QUOTIENT TRIANGULATED CATEGORIES ARISING IN REPRESENTATIONS OF ALGEBRAS

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Abstract. Several kinds of quotient triangulated categories arising naturally in representations of algebras are studied; their relations with the stable categories of Frobenius exact categories are investigated; the derived categories of Gorenstein algebras are explicitly computed inside the stable categories of the graded module categories of the corresponding trivial extension algebras; new descriptions of the singularity categories of Gorenstein algebras are obtained.

1. Introduction

1.1. Throughout A is a finite-dimensional associative algebra with unit over a field k, all (left) A-modules considered are unitary and finite-dimensional. Denote by A-mod the category of such A-modules, and by $D^b(A) := D^b(A\text{-mod})$ the bounded derived category of A-mod. Let $K^b(A\text{-inj})$ (resp., $K^b(A\text{-proj})$) be the bounded homotopy category of injective (resp., projective) A-modules, which are triangulated subcategories of $D^b(A)$, and both are thick (= épaisse; see [V1]). Then one has the quotient triangulated categories

$$\mathcal{D}_I(A) := D^b(A)/K^b(A\text{-inj}) \quad \text{and} \quad \mathcal{D}_P(A) := D^b(A)/K^b(A\text{-proj}).$$

(They are also called the singularity categories, see e.g. [O1]; or the stable derived categories, see e.g. [Kr]). Since $\mathcal{D}_I(A) = 0$ (resp. $\mathcal{D}_P(A) = 0$) if and only if gl.dimA $< \infty$, it follows that for an algebra A of infinite global dimension it is of interest to study $\mathcal{D}_I(A)$ and $\mathcal{D}_P(A)$. A beautiful result in this direction has been obtained by Happel ([Hap2], Theorem 4.6) for Gorenstein algebras, which generalizes an earlier result of Rickard for self-injective algebras ([Ric2], Theorem 2.1).

In the same way, for an algebraic variety X one has the quotient triangulated category $\mathbf{D}_{Sg}(X) := \mathbf{D}^b(coh(X))/\mathsf{perf}(X)$, where $\mathbf{D}^b(coh(X))$ is the bounded derived category of coherent sheaves on X, and $\mathsf{perf}(X)$ is its full subcategory of perfect complexes. Note that $\mathbf{D}_{Sg}(X) = 0$ if and only if X is smooth. Thus, for a singular variety X it is of interest to study $\mathbf{D}_{Sg}(X)$. Similarly, $\mathbf{D}_{Sg}(X)$ for X being Gorenstein has been studied by Orlov ([O1], also in [O2] for the graded case). See also a recent work of Krause [Kr].

1.2. For a self-orthogonal A-module T (i.e., $\mathrm{Ext}^i_A(T,T) = 0$ for each $i \geq 1$), let add$T$ denote the full subcategory of A-mod whose objects are the direct summands of finite direct sum of copies of T. Then $K^b(\mathrm{add}T)$ is a triangulated subcategory of $D^b(A)$ (see [Hap1], p.103, or 1.9 below). Since both $K^b(\mathrm{add}T)$ and $D^b(A)$ are Krull-Schmidt categories (see

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[KV], or [BD]), i.e., each object can be uniquely decomposed into a direct sum of (finitely many) indecomposables, it follows that $K^b(\mathrm{add}T)$ is closed under direct summands, that is, $K^b(\mathrm{add}T)$ is thick in $D^b(A)$ (see [V2], or Proposition 1.3 in [Ric2]), and hence one has the quotient triangulated category

$$\mathcal{D}_T(A) := D^b(A)/K^b(\mathrm{add}T).$$

In the view of the tilting theory (see e.g. [APR], [BB], [HR1], [HR2], [B], [Rin1], [Hap1], [AR1], [M]), $\mathcal{D}_I(A)$ (resp., $\mathcal{D}_P(A)$) is just the special case of $\mathcal{D}_T(A)$ when $T = {}_AD(A_A)$, or any generalized cotilting module (cf. 1.9 below) (resp., $T = {}_AA$, or any generalized tilting module). This encourages us to understand $\mathcal{D}_I(A)$ and $\mathcal{D}_P(A)$ in terms of generalized cotilting and tilting modules, respectively, and $\mathcal{D}_T(A)$ for self-orthogonal modules T, in general. If $(A, {}_AT_B, B)$ is a generalized (co)tilting triple, then by a theorem due to Happel ([Hap1], Theorem 2.10, p.109, for finite global dimension case), and due to Cline, Parshall, and Scott ([CPS], Theorem 2.1, for general case. See also Rickard [Ric1], Theorem 6.4), this permits us to understand the singularity category $\mathcal{D}_P(B)$ in terms of ${}_AT$.

1.3. On the other hand, Auslander and Reiten ([AR1]) have introduced several interesting full subcategories of A-mod, and established fundamental relations between them. These turn out to arise naturally and surprisingly in the investigation of $\mathcal{D}_T(A)$. For a self-orthogonal module T, by using ${}^\perp T$, $T^\perp$, $\underbrace{\mathrm{add}T}$, $\overbrace{\mathrm{add}T}$, $\mathcal{X}_T$, ${}_T\mathcal{X}$, $\underset{\vee}{\mathrm{add}}T$, $\widehat{\mathrm{add}T}$, introduced in [AR1], and their stable categories modulo $\mathrm{add}T$, and calculus on right fractions, we obtain in Section 2 some full subcategories of $\mathcal{D}_T(A)$ (Theorems 2.1 and 2.4). We also obtain some dense functors to $\mathcal{D}_T(A)$ (Theorems 2.12 and 2.13): the proof need a sufficient condition for "$K^{+,b}(\mathrm{add}T) \simeq D^b(A)$" (Proposition 2.11), which seems to be of independent interest. The proof of Proposition 2.11 needs using Theorem 2.1 in [CPS].

These "fully-faithful" and "dense" results measure how large $\mathcal{D}_T(A)$, $\mathcal{D}_I(A)$, and $\mathcal{D}_P(A)$ are. By combining these results, we can describe, in particular for Gorenstein algebras, $\mathcal{D}_I(A) = \mathcal{D}_T(A) = \mathcal{D}_P(A)$ in terms of any generalized cotilting or generalized tilting module T (Theorem 2.16).

As consequences, for a Gorenstein algebra A, the class of generalized cotilting A-modules coincides with the one of generalized tilting A-modules (Corollary 2.10). Also, we have ${}^\perp T \cap T^\perp = \mathrm{add}T$ for any generalized cotilting (= generalized tilting) module T of an algebra of finite global dimension (Corollary 2.19), which is an analogue with one of the properties of the characteristic modules of quasi-hereditary algebras established by Ringel (see [Rin2], Corollary 4).

1.4. Denote by $T(A) := A \oplus D(A)$ the trivial extension of A, where $D = \mathrm{Hom}_k(-, k)$. It is $\mathbb{Z}$-graded with $\deg A = 0$ and $\deg D(A) = 1$. Denote by $T(A)^{\mathbb{Z}}$-mod the category of finite-dimensional $\mathbb{Z}$-graded $T(A)$-modules with morphisms of degree 0. This is a Frobenius abelian category, and hence its stable category $T(A)^{\mathbb{Z}}$-$\underline{\mathrm{mod}}$ modulo projectives is a triangulated category. A theorem of Happel says that there exists a fully-faithful exact functor $F : D^b(A) \longrightarrow T(A)^{\mathbb{Z}}$-$\underline{\mathrm{mod}}$ ([Hap1], p.88, plus p.64); and F is dense if and only gl.dimA $< \infty$ ([Hap3]). There is a natural embedding $i :$ A-mod $\hookrightarrow T(A)^{\mathbb{Z}}$-$\underline{\mathrm{mod}}$ such that each A-module M is a graded $T(A)$-module concentrated at degree 0. Denote by $\mathcal{N}$, $\mathcal{M}_P$, and $\mathcal{M}_I$ the triangulated subcategories of $T(A)^{\mathbb{Z}}$-$\underline{\mathrm{mod}}$ generated by A-mod, A-proj, and A-inj, respectively. Then the first part of Happel's theorem above reads as: there are equivalences of triangulated categories

$$F : D^b(A) \simeq \mathcal{N}, \quad F : K^b(A\text{-inj}) \simeq \mathcal{M}_I, \quad \text{and} \quad F : K^b(A\text{-proj}) \simeq \mathcal{M}_P.$$

Since both $T(A)^{\mathbb{Z}}$-$\underline{\text{mod}}$ and $\mathcal{N}$ are Krull-Schmidt, it follows that $\mathcal{N}$ is closed under direct summands, that is, $\mathcal{N}$ is thick in $T(A)^{\mathbb{Z}}$-$\underline{\text{mod}}$; and so are $\mathcal{M}_P$ and $\mathcal{M}_I$. Again one has the quotient triangulated categories:

$$\mathcal{D}_F(A) := T(A)^{\mathbb{Z}}\text{-}\underline{\text{mod}}/\mathcal{N}, \quad T(A)^{\mathbb{Z}}\text{-}\underline{\text{mod}}/\mathcal{M}_I, \quad \text{and} \quad T(A)^{\mathbb{Z}}\text{-}\underline{\text{mod}}/\mathcal{M}_P.$$

Now the second part of Happel's theorem cited above reads as:

$$\mathcal{D}_F(A) = 0 \text{ if and only if gl.dim } A < \infty.$$

Since $\mathcal{D}_I(A) \simeq \mathcal{N}/\mathcal{M}_I$ and $\mathcal{D}_P(A) \simeq \mathcal{N}/\mathcal{M}_P$, it follows that one has the exact sequences of triangulated categories (in the sense of [Rou], p.23):

$$0 \longrightarrow \mathcal{D}_P(A) \longrightarrow T(A)^{\mathbb{Z}}\text{-}\underline{\text{mod}}/\mathcal{M}_P \longrightarrow \mathcal{D}_F(A) \longrightarrow 0,$$

and

$$0 \longrightarrow \mathcal{D}_I(A) \longrightarrow T(A)^{\mathbb{Z}}\text{-}\underline{\text{mod}}/\mathcal{M}_I \longrightarrow \mathcal{D}_F(A) \longrightarrow 0.$$

It should be of interest to study the quotient triangulated categories

$$\mathcal{D}_F(A), \quad T(A)^{\mathbb{Z}}\text{-}\underline{\text{mod}}/\mathcal{M}_I, \quad \text{and} \quad T(A)^{\mathbb{Z}}\text{-}\underline{\text{mod}}/\mathcal{M}_P,$$

although this is not done in the present paper.

1.5. In Section 3 we describe all the triangulated subcategories of the stable category of a Frobenius exact category (Theorem 3.3). With this description and the $\mathbb{Z}$-graded representations of $T(A)$ ([Hap1]), we can describe explicitly the bounded derived category of a Gorenstein algebra A inside $T(A)^{\mathbb{Z}}$-$\underline{\text{mod}}$ (Theorem 4.1). Also, by an interesting formula in a Frobenius exact category, one can describe the perpendicular of a triangulated subcategory of the stable category of a Frobenius exact category (Proposition 3.4).

1.6. In Section 5 we study the stable category $\underline{\mathfrak{a}(T)}$ of the Frobenius exact category $\mathfrak{a}(T) := \mathcal{X}_T \cap {}_T\mathcal{X}$, where T is a self-orthogonal module. By using an observation for distinguished triangles in the stable category of a Frobenius exact category (Lemma 3.2), the calculus on right fractions, and an analogue of the Comparison-Theorem in homological algebra, we can naturally embed $\underline{\mathfrak{a}(T)}$ into $\mathcal{D}_T(A)$ as a triangulated subcategory (Theorem 5.2); and under a modest condition identify $\underline{\mathfrak{a}(T)}$ with $K^{ac}(T)$ as triangulated categories (Theorem 5.3), where $K^{ac}(T)$ is the triangulated subcategory of the (unbounded) homotopy category $K(A)$ of the acyclic complexes with components in add$T$. In particular, we get another description of the singularity category $\mathcal{D}_P(A)$ for a Gorenstein algebra (Corollary 5.4). This relates Section 5 in [Kr] on stable derived categories.

1.7. Recall that by definition an algebra A is Gorenstein if proj.dim ${}_AD(A_A) < \infty$ and inj.dim ${}_AA < \infty$. Self-injective algebras and algebras of finite global dimensions are such examples; also, the tensor product $A \otimes_k B$ is Gorenstein if and only if so are both A and B ([AR2], Proposition 2.2). Note that A is Gorenstein if and only if $K^b(A\text{-proj}) = K^b(A\text{-inj})$ inside $D^b(A)$ ([Hap2], Lemma 1.5). Thus, by Theorem 6.4 and Proposition 9.1 in [Ric1], if algebras A and B are derived equivalent, then A is Gorenstein if and only if so is B.

1.8. For basic results on triangulated categories and derived categories we refer to [V1] and [Har]. Following [BBD], the n-th shift of an object X in a triangulated category is denoted by $X[n]$. By a functor we always mean a covariant additive functor. A functor $F : \mathcal{C} \longrightarrow \mathcal{D}$ of triangulated categories is said to be exact provided that there is a natural isomorphism $\alpha : F \circ [1] \longrightarrow [1] \circ F$, such that F preserves distinguished triangles. An exact functor which is an equivalence of categories is said to be a triangle-equivalence. Note that the inverse of a triangle-equivalence is again exact.

