$\mathrm{G_1T}$-MODULES, AR-COMPONENTS, AND GOOD FILTRATIONS

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1. G_1T-modules

Let $\mathfrak{g}$ be a finite dimensional Lie algebra over an algebraically closed field k. The structure of $\mathfrak{g}$ and its modules is usually analyzed by considering weight space decompositions relative to a Cartan subalgebra $\mathfrak{h} \subset \mathfrak{g}$:

$$\mathfrak{g} = \mathfrak{h} \oplus \bigoplus_{\alpha\in R} \mathfrak{g}_\alpha \quad ; \quad M = \bigoplus_{\lambda\in X(M)} M_\lambda.$$

Here $R \subset \mathfrak{h}^* \setminus \{0\}$ and $X(M) \subset \mathfrak{h}^*$ are finite subsets, and $\mathfrak{g}_\alpha, M_\lambda$ denote the root spaces and weight spaces of $\mathfrak{g}$ and M, respectively.

In the classical situation, that is, when $\mathfrak{g}$ is semi-simple and $\mathrm{char}(k) = 0$, these decompositions define gradings of $\mathfrak{g}$ and M relative to a finitely generated subgroup $Q \subset \mathfrak{h}^*$. This group is torsion free and hence free. By contrast, one obtains a grading relative to a p-elementary abelian group, whenever $\mathrm{char}(k) = p > 0$.

From now on we assume that $\mathrm{char}(k) = p \geq 3$ and let $\mathfrak{g} = \mathrm{Lie}(G)$ be the Lie algebra of a reductive group G. For instance, the classical groups $\mathrm{GL}(n)$, $\mathrm{O}(n)$, $\mathrm{Sp}(2n)$ are of this type.

We fix a Borel subgroup $B \subset G$, a maximal torus $T \subset B$ and define $\mathfrak{h} := \mathrm{Lie}(T)$, $\mathfrak{b} := \mathrm{Lie}(B)$. The algebra $\mathfrak{g}$ is a restricted Lie algebra, that is, there exists a map

$$\mathfrak{g} \longrightarrow \mathfrak{g} \ \ ; \ \ x \mapsto x^{[p]}$$

that possesses properties derived from those of the p-power map of an associative algebra. The subalgebras $\mathfrak{h} \subset \mathfrak{b}$ are stable under the p-map.

We consider the restricted enveloping algebra

$$U_0(\mathfrak{g}) := U(\mathfrak{g})/(\{x^p - x^{[p]} \ ; \ x \in \mathfrak{g}\}),$$

which is a $p^{\dim_k \mathfrak{g}}$-dimensional quotient of the ordinary enveloping algebra $U(\mathfrak{g})$. Given an algebra homomorphism $\lambda : U_0(\mathfrak{b}) \longrightarrow k$, one defines the corresponding *baby Verma module* via

$$Z(\lambda) := U_0(\mathfrak{g}) \otimes_{U_0(\mathfrak{b})} k_\lambda.$$

The algebra $U_0(\mathfrak{g})$ shares many important properties with the group algebra of a finite group. As a result, much of the early work was concerned with the discovery of analogs. In 1971 Humphreys showed that, subject to restrictions on p, some results by Brauer-Nesbitt concerning Cartan matrices of group algebras can be transferred to this context. He used the modules $Z(\lambda)$ to define "decomposition matrices" D that provided a presentation $C = D^tD$ of the Cartan matrix of $U_0(\mathfrak{g})$.

In their famous 1976 paper, Bernstein-Gel'fand-Gel'fand obtained similar results for the category $\mathcal{O}$ of a complex semi-simple Lie algebra. When Jantzen tried to transfer their methods to algebras of distributions of higher Frobenius kernels, he encountered the aforementioned problems concerning the weights. He overcame these obstacles by defining the category of $\mathrm{mod}\, G_1T$ of G_1T-modules.

