

---

# AUTOMATIC MEROMORPHY IN NON-ARCHIMEDEAN GEOMETRY

by

Antoine Ducros

---

**Abstract.** — In this article, we prove that if  $U$  is a Zariski-open subset of a reduced non-archimedean analytic space  $X$  and  $f$  is an analytic function on  $U$  whose zero-locus is equal to  $Z \cap U$  for some Zariski-closed subset  $Z$  of  $X$ , then  $f$  extends to a meromorphic function on  $X$  (unique if  $U$  is dense). As a corollary, we prove that if  $\mathcal{X}$  is a reduced scheme locally of finite type over an affinoid algebra, every analytic function on  $\mathcal{X}^{\text{an}}$  with algebraic zero-locus is algebraic.

## 1. Introduction

The purpose of this text is to prove the following theorem and its corollary; analytic spaces here have to be understood in the sense of Berkovich.

**1.1. Theorem.** — *Let  $k$  be a complete, non-archimedean field and let  $X$  be a reduced  $k$ -analytic space. Let  $U$  be a Zariski-open subset of  $X$ . Let  $f$  be an analytic function on  $U$ . The following are equivalent :*

- (i) *The function  $f$  can be extended to a meromorphic function on  $X$ .*
- (ii) *The zero-locus of  $f$  is of the form  $Z \cap U$  for  $Z$  a Zariski-closed subset of  $X$  (e.g.,  $f$  is invertible on  $U$ ).*

**1.2. Comments.** — Implication (i) $\Rightarrow$ (ii) is easy. So the actual content of Theorem 1.1 is (ii) $\Rightarrow$ (i). Note that for an analytic function  $f$  on  $U$  to fulfill condition (ii) it suffices that the zero-locus  $T$  of  $f$  in  $U$  be closed in  $X$ . Indeed, if this is the case,  $T = T \cap U$  is Zariski-closed in  $X$  as checked on the open cover of  $X$  by  $U$  and  $X \setminus T$ .

**1.3. Corollary.** — *Let  $X$  be a reduced scheme locally of finite type over an affinoid algebra and let  $f$  be an analytic function on  $X^{\text{an}}$ . The following are equivalent:*

- (i) *The function  $f$  is algebraic; i.e., it belongs to (the image of)  $\mathcal{O}_X(X)$ .*
- (ii) *The zero-locus of  $f$  is algebraic, i.e., of the form  $Y^{\text{an}}$  for  $Y$  a Zariski-closed subset of  $X$  (e.g.,  $f$  is an invertible analytic function on  $X^{\text{an}}$ ).*

**1.4. Comments.** — Implication (i) $\Rightarrow$ (ii) is obvious. So the actual content of Corollary 1.3 is (ii) $\Rightarrow$ (i). Note that for an analytic function  $f$  on  $X^{\text{an}}$  to fulfill condition (ii) it suffices that  $X$  be separated and that the zero-locus  $T$  of  $f$  in  $X^{\text{an}}$  be compact. Indeed, if this is the case, then by compactness  $T$  is contained in  $U^{\text{an}}$  for some quasi-compact open subscheme  $U$  of  $X$ . We can choose by Nagata's Theorem a proper compactification  $\overline{U}$  of  $U$  over  $A$ . Then  $T$  is a Zariski-closed subset of  $\overline{U}^{\text{an}}$  as checked on the open cover of  $\overline{U}^{\text{an}}$  by  $U^{\text{an}}$  and  $\overline{U}^{\text{an}} \setminus T$ , and so  $T$  is algebraic by GAGA, see for example [Poi10, Appendix A].

**1.5. Remark.** — In Theorem 1.1, the meromorphic extension of  $f$  is not unique in general (think of the case  $U = \emptyset$  !), but it is as soon as  $U$  is dense in  $X$ .

**1.6. Remark.** — Theorem 1.1 exhibits a typical non-archimedean feature. Indeed, it fails definitely in the complex-analytic setting, as well as Corollary 1.3, as witnessed by the exponential function on  $\mathbf{A}_C^{1,\text{an}}$ .

**1.7. Remark.** — The reducedness assumption is necessary for Theorem 1.1 and Corollary 1.3 to hold, at least when  $k$  is not trivially valued. Indeed, assume that  $|k^\times| \neq \{1\}$  and choose  $f = \sum a_i t^i \in k[[t]]$  with infinite radius of convergence which is not a polynomial. Then  $1 + \varepsilon f$  is an invertible analytic function on the  $k$ -analytic space  $\mathbf{A}_{k[\varepsilon]}^{1,\text{an}}$  with  $\varepsilon$  a non-zero nilpotent element. And  $f$  is not algebraic, nor does it admit any meromorphic extension to  $\mathbf{P}_{k[\varepsilon]}^{1,\text{an}}$ .

If  $k$  is trivially valued, Theorem 1.1 and Corollary 1.3 also fail in general for non-reduced spaces, as witnessed by the former example over the field  $K := k((s))$  (equipped with any  $s$ -adic absolute value), the point being that  $K[\varepsilon]$  is a (non-strict)  $k$ -affinoid algebra. But if  $X$  is any (possibly non-reduced)  $k$ -scheme locally of finite type over  $k$  itself (rather than over a  $k$ -affinoid algebra), then every analytic function on  $X^{\text{an}}$  is algebraic – one checks it on affine schemes by reducing to the case of the  $n$ -dimensional affine space, for which this is obvious.

**1.8. Remark.** — When the ring of integers  $k^\circ$  is a discrete valuation ring, Piotr Achinger has suggested the following possible analogue of Theorem 1.1 for a special formal scheme  $\mathfrak{X}$  over  $k^\circ$  (see [Ber96], §1 for a definition) : let  $f$  be an analytic function on  $\mathfrak{X}_\eta$  whose zero-locus is of the form  $\mathfrak{Y}_\eta$  for a Zariski-closed formal subscheme  $\mathfrak{Y}$  of  $\mathfrak{X}$ ; then  $f$  is bounded – thus in some sense meromorphic from the formal viewpoint. This would be a beautiful and

natural result, but we unfortunately do not know whether it holds. Adapting our strategy to this situation would require one to establish formal analogues of two results of Bartenwerfer that play a key role in our proof (see Proposition 2.10).

**Strategy of the proof.** — As noticed by one of the referees, the proof ultimately consists in establishing a slightly easier particular case of Theorem 1.1 and then in showing that meromorphy "can be detected curvewise" and even "discwise". We shall more precisely prove the following results, from which Theorem 1.1 will follow straightforwardly; we will then get Corollary 1.3 by combining Theorem 1.1 and GAGA results about meromorphic functions.

