- $K_{\varepsilon,\eta} = (\mathrm{Log}\,|g|)^{-1}(\overline{\Pi_\varepsilon \setminus \Pi_\eta}) \subset U(C)$, for ε any element of R lying on $[0, a]$ and η any element of R lying on $(0, \varepsilon)$.

We fix two subsets I and J of $\{1, \ldots, m\}$ of cardinality n. For every standard real number A, we shall need the following extra notation:

- V_ε^A (resp., $V_{\varepsilon,\eta}^A$) for the intersection of V_ε (resp., $V_{\varepsilon,\eta}$) with the closed analytic domain of U^{an} defined by the inequalities $\mathrm{Log}_\flat |f_i|_\flat \geq A$ for all $i \in I$;
- K_ε^A (resp., $K_{\varepsilon,\eta}^A$) for the intersection of K_ε (resp., $K_{\varepsilon,\eta}$) with the closed semi-algebraic subset of $U(C)$ defined by the inequalities $\mathrm{Log}\,|f_i| \geq A$ for all $i \in I$.

The preimage of $V_{I,J} \setminus V'_{I,J}$ (the notation is introduced in the second paragraph of Section 9.1.2) in K_a under $(\mathrm{Log}\,|f_i|)_{1\leq i\leq m}$ is definably compact, and none of the functions f_i with $i \in I$ vanishes on it; thus there exists some standard real number A such that every point of K_a at which at least one of the $\mathrm{Log}\,|f_i|$ is smaller than A belongs to the preimage of $V'_{I,J}$, so $\varphi_{I,J}(\mathrm{Log}\,|f|)$ vanishes at such a point. Using mutatis mutandis the same argument and up to decreasing A if necessary, we can ensure that $\varphi_{I,J}(\mathrm{Log}_\flat\,|f|_\flat)$ vanishes at every point of V_a at which at least one the $\mathrm{Log}_\flat\,|f_i|_\flat$ is smaller than A.

Otherwise said, there exists a standard real number A such that for every element ε of R lying on $[0,a]$, the function $\varphi_{I,J}(\mathrm{Log}\,|f|)$ vanishes on $K_\varepsilon \setminus K_\varepsilon^A$ and the function $\varphi_{I,J}(\mathrm{Log}_\flat\,|f|_\flat)$ vanishes on $V_\varepsilon \setminus V_\varepsilon^A$.

We are now going to show that $\mathrm{Vol}(\mathrm{Log}_\flat\,|f_I|_\flat(V_\varepsilon^A \setminus V_0^A))$ tends to zero when ε tends to zero, which is the core of the proof of (3)(c). Our method for proving this claim consists in describing $\mathrm{Log}_\flat\,|f_I|_\flat(V_\varepsilon^A)$ more or less as the image under $\mathrm{Log}_\flat\,|f_I|_\flat$ of a *piecewise linear subset* of V_a^A, which allows us to get rid of non-Archimedean geometry and only deal with usual real integration.

Recall that the skeleton of $\mathbf{G}_{\mathrm{m}}^{I,\mathrm{an}}$ is the closed subspace of $\mathbf{G}_{\mathrm{m}}^{I,\mathrm{an}}$ homeomorphic to $\mathbf{R}^I$ via the mapping $\mathrm{sk}: \mathbf{R}^I \to \mathbf{G}_{\mathrm{m}}^{I,\mathrm{an}}$ sending $(\log(r_i))_{i\in I}$ to the seminorm assigning the real number $\max_{m\in\mathbf{Z}^I} |a_m| \prod_{i\in I} r_i^{m_i}$ to a Laurent polynomial $\sum_{m\in\mathbf{Z}^I} a_m T^m$. Let Σ be the preimage of the skeleton of $\mathbf{G}_{\mathrm{m}}^{I,\mathrm{an}}$ under $f_I|_{V_a^A}$. This is a skeleton of V_a^A in the sense of [8, Section 4.6] (see Théorème 5.1 there; a mistake in this paper is corrected in the erratum); in particular, it inherits a canonical piecewise linear structure and $(\mathrm{Log}_\flat\,|f_I|_\flat)|_\Sigma$ is piecewise linear. Moreover, if W is any compact analytic domain of V_a^A, then the intersection $\Sigma \cap W$ is a piecewise linear subset of Σ and $\mathrm{Log}_\flat\,|f_I|_\flat(W)^{(n)} = \mathrm{Log}_\flat\,|f_I|_\flat((\Sigma \cap W)^{(n)})$, where the superscript ${}^{(n)}$ denotes the pure n-dimensional part of a piecewise linear set (this last equality is a lemma which is shown in a forthcoming version of [6]; its proof is not difficult and rests on the description of a skeleton in terms of tropical dimension; see [6, Section 2.3.3]); in particular, the volume of $\mathrm{Log}_\flat\,|f_I|_\flat(W)$ is equal to that of $\mathrm{Log}_\flat\,|f_I|_\flat(W \cap \Sigma)$.

Choose $\varepsilon \in (0,a]$. From the equality $V_\varepsilon^A \setminus V_0^A = \bigcup_{0<\eta<\varepsilon} V_{\varepsilon,\eta}^A$, we get

$$\begin{aligned}\mathrm{Vol}(\mathrm{Log}_\flat\,|f_I|_\flat\big((V_\varepsilon^A \setminus V_0^A)\big) &= \sup_{0<\eta<\varepsilon} \mathrm{Vol}\big(\mathrm{Log}_\flat\,|f_I|_\flat(V_{\varepsilon,\eta}^A)\big)\\ &= \sup_{0<\eta<\varepsilon} \mathrm{Vol}\big(\mathrm{Log}_\flat\,|f_I|_\flat(\Sigma \cap V_{\varepsilon,\eta}^A)\big)\\ &= \mathrm{Vol}(\mathrm{Log}_\flat\,|f_I|_\flat\big((\Sigma \cap V_\varepsilon^A) \setminus (\Sigma \cap V_0^A)\big).\end{aligned}$$

Now $(\Sigma \cap V_\varepsilon^A)_{0<\varepsilon\leq a}$ is a nonincreasing family of compact piecewise linear subsets of Σ with intersection $\Sigma \cap V_0^A$, and $\mathrm{Log}_\flat\,|f_I|_\flat|_\Sigma$ is piecewise linear. Since $\dim \Sigma \leq n$, this implies that $\mathrm{Vol}(\mathrm{Log}_\flat\,|f_I|_\flat(\Sigma \cap V_\varepsilon^A \setminus \Sigma \cap V_0^A))$ tends to zero when ε tends to zero. By the above, this means that $\mathrm{Vol}(\mathrm{Log}_\flat|f_I|_\flat(V_\varepsilon^A \setminus V_0^A))$ tends to zero, as announced.

In order to end the proof of (3)(c), we now have to understand the consequences in the nonstandard world of the limit statement above (which involves only standard objects); this step rests in a crucial way on DOAG-definability.

For every standard $\varepsilon \in (0, a]$, the set $\Lambda_\varepsilon := \mathrm{Log}_\flat|f_I|_\flat(V_\varepsilon^A \setminus V_0^A)$ is DOAG-definable, and depends DOAG-definably on ε. Thus $\Lambda_{\varepsilon,R}$ makes sense for every element $\varepsilon \in R$ with $0 < \varepsilon \leq a$.

Let D be the set of positive elements $x \in R$ such that $x < a/2$ and

$$\mathrm{Log}\,|f_I|(K_\varepsilon^A \setminus K_x^A) \subset \Lambda_{2\varepsilon,R}$$

for all $\varepsilon \in (x, \frac{a}{2})$. An element x of R belongs to D if and only if the implication

$$\left(\left|g(z)\right| \in |t|^{-(\Pi_\varepsilon \setminus \Pi_x)} \text{ and } \left|f_I(z)\right| \in |t|^{[A,+\infty)^I}\right) \Rightarrow \left|f_I(z)\right| \in |t|^{-\Lambda_{2\varepsilon,R}}$$

holds for all $z \in U(C)$. It thus follows from Section 7.1.2 that $|t|^D$ is definable; but since it is 1-dimensional, it is a finite union of intervals by o-minimality, so D is also such a union, hence is definable as well. Moreover, it contains by definition every bounded x whose standard part belongs to $(0, \frac{a}{2}]$. As a consequence, D contains $[\alpha, \frac{a}{2}]$ for some positive negligible element α.