Recall that the maximal torus T acts on $\mathfrak{g}$ via the adjoint representation

$$\mathrm{Ad} : T \longrightarrow \mathrm{GL}(\mathfrak{g}).$$

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In fact, T operates via automorphisms of the restricted Lie algebra $(\mathfrak{g}, [p])$ and the subalgebras $\mathfrak{h}$ and $\mathfrak{b}$ are T-invariant. Consequently, T also acts on the corresponding restricted enveloping algebras, so that we obtain an operation

$$\mathrm{Ad} : T \longrightarrow \mathrm{Aut}_k(U_0(\mathfrak{g}))$$

of T on $U_0(\mathfrak{g})$

Let $X(T) := \mathrm{Hom}(T, k^\times)$ be the *character group* of T. Since T is a torus, $X(T)$ is a finitely generated torsion free abelian goup. If V is a finite dimensional T-module, then

$$V = \bigoplus_{\lambda \in X(T)} V_\lambda,$$

where $V_\lambda = \{v \in V \ ; \ t \cdot v = \lambda(t)v \ \ \forall\, t \in T\}$. If $V = \mathfrak{g}$, then the adjoint representation yields

$$\mathfrak{g} = \bigoplus_{\alpha \in R \cup \{0\}} \mathfrak{g}_\alpha \quad ; \quad R \subset X(T) \setminus \{0\}$$

the root space decomposition of $\mathfrak{g}$ relative to T. Since G is reductive, we have $\mathfrak{g}_0 = \mathrm{Lie}(T) = \mathfrak{h}$ as well as $\dim_k \mathfrak{g}_\alpha = 1$ for all $\alpha \in R$.

Here is Jantzen's definition of a G_1T-module:

Definition. A finite dimensional k-vector space V is a *G_1T-module* if
(a) V is a $U_0(\mathfrak{g})$-module,
(b) V is a T-module,
(c) we have

$$t(uv) = \mathrm{Ad}(t)(u)(tv) \qquad \forall\, t \in T,\ u \in U_0(\mathfrak{g}),\ v \in V,$$

(d) the differential $\mathfrak{h} \longrightarrow \mathrm{gl}(V)$ of the T-action coincides with the action of $\mathfrak{h}$ coming from (a).

Remark. The terminology derives from the equivalence

$$\mathrm{mod}\, U_0(\mathfrak{g}) \cong \mathrm{mod}\, G_1$$

between the category $\mathrm{mod}\, U_0(\mathfrak{g})$ of finite dimensional $U_0(\mathfrak{g})$-modules and the corresponding category $\mathrm{mod}\, G_1$ for the first Frobenius kernel of G. The conditions ensure that the actions of G_1 and T extend to an operation of the algebraic group

$$G_1T \cong (G_1 \rtimes T)/(G_1 \cap T).$$

2. Auslander-Reiten Components

The following result combines work of Gordon and Green on the AR-Theory of $\mathbb{Z}^n$-graded modules with the block theory of $\mathrm{mod}(G_1 \rtimes T)$:

Theorem 2.1. (1) *The category* $\mathrm{mod}\, G_1T$ *has almost split sequences.*

(2) *The forgetful functor* $\mathcal{F} : \mathrm{mod}\, G_1T \longrightarrow \mathrm{mod}\, U_0(\mathfrak{g})$ *sends almost split sequences to almost split sequences.*

Since $\mathrm{mod}\, G_1T$ is a Frobenius category, we can speak of the stable Auslander-Reiten quiver $\Gamma_s(G_1T)$ of $\mathrm{mod}\, G_1T$. We want to study $\Gamma_s(G_1T)$ and the stable AR-quiver $\Gamma_s(\mathfrak{g})$ of $U_0(\mathfrak{g})$ by means of rank varieties.