**1.9. Lemma (Particular case of Theorem 1.1).** — *Let  $D$  be a (closed or open) one-dimensional disc and let  $U \subset D$  be the complement of the origin. Let  $f$  be an analytic function on  $U$ . The following are equivalent:*

- (i) *the function  $f$  can be extended to a meromorphic function on  $D$ ; i.e.,  $f$  has no essential singularity at the origin;*
- (ii) *the zero-locus of  $f$  is of the form  $Z \cap U$  for  $Z$  a Zariski-closed subset of  $D$ ; i.e.,  $f$  is invertible on a punctured neighborhood of the origin.*

**1.10. Theorem (Discwise detection of meromorphy)**

*Let  $X$  be a reduced  $k$ -analytic space, let  $U$  be a Zariski-open subset of  $X$  and let  $f$  be an analytic function on  $U$ . The following are equivalent:*

- (i) *the function  $f$  can be extended to a meromorphic function on  $X$  ;*
- (ii) *for every complete extension  $L$  of  $k$ , every one-dimensional (closed or open) disc  $D$  over  $L$ , and every  $k$ -morphism  $\varphi: D \rightarrow X$  such that  $\varphi^{-1}(U) = D \setminus \{0\}$ , the function  $\varphi^*f$  on  $D \setminus \{0\}$  admits an extension to a meromorphic function on  $D$ .*

**1.11. About the proof of Lemma 1.9.** — The direct implication is obvious. For the converse implication one may assume by shrinking  $D$  that  $f$  is invertible on  $U$ . The Lemma then follows easily from the very classical fact that the power series  $f$  admits a dominant monomial. This is in some sense the core argument on which Theorem 1.1 relies, and the only one that is specific to the non-archimedean world, and definitely prevents our method to be adapted over  $\mathbb{C}$ , see Remark 1.6.

**1.12. Quick overview of the proof of Theorem 1.10.** — The direct implication is straightforward. Let us assume (ii) and prove (i). By arguing componentwise one first reduces to the case where  $U$  is dense. One can then argue  $G$ -locally and thus assume that  $X$  is affinoid. Up to performing a suitable radicial ground field extension and modding out by nilpotents we

can moreover assume that the normalization of  $X$  is geometrically normal, and that the reduced irreducible components of  $X \setminus U$  are generically quasi-smooth.

Relying upon these two facts we then reduce, by using normalization and the previously known non-archimedean analogue (due to Bartenwerfer) of the classical extension theorem for arbitrary meromorphic functions through Zariski-closed subsets of codimension  $\geq 2$  on a normal space, to the case where  $X$  is quasi-smooth and where  $U$  is the complement of a quasi-smooth hypersurface  $S$ ; and then, by looking at the local structure of the embedding  $S \hookrightarrow X$ , to the case where  $X$  is a product  $D \times_k Y$  with  $D$  a one-dimensional closed disc and  $Y$  smooth and irreducible, and where  $S = Y \times \{0\}$ . We then pick a Zariski-generic point  $y$  of  $Y$  and denote by  $\varphi: D_{\mathcal{H}(y)} \rightarrow Y$  the embedding of the fiber at  $y$  of the projection map  $X \rightarrow Y$ . By assumption (ii)  $\varphi^*f$  has a meromorphic singularity at the origin of  $D_{\mathcal{H}(y)}$ ; this means that the coefficients of the power series defining  $f$  vanish at  $y$  for sufficiently negative exponents. By genericity of  $y$ , the corresponding coefficients are actually zero and  $f$  is meromorphic.

**1.13. Structure of the paper.** — Section 2 is devoted to the material in non-Archimedean geometry that will be used in our proofs. The latter are presented in section 3; the reader may go directly to section 3 and refer to section 2 when needed.

## 2. Reminders on analytic geometry

**2.1. General references.** — Our framework is that of Berkovich spaces. We refer the reader to the first chapters of [Ber90] and [Ber93] for the basic definitions. [Duc09, section 3] and [Duc18, chapter 2] for the “commutative algebra properties” (like being normal, reduced, etc.), [Duc09, section 4] for the Zariski topology, [Duc09, section 6] and [Duc18, chapter 5] for the notion of quasi-smoothness, [Duc09, section 5] for the normalization, and [Ber93, §2.6] for the analytification of a scheme.

Let us mention that contrary to Berkovich we use the notation  $\mathcal{O}_X$  (rather than  $\mathcal{O}_{X_G}$ ) for the structure sheaf of  $X$  for the  $G$ -topology, so that  $\mathcal{O}_X(V)$  makes sense for any analytic domain  $V$  of  $X$ .

**2.2. Analytic functions on annuli.** — In some sense, the core argument in the proof of Theorem 1.1, and the only step in which we show by kind of an explicit computation that some function is meromorphic, relies on the well-known description of invertible functions on (relative) annuli, which we have chosen to recall here for the reader’s convenience.

Let  $R_1$  and  $R_2$  be two positive real numbers with  $R_1 \leq R_2$ . The annulus  $\{R_1 \leq |T| \leq R_2\} \subset \mathbf{A}_k^{1,\text{an}}$  is an affinoid space whose algebra of analytic functions is the set  $k\{T/R_2, R_1T^{-1}\}$  of power series  $\sum_{i \in \mathbf{Z}} a_i T^i$  with coefficients in  $k$  such that  $|a_i|R_1^i \rightarrow 0$  when  $i \rightarrow -\infty$  and  $|a_i|R_2^i \rightarrow 0$  when  $i \rightarrow +\infty$ , which can be rephrased by saying that  $|a_i|r^i \rightarrow 0$  for  $|i| \rightarrow +\infty$  for every  $r \in [R_1, R_2]$  (the Banach norm of  $k\{T/R_2, R_1T^{-1}\}$  maps  $\sum a_i T^i$  to the maximum of  $|a_i|r^i$  for  $i \in \mathbf{Z}$  and  $r \in [R_1, R_2]$ ). More generally if  $X = \mathcal{M}(A)$  is an affinoid algebra then  $\{r_1 \leq |T| \leq R_2\} \times_k X$  is affinoid and its algebra of analytic functions is  $k\{T/R_2, R_1T^{-1}\} \hat{\otimes}_k A$  which is the set of power series  $\sum a_i T^i$  with coefficients in  $A$  such that  $\|a_i\|_\infty r^i \rightarrow 0$  for  $|i| \rightarrow \infty$  for every  $r \in [R_1, R_2]$ .

Now let  $I$  be an arbitrary non-empty interval of  $\mathbf{R}_+^\times$ . By exhausting  $I$  with compact intervals and using the G-sheaf property of analytic functions, we see that the ring of analytic functions on  $\{|T| \in I\} \subset \mathbf{A}_k^{1,\text{an}}$  is the set of power series  $\sum_{i \in \mathbf{Z}} a_i T^i$  with  $a_i \in k$  for all  $i$  such that  $|a_i|r^i \rightarrow 0$  for  $|i| \rightarrow +\infty$  for every  $r \in I$ ; more generally, the ring of analytic functions on  $X \times_k \{|T| \in I\}$  is the set of power series  $\sum_{i \in \mathbf{Z}} a_i T^i$  with  $a_i \in A$  for all  $i$  such that  $\|a_i\|_\infty^i \rightarrow 0$  for  $|i| \rightarrow +\infty$  for every  $r \in I$ .