We refer to [V1] (also [Har] and [I]) for the construction of a quotient triangulated category. We recall some parts needed. By a triangulated subcategory we always mean that it is a full subcategory; and by a multiplicative system of a triangulated category we always mean that it is compatible with the triangulation.

Given a multiplicative system S of a triangulated category $\mathcal{K}$, one has a quotient triangulated category $S^{-1}\mathcal{K}$ via localization, whose triangulation is natural in the sense that it is induced by the one of $\mathcal{K}$, and in which morphisms are given by right fractions (if one uses left fractions as morphisms then one gets a quotient triangulated category isomorphic to $S^{-1}\mathcal{K}$). Note that $S^{-1}\mathcal{K}$ is the unique triangulated category such that the localization functor $Q : \mathcal{K} \longrightarrow S^{-1}\mathcal{K}$ is an exact functor sending morphisms in S to isomorphisms in $S^{-1}\mathcal{K}$, and that any exact functor $\mathcal{K} \longrightarrow \mathcal{C}$ sending morphisms in S to isomorphisms factors uniquely through Q.

If in addition S is saturated, then $Q(f)$ is an isomorphism in $S^{-1}\mathcal{K}$ implies that $f \in S$.

On the other hand, the class of saturated multiplicative systems of $\mathcal{K}$, is in one-to-one correspondence with the class of thick triangulated subcategories of $\mathcal{K}$. It follows that given a thick triangulated subcategory $\mathcal{B}$, we obtain a quotient triangulated category $\mathcal{K}/\mathcal{B} := S^{-1}\mathcal{K}$, where S the unique saturated multiplicative system determined by $\mathcal{B}$. Note that $\mathcal{K}/\mathcal{B}$ is the unique triangulated category such that the localization functor $Q : \mathcal{K} \longrightarrow \mathcal{K}/\mathcal{B}$ is an exact functor, with $Q(B) = 0$ if and only if $B \in \mathcal{B}$, and that any exact functor $\mathcal{K} \longrightarrow \mathcal{C}$ sending objects in $\mathcal{B}$ to zero factor uniquely through Q.

We emphasize that given a triangulated subcategory $\mathcal{B}$ (not necessarily thick) of $\mathcal{K}$, one can obtain in the same way as above a unique multiplicative system (not necessarily saturated), and then a quotient triangulated category $\mathcal{K}/\mathcal{B} := S^{-1}\mathcal{K}$, in which the localization functor Q is an exact functor sending objects in $\mathcal{B}$ to zero in $\mathcal{K}/\mathcal{B}$, but $Q(B) = 0$ does not imply $B \in \mathcal{B}$ (in fact, such a B is only a direct summand of an object in $\mathcal{B}$). However, we remark that a multiplicative system (not necessarily saturated) does not determine in the natural way a triangulated subcategory, in general.

1.9. Let $\mathcal{A}$ be an abelian category, and $\mathcal{B}$ a full subcategory of $\mathcal{A}$. Let $K^b(\mathcal{B})$ be the full subcategory of $K^b(\mathcal{A})$ whose objects are complexes of objects in $\mathcal{B}$, and $\varphi : K^b(\mathcal{B}) \longrightarrow D^b(\mathcal{A})$ be the composition of the embedding $K^b(\mathcal{B}) \hookrightarrow K^b(\mathcal{A})$ and the localization functor $K^b(\mathcal{A}) \longrightarrow D^b(\mathcal{A})$. If φ is fully-faithful, then $K^b(\mathcal{B})$ is a triangulated subcategory of $D^b(\mathcal{A})$, that is, $K^b(\mathcal{B})$ is a full subcategory of $D^b(\mathcal{A})$, which is closed under the shifts $[1]$ and $[-1]$, and if two terms in a distinguished triangle of $D^b(\mathcal{A})$ lie in $K^b(\mathcal{B})$ then the third term is also in $K^b(\mathcal{B})$ (for this we need the assumption that the map $f \longmapsto f/\mathrm{Id}_{B_1^\bullet}$ gives the isomorphism $\mathrm{Hom}_{K^b(\mathcal{B})}(B_1^\bullet, B_2^\bullet) \cong \mathrm{Hom}_{D^b(\mathcal{A})}(B_1^\bullet, B_2^\bullet)$ of abelian groups, for any objects $B_1^\bullet$ and $B_2^\bullet$ in $K^b(\mathcal{B})$).

Apply the paragraph above to the full subcategory add$T$ of A-mod, where T is a self-orthogonal A-module. Then by Lemma 2.1 in [Hap1], p.103, we know that $K^b(\mathrm{add}T)$ is a triangulated subcategory of $D^b(A)$.

If T is a generalized tilting module, then $K^b(\mathrm{add}T) = K^b(A\text{-proj})$ in $D^b(A)$ (in fact, for any projective module P and any $T' \in \mathrm{add}T$, we have $P \in K^b(\mathrm{add}T)$ and $T' \in K^b(A\text{-proj})$, in $D^b(A)$. Then the assertion follows from the fact that $K^b(A\text{-proj})$ and $K^b(\mathrm{add}T)$ are the triangulated subcategories of $D^b(A)$ generated by add$A$ and by add$T$, respectively).

2. Quotient category $D^b(A)/K^b(\mathrm{add}T)$

Throughout, T is a self-orthogonal A-module. We need eight kinds of full subcategories of A-mod, introduced by Auslander and Reiten [AR1]. We emphasize that a subcategory is closed under isomorphisms and finite direct sums. Following Ringel [Rin2], we do not assume that a full subcategory is closed under direct summands.

2.1. Consider the following full subcategories of A-mod given by

$$T^{\perp} := \{X \mid \mathrm{Ext}^i_A(T, X) = 0, \ \forall \ i \geq 1\},$$

$$\overbrace{\mathrm{add}T} := \{X \mid \exists \text{ an exact sequence } \cdots \longrightarrow T^{-i} \longrightarrow \cdots T^0 \longrightarrow X \longrightarrow 0, \ T^{-i} \in \mathrm{add}T, \forall \ i\},$$

$$_T\mathcal{X} := \{X \mid \exists \text{ an exact sequence } \cdots \longrightarrow T^{-i} \xrightarrow{d^{-i}} T^{-(i-1)} \longrightarrow \cdots \xrightarrow{d^{-1}} T^0 \xrightarrow{d^0} X \longrightarrow 0,$$
$$T^{-i} \in \mathrm{add}T, \ \ \mathrm{Ker}d^{-i} \in \ T^{\perp}, \ \forall \ i \geq 0\},$$

and

$$\widehat{\mathrm{add}T} := \{X \mid \exists \text{ an exact sequence } 0 \longrightarrow T^{-n} \longrightarrow \cdots T^0 \longrightarrow X \longrightarrow 0, \ T^{-i} \in \mathrm{add}T, \ \ \forall \ i \ \}.$$

By definition and the dimension-shifting technique in homological algebra, we have (note that T is self-orthogonal)

$$\widehat{\mathrm{add}T} \subseteq \ _T\mathcal{X} \subseteq \overbrace{\mathrm{add}T} \cap T^{\perp}.$$

Note that $_T\mathcal{X} = \overbrace{\mathrm{add}T}$ if and only if $\overbrace{\mathrm{add}T} \subseteq T^{\perp}$.

If T is exceptional (i.e., proj.dim $T < \infty$ and T is self-orthogonal. Compare [HU]), then by the dimension-shifting technique we have $\overbrace{\mathrm{add}T} \subseteq T^{\perp}$, and hence $_T\mathcal{X} = \overbrace{\mathrm{add}T}$.

If T is a generalized tilting module (i.e., T is exceptional, and there is an exact sequence $0 \longrightarrow {}_AA \longrightarrow T^0 \longrightarrow T^1 \longrightarrow \cdots \longrightarrow T^n \longrightarrow 0$ with each $T^i \in \mathrm{add}T$), then $\overbrace{\mathrm{add}T} = T^{\perp}$, and hence $_T\mathcal{X} = \overbrace{\mathrm{add}T} = T^{\perp}$.

(In fact, by the theory of generalized tilting modules, $X \in T^{\perp}$ can be generated by T, see [M], Lemma 1.8; and then by using a classical argument in [HR2], p.408, one can prove $X \in \overbrace{\mathrm{add}T}$ by induction.)

It is not hard to prove that if gl.dim$A < \infty$ then $\widehat{\mathrm{add}T} = \ _T\mathcal{X} = \overbrace{\mathrm{add}T}$ for any self-orthogonal module T.

2.2. Dually, we have the full subcategories of A-mod given by

$$^{\perp}T := \{X \mid \mathrm{Ext}^i_A(X, T) = 0, \ \forall \ i \geq 1\},$$

$$\underbrace{\mathrm{add}T} := \{X \mid \exists \text{ an exact sequence } \ 0 \longrightarrow X \longrightarrow T^0 \cdots \longrightarrow T^i \longrightarrow \cdots, \ T^i \in \mathrm{add}T, \ \forall \ i\},$$

$$\mathcal{X}_T := \{X \mid \exists \text{ an exact sequence } \quad 0 \longrightarrow X \longrightarrow T^0 \xrightarrow{d^0} \cdots \longrightarrow T^i \xrightarrow{d^i} T^{i+1} \longrightarrow \cdots, \\ T^i \in \mathrm{add}T, \ \mathrm{Im}d^i \in {}^\perp T, \ \forall \ i \geq 0 \ \},$$

and

$$\underset{\vee}{\mathrm{add}}T := \{X \mid \exists \text{ an exact sequence } \ 0 \longrightarrow X \longrightarrow T^0 \cdots \longrightarrow T^n \longrightarrow 0, \ T^i \in \mathrm{add}T, \ \forall \ i\}.$$

By definition and the dimension-shifting technique we have

$$\underset{\vee}{\mathrm{add}}T \subseteq \ \mathcal{X}_T \subseteq \underbrace{\mathrm{add}T} \cap {}^\perp T.$$

Note that $\mathcal{X}_T = \underbrace{\mathrm{add}T}$ if and only if $\underbrace{\mathrm{add}T} \subseteq {}^\perp T$.

If T is co-exceptional (i.e., T is self-orthogonal with inj.dim $T < \infty$), then by the dimension-shifting technique we have $\underbrace{\mathrm{add}T} \subseteq {}^\perp T$, and hence $\mathcal{X}_T = \underbrace{\mathrm{add}T}$.

If T is a generalized cotilting module (i.e., T is co-exceptional, and there is an exact sequence

$$0 \longrightarrow T^{-n} \longrightarrow \cdots \longrightarrow T^{-1} \longrightarrow T^0 \longrightarrow D(A_A) \longrightarrow 0$$

with each $T^{-i} \in \mathrm{add}T$), then $\underbrace{\mathrm{add}T} = {}^\perp T$, and hence $\mathcal{X}_T = \underbrace{\mathrm{add}T} = {}^\perp T$ (see Theorem 5.4(b) in [AR1]).

(In fact, for $X \in {}^\perp T$, since $D(T)_A$ is a generalized tilting right A-module, it follows that $D(X)_A \in D(T)^\perp = \overbrace{\mathrm{add}D(T)_A}$; and then $X \in \underbrace{\mathrm{add}T}$.)

It is not hard to prove that if gl.dim$A < \infty$, then $\underset{\vee}{\mathrm{add}}T = \mathcal{X}_T = \underbrace{\mathrm{add}T}$ for any self-orthogonal module T.

2.3. Consider the natural functor $\mathcal{X}_T \cap T^\perp \longrightarrow D^b(A)/K^b(\mathrm{add}T)$, which is the composition of the embeddings $\mathcal{X}_T \cap T^\perp \hookrightarrow A$-mod and A-mod $\hookrightarrow D^b(A)$, and the localization functor $D^b(A) \longrightarrow D^b(A)/K^b(\mathrm{add}T)$. We have

Theorem 2.1. *Let T be a self-orthogonal module, $M \in \mathcal{X}_T$ and $N \in T^\perp$. Then there is a natural isomorphism of vector spaces*

$$\mathrm{Hom}_A(M, N)/T(M, N) \simeq \mathrm{Hom}_{D^b(A)/K^b(\mathrm{add}T)}(M, N),$$

where $T(M, N)$ is the subspace of A-maps from M to N which factor through add$T$.

In particular, the natural functor $\mathcal{X}_T \cap T^\perp \longrightarrow D^b(A)/K^b(\mathrm{add}T)$ induces a fully-faithful functor

$$\overline{\mathcal{X}_T \cap T^\perp} \longrightarrow D^b(A)/K^b(\mathrm{add}T),$$

where $\overline{\mathcal{X}_T \cap T^\perp}$ is the stable category of $\mathcal{X}_T \cap T^\perp$ modulo add$T$.

Proof. In what follows, a doubled arrow means the corresponding morphism belonging to the saturated multiplicative system determined by the thick triangulated subcategory $K^b(\mathrm{add}T)$ of $D^b(A)$ (see [V1], or [Har]).