Recall that

$$\mathcal{V}_{\mathfrak{g}} := \{x \in \mathfrak{g} \ ; \ x^{[p]} = 0\}$$

is the *nullcone* of $\mathfrak{g}$. Given $M \in \mathrm{mod}\, U_0(\mathfrak{g})$, we define the *rank variety* of M via

$$\mathcal{V}_{\mathfrak{g}}(M) := \{x \in \mathcal{V}_{\mathfrak{g}} \ ; \ M|_{U_0(kx)} \text{ is not projective}\} \cup \{0\}.$$

If $M \in \mathrm{mod}\, G_1T$, then we put $\mathcal{V}_{\mathfrak{g}}(M) := \mathcal{V}_{\mathfrak{g}}(\mathcal{F}(M))$.

Here are some facts concerning rank varieties and stable AR-components:

- A module $M \in \mathrm{mod}\, G_1T$ is projective if and only if $\dim \mathcal{V}_{\mathfrak{g}}(M) = 0$.
- If $\Theta \subset \Gamma_s(G_1T)$ is a component, then we have
$$\mathcal{V}_{\mathfrak{g}}(M) = \mathcal{V}_{\mathfrak{g}}(N) \qquad \forall\ [M],\, [N] \in \Theta.$$
Accordingly, we can speak of the variety $\mathcal{V}_{\mathfrak{g}}(\Theta)$ of the AR-component Θ.

Theorem 2.2. *Let M be an indecomposable G_1T-module, $\Theta \subset \Gamma_s(G_1T)$ be a component.*

(1) *If $\dim \mathcal{V}_{\mathfrak{g}}(M) = 1$, then there exists a root $\alpha_M \in R$ such that $\mathcal{V}_{\mathfrak{g}}(M) = \mathfrak{g}_{\alpha_M}$, and $\tau_{G_1T}(M) \cong M \otimes_k k_{p\,\alpha_M}$.*

(2) *We have $\Theta \cong \mathbb{Z}[A_\infty]$, $\mathbb{Z}[A_\infty^\infty]$, $\mathbb{Z}[D_\infty]$.*

(3) *If $\dim \mathcal{V}_{\mathfrak{g}}(\Theta) \neq 2$, then $\Theta \cong \mathbb{Z}[A_\infty]$.*

Remarks. (1) If $\Theta \subset \Gamma_s(\mathrm{SL}(2)_1T)$ has a rank variety of dimension 2, then $\Theta \cong \mathbb{Z}[A_\infty^\infty]$.

(2) I do not know whether components of tree class D_∞ can occur.

3. Modules with a good filtration

The main advantage of working in $\mathrm{mod}\, G_1T$ rather than $\mathrm{mod}\, U_0(\mathfrak{g})$ rests on $\mathrm{mod}\, G_1T$ being a *highest weight category* in the sense of Cline-Parshall-Scott. The projective indecomposable objects in $\mathrm{mod}\, G_1T$ are indexed by elements of $X(T)$: Given $\lambda \in X(T)$, we let $\hat{P}(\lambda)$ and $\hat{L}(\lambda)$ be the projective indecomposable and the simple G_1T-module of highest weight λ, respectively. We also consider the G_1T-module

$$\hat{Z}(\lambda) := U_0(\mathfrak{g}) \otimes_{U_0(\mathfrak{b})} k_\lambda,$$

whose T-action is induced by the adjoint representation. A filtration of a G_1T-module M with factors of the form $\hat{Z}(\lambda)$ is called a *$\hat{Z}$-filtration.* Z-filtrations of $U_0(\mathfrak{g})$-modules are defined analogously.

