QUOTIENT TRIANGULATED CATEGORIES ARISING IN REPRESENTATION THEORY

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Several quotient triangulated categories arising naturally from the representations of algebras are studied.

1. For a self-orthogonal A-module T, the quotient triangulated category $D^b(A)/K^b(\mathrm{add}T)$.

(It is a funny trianulated category; if T is generalized tilting (resp. cotilting) then it is exactly the singularity category $\mathcal{D}_P(A) := D^b(A)/K^b(A\text{-proj})$ (resp. $\mathcal{D}_I(A) := D^b(A)/K^b(A\text{-inj})$); when A is Gorenstein then this generalizes a beautiful result of Happel, and a recent work of Orlov; this naturally related to a work of Auslander-Reiten, and a work of Ringel; one may expect to get some information on modules from this quotient triangulated category.)

Theorem 0.1. *Let T be a self-orthogonal module, $M \in \mathcal{X}_T$ and $N \in T^\perp$. Then there is a natural isomorphism of vector spaces* $\mathrm{Hom}_A(M,N)/T(M,N) \simeq \mathrm{Hom}_{D^b(A)/K^b(\mathrm{add}T)}(M,N)$, *where $T(M,N)$ is the subspace of A-maps which factor through* $\mathrm{add}T$.

In particular, the natural functor $\mathcal{X}_T \cap T^\perp \longrightarrow D^b(A)/K^b(\mathrm{add}T)$ induces a fully-faithful functor $\underline{\mathcal{X}_T \cap T^\perp} \longrightarrow D^b(A)/K^b(\mathrm{add}T)$, where $\underline{\mathcal{X}_T \cap T^\perp}$ is the stable category of $\mathcal{X}_T \cap T^\perp$ modulo $\mathrm{add}T$.

Corollary 0.2. *Let A be a Gorenstein algebra and T an A-module. Then T is generalized cotilting if and only if T is generalized tilting.*

Theorem 0.3. *Assume that* $\mathrm{inj.dim}\ {}_AA < \infty$. *Let T be a generalized cotilting A-module. Then the natural functor ${}^\perp T \longrightarrow D^b(A)/K^b(\mathrm{add}T) = \mathcal{D}_I(A)$ is dense.*

Moreover, if in addition A is Gorenstein, then the natural functor ${}^\perp T \cap T^\perp \longrightarrow D^b(A)/K^b(\mathrm{add}T)$ is dense.

Theorem 0.4. *Assume that* $\mathrm{inj.dim}\ {}_AA < \infty$. *Let T be a generalized tilting module. Then the natural functor ${}^\perp T \cap T^\perp \longrightarrow \mathcal{D}_I(A)$ is dense.*

In particular, the natural functor ${}^\perp A \longrightarrow \mathcal{D}_I(A)$ is dense.

Theorem 0.5. *Let A be Gorenstein, and T be a generalized cotilting module (= a generalized tilting module). Then the natural functors induce an equivalences of categories*

$$\underline{{}^\perp T \ \cap T^\perp} \simeq \mathcal{D}_I(A) = \mathcal{D}_P(A).$$

Corollary 0.6. *The following are equivalent*

(i) $\mathrm{gl.dimA} < \infty$*;*

(ii) $\mathrm{inj.dim}\ {}_AA < \infty$, *and ${}^\perp T \cap\ T^\perp = \mathrm{add}T$ for any generalized tilting module;*

$(ii)'$ proj.dim $_{A}D(A_A) < \infty$, *and* $^{\perp}T \cap T^{\perp} = \mathrm{add}T$ *for any generalized cotilting module.*

2. Let $T(A) := A \oplus D(A)$ be the the trivial extension algebra of A. It is $\mathbb{Z}$-graded with degA = 0 and degD(A) = 1. Let $T(A)^{\mathbb{Z}}$-mod be the category of finite-dimensional $\mathbb{Z}$-graded $T(A)$-modules with morphisms of degree 0. A theorem of Happel says that there exists a fully-faithful, exact functor $F : D^b(A) \longrightarrow T(A)^{\mathbb{Z}}\text{-}\underline{\mathrm{mod}}$; and F is dense if and only gl.dimA $< \infty$.

Theorem 0.7. *Let A be a Gorenstein algebra. Then under Happel's functor $F : D^b(A) \longrightarrow T(A)^{\mathbb{Z}}\text{-}\underline{\mathrm{mod}}$ we have*

$$D^b(A) \simeq \mathcal{N} := \{\oplus_{n\in\mathbb{Z}} M_n \in T(A)^{\mathbb{Z}}\text{-}\underline{\mathrm{mod}} \mid \text{proj.dim } {}_AM_n < \infty, \ \forall\ n \neq 0\}$$

and

$$K^b(A\text{-}proj) \simeq \mathcal{M}_P := \{\oplus_{n\in\mathbb{Z}} M_n \in T(A)^{\mathbb{Z}}\text{-}\underline{\mathrm{mod}} \mid \text{proj.dim } {}_AM_n < \infty, \ \forall\ n \in \mathbb{Z}\}.$$

3. The stable category $\underline{\mathfrak{a}(T)}$ of the Frobenius exact category $\mathfrak{a}(T) := \mathcal{X}_T \cap {}_T\mathcal{X}$.

Theorem 0.8. *Let T be self-orthogonal such that $\overbrace{\mathrm{add}T} \subseteq T^{\perp}$ and $\underbrace{\mathrm{add}T} \subseteq {}^{\perp}T$. Then there is an equivalence of triangulated categories $K^{ac}(T) \simeq \underline{\mathfrak{a}(T)}$, where $K^{ac}(T)$ is the full subcategory of $K(A)$ consisting of acyclic complexes with components in* add$T$.

The notations of $^{\perp}T$, $T^{\perp}$, $\underbrace{\mathrm{add}T}$, $\overbrace{\mathrm{add}T}$, $\mathcal{X}_T$, ${}_T\mathcal{X}$, $\underset{\vee}{\mathrm{add}}T$, $\widehat{\mathrm{add}T}$ are same as in [AR].

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