Anticyclic operads and Auslander-Reiten translation

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Abstract

We will show that two different constructions lead to the same actions of cyclic groups on some Abelian groups. The first of these constructions lives in the framework of the theory of operads, and more precisely comes from the notion of anticyclic operad. The other construction is provided by the Coxeter transformation, which is the action induced by the Auslander-Reiten functor on the Grothendieck group of a finite-dimensional algebra of finite global dimension.

There are only two examples so far for this relationship. The first one is between the Diassociative operad and the sequence of hereditary algebras of the $\mathbb{A}_n$ quivers. This is of course a very classical setting. The other one is between the Dendriform operad and the sequence of incidence algebras of the Tamari lattices. This is related to some more recent developments, such as the theory of cluster algebras.

1 Introduction

This talk will deal with some hint at a new relationship between two mathematical domains. This is not a new theory, but only two interesting examples.

The first domain is the theory of operads, which originated in algebraic topology in the 1960's and has more recently known a new sequence of development, related to the moduli spaces of curves and algebraic geometry. The other domain is representation theory, more precisely the study of representations of finite dimensional algebras of finite global dimension. Here the main objects are the Abelian categories of modules and the bounded derived categories of these, which are triangulated categories.

In each of these two domains, we will present some construction of free Abelian groups endowed with an action of a cyclic group:

$$\tau \in \mathrm{End}_{\mathbb{Z}}(\mathbb{Z}^N) \text{ with } \tau^n = \mathrm{Id}, \tag{1}$$

for some N and n. The main point of the talk is to observe that these two actions are closely related and to propose some conjectural explanation for this link.

2 Operads

2.1 Basics and examples

Let us first recall briefly the basics of the theory of operads. We will only use so-called non-symmetric operads and just call them operads for short.

The notion of operad can be defined in any monoidal category. We will use the category of Abelian groups.

Definition 1 *An operad $\mathcal{P}$ is the data of a sequence $\{\mathcal{P}(n)\}_{n\in\mathbb{N}}$ of Abelian groups, of a distinguished element $\mathbf{1} \in \mathcal{P}(1)$ and of composition maps $\circ_i$ from $\mathcal{P}(n) \otimes_{\mathbb{Z}} \mathcal{P}(m)$ to $\mathcal{P}(n+m-1)$ for each n, m and each $1 \leq i \leq n$. This data must satisfy some axioms, modelled after the properties of the first example below: unity, associativity of nested compositions and commutativity of disjoint compositions.*

Let us give some examples.

Example 1: the endomorphism operads

Pick any free Abelian group V of finite rank. Let $\mathcal{P}(n) = \mathrm{Hom}_{\mathbb{Z}}(V^{\otimes n}, V)$. Let $\mathbf{1}$ be the identity map in $\mathcal{P}(1)$. Let $\circ_i$ be the composition of multilinear maps defined, for $f \in \mathcal{P}(n)$ and $g \in \mathcal{P}(m)$, by

$$(f \circ_i g)(x_1, \dots, x_{m+n-1}) = f(x_1, \dots, g(x_i, \dots, x_{i+m-1}), \dots, x_{m+n-1}). \tag{2}$$

These data define the so-called endomorphism operad of V.

Example 2: the associative operad

Let $\mathrm{Assoc}(n)$ be the free Abelian group of rank 1 with basis b^n. Let $\mathbf{1}$ be b^1 and let

$$b^n \circ_i b^m = b^{n+m-1}. \tag{3}$$

The axioms are easily checked and this defines the associative operad Assoc.