For all elements ε of R lying on $(\alpha, a/2)$, we have

$$\mathrm{Log}\,|f_I|(K_\varepsilon^A \setminus K_\alpha^A) \subset \Lambda_{2\varepsilon,R}.$$

The inclusion above holds in particular for every positive standard $\varepsilon < a/2$; for such an ε, we thus have

$$\frac{1}{\lambda^n} \int_{|f_I|(K_\varepsilon^A \setminus K_\alpha^A)} \frac{\mathrm{d}\rho_1}{\rho_1} \wedge \cdots \wedge \frac{\mathrm{d}\rho_n}{\rho_n} \leq \mathrm{Vol}(\Lambda_{2\varepsilon}).$$

Since $\mathrm{Vol}(\Lambda_{2\varepsilon}) \longrightarrow 0$ when $\varepsilon \longrightarrow 0$, it follows that

$$\mathrm{std}\Big(\frac{1}{\lambda^n} \int_{|f_I|(K_\varepsilon^A \setminus K_\alpha^A)} \frac{\mathrm{d}\rho_1}{\rho_1} \wedge \cdots \wedge \frac{\mathrm{d}\rho_n}{\rho_n}\Big) \longrightarrow 0$$

when $\varepsilon \longrightarrow 0$. In view of inequality (k) of Section 9.1.2, this implies that

$$\mathrm{std}\Big(\int_{K_\varepsilon^A \setminus K_\alpha^A} \left|\varphi_{I,J}\big(\mathrm{Log}\,|f_1|, \ldots, \mathrm{Log}\,|f_m|\big)\, \mathrm{d}\,\mathrm{Log}\,|f_I| \wedge \mathrm{d}\,\mathrm{Arg}\, f_J\right|\Big) \longrightarrow 0$$

when $\varepsilon \longrightarrow 0$. But by the choice of A the integral

$$\int_{K_\varepsilon^A \setminus K_\alpha^A} \left|\varphi_{I,J}\big(\mathrm{Log}\,|f_1|, \ldots, \mathrm{Log}\,|f_m|\big)\, \mathrm{d}\,\mathrm{Log}\,|f_I| \wedge \mathrm{d}\,\mathrm{Arg}\, f_J\right|$$

is equal to

$$\int_{K_\varepsilon \setminus K_\alpha} \left|\varphi_{I,J}\big(\mathrm{Log}\,|f_1|,\dots,\mathrm{Log}\,|f_m|\big)\,\mathrm{d}\,\mathrm{Log}\,|f_I| \wedge \mathrm{d}\,\mathrm{Arg}\, f_J\right|,$$

so that

$$\mathrm{std}\Big(\int_{K_\varepsilon \setminus K_\alpha} \left|\varphi_{I,J}\big(\mathrm{Log}\,|f_1|,\dots,\mathrm{Log}\,|f_m|\big)\,\mathrm{d}\,\mathrm{Log}\,|f_I| \wedge \mathrm{d}\,\mathrm{Arg}\, f_J\right|\Big) \longrightarrow 0$$

when $\varepsilon \longrightarrow 0$.

The infinitesimal element α above depends a priori on (I,J); but by taking it large enough (and still infinitesimal) we can ensure that it does not. Then

$$\mathrm{std}\Big(\int_{K_\varepsilon \setminus K_\alpha} |\omega|\Big) \longrightarrow 0$$

when $\varepsilon \longrightarrow 0$, which ends the proof of (3)(c) in our particular setting.

9.1.4. Proof of (3)(a) and (3)(b) in our setting

Assertions (3)(a) and (3)(b) involve the form to be integrated ω, which is defined with an explicit formula using the functions f_i, and the domain of integration, whose definition uses another family of functions g and a pseudopolyhedron Π. We will first simplify slightly this set of data, by showing that we may assume that $f = g$ and Π is of the form P_R for some pseudopolyhedron $P \subset (\mathbf{R} \cup \{-\infty\})^\ell$ (and so $\mathrm{std}(\Pi) = P$), with moreover $\mathrm{Log}_\flat\,|f|_\flat((\mathrm{Log}_\flat\,|f|_\flat)^{-1}(P)) = P$. This reduction essentially uses (3)(c) through its consequence Remark 8.2, together with some elementary definability arguments.

Set $h = (f,g)$, $P = \mathrm{std}(\Pi)$, $W = (\mathrm{Log}_\flat\,|g|_\flat)^{-1}(P) \subset V^{\mathrm{an}}$, and $Q = \mathrm{Log}_\flat\,|h|_\flat(W) \subset \mathbf{R}^{m+\ell}$. Then $W = (\mathrm{Log}_\flat\,|h|_\flat)^{-1}(Q)$. We are now going to explain why it is sufficient to prove assertion (3) for (Q_R, h) instead of (Π, g). So we assume that (3)(a) and (b) hold for (Q_R, h).

If ε is a positive real number, then we clearly have $(\mathrm{Log}_\flat\,|h|_\flat)^{-1}(Q_\varepsilon) \subset (\mathrm{Log}_\flat\,|g|_\flat)^{-1}(P_\varepsilon)$. On the other hand, for every $\varepsilon > 0$, the set $(\mathrm{Log}_\flat\,|h|_\flat)^{-1}(Q_\varepsilon)$ is a neighborhood of W, hence contains $(\mathrm{Log}_\flat\,|g|_\flat)^{-1}(P_\eta)$ for some η which can be taken in $(0,\varepsilon]$ (here we use topological properness—recall that $\Pi \in \Theta(g)$). Let $\delta(\varepsilon)$ denote the least upper bound of

$$\big\{\eta \in (0,\varepsilon), \big(\mathrm{Log}_\flat\,|g|_\flat\big)^{-1}(P_\eta) \subset \big(\mathrm{Log}_\flat\,|h|_\flat\big)^{-1}(Q_\varepsilon)\big\};$$

note that by compactness we have $(\mathrm{Log}_\flat\,|h|_\flat)^{-1}(P_{\delta(\varepsilon)}) \subset (\mathrm{Log}_\flat\,|h|_\flat)^{-1}(Q_\varepsilon)$. Then δ is a DOAG-definable function; in view of the fact that $\delta(\varepsilon) \leq \varepsilon$ by definition, this implies that there exists a positive rational number r and a positive real number M such that $\delta(\varepsilon) = M\varepsilon^r$ for ε small enough.

This implies that

$$\left(\mathrm{Log}\,|h|\right)^{-1}(Q_{R,\frac{\varepsilon}{2}}) \subset \left(\mathrm{Log}\,|g|\right)^{-1}(\Pi_\varepsilon) \subset \left(\mathrm{Log}\,|h|\right)^{-1}(Q_{R,\frac{2}{M}\varepsilon^{1/r}})$$

for ε a small enough standard positive real number. Since we assume that (3)(a) and (3)(b) hold for (Q_R, h) (and since (3)(c) has already been proved), it follows from Remark 8.2 that

$$\mathrm{std}\Big(\int_{(\mathrm{Log}\,|g|)^{-1}(\Pi_\varepsilon)} \omega\Big) \longrightarrow \int_{(\mathrm{Log}_\flat\,|g|_\flat)^{-1}(P)} \omega_\flat$$

and

$$\mathrm{std}\Big(\int_{(\mathrm{Log}\,|g|)^{-1}(\Pi_\varepsilon)} |\omega|\Big) \longrightarrow \int_{(\mathrm{Log}_\flat\,|g|_\flat)^{-1}(P)} |\omega_\flat|_\flat$$

when the positive *standard* number ε belongs to $\Lambda(g, \Pi)$ and tends to zero.

Therefore, if the result holds for (Q_R, h), then it holds for (Π, g); we thus can replace Π by Q_R and g by h, and then enlarge f (which is harmless) so that $g = f$. We keep the notation $P = \mathrm{std}(\Pi)$ and

$$W = \left(\mathrm{Log}_\flat\,|g|_\flat\right)^{-1}(P) = \left(\mathrm{Log}_\flat\,|f|_\flat\right)^{-1}(P);$$

note that we have $\Pi = P_R$ and $(\mathrm{Log}_\flat\,|f|_\flat)(W) = P$.

9.1.5. Arguing piecewise on P

To allow for more flexibility in the proof, we shall need to argue piecewise on P. We explain here why it is possible; the key points are once again (3)(c), and the additivity of integrals in both frames.

Assume that we are given a finite covering $(P_i)_{i\in I}$ of P by pseudopolyhedra, and that for every nonempty subset J of I, statements (3)(a) and (3)(b) hold for (P_J, f) with $P_J := \bigcap_{i\in J} P_i$. Then these statements hold for (P, f).

Indeed, for every i set $\Pi_i = P_{i,R}$, and every nonempty subset J of I, set $\Pi_J = P_{J,R}$. For every positive standard ε, we have $\Pi_\varepsilon = \bigcup_i \Pi_{i,\varepsilon}$. Now let J be a nonempty subset of I.

