On the vanishing of Ext and $\varprojlim^1$

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$\boldsymbol{R}$ associative ring with 1. Mod -$\boldsymbol{R}$= right $\boldsymbol{R}$-modules.

$\{\boldsymbol{A_i; f_i^j}\}_{i\leq j\in I}$ direct system of R-modules; $\boldsymbol{M}\in$ Mod -R

- $\mathbf{Hom}_{\boldsymbol{R}}(\varinjlim \boldsymbol{A_i}, \boldsymbol{M}) \cong \varprojlim \mathbf{Hom}_{\boldsymbol{R}}(\boldsymbol{A_i}, \boldsymbol{M})$
- [Auslander, '78] $\boldsymbol{M}$ is pure injective if and only

$$\mathbf{Ext}^{\mathbf{1}}_{\boldsymbol{R}}(\varinjlim \boldsymbol{A_i}, \boldsymbol{M}) \cong \varprojlim \mathbf{Ext}^{\mathbf{1}}_{\boldsymbol{R}}(\boldsymbol{A_i}, \boldsymbol{M})$$

Look for conditions under which

$$\mathbf{Ext}^{\mathbf{1}}_{\boldsymbol{R}}(\boldsymbol{A_i}, \boldsymbol{M}) = \mathbf{0}, \forall \boldsymbol{i} \in \boldsymbol{I} \Rightarrow \mathbf{Ext}^{\mathbf{1}}_{\boldsymbol{R}}(\varinjlim \boldsymbol{A_i}, \boldsymbol{M}) = \mathbf{0}$$

COUNTABLE DIRECT SYSTEMS

$$A_1 \xrightarrow{f_1} A_2 \xrightarrow{f_2} A_3 \longrightarrow \ldots \longrightarrow A_n \xrightarrow{f_n} A_{n+1} \longrightarrow \ldots$$

[Jensen '66]: $0 \longrightarrow \bigoplus_{n\in\mathbb{N}} A_n \xrightarrow{\phi} \bigoplus_{n\in\mathbb{N}} A_n \longrightarrow \varinjlim A_n \longrightarrow 0$

$$\phi\varepsilon_n = \varepsilon_n - \varepsilon_{n+1} f_n$$

$\varepsilon_n : A_n \longrightarrow \bigoplus_{n\in\mathbb{N}} A_n$ denotes the canonical embedding.

$$\phi = \begin{pmatrix} 1 & 0 & \ldots & 0 & \ldots \\ -f_1 & 1 & & & \\ 0 & -f_2 & \ddots & & \\ \vdots & & \ddots & 1 & \\ \vdots & & & -f_n & \ddots \\ \vdots & & & & \ddots \end{pmatrix}.$$

Assume $\mathbf{Ext}^1_R(A_n, M) = 0$, for every $n \in \mathbb{N}$ and write $\mathbf{Hom}_R(-, M) = \mathbf{H}(-, M)$, then:

$$0 \to \varprojlim H(A_n, M) \to \prod_{n\in\mathbb{N}} H(A_n, M) \xrightarrow{H(\phi,M)} \prod_{n\in\mathbb{N}} H(A_n, M) \to \mathbf{Ext}^1_R(\varinjlim A_n, M) \to 0.$$

For a countable inverse system:

$$\dots H_{n+1} \xrightarrow{g_n} H_n \dots \to H_3 \xrightarrow{g_2} H_2 \xrightarrow{g_1} H_1$$

$$0 \to \varprojlim H_n \to \prod_{n\in\mathbb{N}} H_n \xrightarrow{\Delta} \prod_{n\in\mathbb{N}} H_n \to \varprojlim{}^1 H_n \to 0.$$

$$\Delta((h_n)_{n\in\mathbb{N}}) = ((h_n - g_n(h_{n+1})_{n\in\mathbb{N}})$$

$\Delta = \mathbf{H}(\phi, M)$, hence

$$\mathbf{Ext}^1_R(\varinjlim A_n, M) \cong \varprojlim{}^1 H(A_n, M)$$

Look for conditions implying the vanishing of $\varprojlim{}^1 H_n$.

SUFFICIENT CONDITION.

DEFINITION

An inverse system $\{H_i; g_i^j\}_{i \leq j \in I}$ of R-modules satisfies the Mittag-Leffler condition if for every $i \in I$ there exists $j \geq i$ such that

$$\mathrm{Im} g_i^j = \mathrm{Im} g_i^k, \quad \forall k \geq j$$

A countable inverse system:

$$\ldots H_{n+1} \xrightarrow{g_n} H_n \ldots \rightarrow H_3 \xrightarrow{g_2} H_2 \xrightarrow{g_1} H_1$$

satisfies the Mittag-Leffler condition if for every $m \in \mathbb{N}$ the chain:

$$\ldots \mathrm{Im} g_m \supseteq \mathrm{Im} g_m g_{m+1} \supseteq \mathrm{Im} g_m g_{m+1} g_{m+2} \ldots$$

is stationary.