Example 3: the diassociative operad (Loday)

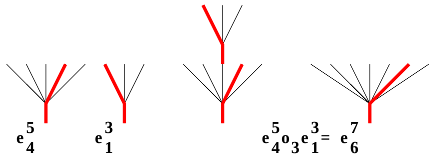
Let $\mathrm{Dias}(n)$ be the free Abelian group of rank n with basis $\{e_1^n, \dots, e_n^n\}$. Let $\mathbf{1}$ be e_1^1. The composition maps are defined by

$$e_k^n \circ_i e_\ell^m = e_j^{n+m-1}, \tag{4}$$

where j is given by the following rule:

$$\begin{cases} k \text{ if } i > k, \\ k + \ell - 1 \text{ if } i = k, \\ k + n - 1 \text{ if } i < k. \end{cases} \tag{5}$$

This is more easily understood using the following graphical description. The basis element e_k^n is associated with the following tree (corolla) endowed with a colored path from the leaf k to the root. Then the composition $e_k^n \circ_i e_\ell^m$ is associated to the grafting of the corolla for e_ℓ^m on the leaf i of the corolla for e_k^n. In the resulting tree, only one leaf is linked to the root by a colored path. The index j of this leaf (numbering leaves of the tree from left to right) provides the result of the composition.



Just like for associative algebras, there exist notions of a free operad generated by a collection $\{E(n)\}_{n\geq 2}$ of Abelian groups, of an ideal in an operad, and of a quotient operad. Therefore one can speak of a presentation by generators and relations of an operad.

Let us give some examples of such presentations.

Example 2: the associative operad The operad Assoc is generated by $b^2 \in \mathrm{Assoc}(2)$. One can compute that

$$b^2 \circ_1 b^2 = b^2 \circ_2 b^2 = b^3. \tag{6}$$

The operad Assoc is presented by the generator b^2 and the relation

$$b^2 \circ_1 b^2 = b^2 \circ_2 b^2. \tag{7}$$

Remark: this relation can be thought of as the axiom of associativity, *i.e.* the fact that

$$(x \cdot y) \cdot z = x \cdot (y \cdot z) \tag{8}$$

holds in any associative algebra $(A, \cdot)$. Indeed, if the product map is denoted by m from $A \otimes A$ to A, then identity (8) can be written in the endomorphism operad of A as $m \circ_1 m = m \circ_2 m$, formally the same as the relation of Assoc.

Example 3: the diassociative operad

The operad Dias is generated by $\mathrm{Dias}(2) = \mathbb{Z}\{e^2_1, e^2_2\}$. One can compute (using the graphical description of Dias) that

$$e^3_1 = e^2_1 \circ_1 e^2_1 = e^2_1 \circ_2 e^2_1 = e^2_1 \circ_2 e^2_2, \tag{9}$$

$$e^3_2 = e^2_2 \circ_2 e^2_1 = e^2_1 \circ_1 e^2_2, \tag{10}$$

$$e^3_3 = e^2_2 \circ_2 e^2_2 = e^2_2 \circ_1 e^2_1 = e^2_2 \circ_1 e^2_2. \tag{11}$$

This provides a presentation of the operad Dias. From this presentation, one can find the axioms of the notion of diassociative algebra, as above for associative algebras.

2.2 Anticyclic operads

We will need the more sophisticated notion of anticyclic operad. This is an operad endowed with additional structure.

Definition 2 *An anticyclic operad $\mathcal{P}$ is an operad $\mathcal{P}$ together with the data of endomorphisms τ_n of $\mathcal{P}(n)$ satisfying*

$$\tau_1(\mathbf{1}) = -\mathbf{1}, \tag{12}$$

$$\tau_n^{n+1} = \mathrm{Id}, \tag{13}$$

$$\tau_{n+m-1}(x \circ_n y) = -\tau_m(y) \circ_1 \tau_n(x), \tag{14}$$

$$\tau_{n+m-1}(x \circ_i y) = \tau_n(x) \circ_{i+1} y, \tag{15}$$

where $x \in \mathcal{P}(n)$, $y \in \mathcal{P}(m)$ and $1 \leq i \leq n-1$.

This notion has been introduced by Getzler and Kapranov.