Morphisms in $D^b(A)/K^b(\mathrm{add}T)$ are denoted by right fractions. Let $M \overset{s}{\Longleftarrow} Z^\bullet \xrightarrow{a} N$ be such a morphism from M to N, where $Z^\bullet \in D^b(A)$. Such a morphism is denoted by a/s. Note that the mapping cone Con(s) of s lies in $K^b(\mathrm{add}T)$. We have a distinguished triangle in $D^b(A)$

$$Z^{\bullet} \overset{s}{\Longrightarrow} M \longrightarrow \mathrm{Con}(s) \longrightarrow Z^{\bullet}[1]. \tag{2.1}$$

Consider the k-map $G : \mathrm{Hom}_A(M, N) \longrightarrow \mathrm{Hom}_{D^b(A)/K^b(\mathrm{add}T)}(M, N)$, given by $G(f) = f/\mathrm{Id}_M$. First, we prove that G is surjective.

By $M \in \mathcal{X}_T$ we have an exact sequence

$$0 \longrightarrow M \xrightarrow{\varepsilon} T^0 \xrightarrow{d^0} T^1 \xrightarrow{d^1} \cdots \longrightarrow T^n \xrightarrow{d^n} \cdots$$

with $\mathrm{Im}d^i \in {}^{\perp}T$, $\forall\ i \geq 0$. Then M is isomorphic in $D^b(A)$ to the complex $T^{\bullet} := 0 \longrightarrow T^0 \longrightarrow T^1 \longrightarrow \cdots$, and then isomorphic to the complex $0 \longrightarrow T^0 \longrightarrow T^1 \longrightarrow \cdots \longrightarrow T^{l-1} \longrightarrow \mathrm{Ker}d^l \longrightarrow 0$ for each $l \geq 1$. The last complex induces a distinguished triangle in $D^b(A)$

$$\sigma^{<l}T^{\bullet}[-1] \longrightarrow \mathrm{Ker}d^l[-l] \overset{s'}{\Longrightarrow} M \xrightarrow{\varepsilon} \sigma^{<l}T^{\bullet}, \tag{2.2}$$

where $\sigma^{<l}T^{\bullet} = 0 \longrightarrow T^0 \longrightarrow T^1 \longrightarrow \cdots \longrightarrow T^{l-1} \longrightarrow 0$, and the mapping cone of s' lies in $K^b(\mathrm{add}T)$. Since $\mathrm{Ker}d^l \in {}^{\perp}T$ and $\mathrm{Con}(s) \in K^b(\mathrm{add}T)$, it follows that there exists $l_0 \gg 0$ such that

$$\mathrm{Hom}_{D^b(A)}(\mathrm{Ker}d^l[-l], \mathrm{Con}(s)) = 0$$

for each $l \geq l_0$.

(In fact, let $\mathrm{Con}(s)$ be of the form $0 \longrightarrow W^{-t'} \longrightarrow \cdots \longrightarrow W^t \longrightarrow 0$ with $t', t \geq 0$, and each $W^i \in \mathrm{add}T$. Consider the distinguished triangle in $D^b(A)$

$$\sigma^{<t}\mathrm{Con}(s)[-1] \longrightarrow W^t[-t] \longrightarrow \mathrm{Con}(s) \longrightarrow \sigma^{<t}\mathrm{Con}(s),$$

Take l_0 to be $t+1$, and apply the functor $\mathrm{Hom}_{D^b(A)}(\mathrm{Ker}d^l[-l], -)$ to this distinguished triangle. Then the assertion follows from $\mathrm{Ker}d^l \in {}^{\perp}T$ and induction.)

Write $E = \mathrm{Ker}d^{l_0}$, and take $l = l_0$ in (2.2). By applying $\mathrm{Hom}_{D^b(A)}(E[-l_0], -)$ to (2.1) we get $h : E[-l_0] \longrightarrow Z^{\bullet}$ such that $s' = s \circ h$. Thus, by the definition of right fractions we have $a/s = (a \circ h)/s'$.

Apply $\mathrm{Hom}_{D^b(A)}(-, N)$ to (2.2), we get an exact sequence

$$\mathrm{Hom}_{D^b(A)}(M, N) \longrightarrow \mathrm{Hom}_{D^b(A)}(E[-l_0], N) \longrightarrow \mathrm{Hom}_{D^b(A)}(\sigma^{<l_0}T^{\bullet}[-1], N).$$

We claim that $\mathrm{Hom}_{D^b(A)}(\sigma^{<l_0}T^{\bullet}[-1], N) = \mathrm{Hom}_{D^b(A)}(\sigma^{<l_0}T^{\bullet}, N[1]) = 0$.

(In fact, apply $\mathrm{Hom}_{D^b(A)}(-, N[1])$ to the following distinguished triangle in $D^b(A)$

$$\sigma^{<l_0-1}T^{\bullet}[-1] \longrightarrow T^{l_0-1}[1-l_0] \longrightarrow \sigma^{<l_0}T^{\bullet} \longrightarrow \sigma^{<l_0-1}T^{\bullet}.$$

Then the assertion follows from induction and the assumption $N \in T^{\perp}$.)

Thus, there exists $f : M \longrightarrow N$ such that $f \circ s' = a \circ h$. Now, again by the definition of right fractions we have $a/s = (a \circ h)/s' = (f \circ s')/s' = f/\mathrm{Id}_M$. This shows that G is surjective.

On the other hand, if $f : M \longrightarrow N$ with $G(f) = f/\mathrm{Id}_M = 0$ in $D^b(A)/K^b(\mathrm{add}T)$, then there exists $s : Z^{\bullet} \Longrightarrow M$ with $\mathrm{Con}(s) \in K^b(\mathrm{add}T)$ such that $f \circ s = 0$. Use the same notation as in (2.1) and (2.2). By the argument above we have $s' = s \circ h$, and hence $f \circ s' = 0$. Therefore, by applying $\mathrm{Hom}_{D^b(A)}(-, N)$ to (2.2) we see that there exists $f' : \sigma^{<l_0}T^{\bullet} \longrightarrow N$ such that $f' \circ \varepsilon = f$.

Consider the following distinguished triangle in $D^b(A)$

$$T^0[-1] \longrightarrow \sigma^{>0}(\sigma^{<l_0})T^{\bullet} \longrightarrow \sigma^{<l_0}T^{\bullet} \xrightarrow{\pi} T^0,$$

where $\sigma^{>0}(\sigma^{<l_0})T^\bullet = 0 \longrightarrow T^1 \longrightarrow T^2 \longrightarrow \cdots \longrightarrow T^{l_0-1} \longrightarrow 0$, and π is the natural morphism. Again since $N \in T^\perp$, it follows that $\mathrm{Hom}_{D^b(A)}(\sigma^{>0}(\sigma^{<l_0})T^\bullet, N) = 0$. By applying $\mathrm{Hom}_{D^b(A)}(-, N)$ to the above triangle we obtain an exact sequence

$$\mathrm{Hom}_{D^b(A)}(T^0, N) \longrightarrow \mathrm{Hom}_{D^b(A)}(\sigma^{<l_0}T^\bullet, N) \longrightarrow 0.$$

It follows that there exists $g : T^0 \longrightarrow N$ such that $g \circ \pi = f'$. Hence $f = g \circ (\pi \circ \varepsilon)$. Since A-mod is a full subcategory of $D^b(A)$, it follows that f factors through T^0 in A-mod. This proves that the kernel of G is $T(M, N)$, which completes the proof. ■

As pointed out in 2.2, if T is a generalized cotilting module then $\mathcal{X}_T = {}^\perp T$, and $K^b(\mathrm{add}T) = K^b(A\text{-inj})$ in $D^b(A)$. It follows from Theorem 2.1 that

Corollary 2.2. *Let T be a generalized cotilting module. Then for $M \in {}^\perp T$ and $N \in T^\perp$, there is a natural isomorphism of vector spaces*

$$\mathrm{Hom}_A(M, N)/T(M, N) \simeq \mathrm{Hom}_{\mathcal{D}_I(A)}(M, N).$$

In particular, the natural functor ${}^\perp T \ \cap \ T^\perp \longrightarrow D^b(A)/K^b(\mathrm{add}T)$ induces a fully-faithful functor

$$\overline{{}^\perp T \ \cap \ T^\perp} \longrightarrow D^b(A)/K^b(\mathrm{add}T) = \mathcal{D}_I(A).$$

By taking $T = {}_AD(A_A)$ in Corollary 2.2 we get

Corollary 2.3. *For $M \in A$-mod and $N \in {}_AD(A_A)^\perp$, there is a natural isomorphism of vector spaces*

$$\mathrm{Hom}_A(M, N)/I(M, N) \simeq \mathrm{Hom}_{\mathcal{D}_I(A)}(M, N),$$

where $I(M, N)$ is the subspace of A-maps from M to N which factor through injective modules.

In particular, the natural functor ${}_AD(A_A)^\perp \longrightarrow \mathcal{D}_I(A)$ induces a fully-faithful functor

$$\overline{{}_AD(A_A)^\perp} \longrightarrow \mathcal{D}_I(A),$$

where $\overline{{}_AD(A_A)^\perp}$ is the stable category of ${}_AD(A_A)^\perp$ modulo injective modules.

By the dual argument with left fractions, we have

Theorem 2.4. *Let T be a self-orthogonal module, $M \in {}^\perp T$ and $N \in {}_T\mathcal{X}$. Then there is a natural isomorphism of vector spaces*

$$\mathrm{Hom}_A(M, N)/T(M, N) \simeq \mathrm{Hom}_{D^b(A)/K^b(\mathrm{add}T)}(M, N).$$

In particular, the natural functor ${}^\perp T \cap {}_T\mathcal{X} \longrightarrow D^b(A)/K^b(\mathrm{add}T)$ induces a fully-faithful functor

$$\underline{{}^\perp T \cap {}_T\mathcal{X}} \longrightarrow D^b(A)/K^b(\mathrm{add}T).$$

where $\underline{{}^\perp T \cap {}_T\mathcal{X}}$ is the the stable category of ${}^\perp T \cap {}_T\mathcal{X}$ modulo $\mathrm{add}T$.

As pointed out in 2.1, if T is a generalized tilting module then ${}_T\mathcal{X} = T^\perp$, and $K^b(\mathrm{add}T) = K^b(A\text{-proj})$. It follows from Theorem 2.4 that

Corollary 2.5. *Let T be a generalized tilting module. Then for $M \in {}^\perp T$ and $N \in T^\perp$, there is a natural isomorphism of vector spaces*

$$\mathrm{Hom}_A(M,N)/T(M,N) \simeq \mathrm{Hom}_{\mathcal{D}_P(A)}(M,N).$$

In particular, the natural functor ${}^\perp T \cap T^\perp \longrightarrow D^b(A)/K^b(\mathrm{add}T)$ induces a fully-faithful functor

$$\underline{{}^\perp T \cap T^\perp} \longrightarrow D^b(A)/K^b(\mathrm{add}T) = \mathcal{D}_P(A).$$

By taking $T = {}_AA$ in Corollary 2.5 we get

Corollary 2.6. *([O2], Proposition 1.10) For $M \in {}^\perp A$ and $N \in A$-mod, there is a natural isomorphism of vector spaces*

$$\mathrm{Hom}_A(M,N)/P(M,N) \simeq \mathrm{Hom}_{\mathcal{D}_P(A)}(M,N),$$

where $P(M,N)$ is the subspace of A-maps from M to N which factor through projective modules.

In particular, the natural functor ${}^\perp A \longrightarrow \mathcal{D}_P(A)$ induces a fully-faithful functor

$$\underline{{}^\perp A} \longrightarrow D^b(A)/K^b(\mathrm{add}T) = \mathcal{D}_P(A),$$

where $\underline{{}^\perp A}$ is the stable category of ${}^\perp A$ modulo projective modules.

Corollary 2.7. *The following are equivalent*

(i) *A is a self-injective algebra;*

(ii) inj.dim ${}_AA < \infty$ *and* ${}^\perp A = {}_AD(A_A)^\perp$*;*

(iii) inj.dim ${}_AA < \infty$ *and* ${}_AA \in {}_AD(A_A)^\perp$*;*

$(ii)'$ proj.dim ${}_AA < \infty$ *and* ${}^\perp A = {}_AD(A_A)^\perp$*;*

$(iii)'$ proj.dim ${}_AA < \infty$ *and* ${}_AD(A_A) \in {}^\perp A$.

