LECTURES ON HALL ALGEBRAS

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Abstract

Let A be an abelian category which satisfies the following finiteness conditions : for any two objects $M, N \in Obj(A)$ and any $j \geq 1$ the sets $Hom(M, N)$ and $Ext^j(M, N)$ are finite, and A is itself of finite global dimension (that is there exists $i \geq 0$ such that $Ext^i(M, N) = 0$ for *any* $M, N \in Obj(A)$.). Such categories abound : for example, one may take A to be the category of representations of a quiver over some finite field $\mathbb{F}_q$, or the category of coherent sheaves over some smooth projective curve C again defined over some finite field $\mathbb{F}_q$. Following an idea of Ringel, who was himself inspired by Hall, Steinitz and others, one associates to such a category A an *associative algebra* H_A, called the *Hall*, or *Ringel-Hall algebra* of A, defined as follows : as a vector space, H_A has a $\mathbb{C}$-basis $\{[M]\}_{M \in \mathcal{I}}$ parametrized by the set $\mathcal{I} = Obj(A)/\sim$ of objects of A, counted up to isomorphism, and with product

$$[M] \cdot [N] = \sum_{Q \in \mathcal{I}} P^Q_{M,N} [Q],$$

where $P^Q_{M,N} = \#\{L \subset Q \mid L \simeq N,\ Q/L \simeq M\}$ is the number of subobjects of Q of type N and cotype M. Thus the Hall algebra encodes the structure of extensions between objects of A, and many properties of the category A can be read off from its Hall algebra. Note that H_A is in general not commutative (as the set of objects which may be obtained as an extension of M by N is usually different from the set of objects which may be obtained as an extension of N by M).

The aim of the course is to study various properties of the algebras H_A, both in general, and for particular choices of abelian category A (such as the above-mentionned examples related to quivers and curves). It turns out that these are often isomorphic to the *quantum groups* associated to certain (Kac-Moody or loop) Lie algebras. Hence Hall algebras throw a bridge between representations of quivers or vector bundles on curves on the one hand, and the structure of Kac-Moody or loop Lie algebras on the other hand. Such a link has proven to be extremely fruitful for both fields of mathematics, as the course will attempt to show.

References

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