Our aim is now to show that the operad Dias can be upgraded to an anticyclic operad. This could be done by defining all the maps τ_n and checking the axioms. It is though much simpler to use the presentation of Dias by generators and relations. One then just has to define τ on the generators and to check compatibility with the relations.

Let us define τ_2 by

$$\tau_2(e_1^2) = -e_1^2 + e_2^2, \tag{16}$$

$$\tau_2(e_2^2) = -e_1^2. \tag{17}$$

Thus the matrix of τ_2 in the basis e^2 is

$$\begin{bmatrix} -1 & -1 \\ 1 & 0 \end{bmatrix}. \tag{18}$$

Theorem 2.1 *The operad* Dias *is an anticyclic operad with* τ_2 *as above. The matrix of* τ_n *in the basis* e^n *is*

$$\begin{bmatrix} -1 & -1 & \dots & -1 \\ 1 & 0 & \dots & 0 \\ 0 & \ddots & \ddots & 0 \\ 0 & \ddots & 1 & 0 \end{bmatrix}. \tag{19}$$

Let us give an example of computation for τ_3:

$$\tau_3(e_2^3) = \tau_3(e_2^2 \circ_2 e_1^2) = -\tau_2(e_1^2) \circ_1 \tau_2(e_2^2) = (e_2^2 - e_1^2) \circ_1 (-e_1^2) = e_3^3 - e_1^3. \tag{20}$$

The reader can check that using $e_2^3 = e_1^2 \circ_1 e_2^2$ instead leads to the same value.

Therefore we have defined an action of the cyclic group $\mathbb{Z}/(n+1)\mathbb{Z}$ on the Abelian group $\mathbb{Z}^n$ for each $n \geq 1$.

Let us now define a similar action in a completely different way.

3 Algebras and Auslander-Reiten translation

Let us consider an algebra Λ of finite dimension over a field k. Let $\operatorname{Mod}\Lambda$ be its category of finite-dimensional modules. This is an Abelian category. We further assume that Λ has finite global dimension. Let also $D\operatorname{Mod}\Lambda$ be the bounded derived category of $\operatorname{Mod}\Lambda$. This is a triangulated category.

Then the theory of Auslander and Reiten provides the existence of a functor τ from $D\operatorname{Mod}\Lambda$ to $D\operatorname{Mod}\Lambda$ which is a self-equivalence. This is the Auslander-Reiten translation.

This functor τ descends on the Grothendieck group $K_0(\operatorname{Mod}\Lambda) = K_0(D\operatorname{Mod}\Lambda)$ and defines a bijective linear map, still denoted by τ, on the Grothendieck group. This map is sometimes called the Coxeter transformation.

This theory has some nice applications to path algebras of quivers. Choose any Dynkin diagram of finite type, in the usual list $(\mathbb{A}_n)_{n\geq 1}, (\mathbb{D}_n)_{n\geq 4}, \mathbb{E}_6, \mathbb{E}_7, \mathbb{E}_8$. Picking any orientation of this Dynkin diagram defines a quiver Q. Let $\operatorname{Mod} kQ$ be the Abelian category of representations of Q.

Then, by classical results of Gabriel and Gelfand & Ponomarev, one knows that $\operatorname{Mod} kQ$ has a finite number of isomorphism classes of indecomposable modules, in bijection with positive roots of the associated root system. Furthermore, the action of τ on the Grothendieck group is exactly the action of a Coxeter element in the corresponding Weyl group. Hence τ has finite order h, the Coxeter number.

Let us look at the case of the equioriented diagram of type $\mathbb{A}_n$:

$$n \to n-1 \to \cdots \to 2 \to 1. \tag{21}$$

We denote by $\operatorname{Mod} \mathbb{A}_n$ the category of modules on this quiver and by S_i the simple module on the vertex i. The action of τ in the basis $\{S_1, S_2, \ldots, S_n\}$ of $K_0(\operatorname{Mod} \mathbb{A}_n)$ has the following matrix

$$\begin{bmatrix} -1 & 1 & 0 & 0 \\ -1 & 0 & \ddots & \ddots \\ \vdots & \vdots & \ddots & 1 \\ -1 & 0 & 0 & 0 \end{bmatrix}. \tag{22}$$

This is clearly the transposed matrix of the map τ_n that was defined purely in terms of operads before.