Proof. It suffices to prove the implication $(iii) \Longrightarrow (i)$. Since inj.dim${}_AA < \infty$, it follows that ${}_AA$ is zero in $\mathcal{D}_I(A)$; since ${}_AA \in {}_AD(A_A)^\perp$ and $\overline{{}_AD(A_A)^\perp}$ is a full subcategory of $\mathcal{D}_I(A)$ (Corollary 2.3), it follows that ${}_AA$ is zero in $\overline{{}_AD(A_A)^\perp}$, that is, ${}_AA$ is injective. ■

2.4. The following corollary can also be proved directly.

Corollary 2.8. *Let T be a self-orthogonal A-module. Then $\underset{\vee}{\mathrm{add}T} \cap T^\perp = \mathrm{add}T$, and ${}^\perp T \cap \widehat{\mathrm{add}T} = \mathrm{add}T$.*

Proof. By duality we only prove the first equality. For $X \in \underset{\vee}{\mathrm{add}T} \cap T^\perp$, there exists an exact sequence $0 \longrightarrow X \longrightarrow T^0 \longrightarrow \cdots \longrightarrow T^n \longrightarrow 0$ with each $T^i \in \mathrm{add}T$. Thus, X is quasi-isomorphic to a bounded complex with components in $\mathrm{add}T$, which implies $X = 0$ in $D^b(A)/K^b(\mathrm{add}T)$. It follows from Theorem 2.1 that $X = 0$ in the stable category $\overline{\mathcal{X}_T \cap T^\perp}$, that is, $X \in \mathrm{add}T$. ■

Lemma 2.9. *Let T be a self-orthogonal A-module. Then*

(i) *T is a generalized cotilting module if and only if $K^b(\mathrm{add}T) = K^b(A\text{-inj})$ in $D^b(A)$.*

(ii) *T is a generalized tilting module if and only if $K^b(\mathrm{add}T) = K^b(A\text{-proj})$ in $D^b(A)$.*

Proof. By duality and 1.9 we only prove the sufficiency of (i). By $T \in K^b(A\text{-inj})$ in $D^b(A)$, we infer that inj.dim $T < \infty$.

By ${}_AD(A_A) \in K^b(\mathrm{add}T)$ in $D^b(A)$, we get a quasi-isomorphism $\varepsilon : T^\bullet \longrightarrow {}_AD(A_A)$ with $T^\bullet \in K^b(\mathrm{add}T)$, where $T^\bullet = 0 \longrightarrow T^{-s} \longrightarrow \cdots \longrightarrow T^{-1} \xrightarrow{d^{-1}} T^0 \xrightarrow{d^0} T^1 \longrightarrow \cdots \longrightarrow T^s \longrightarrow 0$. Then $H^n(T^\bullet) = 0$ for $n \neq 0$, and $\mathrm{Ker}d^0/\mathrm{Im}d^{-1} \simeq {}_AD(A_A)$. Then we have an exact sequence

$$0 \longrightarrow T^{-s} \longrightarrow \cdots \longrightarrow T^{-2} \longrightarrow T^{-1} \xrightarrow{d^{-1}} \mathrm{Ker}d^0 \longrightarrow {}_AD(A_A) \longrightarrow 0.$$

It suffices to show $\mathrm{Ker}d^0 \in \mathrm{add}T$.

First, note that $T^{-1}/\mathrm{Ker}d^{-1} \in \widehat{\mathrm{add}T}$ and thus $T^{-1}/\mathrm{Ker}d^{-1} \in T^\perp$. By seeing the exact sequence $0 \longrightarrow T^{-1}/\mathrm{Ker}d^{-1} \longrightarrow \mathrm{Ker}d^0 \longrightarrow {}_AD(A_A) \longrightarrow 0$ we infer that $\mathrm{Ker}d^0 \in T^\perp$. Secondly, since $0 \longrightarrow \mathrm{Ker}d^0 \longrightarrow T^0 \xrightarrow{d^0} T^1 \longrightarrow \cdots \longrightarrow T^s \longrightarrow 0$ is exact, it follows that $\mathrm{Ker}d^0 \in \underset{\vee}{\mathrm{add}T}$. Therefore by Corollary 2.8 we have $\mathrm{Ker}d^0 \in \underset{\vee}{\mathrm{add}T} \cap T^\perp = \mathrm{add}T$. ■

The following corollary seems to be of independent interest.

Corollary 2.10. *Let A be a Gorenstein algebra and T an A-module. Then T is a generalized cotilting module if and only if T is a generalized tilting module.*

Proof. By Lemma 2.9, T is a generalized cotilting module if and only if T is self-orthogonal and $K^b(\mathrm{add}T) = K^b(A\text{-inj}) = K^b(A\text{-proj})$ in $D^b(A)$ (for the last equality, see [Hap2], Lemma 1.5), if and only if T is a generalized tilting module. ■

2.5. Let T be a self-orthogonal module. Consider the compositions of the following natural functors

$$K^{+,b}(\mathrm{add}T) \hookrightarrow K^{+,b}(A\text{-mod}) \longrightarrow D^+(A), \quad K^{-,b}(\mathrm{add}T) \hookrightarrow K^{-,b}(A\text{-mod}) \longrightarrow D^-(A),$$

where $K^{+,b}(\mathrm{add}T)$ (resp. $K^{-,b}(\mathrm{add}T)$) is the homotopy category of lower bounded (resp. upper bounded) complexes of modules in $\mathrm{add}T$, with only finitely many non-zero cohomologies; similar for $K^{+,b}(A\text{-mod})$ and $K^{-,b}(A\text{-mod})$; and $D^+(A)$ (resp. $D^-(A)$) is the lower bounded (resp. upper bounded) derived category of A-modules. Note that the images of the composition functors lie in $D^b(A)$, and hence we get the natural functors

$$G^+ : K^{+,b}(\mathrm{add}T) \longrightarrow D^b(A), \quad \text{and} \quad G^- : K^{-,b}(\mathrm{add}T) \longrightarrow D^b(A). \tag{2.3}$$

The following result seems to be well-known and of independent interest. We include a proof by using a theorem due to Cline, Parshall, and Scott in [CPS] (however, if $T = {}_AD(A_A)$ (resp. $T = {}_AA$), then (i) (resp. (ii)) below is well-known).

Proposition 2.11. (i) *If T is a generalized cotilting A-module, then G^+ induces an equivalence $K^{+,b}(\mathrm{add}T) \simeq D^b(A)$ as triangulated categories.*

(ii) If T is a generalized tilting A-module, then G^- induces an equivalence $K^{-,b}(\mathrm{add}T) \simeq D^b(A)$ as triangulated categories.

Proof. By duality we only prove (ii). Let $B = (\mathrm{End}_A(T))^{op}$. Then T is a right B-module. Identify $D^b(A)$ with $K^{-,b}(B\text{-proj})$. For any complex $P^\bullet = (P^n, d^n) \in K^{-,b}(B\text{-proj})$, without loss of generality, we may assume that $H^i(P^\bullet) = 0$ for $i \leq 0$. Set $E := \mathrm{Im}d^0$ and proj.dim.$T = r < \infty$. Then we have the exact sequence

$$\cdots \longrightarrow P^{-n} \longrightarrow \cdots \longrightarrow P^0 \longrightarrow E \longrightarrow 0.$$

Then for $j \geq 1$ and $s \geq r$ we have

$$\mathrm{Tor}_j^B(T_B, \mathrm{Ker}d^{-s}) = \mathrm{Tor}_{j+1}^B(T_B, \mathrm{Ker}d^{-(s-1)}) = \cdots = \mathrm{Tor}_{j+r}^B(T_B, \mathrm{Ker}d^{-(s-r)}) = 0.$$

This implies that the complex in $K^-(\mathrm{add}T)$

$$\cdots \longrightarrow T \otimes_B P^{-n} \longrightarrow \cdots \longrightarrow T \otimes_B P^{-(r+2)} \longrightarrow T \otimes_B P^{-(r+1)}$$

is acyclic. It follows that $T \otimes_B P^\bullet \in K^{-,b}(\mathrm{add}T)$.

Now, by Theorem 2.1 in [CPS], the left derived functor $\mathbf{L}^{-,b}(T \otimes_B -): D^b(B) \longrightarrow D^b(A)$ is a triangle-equivalence. While $\mathbf{L}^{-,b}(T \otimes_B -)$ acts on $P^\bullet$ term by term (see e.g. [Har], Theorem 5.1), it follows from the argument above

$$\mathbf{L}^{-,b}(T \otimes_B -)(P^\bullet) = T \otimes_B P^\bullet \in K^{-,b}(\mathrm{add}T),$$

and hence the assertion follows from Cline-Parshall-Scott's theorem (Theorem 2.1 in [CPS]). ■

2.6. Let T be a self-orthogonal module. By Theorem 2.1, it is of interest to know when the natural functor $\mathcal{X}_T \cap T^\perp \longrightarrow D^b(A)/K^b(\mathrm{add}T)$ is dense. However, we only have

Theorem 2.12. *Assume that* inj.dim ${}_AA < \infty$. *Let T be a generalized cotilting A-module. Then the natural functor*

$${}^\perp T \longrightarrow D^b(A)/K^b(\mathrm{add}T) = \mathcal{D}_I(A)$$

is dense.

Moreover, if in addition A is Gorenstein, then the natural functor

$${}^\perp T \cap T^\perp \longrightarrow D^b(A)/K^b(\mathrm{add}T) = \mathcal{D}_I(A)$$

is dense.

Proof. Identify $D^b(A)$ with $K^{-,b}(A\text{-proj})$, the homotopy category of upper bounded complexes of projective A-modules, with only finitely many non-zero cohomologies. Thus any object in $D^b(A)/K^b(\mathrm{add}T)$ is assumed to be a upper bounded complex $P^\bullet = (P^n, d^n)$ of projective modules with $H^n(P^\bullet) = 0$ for $n \leq -l_0$, and $P^m = 0$ for $m \geq 1$. Write $E = \mathrm{Ker}d^{-l_0+1}$. Then $P^\bullet$ is quasi-isomorphic to

$$0 \longrightarrow E \hookrightarrow P^{-l_0+1} \stackrel{d^{-l_0+1}}{\longrightarrow} P^{-l_0+2} \longrightarrow \cdots \longrightarrow P^0 \longrightarrow 0.$$

We claim that $P^\bullet \simeq E[l_0]$ in $\mathcal{D}_I(A)$.

(In fact, we have the following distinguished triangle in $D^b(A)$

$$E[l_0 - 1] \longrightarrow \sigma^{>-l_0} P^\bullet \longrightarrow P^\bullet \longrightarrow E[l_0]$$

where $\sigma^{>-l_0} P^\bullet = 0 \longrightarrow P^{-l_0+1} \longrightarrow \cdots \longrightarrow P^{-1} \longrightarrow P^0 \longrightarrow 0$. Since inj.dim ${}_AA < \infty$, it follows that $K^b(A\text{-proj}) \subseteq K^b(A\text{-inj})$ (see [Hap2], Lemma 1.5), and hence $\sigma^{>-l_0} P^\bullet = 0$ in $\mathcal{D}_I(A)$. It follows the claim.)

Now by Proposition 2.11 there exists a lower bounded complex $T^\bullet = (T^n, \partial^n)$ with each $T^n \in \mathrm{add}T$ and $H^j(T^\bullet) = 0$ for $j \geq t$, such that E is isomorphic to $G^+(T^\bullet)$ in $D^b(A)$, where the functor G^+ is given in (2.3). Note that in $D^b(A)$ we have

$$G^+(T^\bullet) = 0 \longrightarrow \cdots \longrightarrow T^{-1} \longrightarrow T^0 \longrightarrow \cdots \longrightarrow T^{t-1} \twoheadrightarrow \mathrm{Ker}\partial^t \longrightarrow 0.$$

Since two isomorphic complexes in $D^b(A)$ have the same cohomologies in each degree, it follows that $H^n(G^+(T^\bullet)) = 0$ for $n \neq 0$, and $H^0(G^+(T^\bullet)) = E$. Thus we have a short exact sequence of A-modules

$$0 \longrightarrow \mathrm{Im}\partial^{-1} \hookrightarrow \mathrm{Ker}\partial^0 \longrightarrow E \longrightarrow 0,$$

which induces a distinguished triangle in $D^b(A)$

$$\mathrm{Im}\partial^{-1} \longrightarrow \mathrm{Ker}\partial^0 \longrightarrow E \longrightarrow \mathrm{Im}\partial^{-1}[1].$$

Note that $\mathrm{Im}\partial^{-1} \in K^b(\mathrm{add}T)$ in $D^b(A)$ (in fact, the complex $0 \longrightarrow T^{-s} \longrightarrow \cdots \longrightarrow T^{-1} \longrightarrow 0$ for some s is quasi-isomorphic to $\mathrm{Im}\partial^{-1}$). Thus we get $E \simeq \mathrm{Ker}\partial^0$ in $D^b(A)/K^b(\mathrm{add}T) = \mathcal{D}_I(A)$.

Consider the following exact sequence of A-modules (note that one can take $t \geq l_0$)

$$0 \longrightarrow \mathrm{Ker}\partial^0 \longrightarrow T^0 \longrightarrow \cdots \longrightarrow T^{l_0-1} \longrightarrow \mathrm{Ker}\partial^{l_0} \longrightarrow 0, \tag{2.4}$$

which induces a distinguished triangle in $D^b(A)$

$$\mathrm{Ker}\partial^{l_0}[-l_0] \longrightarrow \mathrm{Ker}\partial^0 \longrightarrow T'^{\bullet} \longrightarrow \mathrm{Ker}\partial^{l_0}[-l_0+1]$$

where $T'^{\bullet} = 0 \longrightarrow T^0 \longrightarrow \cdots \longrightarrow T^{l_0-1} \longrightarrow 0$. Again since $T'^{\bullet} \simeq 0$ in $D^b(A)/K^b(\mathrm{add}T)$, it follows that $\mathrm{Ker}\partial^0 \simeq \mathrm{Ker}\partial^{l_0}[-l_0]$, and then $P^{\bullet} \simeq \mathrm{Ker}\partial^{l_0}$ in $D^b(A)/K^b(\mathrm{add}T) = \mathcal{D}_I(A)$. Since $\mathrm{Ker}\partial^{l_0} \in \underbrace{\mathrm{add}T} = {}^{\perp}T$, it follows the first statement.

