INTEGRAL QUADRATIC FORMS AND THE REPRESENTATION TYPE OF AN ALGEBRA

JOSÉ A. DE LA PEÑA

Instituto de Matemáticas, UNAM.
Circuito Exterior.
Ciudad Universitaria.
México 04510, D. F.
México
jap@matem.unam.mx

In these notes k will denote a fixed algebraically closed field.

A will denote a finite-dimensional associative k-algebra with unity. An A-module (if not otherwise stated) is a finitely generated left A-module. According to former lectures in this volume, we may assume that A is a basic algebra and write $A = kQ/I$ where Q is a finite quiver and I is an admissible ideal of the path algebra kQ.

A fundamental problem in the *representation theory of algebras* is the classification of all indecomposable A-modules (up to isomorphism). We say that A is of *finite representation type* if there are only finitely many indecomposable A-modules up to isomorphism. One of the first successes of modern representation theory was the identification by Gabriel of the Dynkin diagrams as the underlying graphs of quivers Q such that kQ is representation-finite. But representation-infinite algebras are common. Already in the 19^{th} century, Kronecker completed work of Weierstrass to classify all indecomposable 'pencils' by means of infinite families of pairwise non-isomorphic normal forms, which in modern terminology corresponds to the classification of the indecomposable modules over the Kronecker algebra. The first explicit recognition that infinite representation type splits in two different classes arises in representations of groups: in 1954, Highman showed that the Klein group has infinitely many representations in characteristic 2 and Hellen and Reiner classified them; in contrast, Krugljak showed in 1963 that solving the classification problem of groups of type (p, p) with $p \geq 3$ implies the classification of the representations of any group of the same characteristic, a task that was recognized as 'wild'.

The first task of these notes is to give precise meaning to the following concepts. The algebra A is *tame* if for every number n, almost every indecomposable A-module of dimension n is isomorphic to a module belonging to a finite number of 1-parameter families of modules. Formally, an algebra A is *tame* if for every $n \in \mathbb{N}$ there is a finite family of $A - k[t]$-bimodules $M_1, \ldots, M_{t(n)}$ with the following properties:

(i) M_i is finitely generated free as a right $k[t]$-module;

(ii) almost every indecomposable left A-module X with $\dim_k X = n$ is isomorphic to a module of the form $M_i \otimes_{k[t]} S_\lambda$ for some $\lambda \in k$.

The algebra A is *wild* if the classification of the indecomposable A-modules implies the classification of the indecomposable modules over the associative algebra $k\langle x, y\rangle$ in two indeterminates. Donovan and Freislich were the first to state the *tame-wild dichotomy* as a conjecture, later made precise and proved by Drozd. Namely,

Dichotomy Theorem of Drozd: Every finite dimensional k-algebra is either tame or wild.

In *Lecture 1* we shall present some important examples of algebras and discuss their representation type: hereditary algebras, local algebras, group algebras. In *Lecture 2* we introduce some fundamental concepts and techniques which are useful for the understanding of tame algebras. Given a basic algebra $A = kQ/I$, for each vector $v \in \mathbb{N}^{Q_0}$, we define a *module variety* $\mathrm{mod}_A(v)$ as a closed subset, relative to the Zariski topology, of an affine space. The notion of tameness for A may be read in different ways in the module varieties $\mathrm{mod}_A(v)$.

Although there is a no general procedure known to decide whether or not a given algebra is tame, there are cases which are well understood. An algebra $A = kQ/I$ is said to be *triangular* if Q has no oriented cycles. For such an algebra the *Tits quadratic form* $q_A \colon \mathbb{Z}^{Q_0} \to \mathbb{Z}$ is introduced by

$$q_A(v) = \sum_{i \in Q_0} v(i)^2 - \sum_{(i \to j) \in Q_1} v(i)v(j) + \sum_{i,j \in Q_0} r(i,j)v(i)v(j)$$

where Q_0 (resp. Q_1) denotes the set of vertices (resp. arrows) of Q and $r(i,j)$ is the number of elements in $R \cap I(i,j)$ whenever R is a minimal set of generators of I contained in $\bigcup\limits_{i,j \in Q_0} I(i,j)$. This quadratic form was introduced by Tits and used systematically by Gabriel and Bernstein-Gelfand-Ponomariev in the study of the representations of hereditary algebras $A = kQ$ of finite representation type.

The *main purpose of these lectures* is to survey the use of the Tits form in representation theory. Special emphasis is made in characterizing the representation type via arithmetic properties of the Tits form. Basic results in this direction are shown in *Lecture 3*. Let A be a triangular algebra, the following holds:

- if A is representation-finite, then q_A is *weakly positive* (that is, $q_A(v) > 0$ for any vector $0 \neq v \in \mathbb{N}^{Q_0}$).
- if A is tame, then q_A is weakly non-negative (that is, $q_A(v) \geq 0$ for any $v \in \mathbb{N}^{(Q_0)}$).

Consideration of special cases where the converses of the above results hold, hence providing combinatorial characterizations of the representation type, is the central issue of *Lecture 3* and *4*. We say that an algebra $B = kQ'/I'$ is a *full subcategory* of $A = kQ/I$ if Q' is a path closed full subquiver of Q and $I' = I \cap kQ$. First, it is shown that an algebra A accepting a preprojective component in the Auslander-Reiten quiver Γ_A is representation-finite if and only if q_A is weakly positive. Moreover, this is equivalent to A not having convex subcategories which are *critical* (an algebra B is critical if Γ_B has a preprojective component, q_B is not weakly positive but every proper restriction of q_B is weakly positive). An algebra B is *hypercritical* if Γ_B has a preprojective component, and the form q_B is not weakly non-negative while every proper restriction of q_B is weakly non-negative. The critical and hypercritical algebras have been classified.

A triangular algebra A is *strongly simply connected* if every convex subcategory of A satisfies the separation condition. Many important examples of algebras satisfy this property. Recently Brüstle-Skowroński and the author have shown that for a strongly simply connnected algebra A the following are equivalent:

(a) A is tame;
(b) q_A is weakly non-negative;
(c) A does not contain a full convex subcategory which is hypercritical.

The *intention* of these lectures is to serve as a source of motivation and information on the main concepts, techniques and results on the topic. While we cannot provide complete proofs of every result, we try to sketch some representative arguments whose proofs are elementary enough not to require other sophisticated parts of the theory.

Notation and conventions.

We fix our notation by recalling basic material on algebras, modules and representation theory which can be found on textbooks and in other lectures at this volume. All algebras in this work are associative k-algebras with an identity. A finite dimensional k-algebra is *basic* if $A/\operatorname{rad} A$ is commutative, where $\operatorname{rad} A$ denotes the Jacobson radical of A.

By mod_A we denote the category of finite dimensional (= finitely generated) left A-modules. Each finite dimensional k-algebra A is Morita equivalent to a basic algebra B, that is, there is an equivalence of categories $\operatorname{mod}_A \xrightarrow{\sim} \operatorname{mod}_B$.

A *quiver* Q is an oriented graph with set of vertices Q_0 and set of arrows Q_1. The *path algebra* kQ has as k-basis the oriented paths in Q, including a trivial path e_s for each vertex $s \in Q_0$, with the product given by concatenation of the paths. A module $X \in \operatorname{mod}_{kQ}$ is a *representation* of Q with a vector space $X(s) = e_s X$ for each vertex $s \in Q_0$ and a linear map $X(\alpha)\colon X(s) \to X(t)$ for each arrow $s \xrightarrow{\alpha} t$ in Q_1.

For a finite dimensional k-algebra A we associate the quiver Q_A in the following way: the set of vertices Q_0 is the set of isoclasses of simple A-modules $\{1, \dots, n\}$. Let S_i be a simple A-module representing the i-th class. Then there are as many arrows from i to j in Q as $\dim_k \operatorname{Ext}^1_A(S_i, S_j)$. By a remark of Gabriel [17], in case A is basic, there is a surjective morphism $kQ \xrightarrow{\nu} A$ such that the ideal $\ker \nu$ is admissible, that is, $(\operatorname{rad} A)^m \subset \ker \nu \subset (\operatorname{rad} A)^2$ for some $m \geq 2$.

We shall identify $A = kQ/I$ with a k-category whose objects are the vertices of Q and whose morphism space $A(s,t)$ is $e_t A e_s$. We say that B is a *convex subcategory* of A if $B = kQ'/I'$ for a path closed subquiver Q' of Q and $I' = I \cap kQ'$. In this view, an A-module X is a k-linear functor $X\colon A \to \operatorname{mod}_k$. The *dimension vector* of X is $\mathbf{dim}\, X = (\dim_k X(s))_{s \in Q_0} \in \mathbb{N}^{Q_0}$ and the *support* of X is $\operatorname{supp} X = \{s \in Q_0 \colon X(s) \neq 0\}$.

For an algebra A, we consider the standard duality $D\colon \operatorname{mod}_A \to \operatorname{mod}_{A^{op}}$ defined as $D = \operatorname{Hom}_k(-, k)$, where A^{op} is the opposite algebra of A. The *Auslander-Reiten translation* $\tau_A = D\mathrm{tr}$ yields a functor $\tau_A\colon \underline{\operatorname{mod}}_A \to \overline{\operatorname{mod}}_A$, where $\underline{\operatorname{mod}}_A$ (resp. $\overline{\operatorname{mod}}_A$) is the category whose objects are A-modules and $\underline{\operatorname{Hom}}_A(X,Y)$ (resp. $\overline{\operatorname{Hom}}_A(X,Y)$) is the quotient of $\operatorname{Hom}_A(X,Y)$ by those morphisms factoring through a projective module (resp. an injective module), satisfying that $\operatorname{Ext}^1_A(X,Y) \xrightarrow{\sim} D\overline{\operatorname{Hom}}_A(Y, \tau_A X)$.

The inverse of τ_A is $\tau_A^- = \operatorname{tr} D$. The *Auslander-Reiten quiver* Γ_A of A has as vertices the isoclasses of indecomposable A-modules and there are n arrows from the class $[Y]$ of the indecomposable module Y to $[X]$ if Y^n, but not Y^{n+1}, is a direct summand of Z for an exact sequence

$$\xi \colon 0 \to \tau_A X \to Z \to X \to 0$$

corresponding to a non-zero element in $\mathrm{Ext}^1_A(X, \tau_A X) \xrightarrow{\sim} D\overline{\mathrm{Hom}}_A(\tau_A X, \tau_A X)$, $\xi \mapsto 1_{\tau_A X}$, in case X is non-projective; or $Z = \operatorname{rad} X$, in case X is projective.

By the Jordan-Hölder theorem, the Grothendieck group $K_0(A)$ of mod$A$ is the free abelian group on the classes $[S_1], \dots, [S_n]$ of simple A-modules, yielding an identification $K_0(A) = \mathbb{Z}$. The class of any A-module M equals $[M] = \sum_{i=1}^{n} [M : S_i][S_i]$, where $[M : S_i]$ is the multiplicity of S_i in the composition series of M (observe that $[M_i : S_i] = \dim_k M(i)$ if i is the vertex of Q_A corresponding to S_i). We shall assume that A has *finite global dimension* (which happens, for example, if A is triangular). Then the classes $[P_1], \dots, [P_n]$ of indecomposable projective covers P_i of S_i, $1 \leq i \leq n$, form another basis of $K_0(A)$. Similarly, the classes $[I_1], \dots, [I_n]$ of indecomposable injective envelopes I_j of S_j, $1 \leq j \leq n$ form a basis of $K_0(A)$. The *homological form* $\langle -, - \rangle_A$ on $K_0(A)$ is the bilinear form

$$\langle [X], [Y] \rangle_A = \sum_{i=0}^{\infty} (-1)^i \dim_k \mathrm{Ext}^i_A(X, Y).$$

Defining the $n \times n$ *Cartan matrix* $C_A = (c_{ij})$ as $c_{ij} = \langle [P_j], [P_i] \rangle$, we get

$$\langle v, w \rangle_A = v C_A^{-t} w^t.$$

The quadratic form $\chi_A(v) = \langle v, v \rangle_A$ is called the *Euler form* of A.

The *Coxeter transformation* φ_A is an automorphism of $K_0(A)$, determined by

$$[P_j]\varphi_A = -[I_j], \qquad \text{for } 1 \leq j \leq n.$$

Therefore $\varphi_A = -C_A^{-t} C_A$ and $\langle v, w\varphi_A \rangle_A = -\langle w, v \rangle_A$ for all $v, w \in K_0(A)$.

In the hereditary case $A = kQ$, for any indecomposable non-projective A-module X, we have

$$[X]\varphi_A = [\tau_A X].$$

In general, the relation between τ_A and φ_A is not so nice, but it will be central for our paper. We recall here the following remarks from [17]:

(a) If $\mathrm{pdim}_A X \leq 1$ and $\mathrm{Hom}_A(X, A) = 0$, then $[\tau_A X] = [X]\varphi_A$.

(b) If $\mathrm{pdim}_A X \leq 2$ and $\mathrm{idim}_A X \leq 2$ then for some injective A-module I we get $[\tau_A X] = [X]\varphi_A + [I]$.

The role of the Coxeter transformation φ_A clarifies with the consideration of the derived category $D^d(\mathrm{mod}_A)$ of the module category mod_A, a construction that we shall not use in these lectures. Namely $[X^\bullet]\varphi_A = [\tau_{D^b(A)} X^\bullet]$ in the Grothendieck group $K_0(D^b(\mathrm{mod}_A)) \cong K_0(A)$, where $\tau_{D^b(A)} X^\bullet$ denotes the Auslander-Reiten translation of the complex $X^\bullet$ in $D^b(\mathrm{mod}_A)$.

Lecture 1. The tame-wild dichotomy.

§1. Examples.

Hereditary algebras.

Let Δ be a quiver without oriented cycles and consider the associated *hereditary algebra* $A = k\Delta$. We assume Δ is connected.

Let $\Delta_0 = \{1, \dots, n\}$ be the set of vertices of Δ and

$$M_\Delta = (m_{ij}) \text{ the } \textit{Cartan matrix} \text{ of } \Delta,$$

$$m_{ij} = \begin{cases} 2, & \text{if } i = j \\ -\#\text{ edges between } i \text{ and } j, & \text{if } i \neq j \end{cases}$$

Consider $V^+ = \{v \in V : v(i) \geq 0, \forall\, i\}$ the positive cone

Lemma. $M_\Delta^{-1}(V^+) \cap \partial V^+ = \{0\}$.

Proof. Assume that $0 \neq y \in M_\Delta^{-1}(V^+) \cap \partial V^+$.