If $P_J = \emptyset$, then for ε small enough we have $\bigcap_{i\in J} \Pi_{i,\varepsilon} = \emptyset$. If $P_J \neq \emptyset$, then by definability and compactness there exist two positive real numbers A and η such that

$$P_{J,\varepsilon} \subset \bigcap_{i\in J} P_{i,\varepsilon} \subset P_{J,A\varepsilon}$$

for all positive real numbers $\varepsilon < \eta$, which implies (by model-completeness of DOAG) that

$$\Pi_{J,\varepsilon} \subset \bigcap_{i \in J} \Pi_{i,\varepsilon} \subset \Pi_{J,A\varepsilon}$$

for every positive $\varepsilon < \eta$ in R

The difference

$$\int_{(\mathrm{Log}\,|f|)^{-1}(\Pi_\varepsilon)} \omega - \sum_{J \neq \emptyset} (-1)^{|J|+1} \int_{(\mathrm{Log}\,|f|)^{-1}(\Pi_{J,\varepsilon})} \omega$$

can be rewritten as

$$\sum_{J \neq \emptyset} (-1)^{|J|+1} \Big(\int_{\bigcap_{i \in J} (\mathrm{Log}\,|f|)^{-1}(\Pi_{i,\varepsilon})} \omega - \int_{(\mathrm{Log}\,|f|)^{-1}(\Pi_{J,\varepsilon})} \omega \Big).$$

It now follows from (3)(c) (which has already been proved) and from the inclusions $\Pi_{J,\varepsilon} \subset \bigcap_{i \in J} \Pi_{i,\varepsilon} \subset \Pi_{J,A\varepsilon}$ (which hold for ε small enough) that

$$\mathrm{std}\Big(\int_{\bigcap_{i \in J} (\mathrm{Log}\,|f|)^{-1}(\Pi_{i,\varepsilon})} \omega - \int_{(\mathrm{Log}\,|f|)^{-1}(\Pi_{J,\varepsilon})} \omega \Big) \longrightarrow 0$$

when $\varepsilon \longrightarrow 0$ (and remains standard). Thus

$$\mathrm{std}\Big(\int_{(\mathrm{Log}\,|f|)^{-1}(\Pi_\varepsilon)} \omega - \sum_{J \neq \emptyset} (-1)^{|J|+1} \int_{(\mathrm{Log}\,|f|)^{-1}(\Pi_{J,\varepsilon})} \omega \Big) \longrightarrow 0$$

when $\varepsilon \longrightarrow 0$. As statements (3)(a) and (3)(b) hold for every P_J, this implies that

$$\mathrm{std}\Big(\int_{(\mathrm{Log}\,|f|)^{-1}(\Pi_\varepsilon)} \omega \Big) \longrightarrow \sum_J (-1)^{|J|+1} \int_{(\mathrm{Log}_\flat\,|f|_\flat)^{-1}(P_J)} \omega_\flat = \int_{(\mathrm{Log}_\flat\,|f|_\flat)^{-1}(P)} \omega_\flat$$

when $\varepsilon \longrightarrow 0$.

We prove in the same way that

$$\mathrm{std}\Big(\int_{(\mathrm{Log}\,|f|)^{-1}(\Pi_\varepsilon)} |\omega| \Big) \longrightarrow \int_{(\mathrm{Log}_\flat\,|f|_\flat)^{-1}(P)} |\omega_\flat|_\flat$$

when $\varepsilon \longrightarrow 0$.

9.1.6

Being allowed to argue piecewise on P, we now would like to cut it into finitely many pieces as nicely as possible. This will be achieved by exhibiting a finite covering (P_i) of P by pseudopolyhedra such that for every i, the following hold:

(i) for every pair (I, J) of subsets of $\{1, \ldots, m\}$ of cardinality n, either $\varphi_{I,J}$ is identically zero on P_i, either for every $(x_1, \ldots, x_m) \in P_i$ and every $j \in I \cup J$, we have $x_j \neq -\infty$;

(ii) there exists a subset E of $\{1,\dots,m\}$ such that:
- for every $(x_1,\dots,x_m)\in P_i$ and every $j\in E$, we have $x_j\neq-\infty$;
- for every pair (I,J) of subsets of $\{1,\dots,m\}$ of cardinality n, there exists a compactly supported smooth function $\psi_{I,J}$ on $\mathbf{R}^E$ such that for every $(x_1,\dots,x_m)\in P_i$ one has $\varphi_{I,J}(x_1,\dots,x_m)=\psi_{I,J}(x_j)_{j\in E}$.

Let us explain how this can be done. Let ξ be a point of P, and let I and J be two subsets of $\{1,\dots,m\}$ of cardinality n. By the very definition of $(I\cup J)$-vanishing reasonably smooth functions, there exists a pseudopolyhedral neighborhood Q of x in P such that:

(i) either $\varphi_{I,J}$ is identically zero on Q, either for every $(x_1,\dots,x_m)\in Q$ and every $j\in I\cup J$ we have $x_j\neq-\infty$;

(ii) there exists a subset E of $\{1,\dots,m\}$ such that:
- we have $x_j\neq-\infty$ for every $(x_1,\dots,x_m)\in Q$ and every $j\in E$;
- there exists a compactly supported smooth function ψ on $\mathbf{R}^E$ such that for every $(x_1,\dots,x_m)\in Q$ one has $\varphi_{I,J}(x_1,\dots,x_m)=\psi(x_j)_{j\in E}$

(note that a priori ψ is a smooth function defined on an open neighborhood of the projection of Q to $\mathbf{R}^E$, but since the latter is compact we can assume that ψ is defined on the whole of $\mathbf{R}^E$ and compactly supported). We now conclude by compactness of P.

9.1.7

In view of Sections 9.1.5 and of 9.1.6, we can assume that there exists a subset E of $\{1,\dots,m\}$ satisfying the following:
- for all $(x_1,\dots,x_m)$ in P and all $j\in E$, we have $x_j\neq-\infty$;
- one can in fact write

$$\omega=\sum_{I,J}\varphi_{I,J}\big(\mathrm{Log}\,|f_j|\big)_{j\in E}\,\mathrm{d}\,\mathrm{Log}\,|f_I|\wedge\mathrm{d}\,\mathrm{Arg}\,f_J$$

and

$$\omega_\flat=\sum_{I,J}\varphi_{I,J}\big(\mathrm{Log}_\flat\,|f_j|_\flat\big)_{j\in E}\,\mathrm{d}'\,\mathrm{Log}_\flat\,|f_I|_\flat\wedge\mathrm{d}''\log|f_J|_\flat,$$

where I and J run through the set of subsets of E of cardinality n, and where the $\varphi_{I,J}$ are smooth, compactly supported functions on $\mathbf{R}^E$.

We note that the functions f_j with $j\in E$ are invertible on the analytic domain W. We set $Q=(\mathrm{Log}_\flat\,|f_E|_\flat)(W)$; this is a compact polyhedron of $\mathbf{R}^E$ which can also be described as the image of P under the projection to $(\mathbf{R}\cup\{-\infty\})^E$.

We denote by ξ the Lagerberg form

$$\sum_{I,J}\Big(\frac{1}{\lambda_\flat}\Big)^n \varphi_{I,J}(x_j/\lambda_\flat)_{j\in E}\,\mathrm{d}'x_E\wedge\mathrm{d}''x_E$$

on $\lambda_\flat Q$; by construction, $\omega_\flat = f_E^*\xi$.

9.1.8
We first consider the case where $\dim Q < n$. In this case the (n,n)-form ξ on $\lambda_\flat Q$ is zero, and it suffices to prove that

$$\operatorname{std}\Big(\int_{(\operatorname{Log}|g|)^{-1}(\Pi_\varepsilon)}|\omega|\Big)\longrightarrow 0$$

when $\varepsilon\longrightarrow 0$. This will follow quite easily from the rough estimates of Section 9.1.2.

Let I be any subset of E of cardinality n. For every positive standard real number ε, let Q_ε^I denote the image of Q_ε under the projection map $\mathbf{R}^E\to\mathbf{R}^I$. The inequality $\dim Q<n$ implies that $\operatorname{Vol}(Q_\varepsilon^I)\longrightarrow 0$ when $\varepsilon\longrightarrow 0$.

Now for every standard positive ε, we have the inclusion

$$\big(\operatorname{Log}|f_I|\big)\big((\operatorname{Log}|f|)^{-1}(\Pi_\varepsilon)\big)\subset Q^I_{2\varepsilon,R}.$$

It follows that

$$\frac{1}{\lambda^n}\int_{|f_I|((\operatorname{Log}|f|)^{-1}(\Pi_\varepsilon))}\frac{\mathrm{d}\rho_1}{\rho_1}\wedge\cdots\wedge\frac{\mathrm{d}\rho_n}{\rho_n}\le\operatorname{Vol}(Q^I_{2\varepsilon}).$$

Since this holds for all I, this implies in view of inequality (k) of Section 9.1.2 that

$$\operatorname{std}\Big(\int_{(\operatorname{Log}|g|)^{-1}(\Pi_\varepsilon)}|\omega|\Big)\longrightarrow 0$$

when $\varepsilon\longrightarrow 0$.

9.1.9
We are now going to describe two general methods which we shall use several times to make the situation simpler. The first one essentially combines the fact that the statements we want to prove can be checked piecewise on P (Section 9.1.5) and the fact that they hold as soon as $\dim Q<n$ (Section 9.1.8); the second one follows easily from Remark 8.2.

9.1.9.1. Arguing cellwise on Q. Let (Q_i) be a finite covering of Q by compact polyhedra whose pairwise intersections are of dimension less than n; for every i, let P_i be the preimage of Q_i in P. Assume that statements (3)(a) and (3)(b) hold for every P_i; then they hold for P. Indeed, let I be any finite set of indices of cardinality at least 2. Then the projection of $\bigcap_{i\in I} P_i$ to $(\mathbf{R}\cup\{-\infty\})^E$ is equal to $\bigcap_{i\in I} Q_i$, so it is of dimension less than n. Therefore, the theorem holds for $\bigcap_{i\in I} P_i$ in view of Section 9.1.8; it now follows from Section 9.1.5 that it holds for P.