If A is Gorenstein, then proj.dim $T = r < \infty$. We may assume that $l_0 \geq r$ in the argument above. Since the sequence (2.4) is exact and T is self-orthogonal, it follows that

$$\mathrm{Ext}_A^i(T, \mathrm{Ker}\partial^{l_0}) \simeq \mathrm{Ext}_A^{i+1}(T, \mathrm{Ker}\partial^{l_0-1}) \simeq \cdots \simeq \mathrm{Ext}_A^{i+l_0}(T, \mathrm{Ker}\partial^0) = 0, \quad i \geq 1,$$

which completes the proof. ■

By the dual argument we have

Theorem 2.13. *Assume that* proj.dim $D(A_A) < \infty$. *Let T be a generalized tilting module. Then the natural functor*

$$T^{\perp} \longrightarrow D^b(A)/K^b(\mathrm{add}T) = \mathcal{D}_P(A)$$

is dense.

Moreover, if in addition A is Gorenstein, then the natural functor

$${}^{\perp}T \cap T^{\perp} \longrightarrow D^b(A)/K^b(\mathrm{add}T) = \mathcal{D}_P(A)$$

is dense.

By taking $T = {}_AD(A_A)$ in Theorem 2.12, and taking $T = {}_AA$ in Theorem 2.13, we get

Corollary 2.14. *([Hap2], Lemma 4.3)* (i) *Assume that* inj.dim ${}_AA < \infty$. *Then the natural functor A-mod $\longrightarrow \mathcal{D}_I(A)$ is dense.*

(ii) *Assume that* proj.dim $D(A_A) < \infty$. *Then the natural functor A-mod $\longrightarrow \mathcal{D}_P(A)$ is dense.*

(iii) *If A is Gorenstein, then both the natural functors*

$${}_AD(A_A)^{\perp} \longrightarrow \mathcal{D}_I(A) = \mathcal{D}_P(A) \quad \textit{and} \quad {}^{\perp}A \longrightarrow \mathcal{D}_P(A) = \mathcal{D}_I(A)$$

are dense.

By combining Corollaries 2.3, 2.14(iii), and 2.6, we get

Corollary 2.15. *([Hap 2], Theorem 4.6) Let A be Gorenstein. Then the natural functor induces equivalences of categories*

$$\overline{{}_AD(A_A)^{\perp}} \simeq \mathcal{D}_I(A) = \mathcal{D}_P(A) \simeq \underline{{}^{\perp}A},$$

where $\overline{{}_AD(A_A)^{\perp}}$ is the stable category of $\{X \in A\text{-mod} \mid \mathrm{Ext}^i_A({}_AD(A_A), X) = 0,\ \forall\ i \geq 1\}$ modulo injective modules; and $\underline{{}^{\perp}A}$ is the stable category of $\{X \in A\text{-mod} \mid \mathrm{Ext}^i_A(X, A) = 0,\ \forall\ i \geq 1\}$ modulo projective modules.

By combining Theorems 2.1 and 2.12, Theorems 2.4 and 2.13, we get

Theorem 2.16. *Let A be Gorenstein, and T be a generalized cotilting module (= a generalized tilting module). Then the natural functors induce an equivalences of categories*

$$\underline{{}^{\perp}T\ \cap T^{\perp}} \simeq \mathcal{D}_I(A) = \mathcal{D}_P(A).$$

We point out that the equivalence in the theorem above is in fact a triangle-equivalence. See Theorem 5.2.

2.7. We have more "dense" type results.

Theorem 2.17. *Assume that* inj.dim ${}_AA < \infty$. *Let T be a generalized tilting module. Then the natural functor ${}^{\perp}T \cap T^{\perp} \longrightarrow \mathcal{D}_I(A)$ is dense.*

In particular, the natural functor ${}^{\perp}A \longrightarrow \mathcal{D}_I(A)$ is dense.

Proof. Set $t :=$ proj.dim T. Since T is a generalized tilting module and inj.dim ${}_AA < \infty$, it follows that

$$K^b(\mathrm{add}T) = K^b(A\text{-proj}) \subseteq K^b(A\text{-inj}),$$

and hence inj.dim $T = s < \infty$.

Identify $D^b(A)$ with $K^{+,b}(A\text{-inj})$. For any object $I^\bullet$ in $\mathcal{D}_I(A)$, without loss of generality, we may assume that

$$I^\bullet = 0 \longrightarrow I^0 \longrightarrow \cdots \longrightarrow I^{l-1} \longrightarrow I^l \xrightarrow{d^l} \cdots \longrightarrow I^{l+r-1} \longrightarrow I^{l+r} \xrightarrow{d^{l+r}} \cdots$$

with $H^n(I^\bullet) = 0$ for $n \geq l$. Set $E := \mathrm{Ker}d^{l+r}$ and $X := \mathrm{Ker}d^l$. Then the complex $0 \longrightarrow I^0 \longrightarrow \cdots \longrightarrow I^{l-1} \longrightarrow I^l \xrightarrow{d^l} \cdots \longrightarrow I^{l+r-1} \longrightarrow E \longrightarrow 0$ is quasi-isomorphic to $I^\bullet$, and hence $I^\bullet \simeq E[-(l+r)]$ in $\mathcal{D}_I(A)$.

We take $r \geq s, t$. By the exact sequence of A-modules

$$0 \longrightarrow X \longrightarrow I^l \longrightarrow \cdots \longrightarrow I^{l+r-1} \longrightarrow E \longrightarrow 0,$$

and proj.dim $T = t < \infty$ and $r \geq t$, we infer that $E \in T^{\perp}$.

By the generalized tilting theory we have $T^{\perp} = \widehat{\mathrm{add}T} = {}_T\mathcal{X}$, and hence we have an exact sequence of A-module

$$0 \longrightarrow W \longrightarrow T^{-(l+r-1)} \longrightarrow \cdots \longrightarrow T^0 \longrightarrow E \longrightarrow 0 \tag{2.5}$$

with each $T^i \in \mathrm{add}T$ and $W \in T^{\perp}$. Since $K^b(\mathrm{add}T) \subseteq K^b(A\text{-inj})$, it follows that the complex $0 \longrightarrow T^{-(l+r-1)} \longrightarrow \cdots \longrightarrow T^0 \longrightarrow 0$ is in $K^b(A\text{-inj})$, and hence $E = W[l+r]$ in $\mathcal{D}_I(A)$. Since T is self-orthogonal with inj.dim $T = s$ and $r \geq s$, by (2.5) we infer that $W \in {}^{\perp}T$. Thus $X = W$ in $\mathcal{D}_I(A)$ with $I^\bullet = W \in {}^{\perp}T \cap T^{\perp}$. ■

By the dual argument we have

Theorem 2.18. *Assume that* proj.dim $_D(A_A) < \infty$. *Let T be a generalized cotilting module. Then the natural functor* $^{\perp}T \cap T^{\perp} \longrightarrow \mathcal{D}_P(A)$ *is dense.*

In particular, the natural functor $_AD(A_A)^{\perp} \longrightarrow \mathcal{D}_P(A)$ *is dense.*

Corollary 2.19. *The following are equivalent*

(i) gl.dimA $< \infty$;

(ii) inj.dim $_AX < \infty$ *for any* $X \in A$*-mod;*

(iii) $\mathcal{D}_I(A) = 0$;

(iv) inj.dim $_AA < \infty$, *and* $^{\perp}T \cap T^{\perp} = \mathrm{add}T$ *for any generalized tilting module;*

$(ii)'$ proj.dim $_AX < \infty$ *for any* $X \in A$*-mod;*

$(iii)'$ $\mathcal{D}_P(A) = 0$;

$(iv)'$ proj.dim $_AD(A_A) < \infty$, *and* $^{\perp}T \cap T^{\perp} = \mathrm{add}T$ *for any generalized cotilting module.*

Proof. The equivalences of $(i), (ii)$, and (iii) are well-known. The implication of $(i) \Longrightarrow (iv)$ follows from Theorem 2.16; and the implication of $(iv) \Longrightarrow (iii)$ follows from Theorem 2.17, since $K^b(\mathrm{add}T) = K^b(A\text{-proj}) \subseteq K^b(A\text{-inj})$. ■

Let A be an algebra of finite global dimension. The corollary above implies that for any generalized cotilting A-module T (= a generalized tilting A-module, by Corollart 2.10), one has $^{\perp}T \cap T^{\perp} = \mathrm{add}T$, which is an analogy with one of the properties of the characteristic modules over quasi-hereditary algebras established by Ringel (see [Rin2], Corollary 4).

3. Triangulated subcategories of a stable category

3.1. Let $\mathcal{A}$ be a Frobenius exact category, that is, $\mathcal{A}$ is a full subcategory of an abelian category, which is closed under extensions and direct summands; and in which there are enough (relatively) injective objects and (relatively) projective objects, such that the injective objects coincide with the projective objects. For the reason requiring that $\mathcal{A}$ is closed under direct summands see Lemma 3.1 below. Compare p.10 in [Hap1], Appendix A in [K], or [Q]. Denote by $\underline{\mathcal{A}}$ its stable category: the objects of $\underline{\mathcal{A}}$ are exactly the ones of $\mathcal{A}$, and the morphism set $\mathrm{Hom}_{\underline{\mathcal{A}}}(X, Y)$ is the quotient group $\mathrm{Hom}_{\mathcal{A}}(X, Y)/I(X, Y)$, where $I(X, Y)$ is the subgroup of the morphisms from X to Y which factor through injective objects. For a morphism $u: X \longrightarrow Y$ in $\mathcal{A}$, denote its image in $\underline{\mathcal{A}}$ by $\underline{u}$.

We need the following well-known fact. For convenience we include a proof.

Lemma 3.1. *Let $\mathcal{A}$ be a Frobenius exact category. Then $X \simeq Y$ in $\underline{\mathcal{A}}$ if and only if there are injective objects I and J such that $X \oplus J \simeq Y \oplus I$ in $\mathcal{A}$.*

Proof. Let $\underline{f}: X \longrightarrow Y$ be an isomorphism in $\underline{\mathcal{A}}$. Then there exists $g: Y \longrightarrow X$ such that $gf - \mathrm{Id}_X$ factors through an injective object I. Suppose $a: X \longrightarrow I$ and $b: I \longrightarrow Y$ such that $gf - \mathrm{Id}_X = ba$. Consider $\binom{f}{a}: X \longrightarrow Y \oplus I$ and $(g, -b): Y \oplus I \longrightarrow X$. Then $(g, -b) \circ \binom{f}{a} = \mathrm{Id}_X$, which implies that there exists J in $\mathcal{A}$ (here we need the assumption that $\mathcal{A}$ is closed under direct summands) and $h: X \oplus J \simeq Y \oplus I$, such that $\binom{f}{a} = h\binom{1}{0}$ in $\mathcal{A}$. Thus $\underline{\binom{1}{0}}: X \longrightarrow X \oplus J$ is an isomorphism in $\underline{\mathcal{A}}$, then by an easy matrix calculation we have $\mathrm{Id}_{\underline{J}} = 0$, which implies that Id_J factors through an injective object, and hence J is an injective object. ■

3.2. Let $\mathcal{A}$ be a Frobenius exact category. Recall the triangulated structure in $\underline{\mathcal{A}}$ (for details see [Hap1], Chapter 1, Section 2). The shift functor $[1] : \underline{\mathcal{A}} \longrightarrow \underline{\mathcal{A}}$ is defined such that for each object X in $\mathcal{A}$, there is an exact sequence in $\mathcal{A}$

$$0 \longrightarrow X \xrightarrow{i_X} I(X) \xrightarrow{\pi_X} X[1] \longrightarrow 0, \tag{3.1}$$

where $I(X)$ is an injective object (note that by Lemma 3.1 if $X \simeq Y$ in $\underline{\mathcal{A}}$ then $X[1] \simeq Y[1]$ in $\underline{\mathcal{A}}$; and that as an object in $\underline{\mathcal{A}}$, $X[1]$ does not depend on the choice of $0 \longrightarrow X \xrightarrow{i_X} I(X) \xrightarrow{\pi_X} X[1] \longrightarrow 0$); and for any morphism $u: X \longrightarrow Y$ in $\mathcal{A}$, the standard triangle $X \xrightarrow{\underline{u}} Y \xrightarrow{\underline{v}} C_u \xrightarrow{\underline{w}} X[1]$ in $\underline{\mathcal{A}}$ is defined by the pushout diagram

$$\begin{tikzcd}
0 \arrow[r] & X \arrow[r, "i_X"] \arrow[d, "u"] & I(X) \arrow[r, "\pi_X"] \arrow[d, "u'"] & X[1] \arrow[r] \arrow[d, equal] & 0 \\
0 \arrow[r] & Y \arrow[r, "v"] & C_u \arrow[r, "w"] & X[1] \arrow[r] & 0;
\end{tikzcd}$$

and then the distinguished triangles in $\underline{\mathcal{A}}$ are defined to be the triangles isomorphic to the standard ones.

We need the following fact, which says that the distinguished triangles in $\underline{\mathcal{A}}$ are given by the short exact sequences in $\mathcal{A}$ in some sense. It is partially given in [Hap1], Lemma 2.7, p.22. For convenience we include a proof.

