

DEFORMATION THEORY OF FINITE DIMENSIONAL MODULES AND ALGEBRAS

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INTRODUCTION

Many mathematical structures can be deformed:

- Manifolds with possibly an extra (e.g. Poisson) structure
- Abelian or triangulated categories
- Lie algebras and their universal enveloping algebras
- Finite dimensional (associative) algebras and modules

In many cases this provides interesting insights into the deformed objects.

The last case of our list is possibly the easiest and will be studied in these notes.

In section 1 we present the basic definitions of deformation theory for the case of associative algebras. In section 2 we present the Hochschild Complex together with the Gerstenhaber bracket as the natural context for the Maurer-Cartan equation. This leads to abstract deformation theory associated to a dg-Lie algebra. We follow here Keller's exposition [11, section 2]. In section 3 we discuss briefly the deformation theory of modules in the same spirit. In section 4 we prepare the ground for the discussion of some geometric applications of deformation theory of algebras in the last two chapters. More specifically, we present in 5 a version of Voigt's lemma for algebras and show that the vanishing of the third Hochschild cohomology of an algebra implies that the corresponding point in the scheme of algebras is smooth. In section 6 we present analogous results for modules and discuss the decomposition theory of the scheme of module structures from [3]. Finally, we collect for convenience in an appendix the for us relevant definitions from the functorial point of view for schemes.

1. DEFORMATIONS OF ALGEBRAS

1.1. Notation. Let k be a field, and A finite dimensional associative (unitary) k -algebra with underlying vector space $V = k^n$ and multiplication given by $\alpha \in \text{Hom}(V \otimes V, V)$. If $g \in \text{GL}_n(k)$ then $\alpha^g := g^{-1}\alpha(g \otimes g)$ is also an associative multiplication on V . Clearly, all associative algebra structures on V which are isomorphic to α are precisely of this form.

Let R a local commutative k -algebra with maximal ideal \mathfrak{m} and counit $p_R: R \rightarrow k$. Thus we have a canonical decomposition $R = k \cdot 1 \oplus \mathfrak{m}$ as a k -vector space.

Important examples to keep in mind are the ring of formal power series $k[[t]]$ and the truncated polynomial rings $k[t]/(t^n)$. We denote by $k[\epsilon] := k[t]/(t^2)$ the *algebra of dual numbers*.

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As a shorthand we write $V_R := V \otimes R$. All “undecorated” tensor products and Hom-spaces are over k .

1.2. Definition. A R -deformation of α is given by a R -linear map

$$\alpha_R \in \text{Hom}_R(V_R \otimes_R V_R, V_R)$$

that reduces modulo R to α and which is associative, i.e.

$$\alpha_R(\mathbb{1}_{V_R} \otimes_R \alpha_R) = \alpha_R(\alpha_R \otimes_R \mathbb{1}_{V_R}).$$

In case $R = k[[t]]$ we speak of a *formal* deformation and in case $R = k[\epsilon]$ of an *infinitesimal* deformation.

We say that an infinitesimal deformation $\alpha_{k[\epsilon]}$ of α is *integrable* if there exists a formal deformation $\alpha_{k[[t]]}$ of α which reduces via the projection $p_{t,\epsilon}: k[[t]] \rightarrow k[\epsilon]$ to $\alpha_{k[\epsilon]}$.

1.3. Remarks. By linearity a R -deformation of α is uniquely determined by its k -linear component $\alpha_m: V \otimes V \rightarrow V \otimes m$, and by some abuse of notation we may write $\alpha_R = \alpha + \alpha_m$. The associativity of α_R may then be expressed by the following equation of maps $V \otimes V \rightarrow V \otimes m$ for α_R

$$(1.1) \quad \alpha(\alpha_m \otimes \mathbb{1}_V - \mathbb{1}_V \otimes \alpha_m) + \alpha_m(\alpha \otimes \mathbb{1}_V - \mathbb{1}_V \otimes \alpha) + \alpha_m(\alpha_m \otimes \mathbb{1}_V - \mathbb{1}_V \otimes \alpha_m) = 0.$$

Let us write $\text{MC}(\alpha, R)$ for the set of solutions of this equation.

For example in case $R = k[\epsilon]$ we have $\alpha_{k[\epsilon]} = \alpha + \epsilon \alpha_1$ for some $\alpha_1 \in \text{Hom}_k(V \otimes V, V)$, and we get for α_1 the (linear) condition

$$\alpha_1 \in Z^2(\alpha) := \{\zeta \in \text{Hom}(V \otimes V, V) \mid \zeta(\mathbb{1}_V \otimes \alpha - \alpha \otimes \mathbb{1}_V) + \alpha(\mathbb{1}_V \otimes \zeta - \zeta \otimes \mathbb{1}_V) = 0\}$$

since $\epsilon^2 = 0$. So, $\text{MC}(\alpha, k[\epsilon]) \cong Z^2(\alpha)$ is a vector space.

In case $R = k[[t]]$ we have $m = (t)$ and

$$\alpha_m = \alpha_1 \cdot t + \alpha_2 \cdot t^2 + \cdots + \alpha_i \cdot t^i + \cdots$$

for a sequence of k -linear maps $\alpha_1, \alpha_2, \alpha_3, \dots \in \text{Hom}(V \otimes V, V)$. The above equation (1.1) means then explicitly that

$$(A_n) \quad \alpha(\mathbb{1}_V \otimes \alpha_n - \alpha_n \otimes \mathbb{1}_V) + \sum_{i+j=n} \alpha_i(\mathbb{1}_V \otimes \alpha_j - \alpha_j \otimes \mathbb{1}_V) + \alpha_n(\mathbb{1}_V \otimes \alpha - \alpha \otimes \mathbb{1}_V) = 0$$

holds for all $n \in \mathbb{N}_{>0}$. In particular, for $n = 1$ we get again $\alpha_1 \in Z^2(\alpha)$.

1.4. Equivalence of deformations. A R -linear automorphism g of V_R reduces modulo m to $\mathbb{1} = \mathbb{1}_V$ if and only if $g = \mathbb{1} + g_m$ for some $g_m: V \rightarrow V \otimes m$ (with the same abuse of notation as above). In fact, the endomorphisms of this type form a subgroup $E_n(R)$ of $\text{GL}_n(R)$. For example, the elements of $E_n(k[[t]])$ are precisely of the form

$$\mathbb{1} + g_1 \cdot t + g_2 \cdot t^2 + \cdots$$

for any sequence of k -linear endomorphisms g_1, g_2, \dots of V .