**2.3. Lemma.** — *Let  $I$  be a non-empty interval of  $\mathbf{R}_+^\times$  and let  $f = \sum a_i T^i$  be an analytic function on  $\{|T| \in I\} \subset \mathbf{A}_k^{1,\text{an}}$ . The following are equivalent:*

- (i) *there exists  $j$  such that  $|a_i|r^i < |a_j|r^j$  for all  $i \neq j$  and all  $r \in I$ ;*
- (ii)  *$f$  is invertible.*

*Proof.* — If (i) holds then  $f$  can be written  $a_j T^j (1+u)$  with  $|u| < 1$  everywhere on  $\{|T| \in I\}$ , so  $f$  is invertible (and  $f^{-1} = a_j^{-1} T^{-j} \sum_\ell u^\ell$ ). Now assume that  $f$  is invertible, and let us prove (i). We start by handling the particular case where  $I$  is a singleton  $\{r\}$ . In order to prove (i) we may enlarge the ground field and rescale  $T$  and  $f$ , so we can assume that  $r = 1$  and then that  $\|f\| = \max_i |a_i| = 1$ . As  $f$  is invertible, its image under the reduction map  $k\{T, T^{-1}\} \rightarrow \tilde{k}[T, T^{-1}]$  is invertible as well, so is of the form  $\alpha T^i$  for some  $i \in \mathbf{Z}$  and  $\alpha \in \tilde{k}^\times$ . Then  $|a_i| = 1$  (and  $\tilde{a}_i = \alpha$ ) and  $|a_j| < 1$  for every  $j \neq i$ , whence (i).

Now let us deal with general  $I$ . By the above for every  $r \in I$  there is some integer  $i(r)$  such that  $|a_{i(r)}|r^{i(r)} > |a_j|r^j$  for every  $j \neq i(r)$ , and by connectedness of  $I$  it suffices to prove that  $r \mapsto i(r)$  is locally constant. But this is a straightforward consequence of the fact that  $|a_i|s^i \rightarrow 0$  when  $|i| \rightarrow \infty$  for every  $s \in I$ .  $\square$

The following lemma describes the local structure of a pair  $(S, X)$  where  $X$  is a quasi-smooth affinoid domain and  $S$  a quasi-smooth closed analytic subspace of  $X$ . It is certainly well-known (such a description is for instance carried out in the proof of [Ber99, Theorem 9.1]), but to our knowledge it

is not stated explicitly in the litterature, so we have chosen to write it down here.

**2.4. Lemma.** — *Let  $X$  be a quasi-smooth affinoid space over a non-archimedean field  $k$ , and let  $S$  be quasi-smooth closed analytic subspace of  $X$ . Let  $x \in S$  and let  $d$  be the codimension of  $S$  in  $X$  at  $x$ . There exists an affinoid neighborhood  $V$  of  $x$  in  $X$ , a one-dimensional closed disc  $D$ , and an isomorphism  $V \simeq D^d \times_k (S \cap V)$  whose restriction to  $S \cap V$  is the composition  $(S \cap V) \simeq \{0\} \times_k (S \cap V) \hookrightarrow D^d \times_k (S \cap V)$ .*

*Proof.* — Let  $I$  be the ideal defining  $S$  and set  $n = \dim_x X$ . The  $\mathcal{H}(x)$ -vector space  $\Omega_{S/k} \otimes \mathcal{H}(x)$  is of dimension  $n - d$ , and is the quotient of the  $n$ -dimensional space  $\Omega_{X/k} \otimes \mathcal{H}(x)$  by the subspace generated by the  $dg \otimes 1$  for  $g \in I$ . Therefore the latter subspace is  $d$ -dimensional and generated by  $dg_1 \otimes 1, \dots, dg_d \otimes 1$  for some  $g_1, \dots, g_d$ . Let  $S'$  be the Zariski closed subspace of  $X$  defined by the ideal  $(g_1, \dots, g_d)$ . It contains  $x$  and is of dimension at least  $n - d$  at  $x$  by the *Hauptidealsatz*, and  $\Omega_{S'/k} \otimes \mathcal{H}(x)$  is of dimension  $n - d$  by construction. Therefore  $S'$  is quasi-smooth of dimension  $n - d$  at  $x$ , so there exists an affinoid neighborhood  $V$  of  $x$  such that  $S' \cap V$  is quasi-smooth and irreducible of dimension  $n - d$ . Then  $S \cap V$  is a closed analytic subspace of  $S' \cap V$  of dimension  $n - d$  at  $x$ , which forces the equality  $S \cap V = S' \cap V$ . Otherwise said  $S \cap V$  is defined as a closed analytic subspace of  $V$  by the equations  $g_1 = 0, \dots, g_d = 0$ . Pick analytic functions  $g_{d+1}, \dots, g_n$  on  $V$  such that the  $dg_i \otimes 1$  generate  $\Omega_{S/k} \otimes \mathcal{H}(x)$ . Then  $dg_1 \otimes 1, \dots, dg_n \otimes 1$  generate  $\Omega_{X/k} \otimes \mathcal{H}(x)$ . The family  $(g_1, \dots, g_n)$  defines a map  $p$  from  $V$  to  $\mathbf{A}_k^{n, \text{an}}$ , which by compactness takes value in  $E^n$  for some suitable one-dimensional closed disc  $E$ . Since  $S \cap V$  is described by the equations  $g_1 = 0, \dots, g_d = 0$ , one has  $S \cap V = p^{-1}(\{0\} \times E^{n-d})$ .