Lemma 3.2. *Given a short exact sequence $0 \longrightarrow X \xrightarrow{u} Y \xrightarrow{v} Z \longrightarrow 0$ in $\mathcal{A}$, then the induced triangle $X \xrightarrow{\underline{u}} Y \xrightarrow{\underline{v}} Z \xrightarrow{-\underline{w}} X[1]$ is a distinguished triangle in $\underline{\mathcal{A}}$, where w is an A-map such that the following diagram is commutative*

$$\begin{tikzcd}
0 \arrow[r] & X \arrow[r, "u"] \arrow[d, equal] & Y \arrow[r, "v"] \arrow[d, dotted, "\sigma"] & Z \arrow[r] \arrow[d, dotted, "w"] & 0 \\
0 \arrow[r] & X \arrow[r, "i_X"] & I(X) \arrow[r, "\pi_X"] & X[1] \arrow[r] & 0.
\end{tikzcd} \tag{3.2}$$

(Note that any two such maps w and w' give the isomorphic triangles.)

Conversely, any distinguished triangle in $\underline{\mathcal{A}}$ is given in this way. That is, given a distinguished triangle $X' \xrightarrow{\underline{u'}} Y' \xrightarrow{\underline{v'}} Z' \xrightarrow{-\underline{w'}} X'[1]$ in $\underline{\mathcal{A}}$, then there is a short exact sequence $0 \longrightarrow X \xrightarrow{u} Y \xrightarrow{v} Z \longrightarrow 0$ in $\mathcal{A}$, such that the induced distinguished triangle $X \xrightarrow{\underline{u}} Y \xrightarrow{\underline{v}} Z \xrightarrow{-\underline{w}} X[1]$ is isomorphic to the given one, where w is an A-map such that (3.2) *is commutative.*

Proof. The pullback square

$$\begin{tikzcd}
Y \arrow[r, "v"] \arrow[d, "\sigma"] & Z \arrow[d, "w"] \\
I(X) \arrow[r, "\pi_X"] & X[1]
\end{tikzcd}$$

induces the second exact column in the following diagram, and then we have the following diagram with exact rows and columns:

$$\begin{tikzcd}
 & 0 \arrow[d] & 0 \arrow[d] & & \\
0 \arrow[r] & X \arrow[r, "u"] \arrow[d, "i_X"'] & Y \arrow[r, "v"] \arrow[d, "\binom{v}{\sigma}"'] & Z \arrow[r] \arrow[d, equal] & 0 \\
0 \arrow[r] & I(X) \arrow[r, "\binom{0}{1}"] \arrow[d, "\pi_X"'] & Z \oplus I(X) \arrow[r, "(1\ 0)"] \arrow[d, "(-w\ \pi_X)"'] & Z \arrow[r] & 0 \\
 & X[1] \arrow[r, equal] \arrow[d] & X[1] \arrow[d] & & \\
 & 0 & 0. & &
\end{tikzcd}$$

This means that upper left square is a pushout, and then by definition the image in $\underline{\mathcal{A}}$ of the following triangle in $\mathcal{A}$

$$X \xrightarrow{u} Y \xrightarrow{\binom{v}{\sigma}} Z \oplus I(X) \xrightarrow{(-w\ \pi_X)} X[1]$$

is a distinguished triangle in $\underline{\mathcal{A}}$, that is, $X \xrightarrow{\underline{u}} Y \xrightarrow{\underline{v}} Z \xrightarrow{-\underline{w}} X[1]$ is a distinguished triangle in $\underline{\mathcal{A}}$.

Conversely, by definition a given distinguished triangle is isomorphic to a standard triangle $X \xrightarrow{\underline{u}} Y \xrightarrow{\underline{v}} C_u \xrightarrow{\underline{w}} X[1]$ in $\underline{\mathcal{A}}$, then by construction we have the short exact sequence in $\mathcal{A}$:

$$0 \longrightarrow X \xrightarrow{\binom{u}{i_X}} Y \oplus I(X) \xrightarrow{(v\ -u')} C_u \longrightarrow 0 \tag{3.3}$$

and the commutative diagram

$$\begin{tikzcd}
0 \arrow[r] & X \arrow[r, "\binom{u}{i_X}"] \arrow[d, equal] & Y \oplus I(X) \arrow[r, "(v\ -u')"] \arrow[d, "(01)"] & C_u \arrow[r] \arrow[d, "-w"] & 0 \\
0 \arrow[r] & X \arrow[r, "i_X"] & I(X) \arrow[r, "\pi_X"] & X[1] \arrow[r] & 0.
\end{tikzcd}$$

This shows that the distinguished triangle induced by (3.3) is isomorphic to the standard triangle. ■

Theorem 3.3. *Let $\mathcal{A}$ be a Frobenius exact category. Then the stablization gives a bijection between the the class of the full subcategories $\mathcal{B}$ of $\mathcal{A}$, where $\mathcal{B}$ contains all the injective objects of $\mathcal{A}$, such that if two terms in a short exact sequence in $\mathcal{A}$ lie in $\mathcal{B}$, then the third term also lies in $\mathcal{B}$, and the class of triangulated subcategories of $\underline{\mathcal{A}}$.*

Proof. If $\mathcal{B}$ is such a full subcategory of $\mathcal{A}$, then by Lemmas 3.2 and 3.1 $\underline{\mathcal{B}}$ is a triangulated subcategory of $\underline{\mathcal{A}}$. Conversely, let $\mathcal{D}$ be a triangulated subcategory of $\underline{\mathcal{A}}$. Set

$$\mathcal{B} := \{X \in \mathcal{A} \mid \text{there exists } Y \in \mathcal{D} \text{ such that } X \simeq Y \text{ in } \underline{\mathcal{A}}\}.$$

Then $\mathcal{D} = \underline{\mathcal{B}}$. Since $\mathcal{D}$ contains zero object, it follows that $\mathcal{B}$ contains all the injective objects of $\mathcal{A}$; and by Lemma 3.1 $\mathcal{B}$ has the required property. If $\mathcal{B}$ and $\mathcal{B}'$ are two such a different full subcategories of $\mathcal{A}$, then by Lemma 3.1 $\underline{\mathcal{B}}$ are $\underline{\mathcal{B}'}$ also different in $\underline{\mathcal{A}}$. ■

3.3. Let $\mathcal{C}$ be a triangulated subcategory of a triangulated category $\mathcal{D}$. Recall the right perpendicular to $\mathcal{C}$ is a full subcategory of $\mathcal{D}$ given by (see e.g. [BK]; also [Rou], p.23)

$$\mathcal{C}_{\mathcal{D}}^{\perp} := \{M \in \mathcal{D} \mid \mathrm{Hom}_{\mathcal{D}}(N, M) = 0, \ \forall \ N \in \mathcal{C}\}.$$

Then $\mathcal{C}^{\perp}$ is also a triangulated subcategory of $\mathcal{D}$. Note that for $N \in \mathcal{C}_{\mathcal{D}}^{\perp}$ and any object M in $\mathcal{D}$ we have a natural isomorphism $\mathrm{Hom}_{\mathcal{D}}(M, N) \simeq \mathrm{Hom}_{\mathcal{D}/\mathcal{C}}(M, N)$ (one can proves this routinely by calculus on right fractions). In particular, the natural functor induces a fully-faithful functor $\mathcal{C}_{\mathcal{D}}^{\perp} \longrightarrow \mathcal{D}/\mathcal{C}$. If in addition this fully-faithful functor is an equivalence of categories, then $\mathcal{C}$ is called a Bousfield subcategory of $\mathcal{D}$. Note that $\mathcal{C}$ is a Bousfield subcategory if and only if the embedding functor $\mathcal{C} \hookrightarrow \mathcal{D}$ has a right adjoint, if and only if for any object D in $\mathcal{D}$ there exists a distinguished triangle $C \longrightarrow D \longrightarrow C^{\perp} \longrightarrow C[1]$ with $C \in \mathcal{C}$ and $C^{\perp} \in \mathcal{C}_{\mathcal{D}}^{\perp}$, if and only if the localization functor $\mathcal{D} \longrightarrow \mathcal{D}/\mathcal{C}$ has a right adjoint. See [N], or 5.2 in [Rou], or 1.1 in [O2].

Proposition 3.4. *Let $\mathcal{A}$ be a Frobenius exact category. Then*

(i) For any objects X, Y in $\mathcal{A}$ and $n \geq 1$ we have

$$\mathrm{Hom}_{\underline{\mathcal{A}}}(X, Y[n]) = \mathrm{Ext}^n_{\mathcal{A}}(X, Y), \tag{3.4}$$

where $Y[n]$ is the n-th shift of Y in $\underline{\mathcal{A}}$ (cf. (3.1)).

(ii) Let $\underline{\mathcal{B}}$ be a triangulated subcategory of $\underline{\mathcal{A}}$. Then we have

$$\begin{aligned} \underline{\mathcal{B}}_{\underline{\mathcal{A}}}^{\perp} &= \{M \in \underline{\mathcal{A}} \mid \mathrm{Ext}^i_{\mathcal{A}}(N, M) = 0, \ \forall \ N \in \mathcal{B}, \ \forall \ i \geq 1\} \\ &= \{M \in \underline{\mathcal{A}} \mid \mathrm{Ext}^1_{\mathcal{A}}(N, M) = 0, \ \forall \ N \in \mathcal{B}\}. \end{aligned} \tag{3.5}$$

Proof. (i) By the definition of $\mathrm{Ext}^n_{\mathcal{A}}$ and using the injective objects being also projective objects in a Frobenius category, one can easily get the formula (3.4).
(ii) This follows from (3.4). ■

One may write out the formula (3.4), in particular for the category of complexes of an abelian category, and for the module category of a self-injective algebra.

4. Bounded derived categories of Gorenstein algebras

4.1. Keep the notation in 1.4 throughout this section, in particular for $\mathcal{N}$ and $\mathcal{M}_P$. Note that $T(A)^{\mathbb{Z}}$-mod is a Krull-Schmidt, Frobenius abelian category (see [Hap1], I. 3.1, II. 2.2, II. 2.4), so we can freely apply results in Section 3. An object in $T(A)^{\mathbb{Z}}$-mod and $T(A)^{\mathbb{Z}}$-$\underline{\text{mod}}$ is denoted by $M = \oplus_{n\in\mathbb{Z}} M_n$ with each M_n an A-module and $D(A).M_n \subseteq M_{n+1}$.

Theorem 4.1. *Let A be a Gorenstein algebra. Then under Happel's functor $F: D^b(A) \longrightarrow T(A)^{\mathbb{Z}}$-$\underline{\text{mod}}$ we have*

$$D^b(A) \simeq \mathcal{N} = \{\oplus_{n\in\mathbb{Z}} M_n \in T(A)^{\mathbb{Z}}\text{-}\underline{\mathrm{mod}} \mid \text{proj.dim}\ {}_A M_n < \infty, \ \forall \ n \neq 0\}$$

and

$$K^b(A\text{-proj}) \simeq \mathcal{M}_P = \{\oplus_{n\in\mathbb{Z}} M_n \in T(A)^{\mathbb{Z}}\text{-}\underline{\mathrm{mod}} \mid \text{proj.dim}\ {}_A M_n < \infty, \ \forall \ n \in \mathbb{Z}\}.$$

Corollary 4.2. *Let A be a self-injective algebra. Then we have*

$$D^b(A) \simeq \mathcal{N} = \{M = \oplus_{n\in\mathbb{Z}} M_n \mid {}_A M_n \text{ is projecive}, \ \forall \ n \neq 0\}$$

and

$$K^b(A\text{-proj}) \simeq \mathcal{M}_P = \{M = \oplus_{n\in\mathbb{Z}} M_n \mid {}_AM_n \textit{ is projective}, \ \forall\ n \in \mathbb{Z}\}.$$

4.2. Before proving Theorem 4.1 we need some preparations. For each $n \in \mathbb{Z}$, given an indecomposable projective A-module P, then the $\mathbb{Z}$-graded $T(A)$-module

$$\mathrm{proj}(P, n, n+1) = \oplus_{i\in\mathbb{Z}} M_i \ \text{ with } \ M_i = \begin{cases} P, & i = n; \\ D(A) \otimes_A P, & i = n+1; \\ 0, & \text{otherwise.} \end{cases} \tag{4.1}$$

is an indecomposable projective $\mathbb{Z}$-graded $T(A)$-module, and any indecomposable projective $\mathbb{Z}$-graded $T(A)$-module is of this form; and given an indecomposable injective A-module I, then the $\mathbb{Z}$-graded $T(A)$-module

$$\mathrm{inj}(I, n-1, n) = \oplus_{i\in\mathbb{Z}} M_i \ \text{ with } \ M_i = \begin{cases} \mathrm{Hom}_A(D(A), I), & i = n-1; \\ I, & i = n; \\ 0, & \text{otherwise.} \end{cases} \tag{4.2}$$

is an indecomposable injective $\mathbb{Z}$-graded $T(A)$-module; and any indecomposable injective $\mathbb{Z}$-graded $T(A)$-module is of this form. Note that

$$\mathrm{proj}(P, n, n+1) \simeq \mathrm{inj}(D(A) \otimes_A P, n, n+1)$$

and

$$\mathrm{inj}(I, n-1, n) \simeq \mathrm{proj}(\mathrm{Hom}_A(D(A), I), n-1, n).$$

Any homogeneous $\mathbb{Z}$-graded $T(A)$-module $M = M_n$ of degree n has the injective hull $\mathrm{inj}(I_A(M_n), n-1, n)$, and the projective cover $\mathrm{proj}(P_A(M_n), n, n+1)$, where $I_A(M_n)$ and $P_A(M_n)$ are respectively the injective hull and the projective cover of M_n as an A-module (see [Hap1], II. 4.1).