THEOREM [Grothendieck, '60]

If $\{H_n; g_n\}_{n\in\mathbb{N}}$ satisfies the Mittag-Leffler condition then $\varprojlim{}^1 H_n = 0$

REMARK: If $\{H_n; g_n\}_n$ satisfies the Mittag-Leffler condition, so does $\{H_n^{(X)}; g_n^{(X)}\}_n$ for every set X.

THEOREM

[Emmanouil '96, Bass '61, Azumaya '87]

The following are equivalent;

(1) $\{H_n; g_n\}_n$ satisfies the Mittag-Leffler condition;

(2) $\varprojlim{}^1 H_n^{(\mathbb{N})} = 0$;

(3) $\varprojlim{}^1 H_n^{(X)} = 0, \quad \forall X$.

• A_n **finitely generated**, then

$H(A_n, M^{(X)}) \cong H(A_n, M)^{(X)}$.

• A_n **finitely presented** and N **pure** in M.

$\{H(A_n, M)\}_n$ Mittag-Leffler $\Rightarrow$ $\{H(A_n, N)\}_n$ Mittag-Leffler.

THEOREM [B, Herbera '05]

Assume A_n finitely presented and $\mathrm{Ext}^1_R(A_n, M) = 0$, for every $n \in \mathbb{N}$. **TFAE**:

(1) $\mathrm{Ext}^1_R(\varinjlim A_n, M^{(\mathbb{N})}) = 0$;

(2) $\varprojlim{}^1 H(A_n, M)^{(\mathbb{N})} = 0$;

(3) $\{H(A_n, M)\}_n$ satisfies the Mittag-Leffler condition.

Moreover,

$\mathrm{Ext}^1_R(\varinjlim A_n, M^{(\mathbb{N})}) = 0 \Rightarrow \mathrm{Ext}^1_R(\varinjlim A_n, N) = 0$

for every **pure submodule** N of M.

APPLICATION TO TILTING THEORY.

DEFINITION [Angeleri-Hügel, Coelho '01]

A right R-module T is $\boldsymbol{n}$-tilting if and only if the following three conditions hold

[$\mathbf{(T1)}$] p.d.$T \leq n$;

[$\mathbf{(T2)}$] $\mathrm{Ext}^i_R(T, T^{(\lambda)}) = 0, \forall i \geq 1, \forall \lambda$ cardinals;

[$\mathbf{(T3)}$] $0 \to R \to T_0 \to T_1 \to \ldots \to T_r \to 0,$

$\boldsymbol{T_i \in \mathbf{Add}\, T}$, Add$\boldsymbol{(T)}$= direct summands of direct sums of copies of T.

$$\boldsymbol{T^{\perp} = \{M \in \text{Mod-}R \mid \mathbf{Ext}^i_R(T, M) = 0, \forall i \geq 1\}}$$

is called $\boldsymbol{n}$-tilting class.

$\boldsymbol{T^{\perp}}$ is closed under direct sums.

$$\boldsymbol{\mathcal{A} = \{A \mid \mathbf{Ext}^i_R(A, M) = 0, \forall i \geq 1, \forall M \in T^{\perp}\}}$$

$\boldsymbol{(\mathcal{A}, T^{\perp})}$ is a cotorsion pair.

Classical tilting modules are finitely presented.

Link between the finite and the infinite case is given by the following notion:

DEFINITION [Angeleri, Herbera, Trlifaj '03] A tilting module T is of finite type if there exists a set $\mathcal{S} \subseteq \text{mod}R$ of modules such that $T^{\perp} = \mathcal{S}^{\perp}$.

($M \in \text{mod}R$ if $\ldots \to P_n \to \ldots \to P_1 \to P_0 \to M \to 0$,

P_i projective finitely generated.)

THEOREM

[B, Eklof, Herbera, Šťovíček, Trlifaj '04-'05]

Every n-tilting module is of finite type.

Proved in many steps, as follows:

THEOREM [B, Eklof, Trlifaj '04]

$\boldsymbol{T}$ 1-tilting module.

(a) $\boldsymbol{T}^{\perp} = \boldsymbol{\mathcal{S}}^{\perp}$, where $\boldsymbol{\mathcal{S}}$ is a set of countably presented modules.