9.1.9.2. Affine change of coordinates. Let $M = (m_{ij})$ be a matrix belonging to $\mathrm{M}_E(\mathbf{Z})$ with nonzero determinant, and let $v = (v_j)_j \in \mathbf{R}^E$. For every point $x = (x_1, \dots, x_m)$ in P, we set $Mx = (y_1, \dots, y_m)$ with $y_i = x_i$ if $i \notin E$, and $y_i = \sum_{j\in I} m_{ij} x_j$ otherwise. For $i \notin E$, we set $h_i = f_i$; for $i \in E$, we set $h_i = |t|^{v_i} \prod_{j\in I} f_j^{m_{ij}}$.

Set $P' = MP + v$; this is a pseudopolyhedron. By expressing $\operatorname{Log}|h|$, $\mathrm{d}\operatorname{Log}|h|$, and $\mathrm{d}\arg h$ in terms of $\operatorname{Log}|f|$, $\mathrm{d}\operatorname{Log}|f|$, and $\mathrm{d}\arg f$, and the same with $\operatorname{Log}_\flat$ instead of Log and $|\cdot|_\flat$ instead of $|\cdot|$, we get equalities

$$\omega = \sum_{I,J} \psi_{I,J}\big(\operatorname{Log}|h_1|, \dots, \operatorname{Log}|h_m|\big)\, \mathrm{d}\operatorname{Log}|h_I| \wedge \mathrm{d}\operatorname{Arg} h_J$$

and

$$\omega_\flat = \sum_{I,J} \psi_{I,J}\big(\operatorname{Log}_\flat|h_1|_\flat, \dots, \operatorname{Log}_\flat|h_m|_\flat\big)\, \mathrm{d}'\operatorname{Log}_\flat|h_I|_\flat \wedge \mathrm{d}''\log|h_J|_\flat.$$

Assume that statements (3)(a) and (3)(b) hold for (P'_R, h). We are going to prove that they hold for (Π, f).

There exist two standard positive real numbers A and B with $A < B$ such that

$$\big(\operatorname{Log}|h|\big)^{-1}(P'_{R,A\varepsilon}) \subset \big(\operatorname{Log}|f|\big)^{-1}(\Pi_\varepsilon) \subset \big(\operatorname{Log}|h|\big)^{-1}(P'_{R,B\varepsilon})$$

for ε small enough. Then

$$\operatorname{std}\Big(\int_{(\operatorname{Log}|f|)^{-1}(\Pi_\varepsilon)} \omega\Big) \longrightarrow \int_{(\operatorname{Log}_\flat|h|_\flat)^{-1}(P')} \omega_\flat = \int_{(\operatorname{Log}_\flat|f|_\flat)^{-1}(P)} \omega_\flat$$

and

$$\operatorname{std}\Big(\int_{(\operatorname{Log}|f|)^{-1}(\Pi_\varepsilon)} |\omega|\Big) \longrightarrow \int_{(\operatorname{Log}_\flat|h|_\flat)^{-1}(P')} |\omega_\flat|_\flat = \int_{(\operatorname{Log}_\flat|f|_\flat)^{-1}(P)} |\omega_\flat|_\flat$$

by Remark 8.2.

9.1.10

We assume now that $(\omega_\flat)|_W = 0$, which means that the form ξ on $\lambda_\flat Q$ is zero, and we are going to prove (3)(a) and (3)(b) under this assumption. We will use the fact that these statements hold whenever $\dim Q < n$ (Section 9.1.8), that they can be checked cellwise on Q (Section 9.1.9.1), that they can be proved after an affine change of coordinates (Section 9.1.9.2), and that J acts trivially on $\mathsf{A}^{n,n}$; and then we will ultimately rely on the estimates in Section 9.1.2.

We want to prove that

$$\operatorname{std}\Big(\int_{(\operatorname{Log}|f|)^{-1}(\Pi_\varepsilon)} |\omega|\Big) \longrightarrow 0$$

when $\varepsilon \longrightarrow 0$. By considering a cell decomposition of Q and using Section 9.1.9.1, we reduce to the case where Q is a cell. If $\dim Q < n$, then we already know that the required statement holds (Section 9.1.8); we can thus assume that $\dim Q = n$. And in view of Section 9.1.9.2, we are allowed to perform an affine change of the coordinates indexed by E with integral linear part; hence we can assume that there exists a subset E_0 of E of cardinality n such that Q is contained in the subspace defined by the equations $x_i = 0$ for i running through $E \setminus E_0$. The assumption that $\xi = 0$ now simply means that $\varphi_{E_0,E_0}|_Q = 0$.

We fix two subsets I and J of E, both of cardinality n. Let $\omega_{I,J}$ be the form $\varphi_{I,J}(\operatorname{Log}|f_1|,\dots,\operatorname{Log}|f_m|)\,\mathrm{d}\operatorname{Log}|f_I| \wedge \mathrm{d}\operatorname{Arg} f_J$. It suffices to prove that

$$\operatorname{std}\Big(\int_{(\operatorname{Log}|f|)^{-1}(\Pi_\varepsilon)} |\omega_{I,J}|\Big) \longrightarrow 0$$

when $\varepsilon \longrightarrow 0$.

9.1.10.1. The case where $I = J = E_0$. We then have $\varphi_{IJ}|_Q = 0$. Let P' be the preimage of ∂Q on P. Since $\varphi_{I,J}|_Q = 0$, we have

$$\operatorname{std}\Big(\int_{(\operatorname{Log}|f|)^{-1}(\Pi_\varepsilon)} |\omega_{I,J}|\Big) = \operatorname{std}\Big(\int_{(\operatorname{Log}|f|)^{-1}((P'_R)_\varepsilon)} |\omega_{I,J}|\Big)$$

for all ε, and since $\dim \partial Q < n$, the result follows from Section 9.1.8.

9.1.10.2. The case where $I \neq E_0$. Choose $i \in I \setminus E_0$. Then since x_i vanishes identically on P, we have for every ε

$$|f_i|\big((\operatorname{Log}|f|)^{-1}(\Pi_\varepsilon)\big) \subset \big[|t|^{2\varepsilon}, |t|^{-2\varepsilon}\big].$$

Therefore, there exists some positive standard real number A such that

$$|f_I|\big((\operatorname{Log}|f|)^{-1}(\Pi_\varepsilon)\big) \subset \big[|t|^{2\varepsilon}, |t|^{-2\varepsilon}\big]^{\{i\}} \times \big[|t|^{A}, |t|^{-A}\big]^{I\setminus\{i\}}$$

for ε small enough (see Section 9.1.2). In view of inequality (k) of Section 9.1.2, it follows that

$$\operatorname{std}\Big(\int_{(\operatorname{Log}|f|)^{-1}(\Pi_\varepsilon)} |\omega_{I,J}|\Big) \longrightarrow 0$$

when $\varepsilon \longrightarrow 0$.

9.1.10.3. The case where $J \neq E_0$. Since the operator J acts trivially on $\mathsf{A}^{n,n}$, we have

$$\begin{aligned}\omega_{I,J} &= \mathrm{J}(\omega_{I,J})\\ &= (-1)^n \varphi_{I,J}\big(\mathrm{Log}\,|f_1|,\dots,\mathrm{Log}\,|f|_m\big)\,\mathrm{d}\,\mathrm{Arg}\, f_I \wedge \mathrm{d}\,\mathrm{Log}\,|f_J|\\ &= (-1)^{n^2+n}\varphi_{I,J}\big(\mathrm{Log}\,|f_1|,\dots,\mathrm{Log}\,|f|_m\big)\,\mathrm{d}\,\mathrm{Log}\,|f_J| \wedge \mathrm{d}\,\mathrm{Arg}\, f_I\\ &= \varphi_{I,J}\big(\mathrm{Log}\,|f_1|,\dots,\mathrm{Log}\,|f|_m\big)\,\mathrm{d}\,\mathrm{Log}\,|f_J| \wedge \mathrm{d}\,\mathrm{Arg}\, f_I\,.\end{aligned}$$

Hence we reduce to the case considered in Section 9.1.10.2.

9.1.11. Proof of (3)(a) and (3)(b) in the general case

Now comes the core of our proof; this is the only step in which one uses the actual definition of the non-Archimedean integrals (the former ones used only basic properties like additivity or obvious norm estimates). Using once again the flexibility allowed by the former steps (which enables us to argue cellwise, see Section 9.1.9.1; or to modify the explicit writing of ω, provided that $(\omega_\flat)|_W$ remains unchanged, see Section 9.1.10), we will simplify slightly our assumptions, and then reduce to the case in which the integral $\int_W \omega_\flat$ can be computed by an explicit formula. The latter involves a classical real integral and the degree d of an étale map between Berkovich spaces over some skeleton Σ, and the main point of our reasoning consists in interpreting this degree d in the nonstandard Archimedean world; this is achieved by showing that our étale map also has degree d above "sufficiently many" C-points (over which the degree is now simply the naive one, namely, the cardinality of the fibers, which makes sense in our nonstandard Archimedean world as well).