Lemma 4.3. *Let A be a Gorenstein algebra. Then the full subcategories given by*

$$\{\oplus_{n\in\mathbb{Z}} M_n \in T(A)^{\mathbb{Z}}\text{-}\underline{\mathrm{mod}} \mid \mathrm{proj.dim}\ {}_AM_n < \infty, \ \forall\ n \neq 0\}$$

and

$$\{\oplus_{n\in\mathbb{Z}} M_n \in T(A)^{\mathbb{Z}}\text{-}\underline{\mathrm{mod}} \mid \mathrm{proj.dim}\ {}_AM_n < \infty, \ \forall\ n \in \mathbb{Z}\}$$

are triangulated subcategories of $T(A)^{\mathbb{Z}}$-$\underline{\mathrm{mod}}$.

Proof. Since the homogeneous components of any injective object in $T(A)^{\mathbb{Z}}$-mod are direct sums of injective A-modules and projective A-modules, it follows from that A is Gorenstein that the two subcategories above contain all the injective modules in $T(A)^{\mathbb{Z}}$-mod (we need to use the fact: If A is Gorenstein, then for any A-module X, proj.dim $X < \infty$ if and only if inj.dim $X < \infty$).

Given a short exact sequence $0 \longrightarrow M \longrightarrow N \longrightarrow L \longrightarrow 0$ in $T(A)^{\mathbb{Z}}$-mod, then for each n we have an exact sequence of A-modules

$$0 \longrightarrow M_n \longrightarrow N_n \longrightarrow L_n \longrightarrow 0.$$

Note that if any two terms of the short exact sequence above have finite projective dimensions, then the other one also has finite projective dimension. Now the assertion follows from Theorem 3.3. ■

4.3. Proof of Theorem 4.1: We only prove

$$\mathcal{N} = \{\oplus_{n\in\mathbb{Z}} M_n \mid \text{proj.dim}\ {}_AM_n < \infty,\ \forall\ n \neq 0\}. \tag{4.3}$$

The another equality can be similarly proven. Since A is Gorenstein, it follows that

$$\{\oplus_{n\in\mathbb{Z}} M_n \mid \text{proj.dim}\ {}_AM_n < \infty,\ \forall\ n \neq 0\} = \{\oplus_{n\in\mathbb{Z}} M_n \mid \text{inj.dim}\ {}_AM_n < \infty,\ \forall\ n \neq 0\}.$$

By Lemma 4.3 the right hand side in (4.3) is a triangulated subcategory of $T(A)^{\mathbb{Z}}$-$\underline{\text{mod}}$ containing all the A-modules, while by definition $\mathcal{N}$ is the triangulated subcategory of $T(A)^{\mathbb{Z}}$-$\underline{\text{mod}}$ generated by A-mod. It follows that $\mathcal{N} \subseteq \{M = \oplus_{n\in\mathbb{Z}} M_n \mid \text{proj.dim}\ {}_AM_n < \infty,\ \forall\ n \neq 0\}$.

For the other inclusion, first, consider all the objects of the form $M = \oplus_{i\geq 0} M_i$ in the right hand side of (4.3). We claim that such an M lies in $\mathcal{N}$. We prove this claim by induction on $l(M) := max\{\ i \mid M_i \neq 0\}$. Assume that $\mathcal{N}$ already contains all such objects M with $l(M) < n,\ \ n \geq 1$. Now, we use induction on $m := \text{inj.dim}\ {}_AM_n$ to prove that $M = \oplus_{i=0}^n M_i \in \mathcal{N}$, where $\text{inj.dim}\ {}_AM_i < \infty,\ \forall\ i \neq 0$.

If $m = 0$, i.e., M_n (which is of degree n) is injective as an A-module, then consider the exact sequences in $TA^{\mathbb{Z}}$-mod

$$0 \longrightarrow M_n \longrightarrow M \longrightarrow M/M_n \longrightarrow 0. \tag{4.4}$$

and (see (4.2))

$$0 \longrightarrow M_n \longrightarrow \text{inj}(M_n, n-1, n) \longrightarrow M_n[1] \longrightarrow 0.$$

By induction we have $M/M_n,\ M_n[1] \in \mathcal{N}$, and hence $M_n \in \mathcal{N}$. Now by Lemma 3.2 the short exact sequence (4.4) induces a distinguished triangle in $T(A)^{\mathbb{Z}}$-$\underline{\text{mod}}$

$$M_n \longrightarrow M \longrightarrow M/M_n \longrightarrow M_n[1] \tag{4.5}$$

with $M_n,\ M/M_n \in \mathcal{N}$. Since $\mathcal{N}$ is a triangulated subcategory of $T(A)^{\mathbb{Z}}$-$\underline{\text{mod}}$, it follows that $M \in \mathcal{N}$.

Assume that for $n, d \geq 1$, $\mathcal{N}$ already contains all the objects $M = \oplus_{i=0}^n M_i$ in the right hand side of (4.3) with $\text{inj.dim}\ {}_AM_n < d$. We will prove that $\mathcal{N}$ also contains such an object M with $\text{inj.dim}\ {}_AM_n = d$. Take an exact sequence in $T(A)^{\mathbb{Z}}$-mod (see (4.2))

$$0 \longrightarrow M_n \longrightarrow \text{inj}(I_A(M_n), n-1, n) \longrightarrow M_n[1] \longrightarrow 0.$$

Since the n-th component $I_A(M_n)/M_n$ of $M_n[1]$ has injective dimension less than d, it follows from induction that $M_n[1] \in \mathcal{N}$, and hence $M_n \in \mathcal{N}$. Also $M/M_n \in \mathcal{N}$ since $l(M/M_n) < n$. Thus $M \in \mathcal{N}$ by (4.5). This proves the claim.

Dually, any object of the form $M = \oplus_{i\leq 0} M_i$ in the right hand side of (4.3) lies in $\mathcal{N}$.

In general, for $M = \oplus_{n\in\mathbb{Z}} M_n$ in the right hand side of (4.3), set $M_{\geq 0} := \oplus_{n\geq 0} M_n$. Then it is a submodule of M. By the argument above we have $M_{\geq 0} \in \mathcal{N}$ and $M/M_{\geq 0} \in \mathcal{N}$. Consider the short exact sequence in $T(A)^{\mathbb{Z}}$-mod

$$0 \longrightarrow M_{\geq 0} \longrightarrow M \longrightarrow M/M_{\geq 0} \longrightarrow 0$$

which induces a distinguished triangle in $T(A)^{\mathbb{Z}}$-$\underline{\text{mod}}$ by Lemma 3.2. Again since $\mathcal{N}$ is a triangulated subcategory, it follows that $M \in \mathcal{N}$. This completes the proof. ■

5. Stable category $\underline{\mathfrak{a}(T)}$

Throughout this section, A is an arbitrary finite-dimensional k-algebra, although almost everything holds for an arbitrary ring.

5.1. Let T be a self-orthogonal A-module. Consider the full subcategory of A-mod given by $\mathfrak{a}(T) := \mathcal{X}_T \cap {}_T\mathcal{X}$. By Proposition 5.1 in [AR1] we know that $\mathfrak{a}(T)$ is closed under extensions and direct summands. By the definition we immediately have

Lemma 5.1. *The full subcategory of* A-mod *given by* $\mathfrak{a}(T)$ *is a Frobenius exact category, where* add$T$ *is exactly the full subcategory of all the (relatively) projective and injective objects.*

It follows that the stable category of $\mathfrak{a}(T)$ modulo add$T$, denoted by $\underline{\mathfrak{a}(T)}$, is a triangulated category. If A is Gorenstein and T is a generalized cotilting module (= a generalized tilting module, see Corollary 2.10), then $\underline{\mathfrak{a}(T)}$ is exactly the singularity category $\mathcal{D}_P(A) = \mathcal{D}_I(A)$ (cf. 2.1, 2.2, and Theorem 2.16).

5.2. For a short exact sequence $0 \longrightarrow X \xrightarrow{u} Y \xrightarrow{v} Z \longrightarrow 0$ in $\mathfrak{a}(T)$, take an exact sequence

$$0 \longrightarrow X \xrightarrow{i_X} T(X) \xrightarrow{\pi_X} S(X) \longrightarrow 0 \tag{5.1}$$

with $T(X) \in \mathrm{add}T$ and $S(X) \in \mathfrak{a}(T)$. Note that $S(X)$ is the translation of X in $\underline{\mathfrak{a}(T)}$, and that $X \xrightarrow{u} Y \xrightarrow{v} Z \xrightarrow{-w} S(X)$ is a distinguished triangle in $\underline{\mathfrak{a}(T)}$, where w is an A-map such that the following diagram is commutative

$$\begin{tikzcd} 0 \arrow[r] & X \arrow[r, "u"] \arrow[d, equal] & Y \arrow[r, "v"] \arrow[d, "\rho"] & Z \arrow[r] \arrow[d, "w"] & 0 \\ 0 \arrow[r] & X \arrow[r, "i_X"] & T(X) \arrow[r, "\pi_X"] & S(X) \arrow[r] & 0; \end{tikzcd} \tag{5.2}$$

and any distinguished triangle in $\underline{\mathfrak{a}(T)}$ is given in this way (see Lemma 3.2).

On the other hand, a short exact sequence $0 \longrightarrow X \xrightarrow{u} Y \xrightarrow{v} Z \longrightarrow 0$ in $\mathfrak{a}(T)$ induces a distinguished triangle in $D^b(A)$

$$X \xrightarrow{u} Y \xrightarrow{v} Z \xrightarrow{w'} X[1]. \tag{5.3}$$

Denote by $\mathrm{Con}(u)$ the complex $0 \longrightarrow X \xrightarrow{u} Y \longrightarrow 0$ in $D^b(A)$, where Y is at the 0-th position. Then we have natural morphisms of complexes $p_X : \mathrm{Con}(u) \longrightarrow X[1]$ and $v' : \mathrm{Con}(u) \longrightarrow Z$. Note that v' is induced by v, and is a quasi-isomorphism, and that

$$w' = p_X/v' \in \mathrm{Hom}_{D^b(A)}(Z, X[1]) \tag{5.4}$$

as a right fraction.

Denote by $\mathrm{Con}(i_X)$ the complex $0 \longrightarrow X \xrightarrow{i_X} T(X) \longrightarrow 0$ in $D^b(A)$, where $T(X)$ is at the 0-th position. Then we have natural morphism of complexes $p'_X : \mathrm{Con}(i_X) \longrightarrow X[1]$ and $\pi'_X : \mathrm{Con}(i_X) \longrightarrow S(X)$. Note that π'_X is induced by π_X, and is a quasi-isomorphism. Write $\beta_X := -p'_X/\pi'_X \in \mathrm{Hom}_{D^b(A)}(S(X), X[1])$. We claim that $w' = -\beta_X w$ in $D^b(A)$, and hence by (5.3), $X \xrightarrow{u} Y \xrightarrow{v} Z \xrightarrow{-\beta_X w} X[1]$ is a distinguished triangle in $D^b(A)$, and hence it is a distinguished triangle in the quotient triangulated category $D^b(A)/K^b(\mathrm{add}T)$.

In fact, by (5.4) the claim is equivalent to $p_X = -\beta_X(wv')$ in $D^b(A)$. Denote by ρ' the chain map $\mathrm{Con}(u) \longrightarrow \mathrm{Con}(i_X)$ induced by ρ. Then by the multiplication rule of right fractions we have

$$-\beta_X(wv') = (p'_X/\pi'_X)(wv') = p'_X\rho' = p_X,$$

where the second equality follows from the multiplication rule of right fractions and $wv = \pi_X\rho$ in (5.2), and that the last equality follows from the definition.

Note that for each $X \in \mathfrak{a}(T)$, β_X is an isomorphism in the quotient triangulated category $D^b(A)/K^b(\mathrm{add}T)$. This can be seen as follows. By the distinguished triangle

$$X \xrightarrow{i_X} T(X) \longrightarrow \mathrm{Con}(i_X) \xrightarrow{p'_X} X[1]$$

in $D^b(A)$ we have the distinguished triangle in $D^b(A)/K^b(\mathrm{add}T)$

$$X \xrightarrow{i_X} T(X) \longrightarrow \mathrm{Con}(i_X) \xrightarrow{p'_X} X[1].$$

But in $D^b(A)/K^b(\mathrm{add}T)$ we have $T(X) = 0$, it follows that p_X is an isomorphism in $D^b(A)/K^b(\mathrm{add}T)$, so is β_X in $D^b(A)/K^b(\mathrm{add}T)$.