The maps  $p: V \rightarrow E^n$  and  $p|_{S \cap V}: S \cap V \rightarrow \{0\} \times E^{n-d} \simeq E^{n-d}$  are quasi-étale at  $x$  by construction and in view of [Duc18, Lemma 5.4.5], so we can shrink  $V$  around  $x$  so that both maps are quasi-étale. Let  $q$  be the quasi-étale map  $\text{Id} \times p|_{S \cap V}: E^d \times (S \cap V) \rightarrow E^n$ . By construction,  $p^{-1}(\{0\} \times E^{n-d})$  and  $q^{-1}(\{0\} \times E^{n-d})$  are isomorphic as quasi-étale spaces over the closed analytic subspace  $\{0\} \times E^{n-d}$  of  $E^n$ , since they both can be identified with

$$S \cap V \xrightarrow{p|_{S \cap V}} \{0\} \times E^{n-d} .$$

It follows by [Duc23, Lemma 2.7] (which itself relies on the henselian property of a Berkovich space along a Zariski-closed subspace, see [Ber93, Prop. 4.3.4]) that there exists an analytic neighborhood  $U$  of  $\{0\} \times E^{n-d}$  in  $E^n$  such that  $p^{-1}(U)$  and  $q^{-1}(U)$  are  $U$ -isomorphic. Up to shrinking  $U$  we can assume that it is of the form  $D^d \times E^{n-d}$  where  $D$  is a closed one-dimensional disc. We thus get an isomorphism between  $\{v \in V, |g_i(v)| \leq r, i = 1, \dots, d\}$  and  $D^d \times_k (S \cap V)$

whose first component is  $(g_1, \dots, g_d)$  and such that the induced isomorphism between  $(g_1, \dots, g_d)^{-1}(0) = S \cap V$  and  $\{0\} \times_k (S \cap V)$  is the obvious one.  $\square$

**2.5. Meromorphic functions.** — One can define straightforwardly a  $G$ -sheaf of meromorphic functions  $\mathcal{K}_X$  on any analytic space  $X$  [Duc21, 2.23]; there is a natural embedding  $\mathcal{O}_X \hookrightarrow \mathcal{K}_X$ ; if  $V = \mathcal{M}(A)$  is an affinoid domain of  $X$  then  $\mathcal{K}_X(V)$  is the total ring of fractions of  $A$  [Duc21, 2.23.4]. As in the scheme-theoretic case, every meromorphic function  $f$  has a *sheaf of denominators*; this is the coherent sheaf of ideals whose sections on an analytic domain  $V$  consist of those  $g \in \mathcal{O}_X(V)$  such that  $gf \in \mathcal{O}_X(V)$ .

**2.6.** — Let  $A$  be an affinoid algebra, let  $X$  be an  $A$ -scheme of finite type and let  $\mathcal{K}_X$  be the sheaf of meromorphic (or rational) functions on  $X$ . Let  $\mathcal{S}$ , resp.  $\mathcal{T}$ , be the subsheaf of  $\mathcal{O}_X$ , resp.  $\mathcal{O}_{X^{\text{an}}}$ , consisting of those functions whose germ at every point is a regular element of the corresponding local ring. The structure map  $\pi: X^{\text{an}} \rightarrow X$  is a faithfully flat map of locally ringed spaces. This implies that the natural arrow  $\mathcal{O}_X \rightarrow \pi_* \mathcal{O}_{X^{\text{an}}}$  is injective, maps  $\mathcal{S}$  into  $\pi_* \mathcal{T}$ , and induces an injective map of presheaves  $\mathcal{S}^{-1} \mathcal{O}_X \rightarrow \pi_* \mathcal{T}^{-1} \mathcal{O}_{X^{\text{an}}}$ . The presheaf  $\mathcal{T}^{-1} \mathcal{O}_{X^{\text{an}}}$  is separated, thus it embeds into its sheafification  $\mathcal{K}_{X^{\text{an}}}$ , whence an injective map from  $\mathcal{S}^{-1} \mathcal{O}_X \hookrightarrow \pi_* \mathcal{K}_{X^{\text{an}}}$  and eventually by sheafifying once again an injective map  $\mathcal{K}_X \hookrightarrow \pi_* \mathcal{K}_{X^{\text{an}}}$ .

**2.7. Lemma.** — Let  $f \in \mathcal{K}_X(X)$  and let  $\mathcal{I} \subset \mathcal{O}_X$  be its sheaf of denominators. Then  $\mathcal{I}^{\text{an}}$  is the sheaf of denominators of  $f$  viewed as a meromorphic function on  $X^{\text{an}}$ .

*Proof.* — If  $\mathcal{J}$  denotes the sheaf of denominators of  $f$  viewed as a meromorphic function on  $X^{\text{an}}$ , it is clear that  $\mathcal{I}^{\text{an}} \subset \mathcal{J}$ . To show that this inclusion is actually an isomorphism, one may argue locally on  $X$ , and thus assume that  $X$  is affine. One can then write  $f = g/h$  with  $g$  and  $h$  in  $\mathcal{O}_X(X)$  and  $h$  regular. Then  $\mathcal{I}$  is the kernel of the map  $\mathcal{O}_X \rightarrow \mathcal{O}_X/(h)$  induced by multiplication by  $g$ , and  $\mathcal{J}$  is the kernel of the map  $\mathcal{O}_{X^{\text{an}}} \rightarrow \mathcal{O}_{X^{\text{an}}}/(h)$  induced by multiplication by  $g$ , hence  $\mathcal{J} = \mathcal{I}^{\text{an}}$ .  $\square$

**2.8. Lemma.** — One has  $\mathcal{O}_X(X) = \mathcal{K}_X(X) \cap \mathcal{O}_{X^{\text{an}}}(X^{\text{an}})$ .

*Proof.* — The direct inclusion is obvious. For the converse one, let  $f$  be an element of  $\mathcal{K}_X(X) \cap \mathcal{O}_{X^{\text{an}}}(X^{\text{an}})$  and let  $\mathcal{I} \subset \mathcal{O}_X$  be the sheaf of denominators of  $f$  viewed as a meromorphic function on  $X$ . By Lemma 2.7 above,  $\mathcal{I}^{\text{an}}$  is the sheaf of denominators of  $f$  viewed as a meromorphic function on  $X^{\text{an}}$ . The fact that  $f \in \mathcal{O}_{X^{\text{an}}}(X^{\text{an}})$  means that  $1 \in \mathcal{I}^{\text{an}}(X^{\text{an}})$ , which we can reformulate by saying that  $1$  is zero in  $(\mathcal{O}_{X^{\text{an}}}/\mathcal{I}^{\text{an}})(X^{\text{an}})$ . But by faithful flatness of  $X^{\text{an}} \rightarrow X$ , the map  $(\mathcal{O}_X/\mathcal{I})(X) \rightarrow (\mathcal{O}_{X^{\text{an}}}/\mathcal{I}^{\text{an}})(X^{\text{an}})$  is injective. Therefore  $1$  is zero in  $(\mathcal{O}_X/\mathcal{I})(X)$ , so  $1 \in \mathcal{I}(X)$  and  $f \in \mathcal{O}_X(X)$ .  $\square$

**2.9. Proposition** (See also Theorem 8.7 of [Meh22])

Assume that the scheme  $X$  is proper over  $A$ . The natural embedding  $\mathcal{K}_X(X) \hookrightarrow \mathcal{K}_{X^{\text{an}}}(X^{\text{an}})$  is an isomorphism.