By the connectivity of Δ we find an edge i —— j such that $y(i) > 0$ and $y(j) = 0$. Then

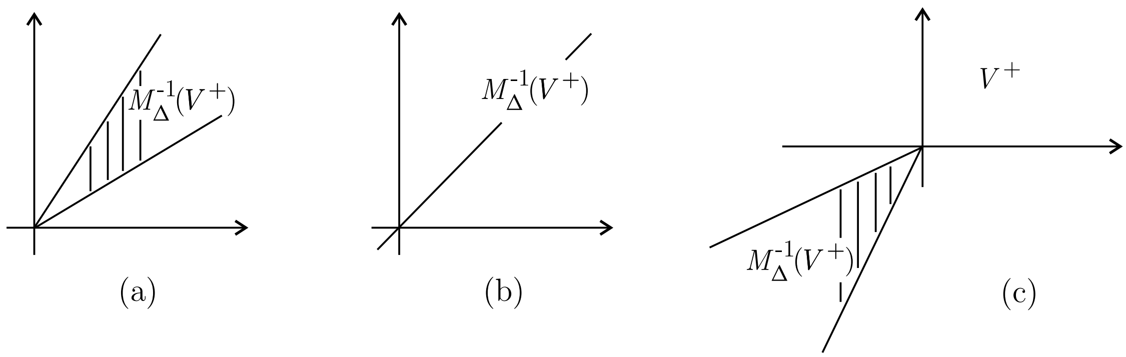
$$\begin{aligned} 0 \leq M(y)(j) &= \sum_k m_{jk} y(k) = m_{jj} y(j) + m_{ji} y(i) + \\ &\quad + \sum_{k \neq i,j} m_{jk} y(k) \leq m_{ji} y(i) < 0, \end{aligned}$$

a contradiction. □

Proposition. *The matrix M_Δ satisfies one and only one of the properties:*

(a) $M_\Delta^{-1}(V^+) \subset V^+$
(b) $M_\Delta^{-1}(V^+) = \mathbb{R}u$ *for some* $u \gg 0$. *In this case* $M_\Delta(u) = 0$
(c) $M^{-1}(V^+) \cap V^+ = \{0\}$

This can be illustrated for $n = 2$:



(a) (b) (c)

Let $q_\Delta \colon \mathbb{Z}^n \to \mathbb{Z}$ be the quadratic form $q_\Delta(v) = \frac{1}{2} v M_\Delta v^t$. Then q_Δ is the *Tits form* associated to the hereditary algebra $A = k\Delta$. Corresponding to the cases distinguished in the above Proposition, we have:

(Elliptic type): q_Δ is positive definite if $M_\Delta^{-1}(V^+) \subset V^+$;

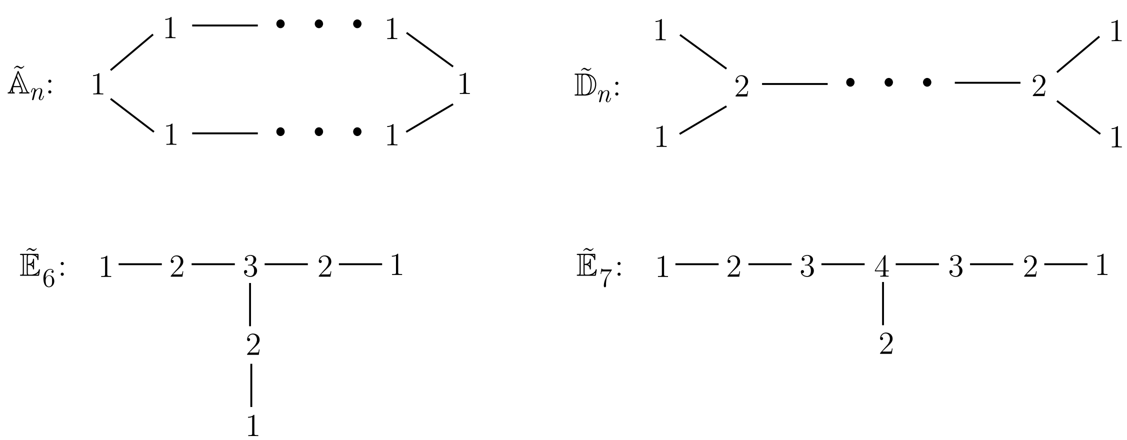
(Parabolic type): q_Δ is non-negative with $\operatorname{corank} q_\Delta = 1$ if $M_\Delta^{-1}(V^+) = \mathbb{R}u$ for some $u \gg 0$;

(Hyperbolic type): q_Δ is indefinite if $M_\Delta^{-1}(V^+) \cap V^+ = \{0\}$.

(details can be completed by the reader as an *exercise*).

In this way, we get three type of quivers. *Classification*:

(1) Let Δ be of parabolic type, $u \gg 0$ be the minimal positive vector with $q_\Delta(u) = 0$. Then the underlying graph $|\Delta|$ is one of the following



$$\tilde{\mathbb{E}}_8: \quad \begin{tikzcd}[column sep=small, row sep=small] 1 \arrow[r, dash] & 2 \arrow[r, dash] & 3 \arrow[r, dash] & 4 \arrow[r, dash] & 5 \arrow[r, dash] & 6 \arrow[r, dash] \arrow[d, dash] & 4 \arrow[r, dash] & 2 \\ & & & & & 3 & & \end{tikzcd}$$

and the indicated vector is u.

These diagrams are called *Euclidean diagrams.*

(2) Δ of elliptic type if and only if $|\Delta|$ does not contain any subgraph of Euclidean type. Hence $|\Delta|$ is one of the following (called *Dynkin diagrams*).

$$\mathbb{A}_n: \quad \begin{tikzcd}[column sep=small] \bullet \arrow[r, dash] & \bullet \arrow[r, dash] & \bullet & \bullet & \bullet \arrow[r, dash] & \bullet \end{tikzcd}$$

$$\mathbb{D}_n: \quad \begin{tikzcd}[column sep=small, row sep=small] \bullet \arrow[dr, dash] & & & & & \\ & \bullet \arrow[r, dash] & \bullet & \bullet & \bullet \arrow[r, dash] & \bullet \\ \bullet \arrow[ur, dash] & & & & & \end{tikzcd}$$

$$\mathbb{E}_6: \quad \begin{tikzcd}[column sep=small, row sep=small] \bullet \arrow[r, dash] & \bullet \arrow[r, dash] & \bullet \arrow[r, dash] \arrow[d, dash] & \bullet \arrow[r, dash] & \bullet \\ & & \bullet & & \end{tikzcd}$$

$$\mathbb{E}_7: \quad \begin{tikzcd}[column sep=small, row sep=small] \bullet \arrow[r, dash] & \bullet \arrow[r, dash] & \bullet \arrow[r, dash] \arrow[d, dash] & \bullet \arrow[r, dash] & \bullet \arrow[r, dash] & \bullet \\ & & \bullet & & & \end{tikzcd}$$

$$\mathbb{E}_8: \quad \begin{tikzcd}[column sep=small, row sep=small] \bullet \arrow[r, dash] & \bullet \arrow[r, dash] & \bullet \arrow[r, dash] \arrow[d, dash] & \bullet \arrow[r, dash] & \bullet \arrow[r, dash] & \bullet \arrow[r, dash] & \bullet \\ & & \bullet & & & & \end{tikzcd}$$

(3) Δ is of hyperbolic type. Then either there are vertices i and j of Δ with $m_{ij} \leq -3$ or $|\Delta|$ contains properly an Euclidean diagram. In the former case

$$q_\Delta(e_i + e_j) = q_\Delta(e_i) + q_\Delta(e_j) + m_{ij} < 0;$$

in the latter case, if Δ' is a full proper subquiver of Δ such that $|\Delta'|$ is Euclidean with a vector $u \gg 0$ such that $q_{\Delta'}(u) = 0$, then for any vertex i of $\Delta \setminus \Delta'$ with i adjacent to Δ', we get

$$q_\Delta(2u + e_i) = 2q_{\Delta'}(u) + 1 + 2 \sum_{u(j) \neq 0} m_{ij} < 0.$$

Local algebras.

(1) Observe that the algebra $A = k[x]/(x^n)$ admits only finitely many indecomposable modules, up to isomorphism. Then A is *representation-finite.*

Indeed, a module $M \in \mathrm{mod}_A$ is a nilpotent matrix, hence M is equivalent to

$$J_{n_1} \oplus \cdots \oplus J_{n_s}$$

where J_i is the $i \times i$ matrix

$$\begin{bmatrix} 0 & & & 0 \\ 1 & \ddots & & \\ & \ddots & \ddots & \\ 0 & & 1 & 0 \end{bmatrix}$$

with $n_i \leq n$. If M is indecomposable, $M \cong J_s$, for some $s \leq n$.

(2) Consider the infinite-dimensional k-algebra $k[x]$.

Let $M \in \mathrm{mod}_{k[x]}$, then M is a $n \times n$ matrix. Let $\chi(T) = \det(TI_n - M)$ be the characteristic polynomial of M. Then M is equivalent to

$$J_{n_1}(\lambda_1) \oplus \cdots \oplus J_{n_s}(\lambda_s)$$

where $\chi(T) = (T - \lambda_1)^{n_1} \ldots (T - \lambda_s)^{n_s}$ is the decomposition of $\chi(T)$ in linear factors (since $k = \bar{k}$) and $J_{n_i}(\lambda_i)$ is the $n_i \times n_i$ Jordan block

$$J_{n_i}(\lambda_i) = \begin{bmatrix} \lambda_i & & & 0 \\ 1 & \ddots & & \\ & \ddots & \ddots & \\ 0 & & 1 & \lambda_i \end{bmatrix}$$

Consider the $k[t] - k[t]$-bimodule given by the $n \times n$ matrix

$$J_n(t) = \begin{bmatrix} t & & & 0 \\ 1 & \ddots & & \\ & \ddots & \ddots & \\ 0 & & 1 & t \end{bmatrix}$$

Let $S_\lambda = k[t]/(t - \lambda)$ be a (one-dimensional) simple $k[t]$-module. Then

$$J_n(t) \otimes_{k[t]} S_\lambda = J_n(\lambda).$$

Therefore, the indecomposable $k[t]$-modules of dimension n are isomorphic to modules in the image of the functor

$$J_n(t) \otimes_{k[x]} - \colon \mathrm{mod}_{k[t]}(1) \to \mathrm{mod}_{k[t]}.$$

(3) The free algebra $k\langle x, y\rangle$ has a 'problematic' behaviour, as shown in the following.

Proposition. *Let B be any finitely generated k-algebra, then there exists a fully faithful functor $F\colon \mathrm{mod}_B \to \mathrm{mod}_{k\langle x,y\rangle}$.*

Proof. Let $b_1, \dots, b_s$ be a system of generators of B. Define the $k\langle x,y\rangle - B$-bimodule M as $M_B = B^{s+2}$ and the structure of left $k\langle x,y\rangle$-module given by the $(s+2)\times(s+2)$-matrices

$$
{}_xM = \begin{bmatrix} 0 & 1 & & & \\ & 0 & \ddots & & \\ & & \ddots & \ddots & \\ & & & 0 & 1 \\ 0 & & & & 0 \end{bmatrix} \qquad {}_yM = \begin{bmatrix} 0 & & & & & 0 \\ 1 & 0 & & & & \\ b_1 & 1 & & & & \\ & \ddots & \ddots & & & \\ & & \ddots & 1 & 0 & \\ 0 & & & b_s & 1 & 0 \end{bmatrix}
$$

We set $F = M \otimes_B \colon \mathrm{mod}_B \to \mathrm{mod}_{k\langle x,y\rangle}$.

Exercise: check that F is full and faithful. □

This means that the representation theory of $k\langle x,y\rangle$ is as complicated as the representation theory of any other algebra.

We say that an algebra A is *wild* if there is a functor $F\colon \mathrm{mod}_{k\langle x,y\rangle} \to \mathrm{mod}_A$ which preserves indecomposable modules and iso-classes. We shall say that the functor F *insets* indecomposable modules.

Group algebras.

Proposition. *Let p be a prime number ≥ 3. Assume k has characteristic p. The group algebra $A = k[\mathbb{Z}_p \times \mathbb{Z}_p]$ is wild.*

Proof. Let $\varphi\colon k[u,v] \to A$, $x \mapsto g-1$, $y \mapsto h-1$, where $\mathbb{Z}_p \times \mathbb{Z}_p = \langle g\rangle \times \langle h\rangle$. Then $A \cong k[u,v]/\ker\varphi = k[u,v]/(u^p, v^p)$.

Moreover $k[u,v]/(u^p,v^p) \twoheadrightarrow k[u,v]/(u,v)^3 = k[u,v]/(u^3,v^3,uv^2,vu^2) =: B$. It is enough to show that B is wild.

Consider the $B - k\langle x,y\rangle$-bimodule M defined as $M_{k\langle x,y\rangle} = k\langle x,y\rangle^4$ and the structure as B-module defined by the matrices

$$
{}_uM = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & x & y & 0 \end{bmatrix} \qquad {}_vM = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 1 & x & 0 \end{bmatrix}
$$

Exercise: check that ${}_BM$ is well defined and

$$
M \otimes_{k\langle x,y\rangle} - \colon \mathrm{mod}_{k\langle x,y\rangle} \to \mathrm{mod}_B
$$

insets indecomposable modules. $\square$

§2. Hereditary algebras and representation type.

(1) The indecomposable modules over the quiver algebra A:

$$\begin{tikzcd} \bullet \arrow[r, shift left] \arrow[r, shift right] & \bullet \end{tikzcd}$$

were classified by Weierstrass and Kronecker in the following families:

$$\begin{tikzcd}[column sep=huge] k^n \arrow[r, bend left, "{\left[\begin{array}{c|c} I_n & \begin{matrix}0\\ \vdots\\ 0\end{matrix}\end{array}\right]}"] \arrow[r, bend right, "{\left[\begin{array}{c|c} \begin{matrix}0\\ \vdots\\ 0\end{matrix} & I_n\end{array}\right]}"'] & k^{n+1} \end{tikzcd} \qquad \begin{tikzcd}[column sep=huge] k^{n+1} \arrow[r, bend left, "{\left[\begin{array}{c} I_n \\ \hline 0 \ \cdots \ 0\end{array}\right]}"] \arrow[r, bend right, "{\left[\begin{array}{c} 0 \ \cdots \ 0 \\ \hline I_n\end{array}\right]}"'] & k^{n} \end{tikzcd}$$

(preprojective representation) (preinjective representation)

$$R_n(\lambda)\colon \begin{tikzcd} k^n \arrow[r, bend left, "I_n"] \arrow[r, bend right, "J_n(\lambda)"'] & k^n \end{tikzcd} \qquad R_n(\infty)\colon \begin{tikzcd} k^n \arrow[r, bend left, "J_n(0)"] \arrow[r, bend right, "I_n"'] & k^n \end{tikzcd}$$

(regular representations)

with $\lambda \in k$.

Let M_n be the $A - k[t]$-bimodule

$$\begin{tikzcd} k[t]^n \arrow[r, bend left, "I_n"] \arrow[r, bend right, "J_n(t)"'] & k[t]^n \end{tikzcd}$$

then $M_n \otimes_{k[t]} k[t]/(t-\lambda) \cong R_n(\lambda)$.

The corresponding Tits form is $q_A(x,y) = x^2 - 2xy + y^2 = (x-y)^2$ which is of parabolic type.

(2) Consider the hereditary algebra B associated to the quiver

$$\begin{tikzcd} \bullet \arrow[r, bend left] \arrow[r] \arrow[r, bend right] & \bullet \end{tikzcd}$$

We claim that B is wild.

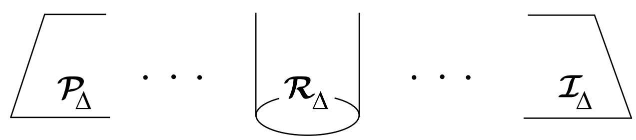
Proof. Consider the $B - k\langle x,y\rangle$-bimodule M given by

$$\begin{tikzcd} k\langle x,y\rangle \arrow[r, bend left, "{[1,0]}"] \arrow[r, "{[x,y]}"] \arrow[r, bend right, "{[0,1]}"'] & k\langle x,y\rangle^2 \end{tikzcd}$$

Exercise: $M \otimes_{k\langle x,y\rangle} -\colon \mathrm{mod}_{k\langle x,y\rangle} \to \mathrm{mod}_B$ insets indecomposable modules. □

The corresponding Tits form is $q_A(x,y) = x^2 - 3xy + y^2 = (x-y)^2 - xy$ which is indefinite.

(3) Let $A = k\Delta$ be a hereditary algebra. The general structure of the Auslander-Reiten quiver Γ_A is as follows:

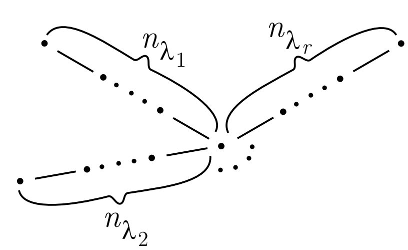


There is a *preprojective component* $\mathcal{P}_\Delta$ (that is, $\mathcal{P}_\Delta$ has no oriented cycles and for every $X \in \mathcal{P}_\Delta$ there is a translate $\tau^n X$, for $n \geq 0$, which is projective). There is *preinjective component* $\mathcal{I}_\Delta$ (that is, $\mathcal{I}_\Delta$ has no oriented cycles and for every $Y \in \mathcal{I}_\Delta$ there is a translate $\tau^{-m}Y$, $m \geq 0$, which is injective). There is a set of *regular* components $\mathcal{R}_\Delta$ (a *component* C is *regular* if for every $X \in C$, $\tau^n X \in C$ is defined for all $n \in \mathbb{Z}$). An indecomposable representation X of Δ is said to be *preprojective*, or *regular*, or *preinjective*, provided it belongs to $\mathcal{P}_\Delta$, or $\mathcal{R}_\Delta$, or $\mathcal{I}_\Delta$, respectively.