By considering a cell decomposition of Q and using Section 9.1.9.1, we reduce to the case where Q is a cell. If $\dim Q < n$, then we already know that the required statement holds (see Section 9.1.8); we can thus assume that $\dim Q = n$. And in view of Section 9.1.9.2 we are allowed to perform an affine change of the coordinates indexed by E with integral linear part, we can assume that there exists a subset E_0 of E of cardinality n such that Q is contained in the subspace defined by the equations $x_i = 0$ for i running through $E \setminus E_0$. Otherwise said, $Q = Q_0 \times \{0\}^{E \setminus E_0}$ for some convex polyhedron Q_0 of $\mathbf{R}^{E_0}$. Since $\dim Q = n$ by our assumption, $\dim Q_0 = n$. Now $\xi|_{\lambda_\flat Q}$ can be written as

$$\frac{1}{\lambda_\flat^n}\varphi\Big(\frac{x_j}{\lambda_\flat}\Big)_{j\in E_0}\,\mathrm{d}' x_{E_0} \wedge \mathrm{d}'' x_{E_0}$$

(with φ smooth). Set

$$\omega' = \varphi\big(\mathrm{Log}\,|f_j|\big)_{j\in E_0}\,\mathrm{d}\,\mathrm{Log}\,|f_{E_0}| \wedge \mathrm{d}\,\mathrm{Arg}\, f_{E_0}$$

and

$$\omega'_\flat = \varphi\big(\mathrm{Log}_\flat |f_j|_\flat\big)_{j\in E_0} \,\mathrm{d}' \mathrm{Log}_\flat |f_{E_0}|_\flat \wedge \mathrm{d}'' \mathrm{Log}_\flat |f_{E_0}|_\flat.$$

Then $(\omega_\flat - \omega'_\flat)|_W = 0$, and in view of Section 9.1.10 this implies that

$$\mathrm{std}\Big(\int_{(\mathrm{Log}\,|f|)^{-1}(\Pi_\varepsilon)} |\omega - \omega'|\Big) \longrightarrow 0$$

when $\varepsilon \longrightarrow 0$. We can thus replace ω with ω', hence reduce to the case where ω is of the form

$$\omega = \varphi\big(\mathrm{Log}\,|f_j|\big)_{j\in E_0} \,\mathrm{d}\,\mathrm{Log}\,|f_{E_0}| \wedge \mathrm{d}\,\mathrm{Arg}\, f_{E_0}.$$

Let $\mu\colon V \to \mathbf{G}_\mathrm{m}^{E_0}$ be the map induced by the functions f_j for $j \in E_0$. Since $\dim Q_\mathbf{0} = n$ the tropical dimension of f_{E_0} is n, which forces μ to be dominant, hence generically étale, because both schemes involved are integral of the same dimension and the ground field is of characteristic 0. Let $\mathscr{Z}$ be a proper Zariski-closed subset of $\mathbf{G}_\mathrm{m}^{E_0}$ such that μ is finite étale over the open complement of $\mathscr{Z}$.

Let D be the affinoid domain of $\mathbf{G}_\mathrm{m}^{E_0,\mathrm{an}}$ defined by the condition $\mathrm{Log}_\flat |T|_\flat \in Q_\mathbf{0}$, and let D' be the open subset of D defined by the condition $\mathrm{Log}_\flat |T|_\flat \in \mathring{Q}_\mathbf{0}$. Also, let sk denote the canonical homeomorphism between $\mathbf{R}^{E_0}$ and the skeleton of $\mathbf{G}_\mathrm{m}^{E_0,\mathrm{an}}$. The images under $\mathrm{Log}_\flat |f_{E_0}|_\flat$ of the boundary of D and the image of $\mathscr{Z}^\mathrm{an}$ under $\mathrm{Log}_\flat |T|_\flat$ are of dimension less than n. Since we can argue cellwise on Q (see Section 9.1.9.1), we may thus assume the following:

- $(\mathrm{Log}_\flat |f_{E_0}|_\flat)(\partial W) \subset \partial Q_\mathbf{0}$;
- the morphism $W \times_{\mathbf{G}_\mathrm{m}^{E_0,\mathrm{an}}} D' \to D'$ is finite étale.

These two conditions imply that $\mu|_W$ is finite étale above D'. Since the latter is connected (it admits a deformation retraction to $\mathrm{sk}(\lambda_\flat \mathring{Q}_\mathbf{0})$), the degree of $\mu|_W$ over D' is constant; let us denote it by d. The map $\mu|_V$ is in particular finite and flat of degree d above every point of $\mathrm{sk}(\lambda_\flat \mathring{Q}_\mathbf{0})$, whence the equalities

$$\begin{aligned}\int_W \omega_\flat &= (-1)^{n(n-1)/2} \frac{d}{\lambda_\flat^n} \int_{\lambda_\flat Q_0} \varphi\Big(\frac{x_j}{\lambda_\flat}\Big)_{j\in E_0} \mathrm{d}x_{E_0} \\ &= (-1)^{n(n-1)/2} d \int_{Q_0} \varphi(x_j)_{j\in E_0} \,\mathrm{d}x_{E_0}\end{aligned}$$

and

$$\begin{aligned}\int_W |\omega_\flat|_\flat &= \frac{d}{\lambda_\flat^n} \int_{\lambda_\flat Q_0} \Big|\varphi\Big(\frac{x_j}{\lambda_\flat}\Big)_{j\in E_0}\Big| \mathrm{d}x_{E_0} \\ &= d \int_{Q_0} \big|\varphi(x_j)\big|_{j\in E_0} \mathrm{d}x_{E_0}.\end{aligned}$$

9.1.11.1. By construction, every point of $D'(C)$ has d preimages under μ in $W(C)$. We would like to exploit this fact in the nonstandard Archimedean setting. The point is that $D'(C)$ and $W(C)$ are ACVF-definable, but not RCF-definable; so we will first have to "approximate" them by RCF-definable subsets for which this statement remains true.

Let $\mathfrak{n}$ be the set of negligible elements of R. Let η be a positive standard real number, and set $Q_\eta = \mathring{Q}_0 \setminus (\partial Q_0)_\eta$. Let $x \in (\operatorname{Log}|T|)^{-1}(Q_{\eta,R})$. The point x belongs to $(\operatorname{Log}_\flat |T|_\flat)^{-1}(\mathring{Q}_0)$, hence the intersection

$$\mu^{-1}(x) \cap \big(\operatorname{Log}|f|\big)^{-1}(\Pi + \mathfrak{n}^\ell) = \mu^{-1}(x) \cap \big(\operatorname{Log}_\flat |f|_\flat\big)^{-1}(P) = \mu^{-1}(x) \cap W$$

has exactly d elements. Let $m(x)$ and $M(x)$ be, respectively, the greatest lower bound and the least upper bound of the set Θ of those $u \in [1, |t|^{-1}]$ such that

$$\mu^{-1}(x) \cap |f|^{-1}\big(|t|^{-\Pi} \cdot [u^{-1}, u]\big)$$

has exactly d elements. Since Θ is definable, if follows from the above that

$$\operatorname{std}\big(\operatorname{Log} m(x)\big) = 0 \qquad \text{and} \qquad \operatorname{std}\big(\operatorname{Log} M(x)\big) > 0.$$

Now m and M are definable functions; as a consequence, the greatest lower bound of M on $(\operatorname{Log}|T|)^{-1}(Q_{\eta,R})$ is equal to $|t|^{B(\eta)}$ for some $B(\eta)$ with negative standard part, and the least upper bound of m on $(\operatorname{Log}|T|)^{-1}(Q_{\eta,R})$ is equal to $|t|^{b(\eta)}$ for some negative negligible $b(\eta)$.

9.1.11.2. Let δ be a positive real number. Choose η such that the volume of $(\partial Q_0)_{2\eta}$ is smaller than δ. Let ε be a positive real number such that $\varepsilon < \min(B(\eta), \eta)$. Let Π' be the subset of Π_ε consisting of points whose projection to the variables in E_0 belongs to Q_η, and let Π'' be the complement of Π' in Π_ε. One has

$$\int_{(\operatorname{Log}|f|)^{-1}(\Pi_\varepsilon)} \omega = \int_{(\operatorname{Log}|f|)^{-1}(\Pi')} \omega + \int_{(\operatorname{Log}|f|)^{-1}(\Pi'')} \omega.$$

It follows from inequality (k) of Section 9.1.2 that there exists a positive standard real number M (independent of δ, η, ε, and so on) such that $\int_{(\operatorname{Log}|f|)^{-1}(\Pi'')} |\omega| \leq M \operatorname{Vol}((\partial Q_0)_{2\eta}) \leq M\delta$.