Now, it is possible to dualize the anticyclic operad Dias into an anticyclic **cooperad** Dias^*. Then the cyclic group actions become exactly the same. This should be the proper setting.

The next step would now be to define functors

$$\Delta_i : \operatorname{Mod} \mathbb{A}_{n+m-1} \longrightarrow \operatorname{Mod} \mathbb{A}_n \otimes \operatorname{Mod} \mathbb{A}_m \tag{23}$$

satisfying axioms dual to those of the maps $\circ_i$ and appropriate compatibility conditions with the Auslander-Reiten translations, dual to the axioms of an anticyclic operad.

Using the presentation of the category $\operatorname{Mod} \mathbb{A}_n$ given by the Auslander-Reiten quiver and the mesh relations, one can describe a candidate functor as the direct sum of the three following functors.

$$\begin{tikzcd}
 & & {[1,i-1] \otimes [n]} \arrow[dr] & & . \\
 & {[1,i-1] \otimes [1,n]} \arrow[ur] \arrow[dr] & \cdots & {[i-1] \otimes [n]} & \\
{[1] \otimes [1,n]} \arrow[ur] & \cdots & {[i-1] \otimes [1,n]} \arrow[ur] & &
\end{tikzcd} \tag{24}$$

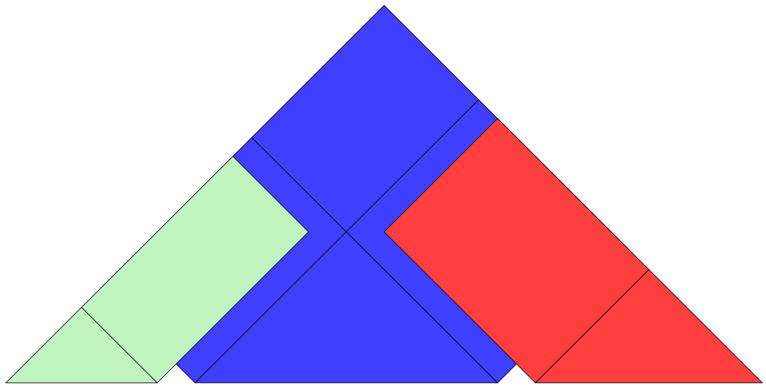


Figure 1: How to take the direct sum

$$\begin{tikzcd}
 & & {[1,m]\otimes[1,n]} \arrow[dr] & & \\
 & {[1,i]\otimes[1,n]} \arrow[ur] \arrow[dr] & \cdots & {[i,m]\otimes[1,n]} \arrow[dr] & \\
{[1,i]\otimes[1]} \arrow[ur] \arrow[dr] & \cdots & {[i]\otimes[1,n]} \arrow[ur] \arrow[dr] & \cdots & {[i,m]\otimes[n]} \\
 & {[i]\otimes[1]} \arrow[ur] & \cdots & {[i]\otimes[n]} \arrow[ur] &
\end{tikzcd}. \tag{25}$$

$$\begin{tikzcd}
 & {[i+1,m]\otimes[1]} \arrow[dr] & & \\
{[i+1]\otimes[1]} \arrow[ur] \arrow[dr] & \cdots & {[i+1,m]\otimes[1,n]} \arrow[dr] & \\
 & {[i+1]\otimes[1,n]} \arrow[ur] & \cdots & {[m]\otimes[1,n]}
\end{tikzcd}. \tag{26}$$

The direct sum is taking according to Figure 1.

4 Second example

Let us now introduce the second example of a relationship between operads and quivers. This example is more complicated than the first, but also maybe more interesting. We will introduce another operad and another family of algebras.