If Δ is *elliptic*, then $\mathcal{R}_\Delta = \emptyset$ and $\Gamma_\Delta = \mathcal{P}_\Delta = \mathcal{I}_\Delta$ is a finite quiver.

If Δ is *parabolic*, the $\mathcal{P}_\Delta$ and $\mathcal{I}_\Delta$ are two different infinite components of Γ_Δ and $\mathcal{R}_\Delta = (T_\lambda)_{\lambda \in P_1(k)}$ is a *stable separating tubular family.* Moreover, if $T_\lambda = \mathbb{Z}\mathbb{A}_\infty / \langle n_\lambda \rangle$, then at most three $n_\lambda \neq 1$. Assume $n_{\lambda_1}, \ldots, n_{\lambda_r}$ are those $n_\lambda \neq 1$, the star

$\mathbb{T}_{n_{\lambda_1},\ldots,n_{\lambda_r}}$



is a Dynkin diagram such that $|\Delta|$ is an extension of $\mathbb{T}_{n_{\lambda_1},\dots,n_{\lambda_r}}$.

After the work of Dlab-Ringel [10] we know that for the hereditary algebra $A = k\Delta$ with $|\Delta|$ an Euclidean diagram and for any dimension vector $v \in \mathbb{N}^{\Delta_0}$, there exists an $A - k[t]$-bimodule M_v such that almost any indecomposable A-module X with $\mathbf{dim}\, X = v$ is isomorphic to $M_v \otimes_{k[t]} S_\lambda$ for some $\lambda \in k$. In particular, A is a tame algebra.

If $|\Delta|$ is *hyperbolic*, the components $\mathcal{P}_\Delta$ and $\mathcal{I}_\Delta$ are two different infinite components of Γ_Δ and every component C in $\mathcal{R}_\Delta$ is of the form $\mathbb{Z}\mathbb{A}_\infty$.

(4) The bilinear form $\langle v, w\rangle_A = \sum\limits_{i,j\in\Delta_0} v(i)w(j) - \sum\limits_{i\to j} v(i)w(j)$ satisfies

$$\langle \mathbf{dim}\, X, \mathbf{dim}\, Y\rangle_A = \dim_k \mathrm{Hom}_A(X,Y) - \dim_k \mathrm{Ext}^1_A(X,Y)$$

for any pair of modules $X, Y \in \mathrm{mod}_A$. In particular,

$$q_A(\mathbf{dim}\, X) = \dim_k \mathrm{End}_A(X) - \dim_k \mathrm{Ext}^1_A(X,X)$$

coincides with the Euler form of A.

[*Proof:* Apply $\mathrm{Hom}_A(-,Y)$ to the projective presentation of X.]

A module X with $\mathrm{End}_A(X) = k$ is called a *brick*. Observe that a brick is indecomposable. Moreover, an indecomposable A-module X with $\mathrm{Ext}^1_A(X,X) = 0$ is a brick.

Lemma. *If X is indecomposable not a brick, then X has a submodule which is a brick with self extensions.*

Proof. By induction, it suffices to show that X has a proper submodule which is indecomposable with self extensions.

Let $f \in \mathrm{End}_A(X)$ with $E = \mathrm{Im}\, f$ of minimal dimension > 0. Since X is indecomposable, then f is nilpotent and minimality implies that $f^2 = 0$. Hence $E \subset \ker f = \bigoplus_{i=1}^m K_i$ with K_i indecomposable modules, $i = 1, \dots, m$. Assume $\alpha\colon E \to \ker f \twoheadrightarrow K_j$ is not zero. Then α is mono (by minimality). We have $\mathrm{Ext}^1_A(E, K_j) \neq 0$ since the pushout

$$\begin{array}{ccccccccc} 0 & \to & \bigoplus_{i=1}^{m} K_i & \to & X & \to & E & \to & 0 \\ & & \downarrow\!\!\downarrow & & \downarrow & & \| & & \\ 0 & \to & K_j & \to & Z & \to & E & \to & 0 \end{array}$$

does not split. Finally, α induces a surjection $\mathrm{Ext}^1_A(K_j, K_j) \to \mathrm{Ext}^1_A(E, K_j)$, which shows that K_j is the wanted submodule of X. $\square$

Gabriel's theorem for representation-finite hereditary algebras can now be proved.

Theorem [12, 13]. *Let Δ be a quiver without oriented cycles and $A = k\Delta$ the corresponding path algebra. Then A is representation-finite if and only if $|\Delta|$ is a Dynkin diagram. The correspondence $X \mapsto \mathbf{dim}\, X$ induces a bijection between the isoclasses of indecomposable A-modules and the positive roots of q_A.*

Proof. Assume first that Δ is of Dynkin type, in particular, q_A is positive. Let X be an indecomposable A-module. Then X is a brick, since otherwise there is a brick $Y \subset X$ with self extensions and

$$q(\mathbf{dim}\, Y) = \dim_k \mathrm{End}_A(Y) - \dim_k \mathrm{Ext}^1_A(Y, Y) < 0.$$

Therefore $\mathbf{dim}\, X$ is a positive root of q_A.

An argument of Drozd (see Lecture 3, § 2) shows that q_A admits only finitely many positive roots. Then A is representation-finite.

Injectivity: Assume Y is another indecomposable with $\mathbf{dim}\, X = \mathbf{dim}\, Y$. Then

$$1 = q_A(\mathbf{dim}\, X) = \langle \mathbf{dim}\, X, \mathbf{dim}\, Y \rangle_A = \dim_k \mathrm{Hom}_A(X, Y) - \dim_k \mathrm{Ext}^1_A(X, Y),$$

in particular $\mathrm{Hom}_A(X, Y) \neq 0$. Symmetrically, $\mathrm{Hom}_A(Y, X) \neq 0$. The description of Γ_A in (3) implies that $X \simeq Y$.

Surjectivity is shown in Lecture 3 in a more general context.

Finally, if Δ is not of Dynkin type, then $A = k\Delta$ accepts infinitely many indecomposable modules as shown by the description of the preprojective component Γ_A. $\square$

(5) Let $A = k\Delta$ be a hereditary algebras and $\chi_A(T)$ the characteristic polynomial of its Coxeter transformation. We collect the relevant information about $\chi_A(T)$ in a table:

Δ of type		Coxeter polynomial	roots $\neq 1$	period $(=p)$
Dynkin	$\mathbb{A}_n$	$V_{n+1} = \prod_{2 \leq m \mid n+1} \phi_m$	$\exp(2i\pi m_{j/p})$ $m_1, \ldots, m_n$ integers $1 \leq m_j \leq p-1$	$n+1$
	$\mathbb{D}_n$, $n \geq 4$	$\phi_2 \prod_{n \leq m \mid 2n} \phi_m$		$2(n-1)$
	$\mathbb{E}_6$	$\phi_3 \phi_{12}$		12
	$\mathbb{E}_7$	$\phi_2 \phi_{18}$		18
	$\mathbb{E}_8$	$\phi_2 \phi_{10} \phi_{30}$		30
$\tilde{\Gamma}$: affine	$\tilde{\mathbb{A}}_{p,q}$	$(T-1)^2 V_p V_q$	$\exp(2i\pi m_{j/p'})$ $1 \leq m_j \leq p'$ integers $p' =$ period of Γ	
	$\tilde{\mathbb{D}}_n$	$(T-1)^2 V_2^2 V_{n-2}$		
	$\tilde{\mathbb{E}}_n$, $n=6,7,8$	$(T-1)^2 V_2 V_3 V_{n-3}$		

Notation: $V_n = (T^n - 1)/(T-1)$ and $\phi_m = V_m / \prod_{d \mid m, 1<d<m} \phi_d$ is the m-th cyclotomic polynomial. Moreover, the *period* (Coxeter number) indicates the minimal number n such that $\varphi_A^n = \mathrm{id}$.

For $A = k\Delta$, let $\rho(\varphi_A)$ (also denoted by ρ_Δ) be the *spectral radius* of φ_A, that is, $\rho(\varphi_A) = \max\{|\lambda| \colon \lambda \text{ a root of } \chi_A(t)\}$. If Δ is of Dynkin or affine type, then $\rho(\varphi_A) = 1$, as can be seen in the table above.

In case A is wild, it is known that $1 < \rho(\varphi_A)$ is a simple root of the Coxeter polynomial $\chi_A(T)$, [35]. Then by [33], there is a vector $y^+ \in K_0(A) \otimes_{\mathbb{Z}} \mathbb{R}$ with positive coordinates such that $y^+ \varphi_A = \rho(\varphi_A) y^+$. Since $\chi_A(T)$ is self reciprocal, there is a vector $y^- \in K_0(A) \otimes_{\mathbb{Z}} \mathbb{R}$ with positive coordinates such that $y^- \varphi_A = \rho(\varphi_A)^{-1} y^-$. The vectors y^+, y^- play an important role in the representation theory of $A = k\Delta$. Namely (see [33]), for an indecomposable A-module X:

(a) X is a preprojective A-module if and only if $\langle y^-, [X] \rangle_A < 0$

(b) X is a preinjective A-module if and only if $\langle [X], y^+ \rangle_A < 0$.

(c) X is regular if and only if $\langle y^-, [X] \rangle > 0$ and $\langle [X], y^+ \rangle > 0$.

(d) If X is preprojective or regular, then $\lim_{n\to\infty} \frac{1}{\rho(\varphi_A)^n}[\tau_A^{-n} X] = \lambda_X^- y^-$, for some $\lambda_X^- > 0$.

(e) If X is preinjective or regular, then $\lim_{n\to\infty} \frac{1}{\rho(\varphi_A)^n}[\tau_A^{n} X] = \lambda_X^+ y^+$, for some $\lambda_X^+ > 0$.

§3. Tilted algebras. Let $A = kQ/I$ be a basic finite dimensional k-algebra. A module ${}_AT$ is called a *tilting module* if it satisfies:

(T1) $\mathrm{Ext}_A^2(T,-) = 0$

(T2) $\mathrm{Ext}_A^1(T,T) = 0$

(T3) The number of non isomorphic indecomposable direct summands of ${}_AT$ is the rank of the Grothendieck group $K_0(A)$.

Let $B = \mathrm{End}_A(T)$. Then ${}_AT$ defines a torsion theory $(\mathcal{F},\mathcal{G})$ in mod_A and a torsion theory $(\mathcal{Y},\mathcal{X})$ in mod_B as follows:

$$\mathcal{F} = \mathcal{F}(T) = \{{}_AX : \mathrm{Hom}_A(T,X) = 0\}, \qquad \mathcal{G} = \mathcal{G}(T) = \{{}_AX : \mathrm{Ext}_A^1(T,X) = 0\}$$
$$\mathcal{Y} = \mathcal{Y}(T) = \{{}_BN : \mathrm{Tor}_1^B(T,N) = 0\}, \qquad \mathcal{X} = \mathcal{X}(T) = \{{}_BN : T \otimes \ {}_BN = 0\}$$

Then we have equivalences:

$$\Sigma_T = \mathrm{Hom}_A(T,-) \colon \mathcal{G} \to \mathcal{Y} \text{ with inverse } T\otimes_B -$$

and

$$\Sigma'_T = \mathrm{Ext}_A^1(T,-);\ \mathcal{F} \to \mathcal{X} \text{ with inverse } \mathrm{Tor}_1^B(T,-).$$

Given a tilting module ${}_AT$ with $B = \mathrm{End}_A(T)$, there is a linear isomorphism $\sigma_T \colon K_0(A) \to K_0(B)$ given by $(\mathbf{dim}\, X)\sigma_T = \mathbf{dim}\, \Sigma_T X - \mathbf{dim}\, \Sigma'_T X$.

In particular, the following formulae hold:

$$C_A^{-t} = \sigma_T C_B^{-t} \sigma_T^t,\ \langle x,y\rangle_A = \langle x\sigma_T, y\sigma_T\rangle_B.$$

In particular $\chi_A(y) = \chi_B(y\sigma_T)$.

Moreover, if $X \in \mathcal{G}(T)$, then $\chi_A(\mathbf{dim}\, X) = \chi_B(\mathbf{dim}\, \Sigma X)$ and if $X \in \mathcal{F}(T)$, then $\chi_A(\mathbf{dim}\, X) = \chi_B(\mathbf{dim}\, \Sigma' X)$. Finally, also $\Phi_A\sigma_T = \sigma_T\Phi_B$.

In case $A = k\Delta$ is a hereditary algebra and ${}_AT$ is a tilting module, $B = \mathrm{End}_A(T)$ is called a *tilted algebra* of type Δ. Observe that in this case $gl\mathrm{dim}\, B \le 2$ and the Euler and the Tits form of B coincide.

Theorem. *Let $A = k\Delta$ and B be a tilted algebra of type Δ. The following are equivalent:*

(a) *B is tame*

(b) *the Euler form $\chi_B(= q_B)$ is weakly non negative.*

The implication a) $\Rightarrow$ b) is shown in greater generality in Lecture 3. For the converse we need some preparation, namely a better knowledge of the structure of Γ_B.

Let A be a wild hereditary algebra. Let ${}_AT = T_1 \oplus \cdots \oplus T_m$ be a decomposition into indecomposables of a tilting module T. Consider the tilted algebra $B = \mathrm{End}_A(T)$. The following description of $\mathrm{mod}\, B$ is given in [21].

Let $(\mathcal{F}(T), \mathcal{G}(T))$, and $(\mathcal{X}(T), \mathcal{Y}(T))$ be the torsion theories of mod_A and mod_B respectively, corresponding to the tilting module T. Recall that $(\mathcal{X}(T), \mathcal{Y}(T))$ splits.

Let $I = \{1 \leq i \leq n \colon \mathrm{Ext}^1_A(T_i, X) \neq 0$ for infinitely many indecomposables $X \in \mathcal{F}(T)\}$ and $J = \{1 \leq i \leq n \colon \mathrm{Hom}_A(T_i, X) \neq 0$ for infinitely many indecomposables $X \in \mathcal{G}(T)\}$ and define $T_\infty = \bigoplus\limits_{i \in I} T_i$ and ${}_\infty T = \bigoplus\limits_{j \in J} T_j$. Then the *end algebras* are defined as the rings of endomorphisms $B_\infty = \mathrm{End}_A(T_\infty)$ and ${}_\infty B = \mathrm{End}_A({}_\infty T)$. With this notation we have:

Proposition.

(a) ${}_\infty B$ *is a tilted algebra. There exists a convex subalgebra* ${}_\infty A$ *of* A *and a tilting module* $\hat{T}$ *of* ${}_\infty A$ *without preinjective direct summands such that* ${}_\infty B = \mathrm{End}_{{}_\infty A}(\hat{T})$.

(b) *There exists a functor* $\varphi \colon \mathrm{mod}_{{}_\infty A} \to \mathrm{mod}_A$ *such that the restriction* $\varphi_{\mathcal{G}} \colon \mathcal{G}(\hat{T}) \to \mathcal{G}(T)$ *is fully faithful, exact, extension closed and cofinite.*

The formulation corresponding to B_∞ is dual. $\square$

Proof of the Theorem: Assume B is wild. Since $\mathrm{mod}_B = \mathcal{X}(T) \vee \mathcal{Y}(T)$ one of the subcategories $\mathcal{X}(T)$ or $\mathcal{Y}(T)$ is not tame. Say $\mathcal{Y}(T)$. Therefore $\mathcal{G}(T)$ is not a tame subcategory of $\mathrm{mod}\, A$. With the notation above, $\varphi \colon \mathcal{G}(\hat{T}) \to \mathcal{G}(T)$ is cofinite and ${}_\infty A$ is wild.