(Proved in ZFC involving set-theoretic methods. Crucial fact: $\boldsymbol{T}^{\perp}$ is closed under direct sums.)

(b) Every countably presented module $\boldsymbol{A} \in \boldsymbol{\mathcal{A}}$ is a countable direct limit of finitely presented modules in $\boldsymbol{\mathcal{A}}$.

THEOREM [B, Herbera '05]

Every $\mathbf{1}$-tilting module is of finite type.

Having the reduction to the countable case the results on the vanishing of $\varprojlim{}^{1}$, imply that the tilting class is closed under pure submodules, hence it is definable.

By (b) the tilting class is of finite type.

THEOREM [Šťovíček, Trlifaj '05]

$\boldsymbol{T}$ n-tilting module.

$\boldsymbol{T}^{\perp} = \boldsymbol{\mathcal{S}}^{\perp}$, where $\boldsymbol{\mathcal{S}}$ is a set of countably generated modules with countably generated syzygies modules.

THEOREM [B, Šťovíček '05]

$\boldsymbol{T}$ n-tilting module.

Every countably generated module $\boldsymbol{A} \in \boldsymbol{\mathcal{A}}$ is a countable direct limit of finitely presented modules in $\boldsymbol{\mathcal{A}}$.

(Using set theoretic methods to filter the syzygies of modules in $\mathcal{A}$).

Hence, analogously to the 1-tilting case

THEOREM [B, Šťovíček '05]

Every $\boldsymbol{n}$-tilting module is of finite type.

APPLICATION TO BAER MODULES.

R commutative domain.

DEFINITION An R-module B is called a Baer module if $\mathrm{Ext}^1_R(B,T)=0$ for every torsion module T.

[Baer '36]:

PROBLEM: Characterize the abelian groups G such that $\mathrm{Ext}^1_{\mathbb{Z}}(G,T)=0$ for all torsion groups T.

- [Baer '36] countably generated groups G with this property must be free.

[Kaplansky '62]:

PROBLEM: Are Baer modules projective?

- [Kaplansky '62] Baer modules are flat and of projective dimension at most one.

• [Griffith '69] Baer groups are free.

• [Grimaldi '72] Baer modules over Dedekind domains are projective.

Using Shelah's Singular Compactness Theorem:

• [Eklof, Fuchs '88] Baer modules over valuation domains are projective.

• [Eklof, Fuchs, Shelah '90]

Reduction to the countable case.

A module B over an arbitrary domain is a Baer module if and only if

$B = \bigcup_\alpha B_\alpha$

continuous ascending chain of submod. such that the factors $B_{\alpha+1}/B_\alpha$ are countably generated Baer modules.

THEOREM [Angeleri-Hügel, B, Herbera, '05]

Baer modules over arbitrary commutative domains are projective.

SKETCH:

- There is a pure embedding: $0 \longrightarrow R \longrightarrow \prod_{0\neq r\in R} R/rR$

- B countably generated Baer module.

B is flat, proj.dim$B \leq 1$, so B is countably presented, hence there are F_n finitely generated free modules such that

$$0 \longrightarrow \oplus_{n\in\mathbb{N}} F_n \xrightarrow{\phi} \oplus_{n\in\mathbb{N}} F_n \longrightarrow \varinjlim F_n = B \longrightarrow 0.$$

$\mathrm{Ext}^1_R(\varinjlim F_n, T^{(\mathbb{N})}) = 0$ for every torsion module T. If $T = \oplus_{0\neq r\in R} R/rR$, then $\{H(F_n, T)\}_{n\in\mathbb{N}}$ and $\{H(F_n, \prod_{0\neq r\in R} R/rR)\}$ satisfy the Mittag-Leffler condition.

So $\{H(F_n, R)\}_{n\in\mathbb{N}}$ satisfies Mittag-Leffler, hence $\mathrm{Ext}^1_R(\varinjlim F_n, \oplus_{n\in\mathbb{N}} F_n) = 0$ and the sequence

$$0 \longrightarrow \oplus_{n\in\mathbb{N}} F_n \xrightarrow{\phi} \oplus_{n\in\mathbb{N}} F_n \longrightarrow \varinjlim F_n = B \longrightarrow 0$$

splits.

Σ COTORSION MODULES

DEFINITION A module M is cotorsion
if $\mathrm{Ext}^1_R(F, M) = 0$ for every flat module F.
M is Σ-cotorsion if $M^{(X)}$ is cotorsion for every set X.

PROBLEM [Guil-Asensio, Herzog '05]

Is a pure submodule of a Σ-cotorsion module again cotorsion?

(True for Σ-pure injective modules).