Now since $b(\eta) < \varepsilon < B(\eta)$, the map μ induces a d-fold covering

$$\big(\operatorname{Log}|f|\big)^{-1}(\Pi') \longrightarrow \big(\operatorname{Log}|T|\big)^{-1}(Q_{\eta,R}),$$

so

$$\begin{aligned}\int_{(\mathrm{Log}\,|f|)^{-1}(\Pi')}\omega &= \frac{d}{\lambda^n}\int_{(\mathrm{Log}\,|T|)^{-1}(Q_{\eta,R})}\varphi\big(\mathrm{Log}\,|T|\big)\,\mathrm{d}\,\mathrm{Log}\,|T|\wedge\mathrm{d}\,\mathrm{Arg}\,T\\ &= (-1)^{n(n-1)/2}d\int_{Q_{\eta,R}}\varphi(x_j)_{j\in E_0}\,\mathrm{d}x_{E_0}\\ &= (-1)^{n(n-1)/2}d\int_{Q_\eta}\varphi(x_j)_{j\in E_0}\,\mathrm{d}x_{E_0}.\end{aligned}$$

Therefore,

$$\begin{aligned}\Big|\int_{(\mathrm{Log}\,|f|)^{-1}(\Pi')}\omega-\int_W\omega_\flat\Big| &\le \sup_{P_0}|\varphi|d\,\mathrm{Vol}(Q_0\setminus Q_\eta)\\ &\le d\,\mathrm{Vol}\big((\partial Q_0)_{2\eta}\big)\sup_{Q_0}|\varphi|\\ &\le \delta d\sup_{Q_0}|\varphi|.\end{aligned}$$

Hence

$$\Big|\int_{(\mathrm{Log}\,|f|)^{-1}(\Pi_\varepsilon)}\omega-\int_W\omega_\flat\Big|\le\delta\big(M+d\sup_{Q_0}|\varphi|\big).$$

One shows exactly in the same way that

$$\Big|\int_{(\mathrm{Log}\,|f|)^{-1}(\Pi_\varepsilon)}|\omega|-\int_W|\omega_\flat|\Big|\le\delta\big(M+d\sup_{Q_0}|\varphi|\big).$$

We thus have proved that

$$\mathrm{std}\Big(\int_{(\mathrm{Log}\,|f|)^{-1}(\Pi_\varepsilon)}\omega\Big)\longrightarrow\int_{(\mathrm{Log}_\flat\,|f|_\flat))^{-1}(\mathrm{std}(\Pi))}\omega_\flat$$

and

$$\mathrm{std}\Big(\int_{(\mathrm{Log}\,|f|)^{-1}(\Pi_\varepsilon)}|\omega|\Big)\longrightarrow\int_{(\mathrm{Log}_\flat\,|f|_\flat))^{-1}(\mathrm{std}(\Pi))}|\omega_\flat|_\flat$$

when the standard positive real number ε tends to zero.

9.2. Construction of the map $\omega\mapsto\omega_\flat$

It is clear that there is at most one such morphism of sheaves. We are going to prove that there is actually one by using our comparison theorem for integrals and the fact that forms are naturally embedded into currents on Berkovich spaces. Let p and q be two integers. Let U be a Zariski-open subset of X. Let ω be a section of $\mathsf{A}^{p,q}$ on U that can be written as

$$\omega = \sum_{|I|=p,|J|=q} \varphi_{I,J}\big(\mathrm{Log}\,|f_1|,\ldots,\mathrm{Log}\,|f_m|\big)\,\mathrm{d}\,\mathrm{Log}\,|f_I| \wedge \mathrm{d}\,\mathrm{Arg}\, f_J$$

with f_i regular functions on U and $\varphi_{I,J}$ an $(I \cup J)$-vanishing reasonably smooth function in $\mathscr{S}^{I,J,(f_i)}$ for each (I, J) (we shall say for short that ω is *tropical* on U).

Let $\omega_\flat$ be the section

$$\sum_{I,J} \varphi_{I,J}\big(\mathrm{Log}_\flat\,|f_1|_\flat,\ldots,\mathrm{Log}_\flat\,|f_m|_\flat\big)\,\mathrm{d}'\,\mathrm{Log}_\flat\,|f_I|_\flat \wedge \mathrm{d}''\log|f_J|_\flat$$

of $\mathsf{B}^{p,q}$ on U. It suffices to show that $\omega_\flat$ depends only on ω, and not on the particular way we have written it. One immediately reduces to proving that $\omega_\flat = 0$ if $\omega = 0$; for that purpose we suppose that $\omega_\flat \neq 0$, and we are going to prove that $\omega \neq 0$. Since $\omega_\flat \neq 0$ and since U^{an} is boundaryless, there exists a smooth compactly supported $(n-p, n-q)$ form η on U^{an} such that $\int_{U^{\mathrm{an}}} \omega_\flat \wedge \eta \neq 0$ (see [6, Corollaire 4.3.7]). Every point of U^{an} has a basis of affinoid neighborhoods V having the following properties.

- The restriction $\eta|_V$ can be written as

$$\sum_{|I|=n-p,|J|=n-q} \psi_{I,J}\big(\mathrm{Log}_\flat\,|g_1|_\flat,\ldots,\mathrm{Log}_\flat\,|g_\ell|_\flat\big)\,\mathrm{d}'\log|g_I|_\flat \wedge \mathrm{d}''\log|g_J|_\flat$$

  with g_i regular functions on V and $\psi_{I,J}$ compactly supported smooth functions on $\mathbf{R}^\ell$.
- The domain V is a Weierstrass domain of Ω^{an} for some open subscheme Ω of U (see Section 6.2).

Then we can find such a V with $\int_V \omega_\flat \wedge \eta \neq 0$. Since V is a Weierstrass domain in Ω^{an}, and since $\eta|_V$ does not change if we replace each g_i by a function having the same norm on V (see [6, Lemme 3.1.10]), we can assume by approximation that each of the functions g_i comes from a function belonging to $\mathscr{O}(\Omega)$, which we still denote by g_i. Then by replacing Ω by the intersection of the sets $D(g_i)$, we can assume that $g_i \in \mathscr{O}(\Omega)^\times$ for all i.

Now set

$$\eta^\sharp = \sum_{|I|=n-p,|J|=n-q} \psi_{I,J}\big(\mathrm{Log}\,|g_1|,\ldots,\mathrm{Log}\,|g_\ell|\big)\,\mathrm{d}\,\mathrm{Log}\,|g_I| \wedge \mathrm{d}\,\mathrm{Arg}\, g_J.$$

This is a section of $\mathsf{A}^{n-p,n-q}$ on Ω. By Section 9.1, the integral $\int_V \omega_\flat \wedge \eta$ can be expressed as a limit of standard parts of integrals of $\omega|_\Omega \wedge \eta^\sharp$ on suitable definably compact semialgebraic subsets of $\Omega(C)$. Then these integrals cannot be all equal to zero, which implies that $\omega|_\Omega \wedge \eta^\sharp \neq 0$, and a fortiori that $\omega \neq 0$. *We thus are done with the proof in the particular setting of Section 9.1.1*.

9.3. Proof of (3)

We are now going to prove (3) in the general case. The reasoning is tedious, but rather formal; it uses as a crucial input the particular case handled above in Section 9.1, together with the additivity of the integrals in both settings.

For all i, we set $P_i = \mathrm{std}(\Pi_i)$ and $W_i = (\mathrm{Log}_\flat |g_i|_\flat)^{-1}(P_i) \subset V_i^{\mathrm{an}}$; we also set $W = \bigcup_i W_i$.

9.3.1. Reduction to the case where $\Pi_i = P_{i,R}$ for all i

Assume that (3) holds for $(P_{i,R})_i$. Since $\mathrm{std}(\Pi_i) = \mathrm{std}(P_{i,R})$, there exists a positive negligible element a such that $\Pi_i \subset P_{i,R,a}$ and $P_{i,R} \subset \Pi_{i,a}$ for every i. Let ε be a standard positive real number. By the above,

$$P_{i,R,\varepsilon/2} \subset \Pi_{i,\varepsilon} \subset P_{i,R,2\varepsilon}$$

for all i. Then it follows from Remark 8.2 that statements (3)(a) and (3)(b) hold for (Π_i).

We then have for all standard $\varepsilon > 0$ and all i

$$\bigcup_i \Pi_{i,\varepsilon} \setminus \bigcup_i \Pi_{i,\alpha+a} \subset \bigcup_i P_{i,R,2\varepsilon} \setminus \bigcup_i P_{i,R,\alpha},$$

so (3)(c) holds for $(\Pi_i)_i$ with the negligible element $\alpha + a$ instead of α. We assume from now on that $\Pi_i = P_{i,R}$ for all i.

9.3.2

Fix an index i. Let x be a point of W_i. There exists a Zariski-open subset Ω of V_i on which ω is tropical, and such that $x \in \Omega^{\mathrm{an}}$. The point x has a Weierstrass neighborhood Ω' in Ω^{an}; by construction, $\Omega' \cap W_i$ is of the form $(\mathrm{Log}_\flat |h|_\flat)^{-1}(Q)$ for some family $h = (h_1, \dots, h_N)$ of regular functions on Ω' and some pseudopolyhedron Q of $(\mathbf{R} \cup \{-\infty\})^N$.