Consider the finite dimensional algebra $C = k\langle x, y\rangle/(x^2, y^2, xy, yx)$ and a full exact embedding $\xi \colon \mathrm{mod}_C \to \mathrm{mod}_{{}_\infty A}$. Let S be the unique simple C-module and consider its image $X = \xi(S)$. We have $\mathrm{End}_{{}_\infty A}(X) \xrightarrow{\sim} k$ and $\dim_k \mathrm{Ext}^1_{{}_\infty A}(X, X) \geq 2$ in particular X is regular in $\mathrm{mod}_{{}_\infty A}$ and $\chi_{{}_\infty A}(\mathbf{dim}\, X) \leq 0$.

Since $\hat{T}$ does not have preinjective direct summands, there exists an $N \in \mathbb{N}$ such that $Y = \tau_{\infty A}^{N} X \in \mathcal{G}(\hat{T})$. Therefore $Z = \mathrm{Hom}_A(T, \varphi(Y)) \in \mathcal{Y}(T)$ and

$$\chi_B(\mathbf{dim}\, Z) = \chi_A(\mathbf{dim}\, \varphi(Y)) = \chi_{\infty A}(\mathbf{dim}\, Y) = \chi_{\infty A}(\mathbf{dim}\, X) < 0. \qquad \square$$

Lecture 2. The geometric approach.

§1. Some elements of algebraic geometry.

We consider the affine space $V = k^n$ with the *Zariski topology*, that is, closed sets are of the form

$$Z(p_1, \dots, p_s) = \{v \in V \colon p_i(v) = 0, \text{ for all } i = 1, \dots, s\},$$

where $p_i \in k[t_1, \dots, t_n]$ is a polynomial in n indeterminates. The following fundamental facts may be found in any book on algebraic geometry.

- $S \subset k[t_1, \dots, t_n]$, then $Z(S)$ is the zero set of S.
- $Z(S) = Z(\langle S \rangle) = Z(\sqrt{\langle S \rangle})$, where

$$\begin{aligned} \langle S \rangle &= \text{ideal of } k[t_1, \dots, t_n] \text{ generated by } S \\ \sqrt{I} &= (\text{radical of } I) = \{p \in k[t_1, \dots, t_n] \colon p^i \in I \text{ for some } i \in \mathbb{N}\} \end{aligned}$$

- $Z\left(\bigcup_{i \in I} S_i\right) = \bigcap_{i \in I} Z(S_i)$ and $Z(S \cdot S') = Z(S) \cup Z(S')$
- *Hilbert's basis theorem*: $\exists\, p_1, \dots, p_s \in S$ with $Z(S) = Z(p_1, \dots, p_s)$
- *Hilbert's Nullstellensatz*: $\{p \in k[t_1, \dots, t_n] \colon p \equiv 0 \text{ on } Z(S)\} = \sqrt{\langle S \rangle}$

  We say that $Z = Z(S)$ is an *affine variety* and $k[Z] = k[t_1, \dots, t_n]/\sqrt{\langle S \rangle}$ is its *coordinate ring*.

  An affine variety $Z = Z(p_1, \dots, p_s)$ is *reducible* if $Z = Z_1 \cup Z_2$ with proper closed subsets $Z_i \subset Z$. Otherwise Z is *irreducible*.
- There is a finite decomposition of any affine variety $Z = \bigcup_{i=1}^{s} Z_i$ into irreducible subsets $Z_i \subset Z$. If the decomposition is irredundant, we say that $Z_1, \dots, Z_s$ are the *irreducible components* of Z.
- If Z is an irreducible variety, then the maximal length of a chain

$$\emptyset \neq Z_0 \subsetneq Z_1 \subsetneq \cdots \subsetneq Z_s = Z$$

  is called the *dimension* of Z ($=\colon \dim Z$).

  If $Z = \bigcup_{i=1}^{s} Z_i$ is an irreducible decomposition

$$\dim Z = \max_i \dim Z_i.$$

A map $\mu\colon Y \to Z$ between affine varieties is a *morphism* (a *regular map*), if $\mu^*\colon k[Z] \to k[Y]$, $p \mapsto p \circ \mu$ is well-defined. In fact, μ^* is a k-algebra homomorphism.

- Any morphism $\mu\colon Y \to Z$ is continuous in the Zariski topology.
- A map $\mu\colon Y \to Z$ is a morphism if and only if $\exists\, \mu_1, \ldots, \mu_m \in k[t_1, \ldots, t_n]$ such that $\mu(y) = (\mu_1(y), \ldots, \mu_m(y))$, $\forall\, y = (y_1, \ldots, y_n) \in Y \subset k^n$.

Proposition. *Let $\mu\colon Y \to Z$ be a morphism between irreducible affine varieties and assume μ is dominant (i.e. $\overline{\mu(Y)} = Z$). Then for every $z \in Z$ and every irreducible component C of $\mu^{-1}(Z)$ we have*

$$\dim C \geq \dim Y - \dim Z$$

with equality on a dense open set of Z.

In particular, if C is an irreducible component of $Z(p_1, \ldots, p_t) \subset k^n$, we have

$$\dim C \geq n - t$$

A fundamental result is the following

Theorem (Chevalley) *Let $\mu\colon Y \to Z$ be a morphism between affine varieties. Then the function*

$$y \mapsto \dim_y \mu^{-1}(\mu(y)) = \max\{\dim C\colon y \in C \text{ irreducible component of } \mu^{-1}(\mu(y))\}$$

is upper semicontinuous (that is, $d\colon Y \to \mathbb{N}$ has $\{y \in Y\colon d(y) < n\}$ open in Y, for all $n \in \mathbb{N}$).

As illustration consider $\mu\colon \mathbb{C}^3 \to \mathbb{C}^2$ with $\mu(x, y, z) = (x, xy)$. Then

$$\mu^{-1}(\mu(x_0, y_0, z_0)) = \mu^{-1}(x_0, x_0 y_0) = \begin{cases} (x_0, y_0, x) & \text{if } x_0 \neq 0,\ \dim = 1 \\ (0, y, z) & \text{if } x_0 = 0,\ \dim = 2 \end{cases}$$

A general morphism $\mu\colon Y \to Z$ is neither open nor closed, but $\mu(Y)$ is a finite union of locally closed subsets of Z. A finite union of locally closed subsets of a variety Z is called a *constructible* subset.

Proposition. *If $\mu\colon Y \to Z$ is a morphism and $Y' \subset Y$ a constructible subset, then $\mu(Y')$ is also constructible.*

§2. The main example: module varieties.

Let $A = kQ/I$ be a finite dimensional k-algebra and fix a finite set $R \subset \bigcup_{x,y \in Q_0} I(x,y)$ of admissible generators of I. Let $z \in \mathbb{N}^{Q_0}$ be a dimension vector.

The *module variety* $\mathrm{mod}_A(z)$ is the closed subset, with respect to the Zariski topology, of the affine space $k^z = \prod_{x \to y} k^{z(y)z(x)}$ defined by the polynomial equations given by the entries of the matrices

$$m_r = \sum_{i=1}^{t} \lambda_i m_{\alpha i1} \ldots m_{\alpha i s_i}, \text{ where } r = \sum_{i=1}^{t} \lambda_i \alpha_{i1} \ldots \alpha_{is_i} \in R$$

and for each arrow $x \xrightarrow{\alpha} y$, m_α is the matrix of size $z(y) \times z(x)$.

$$m_\alpha = (X_{\alpha ij})_{ij}$$

where $X_{\alpha ij}$ are pairwise different indeterminates. We shall identify points in the variety $\mathrm{mod}_A(z)$ with representations X of A with vector dimension $\mathbf{dim}\, X = z$.

Example: $A = kQ/I$ where $Q\colon \bullet \xrightarrow{\alpha} \bullet \xrightarrow{\beta} \bullet$ and $I = \langle \alpha\beta \rangle$

$$\begin{pmatrix} x_{\alpha 11} & x_{\alpha 12} \\ x_{\alpha 21} & x_{\alpha 22} \end{pmatrix} \begin{pmatrix} x_{\beta 11} & x_{\beta 12} \\ x_{\beta 21} & x_{\beta 22} \end{pmatrix} = \begin{pmatrix} x_{\alpha 11}x_{\beta 11} + x_{\alpha 12}x_{\beta 21} & x_{\alpha 11}x_{\beta 12} + x_{\alpha 12}x_{\beta 22} \\ x_{\alpha 21}x_{\beta 11} + x_{\alpha 22}x_{\beta 21} & x_{\alpha 21}x_{\beta 12} + x_{\alpha 22}x_{\beta 22} \end{pmatrix}$$

$\mathrm{mod}_A(2,2,2) \subset k^{2\times 2} \times k^{2 \times 2} = k^8$ defined by 4 equations.

The group $G(z) = \prod_{i \in Q_0} GL_{z(i)}(k)$ acts on k^z by conjugation, that is, for $X \in k^z$, $g \in G(z)$ and $x \xrightarrow{\alpha} y$, then $X^g(\alpha) = g_y X(\alpha) g_x^{-1}$. By restriction of this action, $G(z)$ also acts on $\mathrm{mod}_A(z)$. Moreover, there is a bijection between the isoclasses of A-modules X with $\mathbf{dim}\, X = z$ and the $G(z)$-orbits in $\mathrm{mod}_A(z)$.

Given $X \in \mathrm{mod}_A(z)$, we denote by $G(z)X$ the $G(z)$-orbit of X. Then

$$\dim G(z)X = \ dim\ G(z) - \ \dim \mathrm{Stab}_{G(z)}(X),$$

where the *stabilizer* $\mathrm{Stab}_{G(z)}(X) = \{g \in G(z) : X^g = X\} = \ \mathrm{Aut}_A(X)$ is the group of automorphisms of X. As $\mathrm{Aut}_A(X)$ is an open subset of the affine variety $\mathrm{End}_A(X)$, then

$$\dim \mathrm{Stab}_{G(x)}(X) = \dim \mathrm{Aut}_A(X) = \dim \mathrm{End}_A(X).$$

Finally, we get

$$\dim G(z)X = \dim G(z) - \dim \mathrm{End}_A(X).$$

Moreover, the orbit $G(z)X$ is *locally closed*, that is $G(z)X$ is open in the closure $G(z)X$ defined in $\text{mod}_A(z)$. In particular, $G(z)X \backslash G(z)X$ is formed by the union of orbits of dimension strictly smaller than $G(z)X$.

Let $X, Y \in \text{mod}_A(z)$. If the orbit $G(z)Y$ is contained in $\overline{G(z)X}$, we say that Y is a *degeneration* of X.

Proposition. *Let $X \in \text{mod}_A(z)$. We have the following.*

(a) *Let $0 \longrightarrow X' \longrightarrow X \longrightarrow X'' \longrightarrow 0$ be an exact sequence. Then $X' \oplus X''$ is a degeneration of X.*

(b) *Consider the semisimple module gr $X = \underset{i \in Q_0}{\oplus} S_i^{z(i)}$, obtained as direct sum of the composition factors of X. Then gr X is a degeneration of X.*

Proof of (a): We may assume that X' is a submodule of X and $X'' = X/X'$. Then for each arrow $i \xrightarrow{\alpha} j$, we have

$$X(\alpha) = \begin{pmatrix} X'(\alpha) & f_\alpha \\ 0 & X''(\alpha) \end{pmatrix},$$

where $f_\alpha : X''(i) \longrightarrow X'(j)$. For each $\lambda \in k$, we may define the representation $X_\lambda \in \text{mod}_A(z)$, with

$$X_\lambda(\alpha) = \begin{pmatrix} X'(\alpha) & \lambda f_\alpha \\ 0 & X''(\alpha) \end{pmatrix}.$$

For $\lambda \neq 0$, we get $X_\lambda \simeq X$. Indeed,

$$g_\lambda = \begin{pmatrix} I_{z'(i)} & 0 \\ 0 & \lambda I_{z''(i)} \end{pmatrix}_i \in G(z)$$

satisfies that $X_\lambda^{g_\lambda} = X$. Therefore

$$X' \oplus X'' = X_0 \in \overline{G(z)X}.$$ □

Corollary. *The orbit $G(z)X$ is closed if and only if X is semisimple.* □

Examples: (a) Let $F = k\langle T_1, \dots, T_m\rangle$ be the free algebra in m indeterminates. Let M be a $A - F$-bimodule which is free as right F-module.

Then the functor $M \otimes_F - : \text{mod}_F \longrightarrow \text{mod}_A$ induces a family of regular maps $f_M^n : \text{mod}_F(n) \to \text{mod}_A(nz)$ for some vector $z \in \mathbb{N}^{Q_0}$ and every $n \in \mathbb{N}$.

Indeed, for each vertex $i \in Q_0$, fix a basis of the free right F-module $M(i)$, set $z(i) = rk_F M(i)$. Then for an arrow $i \xrightarrow{\alpha} j$ in $Q, M(\alpha) : M(i) \longrightarrow M(j)$ is a

$z(j) \times z(i)$-matrix with entries in F. Now, an element $\lambda = (\lambda_1, \ldots, \lambda_m) \in \mathrm{mod}_F(n)$ determines an F-module N_λ with $N_\lambda(T_i) = \lambda_i, i = 1, \ldots, m$. Then

$$M \otimes_F N_\lambda(\alpha) : (k^{z(i)})^n \longrightarrow (k^{z(j)})^n$$

is the matrix $M(\alpha)(\lambda) = (M(\alpha)_{st}(\lambda_1, \ldots, \lambda_m))_{s,t}$. Therefore

$$f_M^n(\lambda) = (M(\alpha)_{st}(\lambda_1, \ldots, \lambda_m))_{s,t}$$

is the induced regular map.

(b) Let C be a finitely generated commutative k-algebra without nilpotent elements and $z \in \mathbb{N}^{Q_0}$. For any regular map $g : \mathrm{mod}\ _C(1) \longrightarrow \mathrm{mod}_A(z)$, there is a $A-C$-bimodule M which is free as right C-module and $rk_C(M)(i) = z(i)$, for each $i \in Q_0$, such that $g = f_M^1$.

Indeed, from Hilbert's theorem $C = k[\mathrm{mod}_C(1)]$ is the affine algebra of regular functions on $\mathrm{mod}_C(1)$. We define $M(i) = C^{z(i)}$, for $i \in Q_0$; for $i \xrightarrow{\alpha} j$ in Q, we put $M(\alpha)$ the matrix corresponding to $g(\alpha) : \mathrm{mod}_C(1) \longrightarrow k^{z(j)z(i)}$. By (a), $f_M^1 = g$.

(c) Consider the subset $\mathrm{ind}_A(z)$ of $\mathrm{mod}_A(z)$ $\mathrm{ind}_A(z)$ is a constructible subset of $\mathrm{mod}_A(z)$. Indeed, the set of pairs.

$$\{(X, f) : X \in \ \mathrm{mod}_A(z), f \in \ \mathrm{End}_A(X) \text{ with } 0 \neq f \neq 1_X \text{ and } f^2 = 1_X\}.$$

is a locally closed subset of $\mathrm{mod}_A(z) \times k^{d^2}$, where $d = \sum_{i \in Q_0} z(i)$. The projection $\pi_1 : \mathrm{mod}_A(z) \times k^{d^2} \longrightarrow \mathrm{mod}_A(z)$ is a regular map with image

$$\mathrm{mod}_A(z) \backslash \mathrm{ind}_A(z).$$

(d) Let $z \in \mathbb{N}^{Q_0}$. Let C be an irreducible component of $\mathrm{mod}_A(z)$. A decomposition $z = w_1 + \cdots + w_s$ with $w_i \in \mathbb{N}^{Q_0}$ determines a constructible subset

$$C(w_1, \ldots, w_s) = \{X \in C : X = X_1 \oplus \cdots \oplus X_s \text{ with } X_i \in \ \mathrm{ind}_A(w_i)\}$$

in C. We say that $(w_1, \ldots, w_s)$ is a *generic decomposition* in C if $C(w_1, \ldots, w_s)$ contains an open and dense subset of C.