We say that two R -deformations α_R and α'_R of α are *equivalent* if there exists $g_R \in E_n(R)$ such that

$$\alpha'_R = \alpha_R^g := g^{-1} \alpha_R (g \otimes_R g).$$

A R -deformation α_R of α is *trivial* if it is equivalent to α (seen as a R -deformation).

Note, that we may define an action of the group $E_n(R)$ on $MC(\alpha, R)$ via

$$\alpha_m^g := (\alpha + \alpha_m)^g - \alpha.$$

Now we can define the set of orbits

$$\text{Defo}(\alpha, R) := MC(\alpha, R) / E_n(R),$$

thus the elements of $\text{Defo}(\alpha, R)$ are the equivalence classes of R -deformations of α .

One of the main goals of deformation theory is to describe the equivalence classes of deformations for a given object.

1.4.1. Exercise. Let $\alpha + \epsilon\alpha_1$ and $\alpha + \epsilon\alpha'_1$ be two infinitesimal deformations of α . Show:

(a) $\alpha + \epsilon\alpha_1$ and $\alpha + \epsilon\alpha'_1$ are equivalent if and only if

$$\alpha'_1 = \alpha_1 + \alpha(g_1 \otimes \mathbb{1}_V) - g_1\alpha + \alpha(\mathbb{1}_V \otimes g_1)$$

for some $g_1 \in \text{Hom}(V, V)$.

(b) In this case, $\alpha + \epsilon\alpha_1$ is integrable if and only if $\alpha + \epsilon\alpha'_1$ is integrable.

2. HOCHSCHILD COMPLEX

We will see that the Hochschild complex of an algebra together with its structure of a differential graded (= dg-) Lie algebra controls the deformation theory of this algebra.

2.1. For a k -vector space V we set $V^{\otimes 0} = k$ and $V^{\otimes n} = V \otimes \cdots \otimes V$ (n factors, $n \geq 1$), and

$$V^{n-1} = C^n = \text{Hom}_k(V^{\otimes n}, V) \quad n \in \mathbb{N}$$

the (Hochschild) n -cochains. We define a bilinear map $\circ: V^m \times V^n \rightarrow V^{m+n}$ by

$$(\alpha, \beta) \mapsto \alpha \circ \beta := \sum_{i=0}^m (-1)^{ni} \alpha(\mathbb{1}_V^{\otimes i} \otimes \beta \otimes \mathbb{1}_V^{\otimes (m-i)})$$

We consider now the graded vector space $V^* = \bigoplus_{i \geq -1} V^i$, and write $|\alpha| := i$ for an homogeneous element $\alpha \in V^i$. The map \circ induces a graded “multiplication” on V^* which is *not* associative, however we have:

2.1.1. Lemma. For $\alpha, \beta, \gamma \in V^*$ homogeneous elements holds

$$\alpha \circ (\beta \circ \gamma) - (\alpha \circ \beta) \circ \gamma = (-1)^{|\beta||\gamma|} (\alpha \circ (\gamma \circ \beta) - (\alpha \circ \gamma) \circ \beta).$$

For a proof see [9, §6].

2.1.2. Definition. A graded vector space $W^* = \bigoplus_{i \in \mathbb{Z}} W^i$ together with a bilinear map $[-, -]: W^* \times W^* \rightarrow W^*$ is a *graded Lie-algebra* if $[W^i, W^j] \subset W^{i+j}$ for all $i, j \in \mathbb{Z}$ and moreover

$$[\alpha, \beta] = -(-1)^{|\alpha||\beta|} [\beta, \alpha] \quad (\text{antisymmetry})$$

$$[\alpha, [\beta, \gamma]] = [[\alpha, \beta], \gamma] + (-1)^{|\alpha||\beta|} [\beta, [\alpha, \gamma]] \quad (\text{Jacobi Identity})$$

for homogeneous elements $\alpha, \beta, \gamma \in W^*$. A *derivation* of degree 1 for a graded Lie algebra $(W^*, [-, -])$ is a graded linear map $d: W^* \rightarrow W^*$ of degree 1 such that

$$d([\beta, \gamma]) = [d(\beta), \gamma] + (-1)^{|\beta|}[\beta, d(\gamma)]$$

for homogeneous elements $\beta, \gamma \in V^*$. If moreover $d^2 = 0$, we call $(W^*, [-, -], d)$ a *differential graded* (= dg-) *Lie algebra*.

One verifies easily with Lemma 2.1.1:

2.1.3. Corollary. V^* becomes with $[\alpha, \beta] := \alpha \circ \beta - (-1)^{|\alpha||\beta|}\beta \circ \alpha$ for homogeneous $\alpha, \beta \in V^*$ a graded Lie algebra.

2.1.4. For $\alpha \in V^1 = \text{Hom}_k(V^{\otimes 2}, V)$, define a graded derivation d_α of degree 1 for $(V^*, [-, -])$, by

$$\begin{aligned} d_\alpha(\beta) &:= [\alpha, \beta] \\ &= \alpha(\beta \otimes \mathbb{1}_V) - \sum_{j=0}^{|\beta|} (-1)^j \beta(\mathbb{1}_V^{\otimes (|\beta|-j)} \otimes \alpha \otimes \mathbb{1}_V^{\otimes j}) + (-1)^{|\beta|} \alpha(\mathbb{1}_V \otimes \beta). \end{aligned}$$

2.1.5. Exercise. Show that $d_\alpha^2 = 0$ if $\alpha \in \text{Hom}(V \otimes V, V)$ is associative. The converse is true if $\text{char } k \neq 2$.

2.1.6. Our considerations have shown that $(V^*, [-, -], d_\alpha)$ is a dg-Lie algebra, in case $\alpha \in \text{Hom}_k(V \otimes V, V)$ is associative, and (V_*, d_α) is the usual Hochschild complex (shifted by one degree) obtained by the bar-resolution for an associative algebra A with multiplication α . Thus we may set

$$\begin{aligned} Z^i(\alpha) &:= \{\zeta \in C^i \mid d_\alpha(\zeta) = 0\} && \text{cycles} \\ B^i(\alpha) &:= d_\alpha(C^{i-1}) && \text{boundaries} \\ H^i(\alpha) &:= Z^i/B^i && \text{Hochschild cohomology} \end{aligned}$$

and $[-, -]$ descends to give $H^*(\alpha) = \bigoplus_{i \in \mathbb{N}} H^i(\alpha)$ the structure of a graded Lie algebra (with the grading shifted by one, so that $[H^i(\alpha), H^j(\alpha)] \subset H^{i+j-1}(\alpha)$).