• Let $F = \varinjlim F_i$ where $\{F_i; f_i^j\}_i$ is a direct system of **finitely generated free** modules, then:.

$$\mathrm{Ext}^n_R(F, M) \cong \varprojlim_I{}^n(\mathrm{Hom}_R(F_i, M))$$

In fact,

[Jensen '72] $\{H_i; g_i^j\}_{i \leq j \in I}$ inverse system of R-modules; there is a complex:

$$0 \to \varprojlim_I H_i \to \prod_{i \in I} H_i \xrightarrow{\Delta^0} \prod_{i_0 \leq i_1} H_{i_0 i_1} \xrightarrow{\Delta^1} \prod_{i_0 \leq i_1 \leq i_2} H_{i_0 i_1 i_2} \xrightarrow{\Delta_2} \cdots$$

$$H_{i_0 i_1 \ldots i_n} = H_{i_0} \quad \forall i_0 \leq i_1 \leq \cdots \leq i_n$$

$$\Delta^0((h_i)_i) = (h_{i_0} - g_{i_0}^{i_1}(h_{i_1}))_{i_0 \leq i_1};$$

$$\Delta^1((h_{i_0 i_1})_{i_0 \leq i_1} = (h_{i_0 i_2} - h_{i_0 i_1} - g_{i_0}^{i_1}(x_{i_1 i_2}))_{i_0 \leq i_1 \leq i_2};$$

$\varprojlim_I{}^n H_i = n^{th}$-cohomology group of the complex

$\{A_i; f_i^j\}_{i\leq j\in I}$ direct system of R-modules;

by the exactness of the direct limit there is an acyclic complex:

$$\cdots \xrightarrow{\delta_2} \bigoplus_{i_0\leq i_1\leq i_2} A_{i_0 i_1 i_2} \xrightarrow{\delta_1} \bigoplus_{i_0\leq i_1} A_{i_0 i_1} \xrightarrow{\delta_0} \bigoplus_{i} A_i \to \varinjlim_I A_i \to 0$$

$$A_{i_0 i_1 \ldots i_n} = A_{i_0} \quad \forall i_0 \leq i_1 \leq \cdots \leq i_n$$

$$\delta_0 \varepsilon_{i_0 i_1}(a) = \varepsilon_{i_0}(a) - \varepsilon_{i_1} f_{i_0}^{i_1}(a); (i_0 \leq i_1)$$

$\varepsilon_{i_0 i_1}$ canonical embedding.

Let $M \in$ Mod -R , we get the complex:

$$0 \to \varprojlim_I H(A_i, M) \to \prod_{i\in I} H(A_i, M) \xrightarrow{\Delta^0} \prod_{i_0\leq i_1} H(A_{i_0 i_1}, M) \xrightarrow{\Delta^1}$$

$$\prod_{i_0\leq i_1\leq i_2} H(A_{i_0 i_1 i_2}, M) \xrightarrow{\Delta_2} \ldots$$

$$\Delta^n = H(\delta_n, M)$$

If $\mathrm{Ext}^n_R(A_i, M) = 0 \; \forall n \in \mathbb{N}, i \in I$,

the n^{th}-cohomology group is isomorphic to $\mathrm{Ext}^n_R(\varinjlim_I A_i, M)$, hence

$$\mathrm{Ext}^n_R(\varinjlim_I A_i, M) \cong \varprojlim_I{}^n \mathrm{Hom}(A_i, M)$$

If M is Σ-cotorsion, then

$$\mathrm{Ext}^1_R(\varinjlim_I F_i, M^{(X)}) \cong \varprojlim_I{}^1 \mathrm{Hom}(F_i, M^{(X)}) = 0$$

for every direct system $\{F_i; f_i^j\}_{i \leq j \in I}$ of **finitely generated free** modules

PROPOSITION The following are equivalent:

(1) M is Σ-cotorsion;

(2) for every direct system $\{F_i; f_i^j\}_{i\leq j\in I}$ of **finitely generated free** modules, $\varprojlim_I{}^1 H(F_i, M)^{(X)} = 0.$

- **Necessary condition**

$\{H_i; g_i^j\}_{i\leq j\in I}$ inverse system of R-modules, if $\varprojlim_I{}^1 H_i^{(X)} = 0$, then every countable subsystem $\{H_{i_n}\}_{n\in\mathbb{N}}$ satisfies the Mittag-Leffler condition.

- **Sufficient condition?**

PROBLEM Find a necessary and sufficient condition for the vanishing of

$$\varprojlim_I{}^1 H(F_i, M)^{(X)}$$

If the condition is inherited by pure submodules, then pure submodules of Σ-cotorsion modules are again cotorsion.