CONSTRUCTING TILTING MODULES

OTTO KERNER AND JAN TRLIFAJ

Let A be an Artin algebra and T be a right A-module. Then T is a *tilting module* provided that (T1) p.dim$T \leq 1$, (T2) $\mathrm{Ext}^1_R(T, T^{(I)}) = 0$ for any set I, and (T3) there is a short exact sequence $0 \to A \to T_0 \to T_1 \to 0$ where T_0 and T_1 are direct summands in a direct sum of (possibly infinitely many) copies of T.

If T is a tilting A-module, $T^\perp$ denotes the full subcategory

$$T^\perp = \{M \in \text{Mod-}A \mid \mathrm{Ext}^1_A(T, M) = 0\},$$

which is a torsion class in Mod-A, the category of all A-modules. If T' is another tilting module then T is said to be *equivalent* to T' if $\{T\}^\perp = \{T'\}^\perp$.

Conbining a recent Theorem of Bazzoni and Herbera with a result of Kerner and Trlifaj, the assignment $T \mapsto T^\perp \cap \text{mod-}A$ defines a bijection between the set of torsion classes in mod-A, containing all injective modules in mod-A, and the set of equivalence classes of tilting A-modules.

This fact motivates the study of A-modules X, which can be completed to a tilting module $T = X \oplus Y$. Restricting to hereditary Artin algebras, on can show:

Theorem. *Let A be a connected hereditary Artin algebra, and X be an H-module, finitely generated over its endomorphism ring and with $\mathrm{Ext}^1_H(X, X^{(I)}) = 0$, for any set I. Then there exists an H-module Y, such that $X \oplus Y$ is a tilting H-module.*

The proof is done on two steps: (a) X is faithful, (b) X is not faithful.

As a consequence one gets for example for hereditary algebras: If X is a *stone*, which means that X is endo-finite, indecomposable and without self-extensions, then X is a direct summand of a tilting module.