4.1 Binary trees

A planar binary tree is a graph drawn in the plane, which is connected and simply connected, has vertices of valence 1 or 3 only, together with the data of a distinguished vertex of valence 1 called the root. The other vertices of valence 1 are called the leaves. The root is drawn at the bottom.

Let Y_n be the set of planar binary trees with $n+1$ leaves.

$$Y_1 = \{\mathrm{Y}\} \qquad Y_2 = \{(\vee|), (|\vee)\} \qquad Y_3 = \{((\vee|)|)\,((|\vee)|)\,(\vee\vee)\,(|(\vee|))\,(|(|\vee))\} \tag{27}$$

The cardinality of Y_n is the Catalan number $c_n = \frac{1}{n+1}\binom{2n}{n}$.

Then there exists an operad Dend such that $\mathrm{Dend}(n) = \mathbb{Z}Y_n$. We will not describe the composition maps $\circ_i$ here. The unit $\mathbf{1}$ is the unique element of Y_1.

This operad is generated by the two trees $(\vee|)$ and $(|\vee)$ in Y_2. The relations are as follows:

$$(\vee|) \circ_2 (\vee|) = (\vee|) \circ_1 (\vee|) + (\vee|) \circ_1 (|\vee), \tag{28}$$

$$(\vee|) \circ_2 (|\vee) = (|\vee) \circ_1 (\vee|), \tag{29}$$

$$(|\vee) \circ_2 (\vee|) + (|\vee) \circ_2 (|\vee) = (|\vee) \circ_1 (|\vee). \tag{30}$$

Theorem 4.1 *There exists a unique structure of anticyclic operad on* Dend *such that*

$$\tau((|\vee)) = (\vee|) \quad \textit{and} \quad \tau((\vee|)) = -((|\vee) + (\vee|)). \tag{31}$$

Let us display the matrix of τ_3 in the basis Y_3 of Dend(3):

$$\begin{bmatrix} -1 & 0 & 1 & 1 & -1 \\ -1 & 0 & 1 & 0 & 0 \\ 0 & -1 & 0 & 1 & 0 \\ -1 & 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 & 0 \end{bmatrix}. \tag{32}$$

In general, the map τ_n seems quite complicated. The multiplicities of the roots of unity as eigenvalues of τ_n are not known.

4.2 Tamari posets

Let us introduce a partial order $\leq$ on Y_n, called the Tamari order or Tamari lattice.

The order relation $\leq$ is defined as the transitive closure of some covering relations. A tree S is covered by a tree T if they differ only in some neighborhood of an edge by the replacement of the configuration $(\vee|)$ in S by the configuration $(|\vee)$ in T.

Then one can consider the incidence algebra of this poset, or equivalently look at the Hasse diagram of this poset as a quiver, and add the relations that all paths with same beginning and end are equal. One gets a finite dimensional algebra Λ_n for each $n \geq 1$, which has finite global dimension.

By the Auslander-Reiten theory, there is a Coxeter transformation τ acting on the Grothendieck group of Λ_n. This Grothendieck group has a basis coming from simple modules, which are labelled by Y_n. Hence one can identify $K_0(\Lambda_n)$ with $\mathrm{Dend}(n)$.

Theorem 4.2 *On the Abelian group* $\mathrm{Dend}(n)$*, one has the relation*

$$\tau_n = (-1)^n \theta^2. \tag{33}$$

The expected explanation of all this should be the existence of appropriate functors

$$\circ_i : \mathrm{Mod}\,\Lambda_n \otimes \mathrm{Mod}\,\Lambda_m \longrightarrow \mathrm{Mod}\,\Lambda_{n+m-1}, \tag{34}$$

satisfying, together with the Auslander-Reiten translation, some version of the axioms of an anticyclic operad.