2.2. Infinitesimal deformations. We conclude from Exercise 1.4.1(a) and 2.1.6: *The equivalence classes of infinitesimal deformations of α are naturally identified with $H^2(\alpha)$.*

Similarly, Exercise 1.4.1(b) means that for $\alpha_1 \in Z^2(\alpha)$ the integrability of the infinitesimal deformation $\alpha + \alpha_1 \epsilon$ depends only on the class of α_1 in $H^2(\alpha)$. In particular, if $H^2(\alpha) = 0$ each infinitesimal deformation of α is integrable.

2.3. Lemma. *If $H^2(\alpha) = 0$ then each formal deformation of α is trivial.*

Proof. Let $\alpha_t = \alpha + \alpha_1 \cdot t + \alpha_2 \cdot t^2 + \dots$ be a formal deformation with $\alpha_1 = \alpha_2 = \dots = \alpha_{n-1} = 0$ and $\alpha_n \neq 0$. Then $\alpha_n \in Z^2(\alpha) = B^2(\alpha)$, thus there exists $g_n \in C^1 = \text{Hom}_k(V, V)$ such that $\alpha_n = d_\alpha(g_n)$ and with $g = \mathbb{1} + g_n \cdot t^n$ we get $\alpha_t^g = \alpha_0 + \alpha'_{n+1} \cdot t^{n+1} + \dots$. \square

2.4. Obstructions. With the setup from 2.1 we may rewrite the equations (A_n) from 1.3 (which express the associativity of a formal deformation $\alpha_t = \alpha + \alpha_1 \cdot t + \alpha_2 \cdot t^2 + \dots$ of α) as

$$(A'_n) \quad d_\alpha(\alpha_n) + \sum_{i=1}^{n-1} \alpha_i \circ \alpha_{n-i} = 0$$

for all $n \in \mathbb{N}_{>0}$.

2.4.1. Lemma. *Suppose $\alpha_0, \alpha_1, \dots, \alpha_{n-1}$ satisfy (A'_i) for $i = 0, 1, \dots, n-1$, then*

$$d_\alpha\left(\sum_{i=1}^{n-1} \alpha_i \circ \alpha_{n-i}\right) = 0.$$

Proof.

$$\begin{aligned} d_\alpha\left(\sum_{i+j=n} \alpha_i \circ \alpha_j\right) &= \sum_{i+j=n} (\alpha_i \circ d_\alpha(\alpha_j) - d_\alpha(\alpha_i) \circ \alpha_j) \\ &= - \sum_{i+j+k=n} (\alpha_i \circ (\alpha_j \circ \alpha_k) - (\alpha_i \circ \alpha_j) \circ \alpha_k) \\ &= - \sum_{\substack{i+j+k=n \\ 1 \leq i, j < k}} \alpha_i \circ (\alpha_j \circ \alpha_k + \alpha_k \circ \alpha_j) - ((\alpha_i \circ \alpha_j) \circ \alpha_k + (\alpha_i \circ \alpha_k) \circ \alpha_j) \\ &= 0. \end{aligned}$$

We leave it as an exercise to show that the first equality holds despite the fact that d_α is *not* a derivation for the (non associative) multiplication \circ . The second equation holds then by hypothesis. For the third equation use the fact that $\beta \circ (\gamma \circ \gamma) = (\beta \circ \gamma) \circ \gamma$ if $|\gamma| = 1$ and note that $|\alpha_i| = 1$ for all i . The last equality holds by Lemma 2.1.1. \square

2.4.2. Corollary. *If $H^3(\alpha) = 0$, each infinitesimal deformation of α is integrable.*

2.5. Deformation functor. We call a commutative local k -algebra with maximal ideal \mathfrak{m} and counit $p_R: R \rightarrow k$ a *test algebra* if \mathfrak{m} is nilpotent. Recall that $R = k1_R \oplus \mathfrak{m}$ as a vector space. Important examples of test algebras are the algebras $k[t]/(t^n)$ for $n \in \mathbb{N}$.

With the definition of $\text{Defo}(\alpha, R)$ from 1.4 it is not hard to see that we obtain in fact a functor

$$\text{Defo}(\alpha, -): \text{Test Algebras} \rightarrow \text{Sets}.$$

If α_t is a formal deformation of α , we obtain for all $n \in \mathbb{N}$ via the reduction $k[[t]] \rightarrow k[t]/(t^n)$ a $k[t]/(t^n)$ -deformation of α . It is not hard to see that we obtain in this way a bijective map from the set of formal deformations of α to the set $\varprojlim \{k[t]/(t^n)\text{-deformations}\}$, which descends even to equivalence classes of deformations. Thus for the understanding of formal deformations it is sufficient to study the functor $\text{Defo}(\alpha, -)$ on the category of test algebras.

2.6. Maurer-Cartan equation. Let $L(\alpha) := (V_*, [-, -], d_\alpha)$ be the differential graded Lie algebra with (V^*, d_α) the Hochschild complex (shifted by one degree), together with the Gerstenhaber bracket $[-, -]$ defined in 2.1.3. For a test algebra R we consider $\alpha_R = \alpha + \alpha_{\mathfrak{m}}$ for some $\alpha_{\mathfrak{m}} \in \text{Hom}_{\mathbf{k}}(\mathbf{k}^n \otimes \mathbf{k}^n, \mathbf{k}^n \otimes \mathfrak{m}) = V_1 \otimes \mathfrak{m}$ thus we may interpret α_R as an element of $\text{Hom}_R(R^n \otimes_R R^n, R^n)$ which reduces modulo \mathfrak{m} to α . The associativity of α_R may thus be expressed in $L_\alpha \otimes \mathfrak{m}$ (where $[\beta \otimes m, \gamma \otimes n] := [\beta, \gamma] \otimes (m \cdot n)$) by the *Maurer-Cartan equation*

$$d_\alpha(\alpha_{\mathfrak{m}}) + \frac{1}{2}[\alpha_{\mathfrak{m}}, \alpha_{\mathfrak{m}}] = 0$$

(if $\text{char}(\mathbf{k}) \neq 2$). To see this, recall that $\alpha_{\mathfrak{m}}$ is an element of degree 1 and thus $[\alpha_{\mathfrak{m}}, \alpha_{\mathfrak{m}}] = 2\alpha_{\mathfrak{m}} \circ \alpha_{\mathfrak{m}}$, and $d_\alpha(\alpha_{\mathfrak{m}}) = [\alpha, \alpha_{\mathfrak{m}}]$, then compare with equation (1.1).