Proposition. *Let C be an irreducible component of* $\mathrm{mod}_A(z)$, *then there exists a unique generic decomposition* $(w_1, \dots, w_s)$ *in* C. *Moreover, there exists an irreducible component* C_i *of* $\mathrm{mod}_A(w_i)$ *such that the generic decomposition in* C_i *is* (w_i) *and the following inequality holds:*

$$\dim G(z) - \dim C \geq \sum_{i=1}^{s} (\dim G(w_i) - \dim C_i).$$

Proof. For each decomposition $z = z_1 + \cdots + z_t$ with $z_i \in \mathbb{N}^{Q_0}$ we get a regular map

$$\varphi_{z_1 \dots z_i} : G(z) \times \mathrm{mod}_A(z_1) \times \cdots \times \mathrm{mod}_A(z_t) \longrightarrow \mathrm{mod}_A(z), (g, (X_i)_i) \longmapsto (\oplus_{i=1}^{t} X_i)^g.$$

Since $\mathrm{ind}_A(z_i) = \{Y \in \mathrm{mod}_A(z_i) : Y$ is indecomposable$\}$ is constructible in $\mathrm{mod}_A(z_i)$, then

$$\mathrm{ind}_A(z_1, \dots, z_t) = \varphi_{z_1, \dots z_t}(G(z) \times \mathrm{ind}_A(z_1) \times \cdots \times \mathrm{ind}_A(z_t))$$

is constructible in $\mathrm{mod}_A(z)$. Moreover, $\mathrm{mod}_A(z) = \cup\{\mathrm{ind}_A(z_1, \dots, z_t) : \sum z_i = z\}$. There is a decomposition $z = w_1 + \cdots + w_s$ such that C equals the closure of the intersection $\mathrm{ind}_A(w_1, \dots, w_s) \cap C$. There is an open dense subset U_C of C contained in $\mathrm{ind}_A(w_1, \dots, w_s)$. Thus $z = w_1 + \cdots + w_s$ is generic in C. The unicity is clear. $\square$

§3. The tangent space.

Suppose $V \subset k^n$ is defined by certain polynomials $f(T_1, \dots, T_n)$. For $x \in V$, define

$$d_x f = \sum_{i=1}^{n} \frac{\partial f}{\partial T_i}(x)(T_i - x_i)$$

the *derivative of f at the point x*. Then the *tangent space* of V at x is the linear variety $T_x(V)$ in the k^n defined by the vanishing of all $d_x f$ as $f(T)$ ranges over the polynomials in the radical ideal $\mathcal{I}(V)$ defining V.

There are more algebraic ways to define tangent spaces: let $R = k[V]$ be the affine algebra associated with V and M_x be the maximal ideal of R vanishing at x. Since R/M_x can be identified with k and M_x is a finitely generated R-module, then then R/M_x-module M_x/M_x^2 is a finite dimensional k-vector space. Then $(M_x/M_x^2)^*$ the dual space over k may be identified with $T_x(V)$.

Some facts and examples:

(a) Let $x \in V$ and C_x be any irreducible component of X containing x. Then we have $\dim_k T_x(V) \geq \dim C_x$. If equality holds, x is called a *simple point of* V. If all points of V are simple, we say that V is *smooth*. An important fact:

- the simple points of V form an open dense subset of V.

(b) Consider the variety $\mathrm{mod}_A(z)$ as a topological space. The orbit $G(z)X$ of a point $X \in \mathrm{mod}_A(z)$ is a smooth space. Indeed, given two points x, y in the orbit, there is an element g of the group $G(z)$ such that $y = gx$. The regular map $\ell_g : G(z)X \longrightarrow G(z)X$ given as right multiplication by g, induces a linear isomorphism $T\ell_g : T_x(G(z)X) \longrightarrow T_y(G(z)X)$. Therefore x is a simple point of the orbit if and only if so is y. Thus (a) implies that $G(z)X$ is smooth.

The following is an important result:

Theorem [40]. *Let* $X \in \mathrm{mod}_A(z)$.

Consider $T_X(G(z)X)$ *as a linear subspace of* $T_X(\mathrm{mod}_A(X))$. *Then there exists a natural linear monomorphism*

$$T_X(\mathrm{mod}_A(X))/T_X(G(z)X) \hookrightarrow \mathrm{Ext}^1_A(X, X).$$

(b) *Assume that* X *satisfies* $\mathrm{Ext}^2_A(X, X) = 0$. *Then the linear morphism*

$$T_X(\mathrm{mod}_A(X))/T_X(G(z)X) \xrightarrow{\sim} \mathrm{Ext}^1_A(X, X).$$

is an isomorphism.

We will observe several consequences:

(a) For any $X \in \mathrm{mod}_A(z)$, let C_X be an irreducible component of $\mathrm{mod}_A(z)$ containing X. Then

$$\begin{aligned} \dim_k \mathrm{Ext}^1_A(X, X) &\geq \dim_k T_X(\mathrm{mod}_A(z)) - \dim_k T_X(G(z)X) \\ &\geq \dim C_X - \dim G(z)X \\ &= \dim C_X - \dim G(z) + \dim_k \mathrm{End}_A(X). \end{aligned}$$

Hence,

$$\dim G(z) - \dim C_X \geq \dim_k \mathrm{End}_A(X) - \dim \mathrm{Ext}^1_A(X, X)$$

(b) The inclusion above is not always an isomorphism, as the following simple example shows:

Let $A = k[T]/(T^2)$. Consider the simple module $S \in \mathrm{mod}_A(1)$. Then $\mathrm{mod}_A(1) = G(1)S = \{S\}$ and T_S $(\mathrm{mod}_A(1))$ is trivial. On the other hand $\mathrm{Ext}^1_A(S,S)$ has dimension 1.

Exercises: (1) Let $X \in \mathrm{mod}_A(z)$. Then $G(z)X$ is open if and only if $T_X(\mathrm{mod}_A(z)) = T_X(G(z)X)$.

(2) Let $n \in \mathbb{N}$, the function

$$e^n \colon \mathrm{mod}_A(z) \to \mathbb{N}, \qquad x \mapsto \dim_k \mathrm{Ext}^n_A(X,X)$$

is upper semicontinuous.

(3) Up to isomorphism, there are only finitely many modules X with $\mathbf{dim} X = z$ and satisfying $\mathrm{Ext}^1_A(X,X) = 0$.

§4. Tame algebras and varieties.

Proposition. *The following conditions are equivalent:*

(T_0): *A is tame.*

(T_1): *For each $z \in \mathbb{N}^{Q_0}$, there is a constructible subset C of $\mathrm{mod}_A(z)$ satisfying $\dim\, C \leq 1$ and $\mathrm{ind}_A(z) \subset G(z)C$.*

(T_2): *For each $z \in \mathbb{N}^{Q_0}$, if C is a constructible subset of $\mathrm{ind}_A(z)$ intersecting each orbit of $G(z)$ in at most one point, then $\dim\, C \leq 1$.*

Proof. (T_0) $\Longrightarrow$ (T_1): Let $z \in N^{Q_0}$. Let $M_1, \ldots, M_s$ be the $A - k[t]$-bimodules such that M_i is a free finitely generated $k[t]$-module and any $X \in \mathrm{ind}_A(z)$ is isomorphic to $M_i \otimes_{k[t]} S$ for some i and some simple $k[t]$-module S. Therefore, the functor $M_i \otimes_{k[t]} (-)$ induces a regular map $f_i : \mathrm{mod}_{k[t]}(1) \longrightarrow \mathrm{mod}_A(z), i = 1, \ldots, s$.

The set

$$C = \bigcup_{i=1}^{s} (\mathrm{Im}\ f_i \cap\ \mathrm{ind}_A(z))$$

is a constructible subset of $\mathrm{ind}_A(z)$ with $\dim\, C \leq 1$ and $G(z)C = \mathrm{ind}_A(z)$.

(T_2) $\Longrightarrow$ (T_0): Assume that A is not tame. Then by the tame-wild dichotomy, the algebra A is *wild*. That is, there exists a $A - k\langle u, v\rangle$-bimodule M which is free finitely generated as right $k\langle u,v\rangle$-module and such that the functor $M \otimes_{k\langle x,y\rangle} (-)$: $\mathrm{mod}_{k\langle u,v\rangle} \longrightarrow \mathrm{mod}_A$ insets indecomposable modules.

Let $z \in N^{Q_0}$, where $z(x)$ is the rank of the free $k\langle u,v\rangle$-module $M(x)$. We get an induced regular map $f_M : \mathrm{mod}_{k\langle u,v\rangle}(1) \longrightarrow \mathrm{mod}_A(z)$. By definition, Im f_M is a constructible subset of $\mathrm{ind}_A(z)$ intersecting each orbit in at most one point. Moreover, f_M is injective and theferefore dim Im $f_M = 2$. □

Corollary. *An algebra can not both tame and wild.* □

Proposition. *Let $A = kQ/I$ be a tame algebra. Then for every $z \in \mathbf{N}^{Q_0}$,*

$$\dim\ \mathrm{mod}_A(z) \leq\ \dim\ G(z)$$

Proof: By (1.4), it is enough to show that dim $G(z)-$ dim $C \geq 0$, for an irreducible component C of $\mathrm{mod}_A(z)$

Since A is tame, we may choose a $A - k[t]$-bimodule M which is free as right $k[T]$-module and the following map is dominant

$$\varphi : G(z) \times \mathrm{Im} f^1_M \longrightarrow C, \qquad (g,X) \longmapsto X^g.$$

Let $X \in$ Im φ be such that dim $\varphi^{-1}(X) =$ dim $G(z)-$ dim $C+$ dim Im f^1_M and $(g,Y) \in \varphi^{-1}(X)$. Then the regular map

$$\mathrm{Aut}_A(Y) \longrightarrow \varphi^{-1}(X), \quad h \longmapsto (hg,Y)$$

is injective. Therefore,

$$0 \leq\ \dim\ \mathrm{Aut}_A(Y) - 1 \leq\ \dim\ G(z) -\ \dim\ C$$ □

Example: The converse of the above results are not true.

Let $A_m = k[\alpha_1,\ldots,\alpha_m]/(\alpha_i\alpha_j : 1 \leq i \leq j \leq m)$ with $m \geq 3$. We will calculate $\dim \mathrm{mod}_{Am}(n)$.

We get

$$\dim\ \mathrm{mod}_{A_m}(n) = \begin{cases} \left(\frac{m+1}{4}\right) n^2 & \text{if } n \text{ even} \\ \left(\frac{m+1}{4}\right)(n^2-1) & \text{if } n \text{ odd.} \end{cases}$$

If $m = 3$, then dim $\mathrm{mod}_{A_3}(n) \leq n^2$, showing that the converse of the above Proposition fails.

Lecture 3. The Tits form of an algebra.

§1. Basic results.

Let $A = kQ/I$ be a *triangular* algebra, that is, Q has no oriented cycles.

Choose R a minimal set of generators of I, such that $R \subset \bigcup_{i,j \in Q_0} I(i,j)$. We have:

- $\dim_k \mathrm{Ext}^1_A(S_i, S_j) = \#$ arrows from i to j
- $r(i,j) = |R \cap I(i,j)|$ is independent of the choice of R
- $r(i,j) = \dim_k \mathrm{Ext}^2_A(S_i, S_j)$

The *Tits form* of A is the quadratic form

$$q_A \colon \mathbb{Z}^{Q_0} \to \mathbb{Z},$$

given by $q_A(v) = \sum_{i \in Q_0} v(i)^2 - \sum_{i \to j} v(i)v(j) + \sum_{i,j \in Q_0} r(i,j)v(i)v(j)$.

Proposition. *Assume $A = kQ/I$ is triangular. Let $z \in N^{Q_0}$. Then for any $X \in$ $mod_A(z)$.*

$$q_A(z) \geq \dim_k \mathrm{End}_A(X) - \dim_k \mathrm{Ext}^1_A(X,X).$$

Proof. Let $X \in \mathrm{mod}_A(z)$. The *local dimension* $\dim_X \mathrm{mod}_A(z)$ is the maximal dimension of the irreducible components of $\mathrm{mod}_A(z)$ containing X. By Krull's Hauptidealsatz, we have

$$\dim_X \mathrm{mod}_A(z) \geq \sum_{(i-j) \in Q_1} z(i)z(j) - \sum_{ij \in Q_0} r(i,j)z(i)z(j).$$

Therefore, we get the following inequalities,

$$\begin{aligned} q_A(z) &\geq \dim G(z) - \dim_X \mathrm{mod}_A(z) \geq \dim G(z) - \dim T_X \geq \\ &\geq \dim_k \mathrm{End}_A(X) - \dim_k \mathrm{End}^1_A(X,X). \end{aligned}$$

□

In 1975, Brenner observed certain connections between properties of q_A and the representation type of A. She wrote about her observations: "…is written in the spirit of experimental science. It reports some regularities and suggests that there should be a theory to explain them".

Theorem. *Let $A = kQ/I$ be a triangular algebra.*

[3]: *If A is representation-finite, then q_A is weakly positive*

[28]: *If A is tame, then q_A is weakly non-negative*

Proof. In general, for $v \in \mathbb{N}^{Q_0}$

$$\begin{aligned} \dim \operatorname{mod}_A(v) &\geq \sum_{i \to j} v(i)v(j) - \sum_{i,j \in Q_0} r(i,j)v(i)v(j) \\ \dim G(v) &= \sum_{i \in Q_0} v(i)^2 \end{aligned}$$

$$q_A(v) \geq \dim G(v) - \dim \operatorname{mod}_A(v)$$

If A is tame, then $q_A(v) \geq 0$.

If A is representation-finite, $\operatorname{mod}_A(v) = \bigcup_{i=1}^{m} G(v)X_i$ where $X_1, \ldots, X_m$ are representatives of the isoclasses of A-modules of $\mathbf{dim} = v$. Hence $\dim \operatorname{mod}_A(v) = \dim G(v)X_j = \dim G(v) - \dim \operatorname{Stab}_{G(v)} X_j \leq \dim G(v) - 1$ and $q_A(v) \geq 1$. $\square$

Consider the algebra A given by the quiver

$$\begin{tikzcd} & & 3 \arrow[dr, "\gamma"] & \\ 1 \arrow[r, bend left, "\alpha'"] \arrow[r, bend right, "\beta'"'] & 2 \arrow[ur, "\alpha"] \arrow[rr, "\beta"'] & & 4 \end{tikzcd}$$

with relations $\gamma\alpha\alpha' = \beta\beta'$ and $\alpha\beta' = 0$. The Tits form q_A is

$$\begin{aligned} q_A(x) &= \sum_{i=1}^{4} x_1^2 - 2x_1x_2 - x_2x_3 - x_2x_4 - x_3x_4 + x_1x_3 - x_1x_4 \\ &= \left(x_1 - x_2 + \frac{1}{2}x_3 + \frac{1}{2}x_4 \right)^2. \end{aligned}$$

and therefore (weakly) non-negative. We shall see later that A is wild.

§2. Modules on preprojective components.

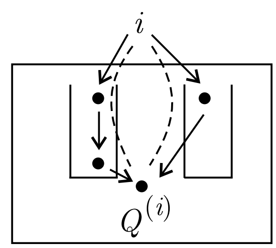
Recall that a component $\mathcal{P}$ of the Auslander-Reiten quiver Γ_A of A is called **preprojective** if it does not contain oriented cycles and for every $X \in \mathcal{P}$ there is a translate $\tau_A^n X$ which is projective. If $X \in \mathcal{P}$ and Y is an indecomposable such that $\operatorname{Hom}_A(Y, X) \neq 0$, then $Y \in \mathcal{P}$.

We give some examples of algebras with preprojective components:

(a) Let $A = k\Delta$ be a hereditary algebra. Then Γ_A has a preprojective component $\mathcal{P}$, and the indecomposable projective modules form a slice.

(b) **Tree algebras** have preprojective components (an algebra $A = kQ/I$ is a tree algebra if the underlying graph $|Q|$ of Q has no cycles). This is a particular case of the following situation.