Now, denote by G the natural fully-faithful functor $\underline{\mathfrak{a}(T)} \longrightarrow D^b(A)/K^b(\mathrm{add}T)$ (cf. Theorem 2.1). Then the discussion above shows that $\beta : G \circ [1] \longrightarrow [1] \circ G$ is a natural isomorphism, where the first $[1] = S$ is the shift of $\underline{\mathfrak{a}(T)}$, and the second $[1]$ is the shift of $D^b(A)/K^b(\mathrm{add}T)$. We conclude the following.

Theorem 5.2. *Let T be a self-orthogonal module. Then the natural embedding $\underline{\mathfrak{a}(T)} \longrightarrow D^b(A)/K^b(\mathrm{add}T)$ is an exact functor.*

5.3. Denote by $K^{ac}(T)$ be the full subcategory of the (unbounded) homotopy category $K(A)$ consisting of acyclic complexes with components in $\mathrm{add}T$ (see also [Kr], Section 5). It is a triangulated subcategory. We have

Theorem 5.3. *Let T be a self-orthogonal module such that $\overbrace{\mathrm{add}T} \subseteq T^{\perp}$ and $\underbrace{\mathrm{add}T} \subseteq {}^{\perp}T$. Then there is an equivalence of triangulated categories $K^{ac}(T) \simeq \underline{\mathfrak{a}(T)}$.*

Together with Theorems 2.16 and 5.2 we have the following result, which gives an another description of the singularity category of a Gorenstein algebra. A similar result on separated noetherian schemes has been given in [Kr], Theorem 1.1(3).

Corollary 5.4. *Let A be Gorenstein, and T be a generalized cotilting module (= a generalized tilting module). Then we have an equivalence of triangulated categories*

$$K^{ac}(T) \simeq \mathcal{D}_I(A) = \mathcal{D}_P(A).$$

To prove Theorem 5.3, we make some preparations.

5.4. Let T be self-orthogonal. Let $X \in {}_T\mathcal{X}$ with an exact sequence

$$\cdots \longrightarrow T^{-i} \xrightarrow{d_T^{-i}} T^{-(i-1)} \longrightarrow \cdots \xrightarrow{d_T^{-1}} T^0 \xrightarrow{d_T^0} X \longrightarrow 0,$$

where each $T^{-i} \in \mathrm{add}T$ and $\mathrm{Ker}d^{-i} \in T^{\perp}$, $i \geq 0$. Let $Y \in A$-mod with a complex

$$\cdots \longrightarrow T'^{-i} \xrightarrow{d_{T'}^{-i}} T'^{-(i-1)} \longrightarrow \cdots \xrightarrow{d_{T'}^{-1}} T'^{0} \xrightarrow{d_{T'}^{0}} Y \longrightarrow 0,$$

where each $T'^{-i} \in \mathrm{add}T$. Denote them by $T^{\bullet} \xrightarrow{d_T^0} X$ and $T'^{\bullet} \xrightarrow{d_{T'}^0} Y$, respectively. The proof of the following fact is similar with the one of the Comparison-Theorem in homological algebra.

Lemma 5.5. *With the notation of $X, Y, T^{\bullet}, T'^{\bullet}$ as above, and any morphism $f : Y \longrightarrow X$, there exists a unique morphism $f^{\bullet} : T'^{\bullet} \longrightarrow T^{\bullet}$ in $K(A)$ such that $fd_{T'}^0 = d_T^0 f^0$.*

5.5. Dually, let $X \in \mathcal{X}_T$ with an exact sequence

$$0 \longrightarrow X \xrightarrow{\varepsilon_X} T^0 \xrightarrow{d_T^0} T^1 \longrightarrow \cdots \longrightarrow T^i \xrightarrow{d_T^i} T^{i+1} \longrightarrow \cdots,$$

where each $T^i \in \mathrm{add}T$ and $\mathrm{Im}d^i \in {}^{\perp}T$, $i \geq 0$. Let $Y \in A$-mod with a complex

$$0 \longrightarrow Y \xrightarrow{\varepsilon_Y} T'^0 \xrightarrow{d_{T'}^0} T'^1 \longrightarrow \cdots \longrightarrow T'^i \xrightarrow{d_{T'}^i} T'^{i+1} \longrightarrow \cdots,$$

where each $T'^i \in \mathrm{add}T$. Denote them by $\varepsilon_X : X \longrightarrow T^{\bullet}$ and $\varepsilon_Y : Y \longrightarrow T'^{\bullet}$, respectively.

Lemma 5.6. *With the notation of $X, Y, T^{\bullet}, T'^{\bullet}$ as above, and any morphism $f : X \longrightarrow Y$, there exists a unique morphism $f^{\bullet} : T^{\bullet} \longrightarrow T'^{\bullet}$ in $K(A)$ such that $\varepsilon_Y f = f^0 \varepsilon_X$.*

5.6. We need the following fact (see p.446 in [Ric1], or p.45 in [KZ]).

Lemma 5.7. *Let $F : \mathcal{C} \longrightarrow \mathcal{D}$ be a full and exact functor of triangulated categories. Then F is faithful if and only if it is faithful on objects, that is, if $F(X) \simeq 0$ then $X \simeq 0$.*

5.7. Proof of Theorem 5.3. Since $\overbrace{\mathrm{add}T} \subseteq T^{\perp}$ and $\underbrace{\mathrm{add}T} \subseteq {}^{\perp}T$, it follows that ${}_T\mathcal{X} = \overbrace{\mathrm{add}T}$ and $\mathcal{X}_T = \underbrace{\mathrm{add}T}$. Thus, for any object $T^{\bullet}$ in $K^{ac}(T)$ we have $\mathrm{Coker}d_T^i \in \overbrace{\mathrm{add}T} \cap \underbrace{\mathrm{add}T} = \mathfrak{a}(T)$, for each $i \in \mathbb{Z}$.

Define a functor $F : K^{ac}(T) \longrightarrow \underline{\mathfrak{a}(T)}$ as follows: for any object $T^{\bullet}$ in $K^{ac}(T)$, define $F(T^{\bullet}) := \mathrm{Coker}d_T^{-1}$; for any morphism $f^{\bullet} : T^{\bullet} \longrightarrow T'^{\bullet}$ in $K^{ac}(T)$, define $F(f^{\bullet})$ to be the image in $\underline{\mathfrak{a}(T)}$ of the unique morphism $\bar{f}^0 : \mathrm{Coker}d_T^{-1} \longrightarrow \mathrm{Coker}d_{T'}^{-1}$ induced by f^0. Note that F is well-defined, dense, and full by Lemmas 5.5 and 5.6.

Note that F is faithful on objects. In fact, if $F(T^{\bullet}) \simeq 0$, then we have $\mathrm{Coker}d_T^{-1} \in \mathrm{add}T$. Since $\mathrm{Coker}d_T^i \in \overbrace{\mathrm{add}T} \cap \underbrace{\mathrm{add}T} \subseteq T^{\perp} \cap {}^{\perp}T$ for each i, it follows that the exact sequence $0 \longrightarrow \mathrm{Coker}d_T^{-2} \longrightarrow T^0 \longrightarrow \mathrm{Coker}d_T^{-1} \longrightarrow 0$ splits, and hence $\mathrm{Coker}d_T^{-2} \in \mathrm{add}T$. Repeating this process, we have the split exact sequence $0 \longrightarrow \mathrm{Coker}d_T^{-i-2} \longrightarrow T^{-i} \longrightarrow \mathrm{Coker}d_T^{-i-1} \longrightarrow 0$, and $\mathrm{Coker}d_T^{-i-2} \in \mathrm{add}T$, for each $i \geq 0$. Similarly, the exact sequence $0 \longrightarrow \mathrm{Coker}d_T^{i-2} \longrightarrow T^i \longrightarrow \mathrm{Coker}d_T^{i-1} \longrightarrow 0$ splits and $\mathrm{Coker}d_T^{i-1} \in \mathrm{add}T$ for each $i \geq 1$. This implies that the identity $\mathrm{Id}_{T^{\bullet}}$ is homotopic to zero, that is, $T^{\bullet}$ is zero in $K^{ac}(T)$.

In order to prove that F is an exact functor, we first need to establish a natural isomorphism $F \circ [1] \longrightarrow [1] \circ F$, where the first $[1]$ is the usual shift of complexes, and the second $[1]$ is the shift functor of the stable category $\underline{\mathfrak{a}(T)}$. In fact, for each $T^{\bullet} \in K^{ac}(T)$, we have a commutative diagram of short exact sequences in A-mod

$$\begin{tikzcd}
0 \arrow[r] & F(T^\bullet) \arrow[r, "i_{T^\bullet}"] \arrow[d, equal] & T^1 \arrow[r, "\pi_{T^\bullet}"] \arrow[d, "\gamma_{T^\bullet}"] & F(T^\bullet[1]) \arrow[r] \arrow[d, "\alpha_{T^\bullet}"'] & 0 \\
0 \arrow[r] & F(T^\bullet) \arrow[r, "i_{F(T^\bullet)}"] & T(F(T^\bullet)) \arrow[r, "\pi_{F(T^\bullet)}"] & F(T^\bullet)[1] \arrow[r] & 0
\end{tikzcd}$$

where $i_{T^\bullet}$ is the natural embedding, $\pi_{T^\bullet}$ is the canonical map, and the exact sequence of the second row is the one defining $F(T^\bullet)[1]$, with $T(F(T^\bullet)) \in \operatorname{add}T$ (see (5.1), where we write $F(T^\bullet)[1]$ for $S(F(T^\bullet)))$. Note that $\underline{\alpha_{T^\bullet}}$ is unique in the stable category $\underline{\mathfrak{a}(T)}$, and that it is easy to verify that $\underline{\alpha} : F \circ [1] \longrightarrow [1] \circ F$ is a natural isomorphism (by using the same argument as in the proof of Lemma 2.2 in [Hap1], p.12). We will show that $(F, \underline{\alpha})$ is an exact functor.

Recall a distinguished triangle in $K^{ac}(T)$ is given by

$$T^\bullet \xrightarrow{f^\bullet} T'^\bullet \xrightarrow{\binom{0}{1}} \operatorname{Con}(f^\bullet) \xrightarrow{(1\ 0)} T^\bullet[1],$$

where the mapping cone $\operatorname{Con}(f^\bullet)$ of $f^\bullet$ is defined by $\operatorname{Con}(f^\bullet)^n = T^{n+1} \oplus T'^n$ with differentials $\begin{pmatrix} -d_T^{n+1} & 0 \\ -f^{n+1} & d^n_{T'} \end{pmatrix}$. Denote by $\theta : F(T'^\bullet) \longrightarrow F(\operatorname{Con}(f^\bullet))$ and $\eta : F(\operatorname{Con}(f^\bullet)) \longrightarrow F(T^\bullet[1])$ the morphisms in A-mod induced by $\binom{0}{1}$ and $(1\ 0)$, respectively. Clearly we have $\eta\theta = 0$. Observe that the following sequence in A-mod

$$0 \longrightarrow F(T^\bullet) \xrightarrow{\binom{\bar{f}^0}{-i_{T^\bullet}}} F(T'^\bullet) \oplus T^1 \xrightarrow{(\theta\ \pi)} F(\operatorname{Con}(f^\bullet)) \longrightarrow 0,$$

is exact, where $i_{T^\bullet}$ is as above and π is the natural map from T^1 to $F(\operatorname{Con}(f^\bullet)) = (T^1 \oplus T'^0)/\operatorname{Im}\begin{pmatrix} -d_T^0 & 0 \\ -f^0 & d_{T'}^{-1} \end{pmatrix}$. This can be seen by directly verifying that $(\theta\ \pi)$ is surjective, $(\theta\ \pi)\binom{\bar{f}^0}{-i_{T^\bullet}} = 0$, and $\operatorname{Ker}(\theta\ \pi) \subseteq \operatorname{Im}\binom{\bar{f}^0}{-i_{T^\bullet}}$. By definition we have $\eta\pi = \pi_{T^\bullet}$, and hence the following diagram of short exact sequences in A-mod commutes

$$\begin{tikzcd}
0 \arrow[r] & F(T^\bullet) \arrow[r, "\binom{\bar{f}^0}{-i_{T^\bullet}}"] \arrow[d, equal] & F(T'^\bullet) \oplus T^1 \arrow[r, "(\theta\ \pi)"] \arrow[d, "(0, -\gamma_{T^\bullet})"] & F(\operatorname{Con}(f^\bullet)) \arrow[r] \arrow[d, "-(\alpha_{T^\bullet}\eta)"] & 0 \\
0 \arrow[r] & F(T^\bullet) \arrow[r, "i_{F(T^\bullet)}"] & T(F(T^\bullet)) \arrow[r, "\pi_{F(T^\bullet)}"] & F(T^\bullet)[1] \arrow[r] & 0.
\end{tikzcd}$$

It follows from Lemma 3.2 that $F(T^\bullet) \xrightarrow{F(f^\bullet)} F(T'^\bullet) \xrightarrow{\theta} F(\operatorname{Con}(f^\bullet)) \xrightarrow{\underline{\alpha_{T^\bullet}\eta}} F(T^\bullet)[1]$ is a distinguished triangle in $\underline{\mathfrak{a}(T)}$. This proves that $F : K^{ac}(T) \longrightarrow \underline{\mathfrak{a}(T)}$ is an exact functor.

Now the theorem follows from Lemma 5.7. ■

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