2.7. Abstract deformation theory. For the rest of this section let \mathbf{k} be a field of characteristic 0. We consider a differential graded Lie algebra $L = (L_*, [-, -], d)$ and let

$$\text{MC}(L) := \{\beta \in L_1 \mid d(\beta) + \frac{1}{2}[\beta, \beta] = 0\}$$

the solutions of the Maurer-Cartan equation. If L_1 is finite dimensional, $\text{MC}(L)$ can be seen as an intersection of quadrics. With the notation from 1.3 we have then $\text{MC}(\alpha, R) = \text{MC}(L(\alpha) \otimes \mathfrak{m})$.

In any case, we define for $\beta \in \text{MC}(L)$ formally the tangent space

$$T_{\beta, \text{MC}(L)} := \{X \in L_1 \mid d(X) + [\beta, X] = 0\}$$

which in case L_1 finite dimensional is actually the scheme-theoretic tangent space of $\text{MC}(L)$ at β (exercise!).

For $\beta \in L_1$ define a linear endomorphism d_β of degree 1 of L_* by $d_\beta(\gamma) = d(\gamma) + [\beta, \gamma]$.

2.7.1. Lemma. *For $\beta \in \text{MC}(L)$ we have*

- (a) $T_{\beta, \text{MC}(L)} = \{X \in L_1 \mid d_\beta(X) = 0\}$
- (b) $d_\beta^2 = 0$
- (c) *For any $Y \in L_0$ we obtain by $\beta \mapsto d_\beta(Y)$ a vector field in $\text{MC}(L)$.*
- (d) *The map $L_0 \rightarrow \text{Lie}(\text{Aff}(L_1)), Y \mapsto (\beta \mapsto d_\beta(Y))$ is a Lie anti-homomorphism.*

Proof. (a) is immediate from the definition, (b) is a straightforward, (c) follows from (a) and (b). For (d) note first that for $Y \in L_0$ the map $\beta \mapsto d_\beta(Y) = d(Y) + [Y, \beta]$ is an affine transformation. ... \square

If L_0 is nilpotent and $\text{ad}(Y) \mid_{L_1}$ is nilpotent for all Y in L_0 then the map from Lemma 2.7.1 (d) integrates to a group anti-homomorphism

$$\exp(L_0) \rightarrow \text{Aff}(L_1)$$

This means that we obtain a right action of the group $\exp(L_0)$ on L_1 via affine automorphisms. This action leaves $\text{MC}(L)$ invariant by 2.7.1 (c), and we obtain a well-defined orbit set $\text{MC}(L)/\exp(L_0)$.

For an arbitrary dg-Lie algebra L and a test algebra R , trivially $L \otimes \mathfrak{m}$ satisfies the above nilpotency hypotheses, and we may define

$$(2.1) \quad \overline{\text{MC}}(L, R) := \text{MC}(L \otimes \mathfrak{m}) / \exp(L_0 \otimes \mathfrak{m})$$

This is motivated by the observation that for $\alpha \in \text{Alg}_n(\alpha)$ there is a functorial bijection

$$\text{Defo}(\alpha, R) \rightarrow \overline{\text{MC}}(L(\alpha), R).$$

2.8. Further Remarks. A morphism of dg-Lie algebras is a linear map of degree 0 which commutes with the respective brackets and differentials. In particular, if $f: L \rightarrow L'$ is a homomorphism of dg-Lie algebras, it induces a homomorphism between the corresponding cohomology groups $H^i(f): H^i(L) \rightarrow H^i(L')$ for all $i \in \mathbb{Z}$. Such a homomorphism is called a *quasi-isomorphism* if $H^i(f)$ is an isomorphism for all $i \in \mathbb{Z}$. One has the following deep result:

2.8.1. Theorem. *Let $f: L \rightarrow L'$ be a quasi-isomorphism of dg-Lie algebras, and R a test algebra. Then f induces a bijection $\overline{\text{MC}}(L, R) \rightarrow \text{MC}(L', R)$.*

2.8.2. Corollary. *Let $\alpha \in \text{Alg}_n(\mathbf{k})$, then each formal deformation of α is equivalent to a formal deformation which has 1_α as a unit.*

Proof. It is sufficient to proof the claim for any test algebra R . Now, let $V_1^i(\alpha)$ be the subspace of $V^i(\alpha)$ consisting of all cochains which vanish if one of their arguments is 1_α . It turns out that the inclusion of $V_1(\alpha)$ into $V(\alpha)$ is a quasi-isomorphism of dg-Lie algebras. So our claim follows from the theorem. \square

3. DEFORMATIONS OF FINITE-DIMENSIONAL MODULES

Deformation theory of modules is quite similar to the deformation theory of algebras, but somehow simpler.

3.1. Notation. We keep the notations from 1.1 and consider an unitary left A -module with underlying vector space $W = \mathbf{k}^d$ and multiplication given by $\mu \in \text{Hom}(V \otimes W, W)$. Thus μ has to fulfill

- (i) $\mu(\mathbb{1}_V \otimes \mu) = \mu(\alpha \otimes \mathbb{1}_W)$,
- (ii) $\mu(1_\alpha \otimes \mathbb{1}_W) = \mathbb{1}_W$.

Note that under (i) condition (ii) is equivalent to

$$(ii') \quad \text{rank } \mu(1_\alpha \otimes \mathbb{1}_W) \geq d.$$

For $g \in \text{GL}_d(\mathbf{k})$ we find that $\mu^g := g^{-1}\mu(\mathbb{1}_V \otimes g)$ defines also a A -module structure on W . In fact, all structures isomorphic to μ are precisely of this form.

3.2. Definition. A R -deformation of μ is an element $\mu_R \in \text{Hom}_R(V_R \otimes_R W_R, W_R)$ which reduces modulo R to μ and fulfills $\mu_R(\mathbb{1}_{V_R} \otimes_R \mu_R) = \mu_R(\alpha \otimes_R \mathbb{1}_{W_R})$. Two R -deformations μ_R and μ'_R are *equivalent* if $\mu'_R = \mu_R^g$ for some $g \in \text{E}_d(R)$ (see 1.4). The concepts of formal, infinitesimal and integrable deformation carry over in the obvious way from the algebra case.