Let R^+ be the set of positive roots of $\mathfrak{g}$, corresponding to our Borel subgroup $B \subset G$. We define a partial ordering on $X(T)$ via

$$\lambda \leq \mu :\Leftrightarrow \mu - \lambda \in \mathbb{N}_0 R^+.$$

Relative to this ordering the modules $\hat{Z}(\lambda) = \Delta(\lambda)$ are the *standard modules* in the highest weight category $\mathrm{mod}\, G_1T$, and a $\hat{Z}$-filtration is a Δ-*good filtration.* The *costandard modules* are given by

$$\nabla(\lambda) := \hat{Z}'(\lambda) := U_0(\mathfrak{g}) \otimes_{U_0(\mathfrak{b}^-)} k_{\lambda - 2(p-1)\varrho},$$

where $\mathfrak{b}^-$ is the opposite Borel subalgebra, and $\varrho := \frac{1}{2}\sum_{\alpha \in R^+} \alpha$. We denote by $F(\Delta)$ and $F(\nabla)$ the full subcategories of $\mathrm{mod}\, G_1T$, whose objects afford a Δ-filtration and a ∇-filtration, respectively. In the representation theory of algebraic groups, the modules belonging to $F(\Delta) \cap F(\nabla)$ are referred to as *tilting modules.*

The first part of the following result, often referred to as BGG duality or Brauer-Humphreys reciprocity, was one of Jantzen's main objectives:

Theorem 3.1. (1) (Jantzen, 1979) *Given $\lambda \in X(T)$, the module $\hat{P}(\lambda)$ has a $\hat{Z}$-filtration and*

$$(\hat{P}(\lambda) : \hat{Z}(\mu)) = [\hat{Z}(\mu) : \hat{L}(\lambda)].$$

(2) (Ringel, 1991) *The subcategory $F(\Delta)$ has relative almost split sequences.*

Strictly speaking, Ringel's result holds for quasi-hereditary algebras, but his arguments also apply in our context.

Lemma 3.2. *Let M be a G_1T-module.*
 (1) *$M \in F(\Delta)$ if and only if $\mathcal{V}_{\mathfrak{g}}(M) \cap \mathfrak{b}^- = \{0\}$.*
 (2) *$M \in F(\nabla)$ if and only if $\mathcal{V}_{\mathfrak{g}}(M) \cap \mathfrak{b} = \{0\}$.*

In particular, the category $F(\Delta)$ is closed under extensions, direct summands, and tensor products.

The following result can be viewed as an interpretation of (3.1(2)). In our particular context, the relative almost split sequences are almost split within the category $\operatorname{mod} G_1T$:

Theorem 3.3. *Let M be an indecomposable G_1T-module, $\Theta \subset \Gamma_s(G_1T)$ and $\Psi \subset \Gamma_s(\mathfrak{g})$ the stable AR-components containing M and $\mathcal{F}(M)$, respectively.*
 (1) *Every vertex of Ψ has a G_1T-structure.*
 (2) *If $M \in F(\Delta)$, then every vertex of Θ belongs to $F(\Delta)$.*
 (3) *If $\mathcal{F}(M)$ has a Z-filtration, so does every vertex of Ψ.*

The third statement illustrates the utility of $\operatorname{mod} G_1T$ in the study of $\operatorname{mod} U_0(\mathfrak{g})$.

Theorem 3.4. *The following statements hold:*
 (1) *The module $\hat{L}(\lambda)$ is either projective, quasi-simple, or it belongs to a component of type $\mathbb{Z}[A_\infty^\infty]$.*
 (2) *The module $\hat{Z}(\lambda)$ is either projective or quasi-simple.*
 (3) *Every component of $\Gamma_s(G_1T)$ contains at most one $\hat{L}(\lambda)$ and at most one $\hat{Z}(\lambda)$, but not both.*

Remark. The proof of (3) employs formal characters and relies on the fact that $\mathbb{Z}[X(T)]$ is an integral domain. Working in $\operatorname{mod} U_0(\mathfrak{g})$ would involve $\mathbb{Z}[X(T)/pX(T)] \cong \mathbb{Z}[X_1, \ldots, X_n]/(X_1^p, \ldots, X_n^p)$. However, using the functor $\mathcal{F}$ one obtains the analogous result for $\Gamma_s(\mathfrak{g})$.

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