*Proof.* — Let  $f$  be a meromorphic function on  $X^{\text{an}}$ . By GAGA for coherent sheaves (see [Poi10, Appendix A]), the sheaf denominators of  $f$  is of the form  $\mathcal{I}^{\text{an}}$  for a (uniquely determined) sheaf of ideals  $\mathcal{I}$  on  $X$ . Multiplication by  $f$  defines a morphism from  $\mathcal{I}^{\text{an}}$  to  $\mathcal{O}_{X^{\text{an}}}$ , which once again by GAGA comes from a unique map from  $\mathcal{I}$  to  $\mathcal{O}_X$ . In order to prove that  $f$  belongs to  $\mathcal{K}_X(X)$ , it suffices to prove that on every affine open subscheme  $U$  of  $X$ , the sheaf  $\mathcal{I}$  has a regular section  $h$  (for if one denotes by  $g$  the image of  $h$  in  $\mathcal{O}(U)$  one will have  $f = g/h$  on  $U^{\text{an}}$ ). So let  $U$  be an affine open subscheme of  $X$ . Suppose that  $\mathcal{I}(U)$  only consists of non-regular elements. This means that  $\mathcal{I}(U)$  is contained in the union of all associated primes of  $\mathcal{O}_X(U)$ , which implies that it is contained in one of them. So there exists a non-zero element  $b$  of  $\mathcal{O}_X(U) \subset \mathcal{O}_{X^{\text{an}}}(U^{\text{an}})$  such that  $hb = 0$  for all  $h \in \mathcal{I}(U)$ . By the definition of  $\mathcal{I}^{\text{an}}$  there is an affinoid  $G$ -covering  $(V_i)$  of  $U^{\text{an}}$  and for every  $i$  a section  $h_i$  of  $\mathcal{I}^{\text{an}}(V_i)$  which is a regular element of  $\mathcal{O}_X(V_i)$ . Since  $h_i$  is a section of  $\mathcal{I}^{\text{an}}$  we have  $h_i b|_{V_i} = 0$ . As  $h_i$  is regular, this implies that  $b|_{V_i} = 0$ . Since the  $V_i$ 's  $G$ -cover  $U^{\text{an}}$  this yields  $b = 0$ , contradiction.  $\square$

For proving Theorem 1.1 we will need three non-archimedean analogues of classical complex-analytic results, all due to Bartenwerfer in the strict case. We will prove them in general by reducing to the strict case. For doing this, we shall use the following notation. If  $r$  is a  $k$ -free polyradius, that is, a finite family  $(r_1, \dots, r_n)$  of positive numbers multiplicatively independent modulo  $|k^\times|$ , we denote by  $k_r$  the set of all power series  $\sum_{I \in \mathbb{Z}^n} a_I T^I$  where the  $a_I$ 's belong to  $k$  and  $|a_I| r^I \rightarrow 0$  when  $|I| \rightarrow \infty$ . The formula  $\sum a_I T^I \mapsto \max_I |a_I| r^I$  is a multiplicative norm that makes  $k_r$  a complete extension of  $k$  (see for instance [Duc07, 1.2]). If  $A$  is a  $k$ -affinoid algebra we set  $A_r = A \hat{\otimes}_k k_r$ . This is the set of power-series  $\sum_{I \in \mathbb{Z}^n} a_I T^I$  where the  $a_I$ 's belong to  $k$  and  $\|a_I\|_\infty r^I \rightarrow 0$  when  $|I| \rightarrow \infty$ .

**2.10. Proposition (Bartenwerfer).** — Let  $X$  be a  $k$ -analytic space and let  $Z$  be a Zariski-closed subset of  $X$  with empty interior; set  $U = X \setminus Z$ .

- (1) If  $X$  is normal any bounded holomorphic function on  $U$  extends to a holomorphic function on  $X$ .
- (2) If  $X$  is reduced and  $Z$  is everywhere of codimension  $\geq 2$  in  $X$ , then every meromorphic function on  $U$  extends to a meromorphic function on  $X$ .
- (3) If  $X$  is normal and  $Z$  is everywhere of codimension  $\geq 2$  in  $X$ , then every holomorphic function on  $U$  extends to a holomorphic function on  $X$ .



*Proof.* — If  $k$  is non-trivially valued and  $X$  is strictly  $k$ -analytic, (1) is [Bar76, §3, Theorem] and (2) and (3) are particular cases of [Bar75, Theorem]. We are just simply going to explain how to extend these statements when  $X$  is not assumed to be strict. Everything being  $G$ -local on  $X$ , we can assume that  $X$  is affinoid; and since for proving (2) one can replace  $X$  with its normalization, we can assume that  $X$  is normal and, by arguing componentwise, irreducible. Let  $A$  be the algebra of analytic functions on  $X$ . Let  $r$  be a  $k$ -free polyradius such that  $|k_r|^\times \neq \{0\}$  and  $A_r$  is  $k_r$ -strict. The ring  $A_r$  is a normal integral domain – this follows for instance from [Duc09, Exemple 3.3, Thm. 3.1 and 3.3], but an elementary proof, can be found in [Duc03, Appendix]. Then Bartenwerfer's statements apply on the  $k_r$ -analytic space  $X_r$ , and for deducing them on  $X$  it suffices to prove the following:

- (a) if  $f = \sum a_I T^I$  is an element of  $A_r$  whose restriction to  $U_r$  belongs to  $\mathcal{O}_X(U)$  then  $f \in A$ .
- (b) If  $g = \sum b_I T^I$  and  $h = \sum c_I T^I$  are two elements of  $A_r$  with  $h \neq 0$  such that the restriction of  $g/h$  to  $U_r$  belongs to  $\mathcal{K}_X(U)$  (where  $\mathcal{K}_X$  is the sheaf of meromorphic functions on  $X$ ), then  $g/h$  belongs to the ring of fractions of  $A$ .

Let us prove (a). If  $V$  is an affinoid domain of  $U$  then by assumption the element  $\sum (a_I|_V) T^I$  of  $\mathcal{O}_X(V)_r$  belongs to  $\mathcal{O}_X(V)$ , which means that  $a_I|_V = 0$  as soon as  $I \neq 0$ . Since this holds for arbitrary  $V$  we see that  $a_I|_U = 0$  as soon as  $I \neq 0$ . As  $X$  is reduced and  $U$  is dense one has  $a_I = 0$  as soon as  $I \neq 0$ , which proves that  $f \in A$ .