3.3. Remarks. As in the algebra case, a R -deformation μ_R of μ is determined by its \mathbf{k} -linear component $\mu_{\mathbf{m}}: V \otimes W \rightarrow W \otimes \mathbf{m}$ and the condition from the definition translates into

$$(3.1) \quad \mu(\mathbb{1}_V \otimes \mu_{\mathbf{m}}) - \mu_{\mathbf{m}}(\alpha \otimes \mathbb{1}_W) + \mu_{\mathbf{m}}(\mathbb{1}_V \otimes \mu) + \mu_{\mathbf{m}}(\mathbb{1}_V \otimes \mu_{\mathbf{m}}) = 0.$$

For example, an infinitesimal deformation $\mu + \epsilon\mu_1$ of μ is given by

$$\mu_1 \in Z^1(\mu) := \{\zeta \in \text{Hom}(V \otimes W, W) \mid \mu(\mathbb{1}_V \otimes \zeta) - \zeta(\alpha \otimes \mathbb{1}_W) + \zeta(\mathbb{1}_V \otimes \mu) = 0\}$$

since $\epsilon^2 = 0$.

Similarly, or a family $\mu_1, \mu_2, \dots \in \text{Hom}_k(V \otimes W, W)$ we see that

$$\mu_t = \mu + \mu_1 \cdot t + \mu_2 \cdot t^2 + \dots$$

is a formal deformation of $\mu \in \text{mod}_A^d(k)$ if and only if for all $n = 1, 2, 3, \dots$ holds

$$(M_n) \quad \mu(\mathbb{1}_V \otimes \mu_n) - \mu_n(\alpha \otimes \mathbb{1}_W) + \mu_n(\mathbb{1}_V \otimes \mu) + \sum_{i+j=n} \mu_i(\mathbb{1}_v \otimes \mu_j) = 0.$$

3.4. Formal deformation theory. We want to exhibit how the deformation theory of a module is controlled by a dg-Lie algebra.

3.4.1. Definition. Set $C^i := \text{Hom}_k(V^{\otimes i} \otimes W, W)$, and equip the graded vector space $C^\bullet := \bigoplus_{i \in \mathbb{N}} C^i$ with an *associative* graded multiplication $*$ by

$$f * g := f(\mathbb{1}_V^{\otimes |f|} \otimes g) \in C^{|f|+|g|}$$

for homogeneous elements f and g . Moreover we have a differential d_μ (of degree one) with

$$d_\mu(f) = \mu(\mathbb{1}_V \otimes f) + \sum_{i=1}^{|f|} (-1)^i f(\mathbb{1}_V^{\otimes(i-1)} \otimes \alpha \otimes \mathbb{1}_V^{\otimes(|f|-i)} \otimes \mathbb{1}_W) - (-1)^{|f|} f(\mathbb{1}_V^{\otimes |f|} \otimes \mu)$$

It is not hard to verify with the defining properties of μ and α that $(C^\bullet, *, d_\mu)$ is a dg-algebra, i.e. $d_\mu^2 = 0$ and

$$d_\mu(f * g) = d_\mu(f) * g + (-1)^{|f|} f * d_\mu(g).$$

Thus we may introduce on C^\bullet also a (graded) Lie-bracket by defining

$$[f, g] := f * g - (-1)^{|f||g|} g * f$$

for homogeneous elements f and g . It is clear that in this way $C(\mu) := (C^\bullet, [-, -], d_\mu)$ becomes a dg-Lie algebra. Finally set

$$\begin{aligned} Z^i(\mu) &:= C^i \cap \text{Ker } d_\mu \\ B^i(\mu) &:= d_\mu(C^{i-1}) \\ H^i(\mu) &:= Z^i(\mu) / B^i(\mu). \end{aligned}$$

3.4.2. Remark. Note that $(C^\bullet, *, d_\mu) = \mathcal{E}nd_A^\infty(\mu)$, the dg-endomorphism ring of μ in the category of A_∞ -modules over A . In particular, $H^i(\mu) = \text{Ext}_A^i(\mu, \mu)$ for $i \in \mathbb{N}$.

We leave it as an exercise to verify that $Z^1(\mu) = T_{\text{mod}_A^d, \mu}$.

3.4.3. Maurer-Cartan equation. With our definitions from 3.4.1 we may rewrite the equations (M_n) in 3.3 as

$$(M'_n) \quad d_\mu(\mu_n) + \sum_{i+j=n} \mu_i * \mu_j = 0.$$

More generally, $\mu + \mu_{\mathfrak{m}}$ is a R -deformation of μ (see 3.3) if and only if $\mu_{\mathfrak{m}}$ fulfills the Maurer-Cartan equation

$$d_\mu(\mu_{\mathfrak{m}}) + \frac{1}{2}[\mu_{\mathfrak{m}}, \mu_{\mathfrak{m}}] = 0$$

in $C(\mu) \otimes \mathfrak{m}$ (provided $\text{char } k \neq 2$), this is just equation (3.1).

We leave it as an exercise to show that if \mathfrak{m} is nilpotent and $\text{char}(\mathbf{k}) = 0$ even $\overline{\text{MC}}(C(\mu), R)$ (equation (2.1)) corresponds bijectively to the equivalence classes of R -deformations of μ .

3.5. Infinitesimal deformations. We find similar results as for algebras: The space $\text{Ext}_A^1(\mu, \mu)$ classifies naturally the equivalence classes of infinitesimal deformations of μ . As a consequence, if $\text{Ext}_A^1(\mu, \mu) = 0$ each formal deformation is trivial, and each infinitesimal deformation is integrable. We leave the details as an exercise. Note that as in the case of algebras, for these results the Lie-structure on the complex (C^\bullet, d_μ) is not needed.

3.6. Obstructions. Let us note that the “obstruction” to extend a $\mathbf{k}[t]/(t^{n+1})$ -deformation of an A -module μ to a $\mathbf{k}[t]/(t^{n+1})$ -deformation is an element of the space $\text{Ext}_A^2(\mu, \mu)$. This follows from the following:

3.6.1. Lemma. *Let $\mu_t = \mu + \mu_1 \cdot t + \cdots + \mu_n \cdot t^n$ be a $\mathbf{k}[t]/(t^{n+1})$ deformation of $\mu \in \text{mod}_A^d(\mathbf{k})$. Then*

$$\sum_{i+j=n+1} \mu_i * \mu_j \in Z^1(\mu).$$

Proof. We calculate

$$\begin{aligned} d_\mu\left(\sum_{i+j=n+1} \mu_i * \mu_j\right) &= \sum_{i+j=n+1} d_\mu(\mu_i) * \mu_j - \mu_i * d_\mu(\mu_j) \\ &\stackrel{\text{hyp.}}{=} \sum_{i+j=n+1} i + j + k(\mu_i * \mu_j) * \mu_k - \mu_i * (\mu_j * \mu_k) \\ &= 0, \end{aligned}$$

since $(C^\bullet, *, d_\mu)$ is an (associative) dg-algebra. \square

3.6.2. Corollary. *If $\text{Ext}_A^2(\mu, \mu) = 0$ each infinitesimal deformation of μ is integrable.*

4. VARIETIES AND SCHEMES

In this section \mathbf{k} will be an algebraically closed field.

