Serre functors, category $\mathcal{O}$, and symmetric algebra

Volodymyr Mazorchuk

This is a joint work with Catharina Stroppel.

Let $\Bbbk$ be a field. A Serre functor on a $\Bbbk$-linear category, $\mathcal{C}$, with finite-dimensional homomorphism spaces is an auto-equivalence, F, of $\mathcal{C}$ which gives isomorphisms

$$\mathrm{Hom}_{\mathcal{C}}(X, FY) \cong \mathrm{Hom}_{\mathcal{C}}(Y, X)^*$$

natural in both X and Y. If A is a finite dimensional algebra, then it is well-known that the bounded derived category $\mathcal{D}^b(A)$ has a Serre functor if and only if gl.dim.$A < \infty$, and if the latter is the case, the Serre functor is just the left derived of the Nakayama functor $A^* \otimes_A -$. In particular, it follows that there is a Serre functor for all blocks of the BGG category $\mathcal{O}$, associated with a semi-simple complex finite-dimensional Lie algebra. However, since in this case the algebra A is not explicitly given, the Serre functor is not easy to compute. One of our results is the following explicit description of the Serre functor on $\mathcal{O}$ (the geometric counterpart of this result was recently obtained by Beilinson, Bezrukavnkov and Mirkovic):

Theorem. Let T_{w_0} be the global Arkhipov's twisting functor on the regular block of the category $\mathcal{O}$. Then $\mathcal{L}\mathrm{T}^2_{w_0}$ is the Serre functor on this block.

Using the recent results of Khomenko on functors, naturally commuting with translation functors, the above theorem allows us to explicitly describe Serre functors on the regular blocks of the parabolic category $\mathcal{O}$ introduced by Rocha-Caridi. Using the connection between the Serre functors and symmetric algebras one obtains that the endomorphism algebra of the basic projective- injective module in the parabolic block is symmetric. This confirms a conjecture of Khovanov.

Department of Mathematics, Uppsala University, Box 480, 75106, Uppsala, Sweden, e-mail: `mazor@math.uu.se`, web: http://www.math.uu.se/~mazor/