Let us prove (b). Let  $E$  denote the set of indices  $I$  with  $c_I \neq 0$  (it is non-empty by assumption). Let  $V$  be a connected and non-empty affinoid domain of  $U$ . As  $X$  is normal  $\mathcal{O}_X(V)$  is an integral domain, and by assumption there exist  $u$  and  $v$  in  $\mathcal{O}(V)$  with  $v \neq 0$  and  $g/h = u/v$  as meromorphic functions on  $V_r$ , which amounts to the equality  $gv = hu$  in  $\mathcal{O}_X(V)_r = \mathcal{O}_{X_r}(V_r)$ . As a consequence, if  $I \in E$  then  $u/v = b_I/c_I$  as meromorphic functions on  $V$  (note that  $c_I|_V \neq 0$  since  $V$  is non-empty and  $X$  is irreducible and reduced) and  $b_I|_V = 0$  if  $I \notin E$ . Therefore all meromorphic functions  $b_I/c_I$  for  $I \in E$  coincide on every irreducible affinoid domain  $V$  of the normal space  $U$ , then coincide on the whole of  $U$ , and then of  $X$  since the latter is reduced ; let  $w$  denote the common value of the  $b_I/c_I$  for  $I \in E$ , as a meromorphic function on  $X$ . If  $I \notin E$  then  $b_I = 0$  on every irreducible affinoid domain  $V$  of the normal space  $U$ , then on the whole of  $U$ , and then on  $X$  since the latter is reduced. Hence  $g/h = w$ , which ends the proof.  $\square$

### 3. The proofs

**3.1. Proof of Lemma 1.9.** — Implication (i) $\Rightarrow$ (ii) is obvious, so let us assume (ii) and prove (i). If  $f = 0$  then it obviously admits a meromorphic extension to  $D$ . If  $f \neq 0$  assumption (ii) implies that  $f$  is invertible on a punctured neighborhood of the origin, and since assertion (i) only concerns the possible singularity of  $f$  at the origin, we can assume up to shrinking  $D$  that  $f$  is an invertible function on  $U$ . Let  $t$  denote the coordinate function on  $D$  and let  $R$  be the radius of  $D$ . The function  $f$  can be written as a power series  $f = \sum_{i \in \mathbf{Z}} a_i t^i$  and since  $f$  is invertible, Lemma 2.3 ensures that there exists some  $j$  such that  $|a_j| r^j > |a_i| r^i$  for every  $i \neq j$  and every  $r \in (0, R)$  (and even  $(0, R]$  if  $D$  is closed). By letting  $r$  tend to zero we see that  $a_i = 0$  for every  $i < j$ . Then  $f = \sum_{i \geq j} a_i t^i$  admits a meromorphic extension to  $d$ .  $\square$

**3.2. Proof of Theorem 1.10.** — Let  $\mathcal{K}_X$  be the sheaf of meromorphic functions on the space  $X$  and let  $(X_i)_{i \in I}$  be the family of its reduced irreducible components. Let  $X'$  be the normalization of  $X$ . We have  $\mathcal{K}_X(X) = \mathcal{K}_{X'}(X')$ : indeed, this can be checked G-locally and thus enables to assume that  $X$  is affinoid, in which case this just comes from the corresponding scheme-theoretic statement. As a consequence,  $\mathcal{K}_X(X) = \prod_i \mathcal{K}_{X_i}(X_i)$ . Let  $J$  be the set of indices  $i$  such that  $X_i \cap U \neq \emptyset$  and set  $V = \bigcup_{i \in J} X_i$ , equipped with its reduced structure. Now, every morphism  $\varphi$  as in (ii) factors through  $V$ , and if  $f|_{V \cap U}$  admits a meromorphic extension  $(g_i)_{i \in J}$  to  $V$ , then  $f$  admits a meromorphic extension to the whole of  $X$ , namely  $(f_i)_{i \in I}$  with  $f_i = g_i$  if  $i \in J$  and  $f_i = 0$  otherwise. Hence for proving our theorem we can replace  $X$  by  $V$  and thus assume that  $U$  is dense.

**3.2.1.** — Assume that (i) holds and let  $\varphi$  as in (ii). Let  $W$  be an affinoid domain of  $X$  containing  $\varphi(x)$ . The pre-image  $\varphi^{-1}(W)$  is an analytic domain of  $D$  containing the origin, so this is a neighborhood of the later in  $D$ . Up to shrinking  $D$  (this is harmless for proving (ii), which is a local property at the origin) we can thus assume that  $\varphi(D) \subset W$ ; otherwise said we can assume that  $X$  is affinoid, say  $X = \mathcal{M}(A)$ . By assumption one can write  $f = g/h$  with  $h$  a non-zero divisor of  $A$ . As  $U$  is dense in  $X$ , the restriction  $h|_U$  is not identically zero, so that  $\varphi^* h|_{D \setminus \{0\}}$  is not identically 0. This ensures that  $\varphi^* h$  is a regular holomorphic function on the reduced irreducible space  $D$ , and in view of the equality  $\varphi^* f \varphi^* h = \varphi^* g$  on  $D \setminus \{0\}$ , the function  $\varphi^* f$  admits the meromorphic extension  $\varphi^* g / \varphi^* h$  to the whole of  $D$ , whence (ii).

**3.2.2.** — We now start the proof of implication (ii) $\Rightarrow$ (i), in several steps. So we assume that (ii) holds. We shall use implicitly several times the following fact: since the triple  $(X, U, f)$  satisfies (ii), for every analytic space  $Y$  defined

over a complete extension  $K$  of  $k$  and every  $k$ -morphism  $\psi: Y \rightarrow X$  the triple  $(Y, \psi^{-1}(U), \psi^*f)$  still satisfies (ii) (over the ground field  $K$ ).

As  $U$  is a dense Zariski-open subset of the reduced space  $X$ , the function  $f$  as at most one meromorphic extension to  $X$ . This uniqueness property also holds on any analytic domain of  $X$ , which will enable us to prove (i) by arguing G-locally: indeed, since G-local meromorphic extensions will be canonical, we will be able to glue them.

We make a first use of our right to argue G-locally by assuming that  $X$  is affinoid, say  $X = \mathcal{M}(A)$ . We denote by  $p$  the characteristic exponent of  $k$ .

**3.2.3.** — Let  $n$  be an integer and set  $L = k^{1/p^n}$ . Assume that the image of  $f$  in  $\mathcal{O}(U_{L,\text{red}})$  (which we shall still denote by  $f$ ) has a meromorphic extension to  $X_{L,\text{red}}$ . We are going to prove that  $f$  has a meromorphic extension to  $X$ . Note that for every affinoid  $k$ -algebra  $B$  one has  $(B \hat{\otimes}_k L)^{p^n} \subset B$  (check it on  $B \otimes_k L$  and use a limit argument), which implies by arguing G-locally that  $\mathcal{O}_X(U_L)^{p^n} \subset \mathcal{O}_X(U)$ .