4.1. Notions from algebraic geometry.

- $\mathbb{A}^n(\mathbf{k}) := \mathbf{k}^n$ the *affine space* of dimension n , with Zariski topology. So *closed subsets* are defined by the vanishing of polynomials. For example the standard parabola in $\mathbb{A}^2(\mathbf{k})$ is the closed subset $V_{(x_1^2 - x_2)}$.
- $X \subset \mathbb{A}^n(\mathbf{k})$ is *locally closed* if it is the intersection of an open subset with a closed subset. Equivalently, X is open in its (Zariski) closure. Note that $\mathbb{A}^2 \setminus \{(t, 0) \mid t \in \mathbf{k}^*\}$ is *not* locally closed.
- For $X \subset \mathbb{A}^n(\mathbf{k})$ locally closed, we say that a continuous function $f: X \rightarrow \mathbf{k} = \mathbb{A}^1$ is *regular* if for each $x \in X$ there exists an open neighborhood U and $g, h \in \mathbf{k}[x_1, \dots, x_n]$ such that the restriction of $f(u) = g(u)/h(u)$ for all $u \in U$ (in particular $h(u) \neq 0$ for all $u \in U$). We write $\mathcal{O}(X)$ for the commutative \mathbf{k} -algebra of regular functions on X .
- A *quasi-affine variety* is a locally closed subset $X \subset \mathbb{A}^n(\mathbf{k})$ together with the datum of $\mathcal{O}(U)$ for all open subsets $U \subset X$.

- A *morphism* of varieties is a continuous map $f: X \rightarrow Y$ such that the composition

$$f^{-1}(U) \xrightarrow{f|_U} U \xrightarrow{g} k$$

is a regular function on $f^{-1}(U)$ for all open subsets $U \subset Y$ and all $g \in \mathcal{O}(U)$. Note, that bijective morphisms are not always isomorphisms of varieties.

- An *affine variety* is a variety which is isomorphic to a closed subset of some $\mathbb{A}^n(k)$.
- A variety X is *irreducible* if $X = X_1 \cup X_2$ for closed subsets X_1, X_2 implies $X_1 = X$ or $X_2 = X$. The *irreducible components* of a variety are its maximal (closed) irreducible subsets. Each variety is the union of its (essentially) unique irreducible components. For example, the affine variety $V(x_1 \cdot x_2) \subset \mathbb{A}^2$ has the two components $V(x_1)$ and $V(x_2)$.

4.2. Remarks on schemes. Schemes are a natural generalization of varieties. A scheme may be identified with its “functor of points”

$$\{\text{commutative } k\text{-algebras}\} \rightarrow \text{Sets}.$$

Affine schemes correspond to representable functors, i.e. to functors of the form $\text{Hom}_{k\text{-alg}}(R, -)$ for some commutative k -algebra R . Thus, the category of affine schemes is anti-equivalent to the category of commutative k -algebras.

Affine algebraic schemes correspond in this way to finitely generated commutative k -algebras, while *reduced affine* schemes correspond to commutative k -algebras without nilpotent elements.

Any algebraic scheme X gives rise to a (not necessarily affine) variety $X(k)$. In this way, algebraic reduced schemes correspond bijectively to varieties. In particular, an affine variety V corresponds to the affine scheme $\text{Hom}_{k\text{-alg}}(\mathcal{O}(V), -)$.

4.3. Examples.

- $\mathbb{A}^n(S) = S^n$ is an affine reduced scheme, it is represented by $k[x_1, \dots, x_n]$.
- $\text{GL}_d(S) := \{M \in \text{Mat}_{n \times n}(S) \mid M \text{ invertible}\}$. This is an algebraic reduced affine scheme. It is represented by $k[t, X_{ij} \mid 1 \leq i, j \leq d] / (t \cdot \det((X_{ij})) - 1)$.
- $\text{Ass}_d(S) := \{\text{associative } S\text{-algebra structures on } S^d\}$. This is an affine algebraic scheme represented by

$$k[X_{ij}^k \mid 1 \leq i, j, k \leq d] / \left(\sum_{s=1}^d (X_{ij}^s X_{sk}^t - X_{is}^t X_{jk}^s) \mid 1 \leq i, j, k, t \leq d \right).$$

This is not reduced, and it seems to be quite difficult to describe the coordinate ring of the corresponding reduced scheme. It is here more natural to work with the non-reduced structures.

- For $r, n \in \mathbb{N}_+$ define the Grassmann scheme by

$$\mathbb{G}_{r,n}(S) := \{\text{direct summands } X \text{ of } S^{r+n} \mid \text{rank } X = r\}.$$

This is in fact a scheme [5, I §1, 3.13], which is reduced but it is not affine. Note that $\mathbb{G}_{1,n}$ is \mathbb{P}^n , the projective n -space.

4.4. Some deformation lemmata. We present here some results which are useful in the context of deformation theory, as we shall see.

4.4.1. Lemma. *Let $f: X \rightarrow Y$ be a morphism between schemes over k and assume that Y is of algebraic and quasi-projective over k . If for every $x \in X(k)$ the restriction*

$$f_{k[[t]]}: (X(p_{k[[t]]}))^{-1}(x) \rightarrow Y(p_{k[[t]]})^{-1}(f(x))$$

is surjective, then $f(X(k))$ is open in $Y(k)$.

This result from [3, Lemma 7.1] is similar to the valuative criteria for separateness and properness of a morphism (where $p_{k[[t]]}$ has to be replaced by the inclusion $\iota: R \rightarrow K$ of a discrete valuation ring R into its field of fractions K , and the corresponding restriction of f_R has to be injective resp. bijective). Sloppily one might state the condition as “each deformation of $f(x)$ comes from a deformation of x .” The next result is Lemma 7.2 from [3].