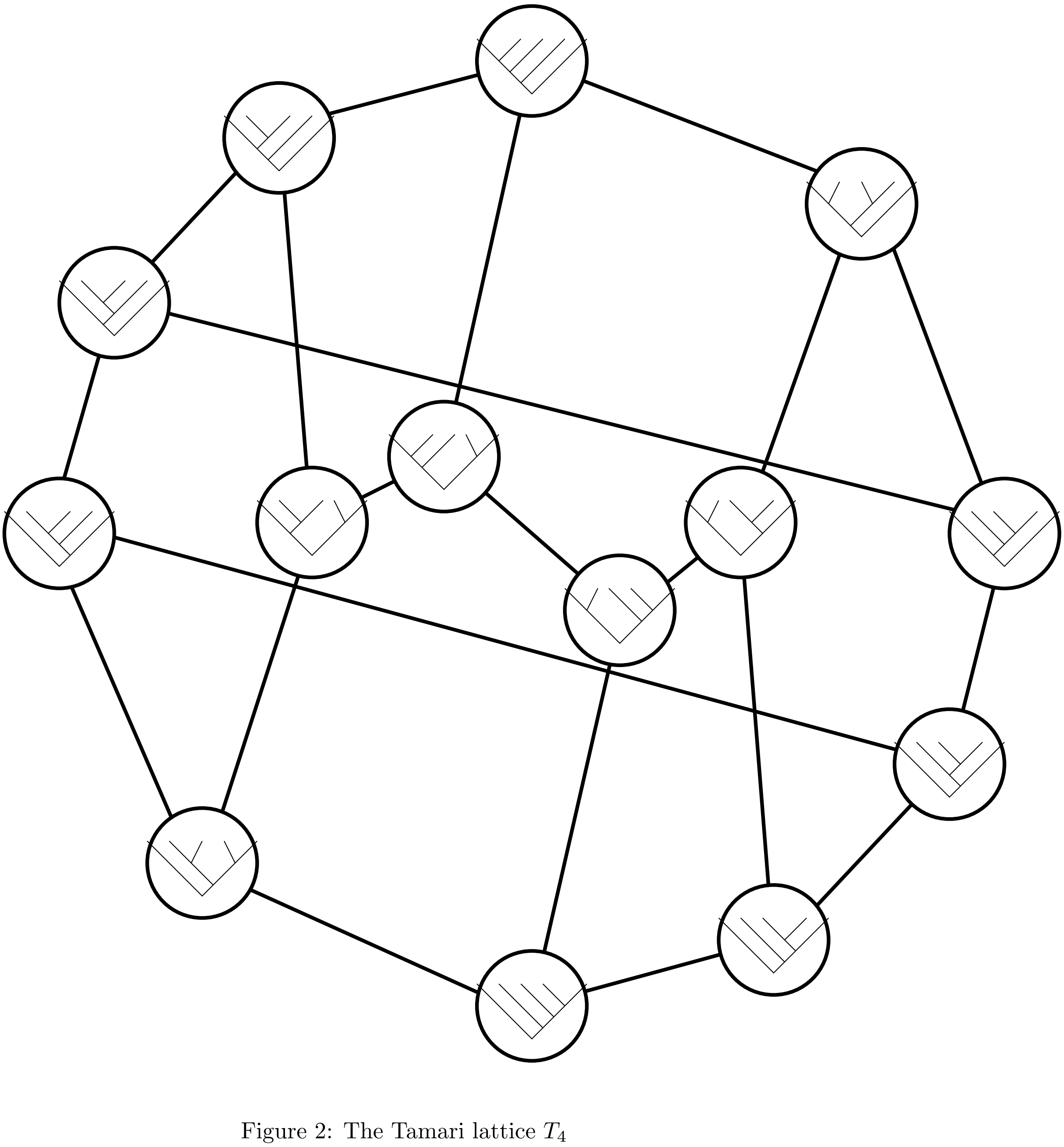


Figure 2: The Tamari lattice T_4