(c) An indecomposable projective P_i is said to have **separated radical** whenever the supports of any two non-isomorphic direct summands of $rad\ P_i$ are contained in different components of the subquiver $Q^{(i)}$ of Q obtained by deleting all vertices in $[\to i] = \{j \in Q_0 : \{j \in Q_0 \colon j \rightsquigarrow i\}$. If for every vertex $i \in Q_0$, P_i has separated radical, then A satisfies the **separation condition**. Note that tree algebras satisfy the separation condition. If A satisfies the separation condition, then Γ_A has a preprojective component.



A representation-finite algebra A such that Γ_A is a preprojective component is said to be **representation-directed**.

Let Q' be a subquiver of Q, we say that Q' is **convex** in Q if Q' is path closed in Q (that is, whenever $i_0 \to i_1 \to \cdots \to i_m$ is a path in Q with $i_0, i_m \in Q'$ then $i_j \in Q'$ for $1 \leq j \leq m-1$).

Lemma. *Suppose that X is an indecomposable lying in a preprojective component $\mathcal{P}$ of Γ_A. Then supp X is convex in Q.*

Proof. Supose that $i_1 \xrightarrow{\alpha_1} i_2 \xrightarrow{\alpha_2} \cdots \xrightarrow{\alpha_{m-1}} i_m$ is a path in Q such that $X(i_1) \neq 0 \neq X(i_m)$ but $X(i_j) = 0$ for $2 \leq j \leq m-1$. Let I' be the ideal of kQ generated by all paths of the form: $\epsilon\gamma$ with $\epsilon, \gamma \in Q_1$ where either $i_1 \xrightarrow{\gamma} i_2$ and ϵ starts at i_2 or γ

ends at i_{m-1} and $i_{m-1} \xrightarrow{\epsilon} i_m$. Let $A' = kQ/(I+I')$. Then X is a A'-module and there is a chain of non-zero morphisms

$$X \longrightarrow I'_{i_m} \longrightarrow S_{m-1} \longrightarrow M_{i_{m-1}}^{i_{m-2}} \longrightarrow S_{m-2} \longrightarrow \cdots \longrightarrow S_{i2} \longrightarrow P'_{i1} \longrightarrow X$$

where M_i^j denotes the indecomposable module $k_i \to k_j$ and I'_{i_m} is the A'-module associated with i_m. Since $X \in \mathcal{P}$, this cycle should lie in $\mathcal{P}$. A contradiction. $\square$

Corollary. *Let X be a preprojective A-module. Then $q_A(\mathbf{dim}\, X) = 1$.*

Proof. We may assume that X is omnipresent in A. Then $p\mathrm{dim}_A X \leq 1$: otherwise there are non-zero maps as in the picture,

$$\begin{tikzcd}
I_s \arrow[dr] & & \bullet \arrow[dr] & \\
& \tau_A X \arrow[ur] & & X \arrow[lllu, bend right=60]
\end{tikzcd}$$

A contradiction. Similarly, $gl\mathrm{dim}\, A \leq 2$. Hence $q_A(\mathbf{dim}\, X) = \mathrm{dim}_k \mathrm{End}_A(X) - \mathrm{dim}_k \mathrm{Ext}_A^1(X,X) = 1$. $\square$

The following basic fact is due to Drozd (in Lecture 1 we already used a particular case of this result):

Lemma. *A weakly positive quadratic form $q : \mathbb{Z}^n \longrightarrow \mathbb{Z}$ has only finitely many positive roots.*

Proof. Consider q as a function $q : \mathbb{R}^n \longrightarrow \mathbb{R}$. By continuity $q(z) \geq 0$ in the positive cone $K = (\mathbb{R}^n)^+$. By induction on n, it can be shown that $q(z) > 0$ for any $0 \neq z \in K$. Let $0 < \gamma$ be the minimal value reached by q on $\{z \in K : \|z\| = 1\}$ (a compact set). Then a positive root z of q satisfies $\gamma \leq q\left(\frac{z}{\|z\|}\right) = \frac{1}{\|z\|^2}$, that is $\|z\| \leq \sqrt{1/\gamma}$. $\square$

Theorem [3]. *Let $A = kQ/I$ be an algebra such that Q has no oriented cycles. Assume that Γ_A has a preprojective component. Then A is representation-finite if and only if the Tits form q_A is weakly positive. In that case, there is a bijection $X \mapsto \mathbf{dim}\, X$ between the isoclasses of indecomposable A-modules and the positive roots of q_A.*

Proof. Assume that q_A is weakly positive. Let $\mathcal{P}$ be a preprojective component of Γ_A. Let $X \in \mathcal{P}$ then $\mathbf{dim}X$ is a root of q_A. Moreover, the map $X \to \mathbf{dim}X$, for $X \in \mathcal{P}$, is injective. Indeed, let $X, Y \in \mathcal{P}$ be such that $\mathbf{dim}\ X = \mathbf{dim}\ Y$. We may assume that X is omnipresent. Then, we get

$$1 = q_A(\mathbf{dim}\ X) = \dim_k \operatorname{Hom}_A(X,Y) - \dim_k \operatorname{Ext}^1_A(X,Y).$$

In particular, $\operatorname{Hom}_A(X,Y) \neq 0$. By symmetry, $\operatorname{Hom}_A(Y,X) \neq 0$ and $X = Y$. It follows that $\mathcal{P}$ is a finite component of Γ_A and $\mathcal{P} = \Gamma_A$.

Finally, let $z \in \mathbb{N}^{Q_0}$ be a root of q_A. Then there is a module $X \in \operatorname{mod}_A(z)$ with the orbit $G(z)X$ of dimension $\dim G(z) - 1$. Since $\dim G(z)X = \dim G(z) - \dim \operatorname{End}_A(X)$, we obtain that $\operatorname{End}_A X = k$. □

We give some **examples**.

(a) The statement of (2.3) may be false if A has no preprojective component. Consider the algebras A_i given by the quiver Q with relations $I_i = \langle \rho_i \rangle$:

$$Q: \begin{tikzcd} & & 1 \arrow[dl, "\alpha_1"'] \arrow[dr, "\beta_1"] & & & & \\ & 2 \arrow[d, "\alpha_2"'] & & 5 \arrow[dl, "\beta_2"] \arrow[r] & 6 \arrow[r] & 7 \arrow[r] & 8 \\ & 3 \arrow[r, "\alpha_3"'] & 4 & & & & \end{tikzcd}$$

$$\begin{aligned} \rho_1 &= (\alpha_3\alpha_2\alpha_1 - \beta_2\beta_1) \\ \rho_2 &= \alpha_3\alpha_2\alpha_1 \end{aligned}$$

Clearly, they have the same Tits form

$$\begin{aligned} q &= \sum_{i=1}^{8} x_i^2 - x_1x_2 - x_2x_3 - x_3x_4 - x_1x_5 - x_4x_5 - x_5x_6 - x_6x_7 - x_7x_8 + x_1x_4 \\ &= \left(x_1 - \frac{1}{2}x_2 + \frac{1}{2}x_4 - \frac{1}{2}x_5\right)^2 + \frac{3}{4}\left(x^2 - \frac{2}{3}x_3 + \frac{1}{3}x_4 - \frac{1}{3}x_5\right)^2 + \\ &+ \frac{2}{3}\left(x_3 - \frac{1}{2}x_4 - \frac{1}{4}x_5\right)^2 + \frac{1}{2}(x_4 - \frac{1}{2}x_5)^2 + \frac{1}{2}\left(x_5 - x_6\right)^2 + \frac{1}{2}(x_6 - x_7)^2 + \\ &+ \frac{1}{2}(x_7 - x_8)^2 + \frac{1}{2}x_8^2 \end{aligned}$$

which is positive.

The algebra A_1 satisfies the separation condition and Bongartz theorem applies. The algebra A_2 is not representation-finite: $\operatorname{mod} A_2$ contains the representations of the Euclidean quiver

$$\begin{array}{ccccccccc} 3 \leftarrow 2 \leftarrow 1 \rightarrow & 5 & \rightarrow 6 \rightarrow 7 \rightarrow 8 \\ & \downarrow & \\ & 4 & \end{array}$$

§3. Critical forms and critical algebras.

We recall some important facts of linear algebra

(a) Let $A = (a_{ij})$ be an $n \times n$-matrix. Let $1 \leq i_1 < i_2 < \cdots < i_s \leq n$ and $1 \leq j_1 < j_2 < \cdots < j_s \leq m$. Form the $s \times s$-matrix

$$A\begin{pmatrix} i_1 & i_2 & \dots & i_s \\ j_1 & j_2 & \dots & j_s \end{pmatrix} = \begin{pmatrix} a_{i_1j_1} & a_{i_1j_2} & \dots & a_{i_1j_s} \\ & \vdots & & \\ a_{i_sa_1} & a_{i_sj_2} & \dots & a_{i_sj_s} \end{pmatrix}$$

The determinant $det\ A\begin{pmatrix} i_1 \dots i_s \\ j_1 \dots j_s \end{pmatrix}$ is called a **minor** of A.

If $i_1 = j_1, \dots, i_s = j_s$, then $A\begin{pmatrix} i_1 \dots i_s \\ j_1 \dots j_s \end{pmatrix}$ is called a **principal submatrix** and $det\ A\begin{pmatrix} i_1 \dots i_s \\ j_1 \dots j_s \end{pmatrix}$ a **principal minor**.

If $s = n-1, \{i_1, \dots, i_s\} = \{1, \dots, \hat{i}, \dots, n\}$ and $\{j_1, \dots, j_s\} = \{1, \dots, \hat{j}, \dots, n\}$, then $A\begin{pmatrix} i_1 \dots i_s \\ j_1 \dots j_s \end{pmatrix}$ is denoted by $A^{i,j}$.

(b) The matrix $ad(A)$ whose (i,j) entry is $(-1)^{i+j}\ detA^{(i,j)}$, is called the **adjoint matrix** of A. It has the property that $A\ ad(A) = (det\ A)\ E_n = \ ad(A)A$.

(c) Let q be the quadratic form associated with a symmetrical real matrix A, that is $q(x) = \frac{1}{2}xAx^t$.

The form q is **positive** if and only if the determinants of the principal submatrices $A\begin{pmatrix} 1 \\ 1 \end{pmatrix}, A\begin{pmatrix} 1 & 2 \\ 1 & 2 \end{pmatrix}, \dots, A\begin{pmatrix} 12\dots n \\ 12\dots n \end{pmatrix} = A$ are positive, or equivalently, if all principal minors are positive.

The form q is **non-negative** if and only if all principal minors of A are non-negative $det\ A\begin{pmatrix} i_1 \dots i_s \\ j_1 \dots j_s \end{pmatrix} \geq 0$ for all $1 \leq i_1 < i_2 < \cdots < i_s \leq n, x = 1, \dots, n$.

(d) **Perron-Frobenious theorem:** Let $A = (a_{ij})$ be a real matrix with $a_{ij} \geq 0$. Then for the **spectral radius** $\rho = \ max \ \{\|\lambda\| : \lambda \text{ is an eigenvalue of } A\}$, there is a vector y with non-negative coordinates such that $yA = \rho y$. Moreover, if $a_{ij} > 0$ for every i, j, then $0 < \rho$ and the coordinates of y are positive.

We say that an integral quadratic form $q(x_1, \ldots, x_n) = \sum\limits_{i=1}^{n} x_i^2 + \sum\limits_{i<j} q_{ij} x_i x_j$ is a *unit form.*

Theorem [41]. *Let $q : \mathbb{Z}^n \to \mathbb{Z}$ be a unit form and let A be the associated symmetric matrix. The following are equivalent:*

(a) *q is weakly positive.*

(b) *For each principal submatrix B of A either $det\ B > 0$ or $ad\ (B)$ is not positive (that is, it has an entry ≤ 0).*

Proof. a) $\Rightarrow$ b): Let B be a principal submatrix of A. Suppose that $ad\,(B)$ is positive. Then there is a positive vector v and a number of $\rho > 0$ such that $v\ ad(B) = \rho v$. Then $0 < q(v) = vBv^t = \rho^{-1}\ ad(B)Bv^t = \rho^{-1}(det\ B)vv^t$. Thus $det\ B > 0$.

b) $\Rightarrow$ a). Let A be a $n \times n$-matrix satisfying (b). We show that q is weakly positive by induction on n.

Since property (b) is inhereted to principal submatrices, we can assume that the quadratic form $q^{(i)}$ associated with each principal submatrix $A^{(i,i)}$ is weakly positive.

Claim: $q^{(i)}$ is positive, $1 \leq i \leq n$ (exercise).

Assume that q is not weakly positive. Therefore, we get a vector $0 \ll y \in \mathbb{N}^n$ such that $q(y) \leq 0$.

In particular, every proper principal submatrix B of A has $det\ B > 0$. Since A is not positive, $det\ A \leq 0$. By hypothesis, $ad\ (A)$ is not positive. Suppose that the j-th row v of $ad(A)$ has some non positive coordinate. Therefore, there exists a number $\lambda \geq 0$ such that $0 \leq \lambda y + v$ is not omnipresent. Therefore

$$\begin{aligned} 0 < q(\lambda y + v) &= \lambda^2 q(y) + \lambda v A y^t + q(v) \leq \lambda(det\ A)y(j) + (det\ A)v(j) \\ &\leq (det\ A)(det\ A^{(j,j)}) \leq 0, \end{aligned}$$

since by the claim $q^{(j)}$ is positive. $\square$

A unit form $q\colon \mathbb{Z}^n \to \mathbb{Z}$ is *critical* if q is not weakly positive but all its restrictions $q^{(i)}$ $(i = 1, \dots, n)$ are weakly positive.

Corollary. *If q is critical, then the set*

$$C_q = \{v \in \mathbb{Z}^n : v(i) \geq 0 \textit{ and } v(j) < 0 \textit{ for some } 1 \leq i, j \leq n \textit{ and } q(v) = 1\}$$

is finite.

Theorem [25]. *Let q be a critical form. Then there exists a Euclidean quiver Δ and an invertible transformation qT of q such that $q_\Delta = qT$. In particular, q is non negative and there is a vector $0 \ll z \in \mathbb{Z}^n$ such that rad $q = \mathbb{Z}z$.*

Proof. Since $n \geq 3$, then $0 < q(e_s \pm e_t) = 2 \pm a_{st}$. Choose $q' = qT$ an invertible transformation of q such that the set $C_{q'}$ has minimal cardinality.

Therefore, $q' = \sum\limits_{i=1}^{n} x_i^2 + \sum\limits_{i \neq j} a'_{ij} x_i x_j$ is critical and $-1 \leq a'_{ij} \leq 0$ for every pair i, j with $i \neq j$. Thus $q' = q_\Delta$ for some quiver Δ. Since q' is critical, Δ is Euclidean. Then $\operatorname{rad} q' = \mathbb{Z}u$ with $u \gg 0$ and $z = T^{-1}(u)$. □

Let $A = k[Q]/I$ be a k-algebra. We say that A is **minimal representation-infinite** it it is representation-infinite but every quotient A/AeA is representation-finite for any idempotent $0 \neq e$ of A.

A minimal representation-infinite algebra A with preprojective component is called **critical**. Observe that a preprojective component of a critical algebra contains all the indecomposable projective modules (and therefore is unique).

Lemma. *Let A be an algebra with a preprojective component containing all projective modules. If e is an idempotent of A, then A/AeA has preprojective components such that their union contains all indecomposable projective A/AeA-modules.* □

Theorem [19]. *Let $A = kQ/I$ be an algebra with preprojective component. Assume that Q has at least 3 vertices. Then the following are equivalent:*

(a) *A is critical;*

(b) *The Tits form q_A is critical;*

(c) *A is tame concealed.*

Proof. Let $\mathcal{P}$ be a preprojective component of Γ_A.

b) $\Rightarrow$ c): Assume that q_A is critical. Therefore, A is representation-infinite. A preprojective component $\mathcal{P}$ of Γ_A should contain all indecomposable projective modules. Moreover, this component $\mathcal{P}$ does not contain injective modules. Therefore, A is tilted.

Assume that $A = \mathrm{End}_B(T)$ where $B = k\Delta$ is an hereditary algebra and ${}_BT$ is a tilting module. Therefore the Euler forms χ_A and χ_B are equivalent. Since $gl\dim A \leq 2$, then $\chi_A = q_A$. Therefore, Δ is a tame quiver.