4.4.2. Lemma. *Let X be a scheme over k and U an open subscheme of X . If $x \in U(k)$ and $x_t \in X(k[[t]])$ such that $X(p_{k[[t]]})(x_t) = x$, then $x_t \in U(k[[t]])$.*

The proof in [8, §1.6] can be easily adapted to show the following:

4.4.3. Lemma. *Let X be an algebraic scheme over k and suppose that $x \in X(k)$ has an open neighborhood U such that for all $y \in U(x)$ the following conditions hold:*

- (i) *For each $y' \in T_{U,y}$ we have $(U(p_{t,\epsilon}))^{-1}(y') \neq \emptyset$ where $p_{t,\epsilon}: k[[t]] \rightarrow k[[\epsilon]]$ is the canonical projection.*
- (ii) *$\dim T_{U,y} = \dim T_{U,x}$.*

Then x is a regular point of X .

5. THE SCHEME OF ALGEBRA STRUCTURES

We consider the k -functor Alg_n of associative unitary algebra structures on a n -dimensional space. Thus it is defined by

$$\text{Alg}_n(S) := \{\alpha \in \text{Ass}_n(S) \mid \alpha \text{ has a } 1\}.$$

This is in fact an open, affine subscheme of Ass_n . We show here however the following weaker result following [2, §1].

- 5.1. Proposition.**
- (a) $\text{Alg}_n(k)$ is an open subset of $\text{Ass}_n(k)$.
 - (b) $e: \text{Alg}_n(k) \rightarrow k^n, \alpha \mapsto 1_\alpha$ is a morphism.
 - (c) $\text{Alg}_n(k)$ is an affine variety.

Proof. (a) For $a \in k^n$ and $\alpha \in \text{Ass}_n(k)$ we define endomorphisms of k^n by $l_{a,\alpha}(v) = \alpha(a \otimes v)$ and $r_{a,\alpha}(v) = \alpha(v \otimes a)$. It is not hard to see that α admits an unit 1_α if and only if for some $a \in k^n$ $l_{a,\alpha}$ and $r_{a,\alpha}$ are invertible, and in this case $1_\alpha = l_{a,\alpha}^{-1}(a)$. Now, define for $a \in k^n$ open (affine) subsets of $\text{Ass}(k)$ by

$$D_a = \{\alpha \in \text{Ass}_n(k) \mid \det(l_{a,\alpha}) \det(r_{a,\alpha}) \neq 0\},$$

then $\text{Alg}_n(k) = \cup_{a \in k^n} D_a(k)$.

- (b) Since 1_α is unique (if it exists), we can define e locally on D_a as $\alpha \mapsto l_{a,\alpha}^{-1}(a)$.

(c) The map $\alpha \mapsto (\alpha, e(\alpha))$ defines an isomorphism from $\text{Alg}_n(\mathbf{k})$ to the affine variety

$$\{(\alpha, v) \in \text{Ass}_n(\mathbf{k}) \times \mathbf{k}^n \mid v = 1_\alpha\}$$

□

GL_n acts on Alg_n via $\alpha^g = g^{-1}\alpha(g \otimes g)$, thus GL_n -orbits correspond bijectively to isoclasses of n -dimensional unitary \mathbf{k} -algebras.

$\text{Alg}_n(\mathbf{k})$ is connected. In fact, the orbit of the unique n -dimensional local algebra with maximal ideal \mathfrak{m} such that $\mathfrak{m}^2 = 0$, belongs to each irreducible component.

5.1.1. Remark. The scheme of algebra structures was already studied in [6]. There, Alg_4 is described in detail, it has 5 components of dimension 16, 14, 13(2), 10. For example, the automorphism group of the semi-simple algebra $\mathbf{k} \times \mathbf{k} \times \mathbf{k} \times \mathbf{k}$ is the symmetric group \mathfrak{S}_4 , so the corresponding GL_4 -orbit in Alg_4 has dimension 16.

Alg_5 has 10 components of dimensions 25, 23, 22(2), 21(3), 19, 17, 13. This was worked out in [12]. In this case, each irreducible component contains a (scheme-theoretically) open GL_5 -orbit. In fact, it is not difficult to verify that for each of the generic algebras A found by Mazzola one has $H^2(A) = 0$.

The calculations in [4] show that for $n \geq 6$ the scheme Alg_n is not generically reduced.

5.2. Connection with Deformation theory. Let R be a local commutative \mathbf{k} -algebra with maximal ideal \mathfrak{m} and $p_R: R \rightarrow R/\mathfrak{m} = \mathbf{k}$ the canonical projection. Then the R -deformation of $\alpha \in \text{Alg}_d(\mathbf{k})$ are just the elements of $(\text{Alg}_d(p_R))^{-1}(\alpha) \subset \text{Alg}_d(R)$. Similarly we have the group $E(R) = (\text{GL}_n(p_R))^{-1}(\mathbb{1}_n) \subset \text{GL}_n(R)$, compare 1.4. Thus the equivalence classes of R -deformations are just the $E(R)$ -orbits on $(\text{Alg}_d(p_R))^{-1}(\alpha)$.

5.3. Infinitesimal deformations. In this case $R = \mathbf{k}[\epsilon] := \mathbf{k}[t]/(t^2)$. Thus the set $Z^2(\alpha)$ of infinitesimal deformations of α (see 1.3 and 2.1) is identified with the tangential space $T_{\text{Alg}_n, \alpha}$. This is in general *not* true for the tangential space of the *variety* of algebra structures.

Next, we may consider the orbit map

$$\omega: \text{GL}_n \rightarrow \text{GL}_n \cdot \alpha, \quad g \mapsto g\alpha(g^{-1} \otimes g^{-1}).$$

We leave it as an exercise to identify the image of the differential $d_{\omega,1}$ with $d_\alpha(C^1)$, where d_α is the differential of the Hochschild complex defined in 2. The kernel of $d_{\omega,1}$ may be identified with the scheme theoretical tangential space of the stabilizer of α (*i.e.* the automorphism group of the corresponding algebra). So an easy dimension count shows that $d_{\omega,1}$ is onto if and only if the stabilizer of α is reduced. Thus we obtain the following version of Voigt's lemma:

5.3.1. Lemma. *For $\alpha \in \text{Alg}_n(\mathbf{k})$ one has a natural surjective linear map*

$$H^2(\alpha) \rightarrow T_{\text{Alg}_n, \alpha} / T_{\text{GL}_n \cdot \alpha, \alpha}.$$

This map is an isomorphism if and only if the stabilizer of α is reduced.

If $\text{char } \mathbf{k} = 0$ each affine group scheme is reduced by Cartier's theorem [5, II §6.1.1]. However, the automorphism group of $\mathbf{k}[x]/(x^2)$ is not reduced if $\text{char}(\mathbf{k}) = 2$. So, at least for fields of characteristic 0 the equivalence classes of infinitesimal deformations

of an finite-dimensional algebra can be identified with the tangential space of the scheme of algebra structures at this algebra modulo the tangential space along the orbit of this algebra.