By compactness, it follows that there exists a finite family (V_{ij}) of Zariski-open subsets of V_i and, for each (i,j), a finite family $h_{ij} = (h_{ijk})_{1 \le k \le \ell_{ij}}$ of regular functions on V_{ij} and a pseudopolyhedron P_{ij} of $(\mathbf{R} \cup \{-\infty\})^{\ell_{ij}}$ such that the following hold:

- for each (i,j), the form ω is tropical on V_{ij};
- $W_i = \bigcup_j W_{ij}$ with $W_{ij} = (\mathrm{Log}_\flat |h_{ij}|_\flat)^{-1}(P_{ij})$.

We set $\Pi_{ij} = P_{ij,R}$; for every nonempty set I of pairs (i,j), we set

- $\ell_I = \sum_{(i,j)\in I} \ell_{ij}$;
- $\Pi_I = \prod_{(i,j)\in I} \Pi_{ij} \subset (R \cup \{-\infty\})^{\ell_I}$;
- $P_I = \prod_{(i,j)\in I} P_{ij} \subset (\mathbf{R} \cup \{-\infty\})^{\ell_I}$;
- $V_I = \bigcap_{(i,j)\in I} V_{ij}$ and $W_I = \bigcap_{(i,j)\in I} W_{ij}$.

We also denote by h_I the concatenation of the functions h_{ij} for $(i,j) \in I$; this is a family of ℓ_I invertible functions on V_I and $W_I = (\mathrm{Log}_\flat |h_I|_\flat)^{-1}(P_I) \subset V_I^{\mathrm{an}}$.

For every I, the form $\omega|_{V_I}$ is tropical. It follows therefore from Section 9.1 that

$$\mathrm{std}\Big(\int_{(\mathrm{Log}\,|h_I|)^{-1}(\Pi_{I,\varepsilon})} \omega\Big) \longrightarrow \int_{(\mathrm{Log}_\flat\,|h_I|_\flat)^{-1}(P_I)} \omega_\flat, \tag{o}$$

$$\mathrm{std}\Big(\int_{(\mathrm{Log}\,|h_I|)^{-1}(\Pi_{I,\varepsilon})} |\omega|\Big) \longrightarrow \int_{(\mathrm{Log}_\flat\,|h_I|_\flat))(P_I)} |\omega_\flat|_\flat \tag{p}$$

when the positive *standard* number ε tends to zero, and that there exists a positive negligible $\alpha \in R$ such that

$$\mathrm{std}\Big(\int_{(\mathrm{Log}\,|g|)^{-1}(\Pi_{I,\varepsilon}\setminus\Pi_{I,\alpha})} |\omega|\Big) \longrightarrow 0 \tag{q}$$

when the positive *standard* number ε tends to zero.

The equality $W = \bigcup_{(i,j)\in I} V_{ij}$ can be rewritten as

$$\bigcup_i \underbrace{\big(\mathrm{Log}_\flat\,|g_i|_\flat\big)^{-1}(P_i)}_{\text{understood as contained in } V_i^{\mathrm{an}}} = \bigcup_{(i,j)} \underbrace{\big(\mathrm{Log}_\flat\,|h_{ij}|_\flat\big)^{-1}(P_{ij})}_{\text{understood as contained in} V_{ij}^{\mathrm{an}}}.$$

If a is a small enough positive real number, then for every i, j the sets $(\mathrm{Log}_\flat\,|g_i|_\flat)^{-1}(P_{i,a})$ and $(\mathrm{Log}_\flat\,|h_{ij}|_\flat)^{-1}(P_{ij,a})$ are compact in view of assertion (1). Hence for a small enough, the infimum $m(a)$ of all positive real numbers b such that

$$\bigcup_i \big(\mathrm{Log}_\flat\,|g_i|_\flat\big)^{-1}(P_{i,a}) \subset \bigcup_{(i,j)} \big(\mathrm{Log}_\flat\,|h_{ij}|_\flat\big)^{-1}(P_{ij,b})$$

is well defined. This is a DOAG-definable function of a that tends to zero when a tends to zero. It follows that there exists a positive rational number ρ such that

$$\bigcup_i \big(\mathrm{Log}_\flat\,|g_i|_\flat\big)^{-1}(P_{i,a}) \subset \bigcup_{(i,j)} \big(\mathrm{Log}_\flat\,|h_{ij}|_\flat\big)^{-1}(P_{ij,\rho a})$$

for a small enough. We can perform the same kind of reasoning for the converse inclusion, and by taking ρ big enough we can thus assume that we also have

$$\bigcup_i \big(\mathrm{Log}_\flat\,|g_i|_\flat\big)^{-1}(P_{i,\rho a}) \supset \bigcup_{(i,j)} \big(\mathrm{Log}_\flat\,|h_{ij}|_\flat\big)^{-1}(P_{ij,a})$$

for a small enough.

We then have for all positive standard real numbers a the inclusions

$$\bigcup_i \big(\mathrm{Log}\,|g_i|\big)^{-1}(\Pi_{i,a}) \subset \bigcup_{(i,j)} \big(\mathrm{Log}\,|h_{ij}|\big)^{-1}(\Pi_{ij,2\rho a})$$

and

$$\bigcup_i \bigl(\operatorname{Log}|g_i|\bigr)^{-1}(\Pi_{i,2\rho a}) \supset \bigcup_{(i,j)} \bigl(\operatorname{Log}|h_{ij}|\bigr)^{-1}(\Pi_{ij,a}).$$

But then by a definability argument (using Section 7.1.2), there exist a positive negligible element $\beta \in R$ and an element $\gamma \in R$ with positive standard part such that the above inclusions hold for all elements $a \in R$ with $\beta \leq a \leq \gamma$.

By the same kind of arguments, we can increase β and ρ and decrease γ so that we have for all I and all $a \in [\beta, \gamma]$ the inclusions

$$\bigcap_{(i,j)\in I} \bigl(\operatorname{Log}|h_{ij}|\bigr)^{-1}(\Pi_{ij,a}) \subset \bigl(\operatorname{Log}|h_I|\bigr)^{-1}(\Pi_{I,\rho a})$$

and

$$\bigcap_{(i,j)\in I} \bigl(\operatorname{Log}|h_{ij}|\bigr)^{-1}(\Pi_{ij,\rho a}) \supset \bigl(\operatorname{Log}|h_I|\bigr)^{-1}(\Pi_{I,a}).$$

Together with (o), (p), and (q) above and with the additivity of both the Archimedean and the Berkovich integrals, this ends the proof of (2).

9.4. End of the proof

It remains to show (1) and (4). The proofs essentially consist in standard computations, once granted the existence of our map of complexes and the comparison theorems (3)(a), (3)(b), and (3)(c) for integrals.

We use the assumptions of (1). Choose a finite open affine cover (U_i) of U. For every i, let $(f_{ij})_j$ be a finite generating family of the C-algebra $\mathscr{O}_X(U_i)$. By our assumption on the support of ω and by Lemma 3.2, there exists $A \in \mathbf{R}$ such that the ω is zero outside the set

$$E_A := \bigcup_i \bigl\{x \in U_i(C), \operatorname{Log}\bigl|f_{ij}(x)\bigr| \leq A \text{ for all } j\bigr\}.$$

We also set

$$E_{A,\flat} = \bigcup_i \bigl\{x \in U_i^{\mathrm{an}}, \operatorname{Log}_\flat \bigl|f_{ij}(x)\bigr|_\flat \leq A \text{ for all } j\bigr\}.$$

9.4.1. Proof of (1)

We are going to prove that $\omega_\flat$ is zero outside $E_{A,\flat}$, which will show that it is compactly supported.

Let y be a point of $U^{\mathrm{an}} \setminus E_{A,\flat}$. The point y belongs to U_i for some i. Let us choose a neighborhood V of y in $U_i^{\mathrm{an}} \setminus E_{A,\flat}$ of the form $\operatorname{Log}_\flat |g|_\flat^{-1}(P)$, where

$g = (g_1, \ldots, g_m)$ is a finite family of regular functions on U_i and where $P \subset (\mathbf{R} \cup \{-\infty\})^m$ is a product of intervals, each of which is either of the form (λ, μ) or of the form $(-\infty, \mu)$. Up to shrinking P, we can assume that for some $\varepsilon > 0$ the preimage $\mathrm{Log}_\flat |g|_\flat^{-1}(P + [0, \varepsilon)^m)$ still avoids $E_{A,\flat}$. Let φ be a reasonably smooth function on $(\mathbf{R} \cup \{-\infty\})^m$ whose support is contained in P, which does not vanish at $g(y)$, and which takes only nonnegative values. We shall prove that the form $\varphi(\mathrm{Log}_\flat |g|_\flat)\omega_\flat \in \mathscr{A}^{p,q}(U_i^{\mathrm{an}})$ is zero; this will ensure that $\omega_\flat$ vanishes around y and thus imply our claim.