By a dual argument, A has a preinjective component with all indecomposable injective modules. Hence A is tame concealed. □

Critical algebras were classified in a list of frames in [19]. With a different approach the list was also obtained in [4]. In fact, we have the equivalent concept given by the following result.

Theorem [4]. *Let $A = k[Q]/I$ be an algebra with preprojective component. Then A is representation-finite if and only if there is no convex subalgebra A_0 of A such that A_0 is critical.* □

§4. Preprojective components of tame algebras.

Let $A = kQ/I$ be a k-algebra and assume that Q has no oriented cycles.

Proposition. *Let $\mathcal{P}$ be a preprojective component of Γ_A. The following are equivalent:*

(a) *The algebra $A_{\mathcal{P}} = \mathrm{End}_A(P)$ is tame, where $P = \bigoplus_{P_x \in \mathcal{P}} P_x$.*

(b) *There exists a constanct $c > 0$ such that for every $x \in Q_0$, $s \in \mathbb{N}$, the inequality*

$$\dim_k \tau_A^{-s} P_s \leq cs$$

is satisfied.

Proof. Consider the algebra $A_{\mathcal{P}}$. Since $\mathcal{P}$ is a preprojective component of $A_{\mathcal{P}}$, we may assume that $A = A_{\mathcal{P}}$. Let τ be the Auslander-Reiten translation in $\mathcal{P}$.

a) $\Rightarrow$ b): Assume that A is tame. Then the Tits form q_A is weakly non negative, which implies the following:

Let $X \in \mathcal{P}$ and $i \in Q_0$. If X is not injective, then

$$|\dim_k \tau^{-1} X(i) - \dim_k X(i)| \leq 2.$$

Let $m = \max\{\dim_k P_x \colon x \in Q_0\}$, then $\dim_k \tau^{-s} P_x \leq 2ns + m$ for every $x \in Q_0$, $s \in \mathbb{N}$ and $n =$ number of vertices of Q.

b) $\Rightarrow$ a): Assume that A is wild. Let $A' = A/AeA$ where $e = \sum\limits_{I_x \in \mathcal{P}} e_x$. Then there is a preprojective component $\mathcal{P}'$ of $\Gamma_{A'}$ (with translation τ') and a number $r \geq 0$ such that for every $x \in Q_0$ and $t \geq r$, the following is satisfied: if the module $X = \tau^{-t} P_x$ exists, then $X \in \mathcal{P}'$ and $\tau'^{-1} X = \tau^{-1} X$.

Therefore, we may assume that $A = A'$, that is, $\mathcal{P}$ is a preprojective component containing all indecomposable projective modules and without injectives. Let $\mathcal{S}$ be a slice in $\mathcal{P}$. Then ${}_A T = \oplus \mathcal{S}$ is an A-tilting module such that $B = \operatorname{End}_A(T)$ is a wild hereditary algebra, say $B = k\Delta$.

Let $\sigma_T \colon K_0(A) \to K_0(B)$ be the isomorphism of Grothendieck groups induced by T. Thus $\phi_A = \sigma_T \varphi_B \sigma_T^{-1}$.

Let $X \in \mathcal{P}$ be such that there is an oriented path from some $Z \in \mathcal{S}$ to X. Then $\mathbf{dim}\, \tau^{-m} X = (\mathbf{dim}\, X) \phi_A^{-m}$ for $m \geq 0$. Let $Y = \Sigma X$, where $\Sigma = \operatorname{Hom}_A(T, -)$. Then Y is a preprojective B-module.

We claim that $\lim\limits_{m \to \infty} \sqrt[m]{\dim_k \tau^{-m} X}$ exists if and only if $\lim\limits_{m \to \infty} \sqrt[m]{\dim_k \tau_B^{-m} Y}$ exists and in that case they are equal. Indeed, let $\sigma_T = (a_{ij})$, $\sigma_T^{-1} = (b_{ij})$ be $n \times n$ matrices. Let $a = \max\{|a_{ij}|, |b_{ij}| \colon 1 \leq i, j \leq n\}$. For a vector $z \in \mathbb{N}^{Q_0}$ we write $|z| = \sum\limits_{i=1}^{n} z(i)$. We get

$$\begin{aligned}
|(\mathbf{dim}\, Y)\varphi_B^{-m}| &= |(\mathbf{dim}\, X)\varphi_A^{-m}\sigma_T| \leq na|(\mathbf{dim}\, X)\varphi_A^{-m}| \text{ and} \\
|(\mathbf{dim}\, X)\varphi_A^{-m}| &= |(\mathbf{dim}\, Y)\varphi_B^{-m}\sigma_T^{-1}| \leq na|(\mathbf{dim}\, Y)\varphi_B^{-m}|.
\end{aligned}$$

This shows the claim.

On the other hand, $\lim\limits_{m \to \infty} \sqrt[m]{\dim_k \tau_B^{-m} Y}$ exists and equals $\rho > 1$, where ρ is the spectral radius of φ_B, that is $\rho = \max\{\|\lambda\| \colon \lambda \text{ is an eigenvalue of } \varphi_B\}$. Therefore A can not satisfy (b). $\square$

The next Proposition completes the discussion on tilted algebras of tame type iniciated in Lecture 1.

Proposition. *Let $\mathcal{P}$ be a preprojective component of Γ_A containing all indecomposable projective A-modules and no injective module. Then the following are equivalent:*

(a) *A is tilted of Euclidean type*
(b) *A is tame*
(c) *The Tits form q_A is non negative*
(d) *q_A is weakly non negative*
(e) *Γ_A has a tube.*

Proof. a) $\Rightarrow$ b): Clear.

a) $\Leftrightarrow$ c): Since $q_A = \chi_A$, A is tilted of a tame hereditary algebra if and only if q_A is non negative.

c) $\Rightarrow$ d): Clear.

b) $\Rightarrow$ e): By Lecture 1, Γ_A has a stable tube.

e) $\Rightarrow$ a): Let $\mathcal{S}$ be a slice in $\mathcal{P}$. Let $T = \oplus \mathcal{S}$ and $B = \mathrm{End}_A(T)$ be a hereditary algebra. Assume that A is wild. Let $X \in \Gamma_A \setminus \mathcal{P}$. As in the proof of the above Proposition, $\lim\limits_{s\to\infty} \sqrt[s]{\dim_k \tau^{-s} X} = \rho > 1$. This implies that X does not lie in a tube in Γ_A. If A is tame, then $|\mathcal{S}|$ is an euclidean diagram and A is a domestic cotubular algebra. $\square$

Lecture 4. Structure of tame algebras and their categories of modules.

§1. Standard tubes in Auslander-Reiten quivers.

Let A be a finite dimensional k-algebra. We recall that two modules X_1, X_2 are said to be *orthogonal* if $\mathrm{Hom}_A(X_1, X_2) = 0 = \mathrm{Hom}_A(X_2, X_1)$.

Let $E_1, \ldots, E_s$ be a family of pairwise orthogonal bricks. Define $\varepsilon(E_1, \ldots, E_s)$ as the full subcategory of mod_A whose objects X admit a filtration $X = X_0 \supset X_1 \supset \cdots \supset X_m = 0$ for some $m \in \mathbb{N}$, with X_i/X_{i+1} isomorphic to some E_j, for any $1 \leq i \leq n$.

Lemma. *The category $\varepsilon = \varepsilon(E_1, \ldots, E_s)$ is an abelian category, with $E_1, \ldots, E_s$ being the simple objects of E.* □

An abelian category ε is said to be *serial* provided any object in E has finite lenght and any indecomposable object in ε has a unique composition series.

Proposition. *Let $E_1, \ldots, E_s$ be pairwise orthogonal bricks in some module category* mod A. *Assume that (a) $\tau E_i \cong E_{i-1}$ for $1 \leq i \leq s$ with $E_0 = E_s$ and (b)* $\mathrm{Ext}^2_A(E_i, E_j) = 0$ *for all $1 \leq i, j \leq n$. Then $\varepsilon = \varepsilon(E_1, \ldots, E_s)$ is serial, it is a standard component of Γ_A of the form $\mathbb{Z}\mathbb{A}_\infty/(n)$.* □

With the notation of the Proposition above: we denote by $E_i[t]$ the unique module in the serial category E which has socle E_i and lenght t.

A family $\mathcal{T} = (T_\lambda)_{\lambda \in L}$ of the Auslander-Reiten quiver of an algebra A is a *standard stable tubular family* if each T_λ is a standard component of the form $\mathbb{Z}\mathbb{A}_\infty/(n_\lambda)$ for some n_λ and for $\lambda \neq \mu$ the components T_λ and T_μ are orthogonal.

Corollary. *Let $\mathcal{T} = (T_\lambda)_{\lambda \in L}$ be a standard stable tubular family in the Auslander-Reiten quiver of A. Then the additive closure add $\mathcal{T}$ of $\mathcal{T}$ in* mod_A *is an abelian category which is serial and is closed under extensions in* mod_A. □

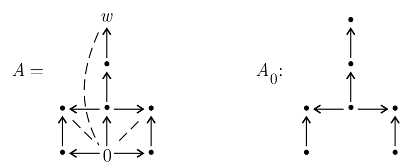
A standard stable tubular family $\mathcal{T} = (T_\lambda)_{\lambda \in L}$ is said to be *separating* if there are full subcategories $\mathcal{P}$ and $\mathcal{I}$ of mod_A satisfying the following conditions:

(i) each indecomposable A-module belongs to one of $\mathcal{P}, \mathcal{T}$ or $\mathcal{I}$;

(ii) for modules $X \in \mathcal{P}, Y \in \mathcal{T}$ and $Z \in \mathcal{I}$ we have $\mathrm{Hom}_A(Z, Y) = 0 = \mathrm{Hom}_A(Z, X)$ and $\mathrm{Hom}_A(Y, X) = 0$.

(iii) each non zero morphism $f \in \operatorname{Hom}_A(X, Z)$ for indecomposable modules $X \in \mathcal{P}, Z \in \mathcal{I}$, factorizes through each component T_λ.

Example: Let A be the algebra given by the quiver with relations below



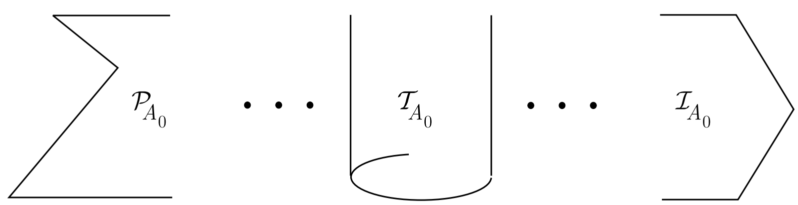
$$\mathbf{dim}\, E_0 : \quad \begin{array}{ccc} & 0 & \\ & 1 & \\ 1 & 1 & 1 \\ 1 & & 1 \end{array}$$

Then A is the *one-point extension* $A_0[E_0]$ as follows

$$A_0[E_0] = \begin{bmatrix} A_0 & E_0 \\ 0 & k \end{bmatrix}$$

with the usual matrix operations and where E_0 is considered as an $A_0 - k$-bimodule. Moreover $\operatorname{rad} P_0 = E_0$.

The algebra A_0 is tame hereditary with an Auslander-Reiten quiver of the shape



where $\mathcal{P}_{A_0}$ is a preprojective component, $\mathcal{I}_{A_0}$ a preinjective component and $\mathcal{T}_{A_0}$ is a separating tubular family of tubular type $(2, 3, 3)$. In $\mathcal{T}_{A_0} = (T_\lambda)_\lambda$ almost all tubes are of rank one with a module on the mouth with dimension vector

$$z_0 : \quad \begin{array}{ccc} & 1 & \\ & 2 & \\ 2 & 3 & 2 \\ 1 & & 1 \end{array}$$

The tubes of rank 2 and rank 3 have modules on the mouths with the unique indecomposable A_0-modules having the indicated dimension vectors:

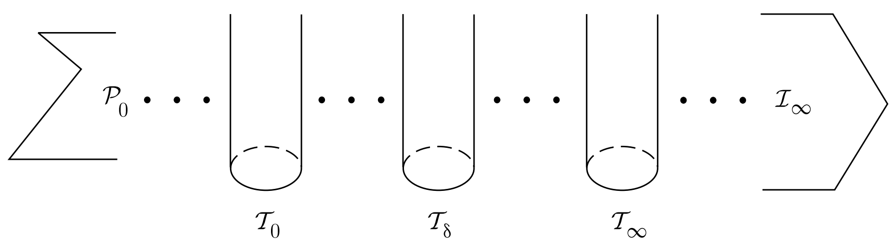
$$E_0:\ \begin{array}{ccc} & 0 & \\ & 1 & \\ 1 & 1 & 1 \\ 1 & & 1 \end{array} \qquad E_1:\ \begin{array}{ccc} & 1 & \\ & 1 & \\ 1 & 2 & 1 \\ 0 & & 0 \end{array}$$

$$X_0:\ \begin{array}{ccc} & 0 & \\ & 1 & \\ 1 & 1 & 0 \\ 0 & & 0 \end{array} \qquad X_1:\ \begin{array}{ccc} & 0 & \\ & 0 & \\ 1 & 1 & 1 \\ 1 & & 0 \end{array} \qquad X_2:\ \begin{array}{ccc} & 1 & \\ & 1 & \\ 0 & 1 & 1 \\ 0 & & 1 \end{array}$$

$$Z_0:\ \begin{array}{ccc} & 0 & \\ & 1 & \\ 0 & 1 & 1 \\ 0 & & 0 \end{array} \qquad Z_1:\ \begin{array}{ccc} & 0 & \\ & 0 & \\ 1 & 1 & 1 \\ 0 & & 1 \end{array} \qquad Z_2:\ \begin{array}{ccc} & 1 & \\ & 1 & \\ 1 & 1 & 0 \\ 1 & & 0 \end{array}$$

and where the Auslander-Reiten translation is given by $\tau_{A_0} E_i = E_{i-1}$, $\tau_{A_0} X_i = X_{i-1}$ and $\tau_{A_0} Z_i = Z_{i-1}$ cyclically.

The structure of Γ_A is given as follows:

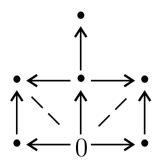


where $\mathcal{T}_0 = \bigvee_{\lambda \neq 2} T_\lambda \vee T_2[E_0]$ is the family of tubes $\mathcal{T}_{A_0}$ with the exception of the tube of rank 2 which appears now 'inserted' with the new projective at the extension vertex 0.

For each positive rational number $\delta = \frac{a}{b}, (a,b), \mathcal{T}_\delta$ is a separating family of tubes of tubular type $(3,3,3)$ with all homogeneous tubes but 2 of rank 3. The homogeneous tubes have modules on the mouths of vector dimension

$$az_0 + bz_\infty$$

where z_∞ is given by

A_∞: 

$$z_\infty: \begin{array}{ccc} & 1 & \\ 1 & 2 & 1 \\ 1 & 1 & 1 \end{array} \qquad \mathbf{dim}\, E_\infty: \begin{array}{ccc} & 1 & \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{array}$$

Observe that A_∞ is tame concealed and $A = [E_\infty]A_\infty$ is a one-point coextension where the module E_∞ lies on a regular tube of Γ_{A_∞}. The algebra A is a typical *tubular algebra* as defined by Ringel [34].