By our assumption there exist  $g$  and  $h$  in  $A_L := A \hat{\otimes}_k L$  with  $h$  regular (i.e., not a zero divisor) in  $A_{L,\text{red}}$  such that  $f = g/h$  in  $\mathcal{O}(U_{L,\text{red}})$ . We have then  $fh = g + N$  in  $\mathcal{O}(U_L)$  for some nilpotent function  $N$  on  $U_L$  (the sheaf of locally nilpotent functions on  $U_L$  is the restriction of the coherent sheaf on  $X_L$  associated with the nilradical of the noetherian ring  $A$ , so its sections are actually nilpotent). It follows that for  $m \geq n$  large enough  $f^{p^m} h^{p^m} = g^{p^m}$  in  $\mathcal{O}(U_L)$ ; but since  $m \geq n$  all functions involved in this equality belong to the subring  $\mathcal{O}(U)$  of  $\mathcal{O}(U_L)$ , so  $f^{p^m} h^{p^m} = g^{p^m}$  should be understood as an equality between analytic functions on  $U$ .

Set  $H = h^{p^m}$  and  $G = g^{p^m}$ , so that we have  $f^{p^m} H = G$  on  $U$ . Both  $G$  and  $H$  belong to  $A$ . Let us show that  $H$  is a regular element of  $A$ . As  $X$  is reduced, it suffices to show that the zero-locus of  $H$  does not contain any irreducible component of  $X$ . But since  $h$  is a regular element of  $A_{L,\text{red}}$ , the zero-locus of  $h$  in  $X_L$ , which is the same as the zero-locus of  $H$ , does not contain any irreducible component of  $X_L$ , and we conclude by noticing that  $X_L \rightarrow X$  induces a homeomorphism for the Zariski topologies on the source and the target.

Since  $f^{p^m} H = G$  one has  $(fH)^{p^m} = GH^{p^m-1}$  on  $U$ ; therefore the analytic function  $fH$  is bounded on  $U$ . By the non-archimedean version of Riemann's extension theorem (essentially due to Bartenwerfer, see Proposition 2.10 (1) for more details),  $fH$  extends to a holomorphic function on the normalization  $X'$  on  $X$ . Therefore  $fH$  extends to a meromorphic function on  $X$  and so does  $f$  since  $H$  is regular.

**3.2.4.** — If  $n$  is large enough, a result by Conrad [Con99, Lemma 3.3.1], see also [Duc09, Thm. 6.10], ensures that the normalization of  $X_{k^{1/p^n}}$  is geometrically normal. In view of 3.2.3, it suffices to check that the image of  $f$

in  $\mathcal{O}(U_{k^{1/p^n}, \text{red}})$  extends to a meromorphic function on  $X_{k^{1/p^n}, \text{red}}$ . As a consequence, we may assume that the normalization  $X'$  of  $X$  is geometrically normal (and then  $X'_L$  is the normalization of  $X_L$  for every complete extension  $L$  of  $k$ , see [Duc09, Prop. 5.20]). Let  $U'$  be the pre-image of  $U$  in  $X'$ .

Another result by Conrad [Con99, Thm. 3.3.8], see also [Duc09, Thm. 6.11], ensures that for large enough  $n$ , the space  $(X'_{k^{1/p^n}} \setminus U'_{k^{1/p^n}})_{\text{red}}$  is geometrically reduced. So using again 3.2.3, we may moreover assume that  $(X' \setminus U')_{\text{red}}$  is geometrically reduced.

**3.2.5.** — By replacing  $X$  with  $X'$  and  $U$  with  $U'$  we can assume that  $X$  is normal. And then by arguing componentwise we can assume that  $X$  is moreover integral. Let  $n$  denote its dimension. As  $U$  is dense,  $X \setminus U$  is a Zariski-closed subset of  $X$  of dimension  $\leq n-1$ . Let  $K$  be the completion of an algebraic closure of  $k$ . Let  $Y$  denote the complement of the quasi-smooth locus of  $X$ . The space  $X_K$  is normal, therefore  $\dim Y_K \leq n-2$ , hence  $\dim Y \leq n-2$ . And the space  $(X \setminus U)_{\text{red}, K}$  is reduced, so it has a dense quasi-smooth locus. Therefore  $(X \setminus U)_{\text{red}}$  has a dense quasi-smooth locus. Let us denote by  $Y'$  be the union of  $Y$ , of all irreducible components of  $X \setminus U$  of dimension  $\leq n-2$ , and of the non-quasi-smooth locus of  $(X \setminus U)_{\text{red}}$ . By construction,  $Y'$  is a Zariski-closed subset of  $X$  of dimension  $\leq n-2$ . As  $X$  is reduced, it follows from an extension theorem essentially due to Bartenwerfer (see Proposition 2.10 (2) for more details) that every meromorphic function on  $X \setminus Y'$  extends to a meromorphic function on  $X$ . It is therefore sufficient to prove that  $f|_{V \cap U}$  extends to a meromorphic function on  $V$  for every affinoid domain  $V$  of  $X \setminus Y'$ . Hence we have reduced to the case where  $X$  is quasi-smooth and  $U$  is the complement of a quasi-smooth hypersurface  $S$ .

**3.2.6.** — For proving the theorem we may once again argue locally on  $X$ . The theorem obviously holds on  $X \setminus S$ , and if  $x$  is a point of  $S$  it follows from Lemma 2.4 that there exists an affinoid neighborhood  $V$  of  $x$  in  $X$  such that  $(V, S \cap V)$  is isomorphic to  $D \times_k (S \cap V) \times D, \{0\} \times_k (S \cap V)$  for some closed one-dimensional disc  $D$ . Moreover we can shrink  $V$  so that the smooth space  $S \cap V$  is connected, hence irreducible (and reduced).

Therefore we can assume that  $X = D \times_k Y$  for some irreducible and reduced analytic space  $Y$  and some closed disc  $D$  and that  $S = \{0\} \times_k Y$ . (The quasi-smoothness of  $S \simeq Y$  was useful for reducing to this product situation, but will not be used anymore; the fact that  $Y$  is irreducible and reduced will be sufficient.)

Let  $t$  be the coordinate function on  $D$ . By hypothesis,  $f$  is an analytic function defined on  $(D \setminus \{0\}) \times_k Y$ , so it can be written  $\sum_{i \in \mathbf{Z}} b_i t^i$  where every  $b_i$  belongs to  $\mathcal{O}(Y)$  (2.2). Choose  $y \in Y$  lying over the generic point of  $\text{Spec } \mathcal{O}(Y)$ . The fiber of  $X = D \times_k Y$  over  $y$  (through the second projection) is canonically isomorphic to  $D_{\mathcal{H}(y)}$ ; let  $\varphi: D_{\mathcal{H}(y)} \rightarrow X$  be the corresponding embedding. By

construction,  $\varphi^{-1}(U) = D_{\mathcal{H}(y)} \setminus \{0\}$ . It thus follows from our assumption (ii) that  $\varphi^*f$  admits a meromorphic extension to  $D_{\mathcal{H}(y)}$ . Since  $\varphi^*f = \sum_i b_i(y)t^i$ , this means that there is some  $j$  such that  $b_i(y) = 0$  for every  $i \leq j$ . As  $y$  is Zariski-generic on the reduced, irreducible space  $Y$ , this implies that  $b_i = 0$  for every  $i < j$ . Thus  $f = \sum_{i \geq j} b_i t^i$  extends to a meromorphic function on  $X$ , which ends the proof of Theorem 1.10.  $\square$

**3.3. Proof of Theorem 1.1.** — Let us assume that (i) holds. Let us choose a meromorphic extension of  $f$ , which we still denote by  $f$ , and let  $\mathcal{J}$  be its sheaf of denominators. The  $\mathcal{J} := \mathcal{J}f$  is a coherent sheaf of ideals on  $X$ , whose restriction to  $U$  is equal to  $(f|_U)$  since  $\mathcal{J}|_U = \mathcal{O}_U$ . Therefore the zero-locus of  $f|_U$  is the intersection of  $U$  and of the zero-locus of  $\mathcal{J}$ , which is a Zariski-closed subset of  $X$ , whence (ii).