Since $\mathrm{Log}_\flat |g|_\flat^{-1}(P + [0, \varepsilon)^m)$ is contained in $U^{\mathrm{an}} \setminus E_{A,\flat}$, the preimage $\mathrm{Log}\,|g|^{-1}(P_R)$ avoids E. As the support of φ is contained in P and as ω vanishes outside E, the form $\varphi(\mathrm{Log}\,|g|)\omega$ vanishes. But this form belongs to $\mathsf{A}^{p,q}(U_i)$ and its image in $\mathsf{B}^{p,q}(U_i)$ is precisely $\varphi(\mathrm{Log}_\flat |g|_\flat)\omega_\flat$. The latter is thus zero, as announced.

9.4.2. Proof of (4)

Assume moreover that $p = q = n$, and let us prove (f) and (g). It follows from (2), (3)(a), and (3)(b) that if the standard positive ε is small enough, then $\int_{E_{A+\varepsilon}} |\omega|$ is bounded, and that

$$\mathrm{std}\Big(\int_{E_{A+\varepsilon}} \omega\Big) \to \int_{E_{A,\flat}} \omega_\flat$$

and

$$\mathrm{std}\Big(\int_{E_{A+\varepsilon}} |\omega|\Big) \to \int_{E_{A,\flat}} |\omega_\flat|_\flat$$

when ε tends to zero (while remaining standard and positive).

But since ω is zero outside E_A, we have

$$\int_{E_{A+\varepsilon}} \omega = \int_{U(C)} \omega \qquad \text{and} \qquad \int_{E_{A+\varepsilon}} |\omega| = \int_{U(C)} |\omega|$$

for any ε as above. And since $\omega_\flat$ is zero outside $E_{A,\flat}$ by Section 9.4.1, we have

$$\int_{E_{A,\flat}} \omega_\flat = \int_{U^{\mathrm{an}}} \omega_\flat \qquad \text{and} \qquad \int_{E_{A,\flat}} |\omega_\flat|_\flat = \int_{U^{\mathrm{an}}} |\omega_\flat|_\flat.$$

Assertion (4) follows immediately.

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References

[1] A. BERARDUCCI and M. OTERO, *An additive measure in o-minimal expansions of fields*, Q. J. Math. **55** (2004), no. 4, 411–419. MR 2104681. DOI 10.1093/qjmath/55.4.411. *(314)*

[2] V. G. BERKOVICH, "A non-Archimedean interpretation of the weight zero subspaces of limit mixed Hodge structures" in *Algebra, Arithmetic, and Geometry, Vol. I*, Progr. Math. **269**, Birkhäuser Boston, Boston, 2009, 49–67. MR 2641170. DOI 10.1007/978-0-8176-4745-2_2. *(314)*

[3] J. BOCHNAK, M. COSTE, and M.-F. ROY, *Géométrie algébrique réelle*, Ergeb. Math. Grenzgeb. (3) **12**, Springer, Berlin, 1987. MR 0949442. *(326, 335, 348)*

[4] S. BOUCKSOM and M. JONSSON, *Tropical and non-Archimedean limits of degenerating families of volume forms*, J. Éc. polytech. Math. **4** (2017), 87–139. MR 3611100. DOI 10.5802/jep.39. *(314)*

[5] L. BROECKER, "On the reduction of semialgebraic sets by real valuations" in *Recent Advances in Real Algebraic Geometry and Quadratic Forms (Berkeley, 1990/1991; San-Francisco, 1991)*, Contemp. Math. **155**, Amer. Math. Soc., Providence, 1991, 75–95. MR 1260702. DOI 10.1090/conm/155/01374. *(331)*

[6] A. CHAMBERT-LOIR and A. DUCROS, *Formes différentielles réelles et courants sur les espaces de Berkovich*, preprint, arXiv:1204.6277v1 [math.AG]. *(313, 315, 320, 321, 355, 362, 366, 379)*

[7] L. DEMARCO, H. KRIEGER, and H. YE, *Uniform Manin-Mumford for a family of genus 2 curves*, Ann. of Math. (2) **191** (2020), no. 3, 949–1001. MR 4088354. DOI 10.4007/annals.2020.191.3.5. *(314)*

[8] A. DUCROS, *Espaces de Berkovich, polytopes, squelettes et théorie des modèles*, Confluentes Math. **4** (2012), no. 4, art. ID 1250007. MR 3020334. DOI 10.1142/S1793744212500077. *Erratum*, Confluentes Math. **5** (2013), no. 2, 43–44. MR 3145032. DOI 10.5802/cml.7. *(366)*

[9] C. FAVRE, *Degeneration of endomorphisms of the complex projective space in the hybrid space*, J. Inst. Math. Jussieu **19** (2020), no. 4, 1141–1183. MR 4120806. DOI 10.1017/s147474801800035x. *(314)*

[10] M. GROMOV, *Groups of polynomial growth and expanding maps*, Publ. Math. Inst. Hautes Études Sci. **53** (1981), 53–73. MR 0623534. *(314)*

[11] M. GROSS, V. TOSATTI, and Y. ZHANG, *Collapsing of abelian fibered Calabi-Yau manifolds*, Duke Math. J. **162** (2013), no. 3, 517–551. MR 3024092. DOI 10.1215/00127094-2019703. *(314)*

[12] ———, *Gromov-Hausdorff collapsing of Calabi-Yau manifolds*, Comm. Anal. Geom. **24** (2016), no. 1, 93–113. MR 3514555. DOI 10.4310/CAG.2016.v24.n1.a4. *()*

[13] M. GROSS and P. M. H. WILSON, *Large complex structure limits of $K3$ surfaces*, J. Differential Geom. **55** (2000), no. 3, 475–546. MR 1863732. *(314)*

[14] E. HRUSHOVSKI and F. LOESER, *Non-Archimedean Tame Topology and Stably Dominated Types*, Ann. of Math. Stud. **192**, Princeton Univ. Press, Princeton, 2016. MR 3445772. DOI 10.1515/9781400881222. *(326)*

[15] H. J. KEISLER, *Ultraproducts and saturated models*, Indag. Math. (N.S.) **26** (1964), 178–186. MR 0168483. *(325)*

[16] M. KONTSEVICH and Y. SOIBELMAN, "Homological mirror symmetry and torus fibrations" in *Symplectic Geometry and Mirror Symmetry (Seoul, 2000)*, World Sci., River Edge, 2001, 203–263. MR 1882331. DOI 10.1142/9789812799821_0007. *(314)*

[17] ———, "Affine structures and non-Archimedean analytic spaces" in *The Unity of Mathematics*, Progr. Math. **244**, Birkhäuser Boston, Boston, 2006, 321–385. MR 2181810. DOI 10.1007/0-8176-4467-9_9. *(314)*

[18] L. KRAMER and K. TENT, *Asymptotic cones and ultrapowers of Lie groups*, Bull. Symb. Log. **10** (2004), no. 2, 175–185. MR 2062416. DOI 10.2178/bsl/1082986261. *(314)*

[19] A. LAGERBERG, *Super currents and tropical geometry*, Math. Z. **270** (2012), no. 3–4, 1011–1050. MR 2892935. DOI 10.1007/s00209-010-0837-8. *(313)*

[20] A. H. LIGHTSTONE and A. ROBINSON, *Nonarchimedean Fields and Asymptotic Expansions*, North-Holland Math. Lib. **13**, North-Holland, Amsterdam, 1975. MR 0414354. *(314)*

[21] W. A. J. LUXEMBURG, *On a class of valuation fields introduced by A. Robinson*, Israel J. Math. **25** (1976), no. 3–4, 189–201. MR 0460102. DOI 10.1007/BF02756999. *(317, 325)*

[22] D. MARKER, *Model Theory: An Introduction*, Grad. Texts in Math. **217**, Springer, New York, 2002. MR 1924282. *(321)*

[23] Y. ODAKA, *Tropical geometric compactification of moduli, II: A_g case and holomorphic limits*, Int. Math. Res. Not. IMRN **2019**, no. 21, 6614–6660. MR 4027562. DOI 10.1093/imrn/rnx293. *(314)*

[24] Y. ODAKA and Y. OSHIMA, *Collapsing K3 surfaces and moduli compactification*, Proc. Japan Acad. Ser. A Math. Sci. **94** (2018), no. 8, 81–86. MR 3859764. DOI 10.3792/pjaa.94.81. *(314)*

[25] A. ROBINSON, *Function theory on some nonarchimedean fields*, Amer. Math. Monthly **80** (1973), no. 6, 87–109. MR 0330126. DOI 10.2307/3038223. *(314)*

[26] D. SUSTRETOV, *Gromov-Hausdorff limits of flat Riemannian surfaces*, preprint, arXiv:1802.03818v3 [math.AG]. *(314)*

[27] K. TENT and M. ZIEGLER, *A Course in Model Theory*, Lect. Notes Log. **40**, Cambridge Univ. Press, Cambridge, 2012. MR 2908005. DOI 10.1017/CBO9781139015417. *(321)*

[28] L. VAN DEN DRIES and A. J. WILKIE, *Gromov's theorem on groups of polynomial growth and elementary logic*, J. Algebra **89** (1984), no. 2, 349–374. MR 0751150. DOI 10.1016/0021-8693(84)90223-0. *(314)*

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