Proposition. *Let $\mathcal{T} = (T_\lambda)_\lambda$ be a standard separating tubular family for the module category* mod_A. *Then*

(a) *For almost every λ, the tube T_λ is homogeneous.*

(b) *Let T_λ be a homogeneous tube of the family $\mathcal{T}$. Let X be a module in the mouth of T_λ and $v = \mathbf{dim}\, X$. Then $q_A(v) = 0$.*

Proof of (b): Let X be a module in the mouth of a homogeneous tube T_λ in $\mathcal{T}$. Let B be the convex closure in A of $\cup \mathrm{supp}\, X$ with $X \in T_\lambda$. Since B is convex in A and $gl\mathrm{dim}\ B \leq 2$, then

$$q_A(\mathbf{dim}\ X) = q_B(\mathbf{dim}\ X) = \dim_k \mathrm{End}_A(X) - \dim_k \mathrm{Ext}^1_A(X, X).$$

Since T_λ is standard and $X \simeq \tau X$, then $\mathrm{Ext}^1_A(X, X) \cong D\mathrm{Hom}_A(X, \tau X)$ and we get $q_A\ (\mathbf{dim}\ X) = 0$. □

Notation: Let $\mathcal{T} = (T_\lambda)_\lambda$ be a standard separating stable tubular family in mod A. Let $r(\lambda)$ be the *period* (or *rank*) of the tube T_λ. Consider those $r(\lambda_1), \ldots, r(\lambda_s)$ which are strictly bigger than 1 (finite number by (1.4)). We define the *star diagram* $\mathbb{T}_r$ of the family $\mathcal{T}$ as the diagram with a unique ramification point and s branches of lengths $r(\lambda_1), \ldots, r(\lambda_s)$.

Theorem [24, 34]. *Let $A = kQ/I$ be a k-algebra. Let n be the number of vertices of Q. Let $\mathcal{T} = (T_\lambda)_{\lambda \in L}$ be a standard separating stable sincere tubular family in mod A. Let $r(\lambda)$ be the rank of the tube T_λ. Then*

$$\sum_{\lambda \in L} (r(\lambda) - 1) = n - 2.$$

Moreover, A is a tame algebra if and only if the star diagram $\mathbf{T}_r$ is a Dynkin or extended Dynkin diagram. □

§2. Tubes and isotropic roots of the Tits form.

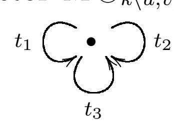
We say that a property P is satisfied by almost every indecomposable in mod_A if for each $d \in \mathbb{N}$, the set of indecomposable A-modules of dimension d which do not satisfy P form a finite set of isomorphism classes. The following is a central fact about the structure of the Auslander-Reiten quiver Γ_A of a tame algebra A.

Theorem [8]. *Let A be a tame algebra. Then almost every indecomposable lies in a homogeneous tube. In particular, almost every indecomposable X satisfies $X \simeq \tau X$.*

Open problem: Is it true that an algebra is of tame type if and only if almost every indecomposable module belongs to a homogeneous tube?

Proposition. *Let A be an algebra such that almost every indecomposable lies in a standard tube. Then A is tame.*

Proof: Our hypothesis implies that almost every indecomposable X satisfies $\dim_k \mathrm{End}_A(X) \leq \dim_k X$. We show that this condition implies the tameness of A.

Indeed, assume that A is wild and let M be a $A - k\langle u, v\rangle$-bimodule which is finitely generated free as right $k\langle u, v\rangle$-module and the functor $M \otimes_{k\langle u,v\rangle}$-insets indecomposables. Consider the algebra B given by the quiver 

and with radical J satisfying $J^2 = 0$. Then there is a $A-B$-bimodule N such that N_B is free and $N \otimes_B - :$ mod $B \longrightarrow$ mod A is fully faithful. Therefore the composition $F = M \otimes_A (N \otimes_B -)$ is faithful and insets indecomposables. Moreover, $\dim_k FX \leq m \dim_k X$ for any $X \in$ mod B if we set $m = \dim_k (M \otimes_A N)$.

Consider also the functor $H : \mathrm{mod}\, A \longrightarrow \mathrm{mod}\, B$ sending X to the space $X' = X \oplus X$ with endomorphisms

$$X'(t_1) = \begin{bmatrix} 0 & X(w) \\ 0 & 0 \end{bmatrix}, X'(t_2) = \begin{bmatrix} 0 & X(v) \\ 0 & 0 \end{bmatrix} \quad \text{and} \quad X'(t_3) = \begin{bmatrix} 0 & 1_X \\ 0 & 0 \end{bmatrix}$$

This functor insets indecomposables. For the simple A-modules X of dimension n, we get indecomposable A-modules $FH(X)$ with

$$\dim_k FH(X) \leq m \dim_k H(X) = 2mn$$

and

$$\dim_k \operatorname{End}_A(FH(X)) \geq \dim_k \operatorname{End}_B(H(X)) = n^2 + \dim_k \operatorname{End}_A(X) = n^2 + 1. \quad \square$$

Let $A = kQ/I$ be a triangular algebra. In case A is tame, we would like to find the dimensions $z \in \mathbb{N}^{Q_0}$ where indecomposable modules X with $\mathbf{dim}\, X = z$ and X in a homogeneous tube exist. A partial result:

Proposition. *Assume that A is tame and $q_A(z) = 0$. Then there is a decomposition $z = w_1 + \cdots + w_s$ with $w_i \in \mathbf{N}^{Q_0}$ and an open subset $\mathcal{U}$ of* $\operatorname{mod}_A(z)$ *satisfying:*

(a) $\dim \mathcal{U} = \dim \operatorname{mod}_A(z)$.

(b) *Every $X \in \mathcal{U}$ has an indecomposable decomposition $X = X_1 \oplus \cdots \oplus X_s$ such that* $\dim X_i = w_i$ *and the module X_i lies in the mouth of a homogeneous tube. Moreover,* $\dim_k \operatorname{Hom}_A(X_i, X_j) = \delta_{ij} = \operatorname{Ext}^1_A(X_i, X_j)$ *for $1 \leq i, j \leq s$.* $\square$

§3. Hypercritical algebras.

Let $q = \sum_{i=1}^{n} x_i^2 + \sum_{i \neq j} a_{ij} x_i x_j$ be a unit form. Let M be the symmetric matrix associated with q.

Proposition. *The following are equivalent:*

(a) *q is weakly non-negative*

(b) *Every critical restriction q^I of q with v the positive generator of rad q^I, satisfies $v^0 M \geq 0$.*

Proof. a) $\Rightarrow$ b): Assume that q^I is critical and $v^0 M$ has its j-th component negative. Then $0 \leq 2v^0 + e_j \in \mathbb{Z}^n$ and $q(2v^0 + e_j) = 2v^0 M e_j^t + 1 < 0$

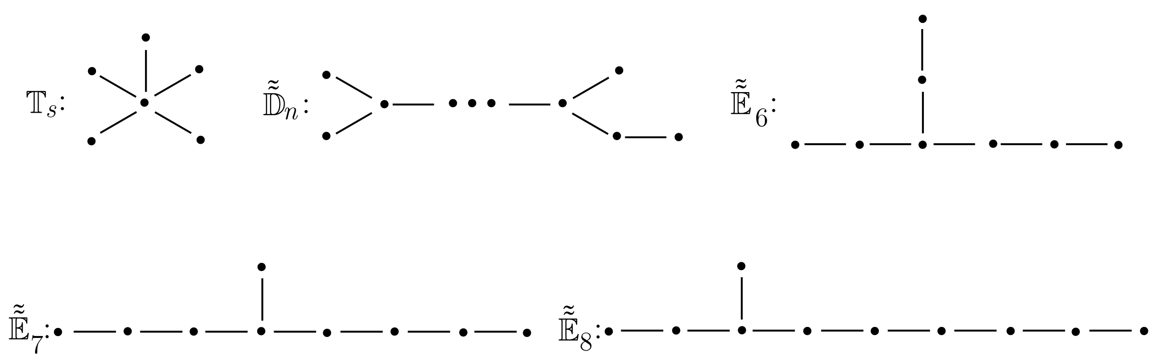
b) $\Rightarrow$ a): Assume that q satisfies (b) but not (a). By induction, we may suppose that $q^{(i)}$ satisfies (a), $1 \leq i \leq n$. Let $0 \ll z$ be such that $q(z) < 0$. Let q^I be a critical restriction. Let v be the positive generator of *rad* q^I. We can find a number $a \leq 0$ such that $0 \leq z + av^0$ and $(z + av^0)(j) = 0$ for some $1 \leq j \leq n$. Then

$$0 \leq q^{(j)}(z + av^0) < av^0 M z^t \leq 0,$$

a contradiction. $\square$

Corollary. *The unit form q is weakly non negative if and only if $0 \leq q(z)$ for every $z \in [0,12]^n$.* □

Following [38] a triangular algebra $A = kQ/I$ is *strongly simply connected* if every convex subcategory B of A satisfies the separation condition. By [27], the Tits form q_A of a strongly simply connected algebra A is weakly non-negative if and only if A does not contain a full convex subcategory which is tilted of a hereditary algebra of one of the tree types



where in the case $\tilde{\tilde{\mathbb{D}}}_n$ the number of vertices is $n+2$, with $4 \leq n \leq 8$. The hereditary algebras corresponding to this list are called *hipercritical algebras.*

Theorem [7]. *Let A be a strongly simply connected algebra, then the following are equivalent:*

(a) *A is tame*

(b) *q_A is weakly non-negative*

(c) *A does not contain a full convex subcategory which is hypercritical.* □

The proof of the Theorem depends on many partial results proved along many years by several people. We give only a superficial idea of the used arguments.

Let $A = kQ/I$ be a strongly simply connected algebra.

- A is of *polynomial growth* if there is a natural number m such that the number of one-parameter families of indecomposable modules is bounded, in each dimension d, by d^m.

- The representation theory of strongly simply connected algebras of polynomial growth is well understood [39] and the structure of the Auslander-Reiten quiver is described via *coils and multicoils* [1].

- A is (tame) of polynomial growth if and only if q_A is weakly non-negative and A does not contain a convex subcategory of a certain list of (the so called, *pg-critical*) algebras [39].

Hence, in order to prove the Theorem, we may assume that:

(i) A contains a convex pg-critical algebra.

(ii) A accepts an indecomposable A-module X so that $X(i) \neq 0$ for every source or sink i in Q.

- In [7], it is proved that A is constructed from (as a suitable pushout glueing of blowups of) extensions of coil algebras and pg-critical algebras (thus A is said to be a $\mathbb{D}$-algebra).

- The category of A-modules is equivalent (up to finitely many indecomposable objects) to the category of A^*-modules, where A^* is canonically constructed.

- A^* degenerates to a special biserial algebra.

- By [17], it is enough to show that special biserial algebras are tame (which is well-known).

References

[1] I. Assem and A. Skowroński. *Coil and multicoil algebras.* In: Representation theory of Algebras and related topics. CMS Conference Proc. 19 (CMS-AMS 1996) 1–24.

[2] R. Bautista, P. Gabriel, A. V. Roiter and L. Salmerón. *Representation-finite algebras and multiplicative bases.* Invent. Math. 81 (1984) 217–285.

[3] K. Bongartz. *Algebras and quadratic forms.* J. London Math. Soc. (2) 28 (1983) 461–469.

[4] K. Bongartz. *Critical simply connected algebras.* Manuscripta math. 46 (1984) 117–136.

[5] S. Brenner. *Quivers with commutative conditions and some phenomenology of forms.* In Representations of Algebras, Springer LNM 488 (1975), 29–53.

[6] Th. Brüstle. *Kit algebras.* J. Algebra 240 (2001), 1–24.

[7] Th. Brüstle, J. A. de la Peña and A. Skowroński. *Tame algebras and Tits quadratic form.* To appear.

[8] W. W. Crawley-Boevey. *On tame algebras and bocses.* Proc. London Math. Soc. III, 56 No. 3 (1988) 451–483.

[9] W. W. Crawley-Boevey and J. Schröer. *Irreducible components of varieties of modules.* J. Reine Angew. Math. 553 (2002), 201–220.

[10] V. Dlab and C. M. Ringel. *Indecomposable representations of graphs and algebras.* Mem. Amer. Math. Soc. 173 (1976).

[11] Ju A. Drozd. *On tame and wild matrix problems.* In Representation Theory II. Proc. ICRA II (Ottawa, 1979), Springer LNM 831 (1980) 242–258.
[12] P. Gabriel. *Unzerlegbare Darstellungen I.* Manuscripta Math. 6 (1972), 71–103.
[13] P. Gabriel. *Indecomposable representation II.* Symposia Math. Inst. Naz. Alta Mat. 11 (1973) 81–104.
[14] P. Gabriel. *Finite representation type is open.* In Representations of Algebras, Springer LNM 488 (1975) 132–155.
[15] P. Gabriel. *Auslander-Reiten sequences and representation-finite algebras.* Proc. ICRA II (Ottawa 1979). In Representation of Algebras, Springer LNM 831 (1980) 1–71.
[16] P. Gabriel and A. V. Roiter. *Representations of finite-dimensional algebras.* Algebra VIII, Encyclopaedia of Math. Sc. Vol 73. Springer (1992).
[17] Ch. Geiss. *On degenerations of tame and wild algebras.* Arch. Math. 64 (1995) 11–16.
[18] Ch. Geiss and J. A. de la Peña. *On the deformation theory of finite dimensional algebras.* Manuscripta Math. 88 (1995) 191–208.
[19] D. Happel and D. Vossieck. *Minimal algebra of infinite representation-type with preprojective component.* Manuscripta math. 42 (1983) 221–243.
[20] R. Hartshorne. *Introduction to algebraic geometry.* Springer Verlag (1977).
[21] O. Kerner. *Tilting wild algebras.* J. London Math. Soc. 39 (1989) 29–47.
[22] H. Kraft. *Geometric Methods in Representation Theory.* In Representations of Algebras. Springer LNM 944 (1981) 180–258.
[23] H. Kraft. *Geometrische Methoden in der Invariantentheorie.* Vieweg. Braunschweig (1985).
[24] H. Lenzing and J. A. de la Peña. *Concealed canonical algebras and separated tubular families.* Proc. London Math Soc (3) 78 (1999), 513–540.
[25] A. Ovsienko. *Integral weakly positive forms.* In Schur matrix problems and quadratic forms. Kiev (1978) 3–17 (in russian).
[26] J. A. de la Peña. *Quadratic forms and the representation type of an algebra.* Sonderforschungsbereich Diskrete Strukturen in der Mathematik. Ergänzungsreihe 343, 90–003. Bielefeld (1990).
[27] J. A. de la Peña. *Algebras with hypercritical Tits form.* In Topics in Algebra. Banach Center Publ. 26 (1990) 353–359.
[28] J. A. de la Peña. *On the dimension of the module-varieties of tame and wild algebras.* Comm. in Algebra 19 (6), (1991) 1795–1807.
[29] J. A. de la Peña. *Sur les degrés the liberté des indecomposables.* C.R. Acad. Sci. Paris, t. 412 (1991), 545–548.
[30] J. A. de la Peña. *Tame algebras: some fundamental notions.* Universität Bielefeld. Ergänzungsreihe 95–010. (1995).
[31] J. A. de la Peña. *The Tits form of a tame algebra.* In Canadian Math. Soc. Conference Proceedings. Vol. 19 (1996) 159–183.
[32] J. A. de la Peña and A. Skowroński. *Substructures of algebras with weakly non-negative Tits form.* To appear.
[33] J. A. de la Peña and M. Takane. *Spectral properties of Coxeter transformations and applications.* Arch. Math. 55 (1990) 120–134.
[34] C. M. Ringel. *Tame algebras.* Representation Theory I. Proc. ICRA II (Ottawa, 1979), Springer LNM 831 (1980) 137–287.
[35] *The Spectral radius of the Coxeter transformation for a generalized Cartan matrix.* Math. Ann. 300 (1994) 331–339.
[36] C. M. Ringel. *Tame algebras and integral quadratic forms.* Springer, Berlin LNM 1099 (1984).
[37] A. Shafarevich. *Basic algebraic geometry.* Graduate Texts Springer 213 (1977).

[38] A. Skowroński. *Simply connected algebras and Hochschild cohomologies.* Proc. ICRA IV (Ottawa, 1992), Can. Math. Soc. Conf. Proc. Vol. 14 (1993) 431–447.

[39] A. Skowroński. *Simply connected algebras of polynomial growth.* Compositio Math. 109 (1997) 99–133.

[40] D. Voigt. *Induzierte Darstellungen in der Theorie der endlichen algebraischen Gruppen.* Springer LNM 592 (1977).

[41] M. Zel'dich. *A criterion for weakly positive quadratic forms.* In Linear algebra and Representation Theory. Kiev (1983) (in russian).