Conversely, assume that (ii) holds. Let  $L$  be a complete extension of  $k$ , let  $D$  be a one-dimensional disc over  $L$  and let  $\varphi: D \rightarrow X$  be a  $k$ -morphism such that  $\varphi^{-1}(U) = D \setminus \{0\}$ . The zero-locus of the function  $\varphi^*f \in \mathcal{O}_D(D \setminus \{0\})$  is then equal to  $\varphi^{-1}(Z) \cap (D \setminus \{0\})$ . Therefore by Lemma 1.9 which we have already proved (3.1),  $\varphi^*f$  admits an extension to a meromorphic function of  $D$ . Since this holds for arbitrary  $(L, D, \varphi)$ , Theorem 1.10 proven in 3.2 ensures that  $f$  admits an extension to a meromorphic function on  $X$ , which ends the proof of Theorem 1.1.  $\square$

**3.4. Proof of Corollary 1.3.** — The direct implication is obvious. So let us assume that the zero-locus of  $f$  is equal to  $Y^{\text{an}}$  for some Zariski-closed subset  $Y$  of the scheme  $X$ , and let us prove that  $f$  is algebraic. This statement is local on  $X$ , which enables us to assume that  $X$  is affine. Let us choose a reduced projective compactification  $\overline{X}$  of  $X$ . The zero-locus of  $f$  on  $X^{\text{an}}$  is then equal to  $\overline{Y}^{\text{an}} \cap X^{\text{an}}$  (where  $\overline{Y}$  is the closure of  $Y$  in  $\overline{X}$ ) so that we can apply Theorem 1.1 proved in 3.3 (taking  $X = \overline{X}^{\text{an}}$  and  $U = X^{\text{an}}$ ) and conclude that the function  $f$  on  $X^{\text{an}}$  “is” then a meromorphic function on  $\overline{X}^{\text{an}}$ . By GAGA for meromorphic functions (see Prop. 2.9),  $f$  “is” a meromorphic function on  $\overline{X}$ . Then  $f$  is a meromorphic function on the scheme  $X$  inducing a holomorphic function on  $X^{\text{an}}$ . By Lemma 2.8,  $f \in \mathcal{O}_X(X)$ .  $\square$

### Acknowledgements

The starting point of this work was a discussion with Marco Maculan who drew my attention to the fact that invertible analytic functions on analytifications of curves are algebraic. This is what led me to Theorem 1.1 and Corollary 1.3, and I would like to thank him warmly for that. I am also very grateful to Piotr Achinger for his careful reading a former version of this

manuscript, his insightful comments and his very interesting suggestion described in Remark 1.8 at the beginning of this text. Finally I am thankful to the two anonymous referees whose remarks, suggestions and comments on several versions of this work helped me to significantly improve the manuscript.

### References

- [Bar75] W. BARTENWERFER – “Die Fortsetzung holomorpher und meromorpher Funktionen in eine  $k$ -holomorphe Hyperfläche hinein”, *Math. Ann.* **212** (1975), p. 331–358 (German).
  - [Bar76] ———, “Der erste Riemannsche Hebbarkeitssatz im nichtarchimedischen Fall”, *J. Reine Angew. Math.* **286/287** (1976), p. 144–163 (German).
  - [Ber90] V. BERKOVICH – *Spectral theory and analytic geometry over non-Archimedean fields*, Mathematical Surveys and Monographs, vol. 33, American Mathematical Society, Providence, RI, 1990.
  - [Ber93] ———, “Étale cohomology for non-Archimedean analytic spaces”, *Inst. Hautes Études Sci. Publ. Math.* **78** (1993), p. 5–161.
  - [Ber96] V. G. BERKOVICH – “Vanishing cycles for formal schemes. II”, *Invent. Math.* **125** (1996), no. 2, p. 367–390 (English).
  - [Ber99] V. BERKOVICH – “Smooth  $p$ -adic analytic spaces are locally contractible”, *Invent. Math.* **137** (1999), no. 1, p. 1–84.
  - [Con99] B. CONRAD – “Irreducible components of rigid spaces”, *Ann. Inst. Fourier (Grenoble)* **49** (1999), no. 2, p. 473–541.
  - [Duc03] A. DUCROS – “Parties semi-algébriques d’une variété algébrique  $p$ -adique”, *Manuscripta Math.* **111** (2003), no. 4, p. 513–528.
  - [Duc07] ———, “Variation de la dimension relative en géométrie analytique  $p$ -adique”, *Compos. Math.* **143** (2007), no. 6, p. 1511–1532 (French).
  - [Duc09] ———, “Les espaces de Berkovich sont excellents”, *Ann. Inst. Fourier (Grenoble)* **59** (2009), no. 4, p. 1443–1552 (French).
  - [Duc18] ———, “Families of Berkovich spaces”, *Astérisque* **400** (2018), vii+262 p.
  - [Duc21] ———, “Dévisser, découper, éclater et aplatir les espaces de Berkovich”, *Compos. Math.* **157** (2021), no. 2, p. 236–302.
  - [Duc23] ———, “Utilisation de l’aplatissement en géométrie de Berkovich”, Preprint, arXiv:2304.02415 [math.AG] (2023), 2023.
  - [Meh22] V. MEHMETI – “Patching over analytic fibers and the local-global principle”, *Math. Ann.* **383** (2022), no. 3-4, p. 1825–1904 (English).
  - [Poi10] J. POINEAU – “Raccord sur les espaces de Berkovich”, *Algebra Number Theory* **4** (2010), no. 3.
-

ANTOINE DUCROS, Sorbonne Université, Université Paris-Diderot, CNRS, Institut de  
Mathématiques de Jussieu-Paris Rive Gauche, Campus Pierre et Marie Curie, case 247,  
4 place Jussieu, 75252 Paris cedex 5, France • *E-mail* : `antoine.ducros@imj-prg.fr`  
*Url* : `https://webusers.imj-prg.fr/~antoine.ducros/`