5.3.2. Lemma. *Let $\alpha \in \text{Alg}_n(k)$. If $H^3(\alpha) = 0$ then α is a smooth point of Alg_n .*

This follows essentially from 4.4.3 together with 2.4.2, taking into account that the dimensions of Hochschild cohomology groups are upper semi-continuous functions on $\text{Alg}_n(k)$. See [8, §1.6] for more details.

5.4. Formal deformations. Discuss relation with curves through α .

6. THE SCHEME OF MODULE STRUCTURES

6.1. Basic construction. Let A be a finitely generated (unitary) k -algebra, say $A = k\langle a_1, \dots, a_n \rangle / J$. For $d \in \mathbb{N}$ we define mod_A^d , the scheme of d -dimensional (unitary) A -modules by

$$\text{mod}_A^d(S) := (m^{(1)}, \dots, m^{(n)}) \in \text{Mat}_{d \times d}(S)^n \mid f(m^{(1)}, \dots, m^{(n)}) = 0 \text{ for all } f \in J,$$

in other words, $\text{mod}_A^d(S)$ consists of the k -algebra homomorphisms $A \rightarrow \text{Mat}_{d \times d}(S)$. This is an affine scheme, represented by $R = P/I$, where

$$P := k[X_{i,j}^{(k)} \mid 1 \leq k \leq n, 1 \leq i, j \leq d].$$

In order to describe I , we set

$$X^{(l)} := \begin{pmatrix} X_{1,1}^{(l)} & \cdots & X_{1,d}^{(l)} \\ \vdots & & \vdots \\ X_{d,1}^{(l)} & \cdots & X_{d,d}^{(l)} \end{pmatrix}$$

and I is the ideal of P which is generated by the $(d \times d)$ entries of $f(X^{(1)}, \dots, X^{(n)})$, where f runs over the elements of J . Note that I is finitely generated even if J is not since P is noetherian.

In general, mod_A^d is not reduced, and in general it would be hopeless to describe the coordinate ring of its reduced structure.

GL_d acts on mod_A^d by conjugation: If $m = (m_1, \dots, m_n) \in \text{mod}_A^d(S)$ and $g \in \text{GL}_d(S)$ then $m^g := (g^{-1}m_1g, \dots, g^{-1}m_ng)$. Thus the $\text{GL}_d(k)$ -orbits on $\text{mod}_A^d(k)$ correspond bijectively to the isoclasses of d -dimensional A -modules.

6.2. Examples. Let $A = k[x_1, x_2]/(x_1x_2)$. Then

$$\text{mod}_A^d(S) := \{(M_1, M_2) \in \text{Mat}_{d \times d}^2 \mid M_1M_2 = 0 = M_2M_1\}.$$

For an arbitrary (finitely generated) commutative k -algebra A it is easy to see that

$$\text{mod}_A^1(-) := \text{Hom}_{k\text{-alg}}(A, -).$$

6.3. Basic properties. In general, mod_A^d is not connected, and the connected components are possibly not irreducible. For example, if A is finite dimensional, then

$$\text{mod}_A^d := \coprod_{S \in \mathcal{S}(d)} \text{mod}_A^{[S]}$$

where $\mathcal{S}(d)$ is a set of representatives for the isoclasses of d -dimensional semisimple S -modules, and $\text{mod}_A^{[S]}(\mathbf{k})$ are the d -dimensional A -modules which have the same simple composition factors as S . In this case the orbit of $[S]$ belongs to all irreducible components of $\text{mod}_A^{[S]}$.

On the other hand, mod_A^d is smooth for all $d \in \mathbb{N}$ if and only if A is hereditary (and in this case each connected component is also irreducible). See [1] for more details and a good account of related material.

6.4. Differentials and infinitesimal deformations. In this paragraph we assume that the underlying vector space V of A is finite dimensional, and write $W := \mathbf{k}^d$. We start with the following remarks which we leave as exercises.

(a) With the definitions from 3.4.1, the functions

$$\begin{aligned} z^{(i)}: \text{mod}_A^d(\mathbf{k}) &\rightarrow \mathbb{N}_0, \mu \mapsto \dim Z^i(\mu) \\ e^{(i)}: \text{mod}_A^d(\mathbf{k}) &\rightarrow \mathbb{N}_0, \mu \mapsto \dim H^i(\mu) \end{aligned}$$

are upper semicontinuous. In particular, if $e^{(i+1)}(\mu) = 0$ there exists an open neighborhood U of μ and $c \in \mathbb{N}_0$ such that $e^{(i+1)}(\mu') = 0$ and $z^{(i)}(\mu') = c$ for all $\mu' \in U$.

(b) The restriction $d_\mu^0: \text{Hom}_{\mathbf{k}}(W, W) \rightarrow \text{Hom}_{\mathbf{k}}(V \otimes W, W)$ of d_μ identifies naturally to the differential $d_{\omega_\mu, 1}$ of the orbit map

$$\omega_\mu: \text{GL}_d \rightarrow \text{Hom}_{\mathbf{k}}(V \otimes W, W), \quad g \mapsto g \cdot \mu(\mathbb{1} \otimes g^{-1}).$$

Similarly we have a natural isomorphism $Z^1(\mu) \cong T_{\text{mod}_A, \mu}$.

(c) The restriction $d_{\omega, 1}: T_{\text{GL}_d, 1} \rightarrow T_{\text{GL}_d \cdot \mu, \mu}$ is always surjective, since the stabilizer $\text{Aut}_A(\mu)$ is an open subscheme of the linear space $\text{End}_A(\mu)$.

From the above remarks (b) and (c) we obtain as in 5.3:

6.4.1. Lemma (Voigt). *For $\mu \in \text{mod}_A^d(\mathbf{k})$ we have $T_{\text{mod}_A, \mu} / T_{\text{GL}_d \cdot \mu, \mu} \cong \text{Ext}_A^1(\mu, \mu)$.*

Note, that the above isomorphism holds in general only for the scheme-theoretic tangential space, for the tangent space to the (reduced) variety one obtains only an inclusion.

Finally, the identifications from (b) together with (a) show that the hypothesis of 4.4.3 are fulfilled, and we obtain:

6.4.2. Corollary. *If $\text{Ext}_A^2(\mu, \mu) = 0$ for some $\mu \in \text{mod}_A^d(\mathbf{k})$, then this is a smooth point of mod_A^d .*

6.5. Decomposition theory. Inspired by the work of Kac and Schofield on representations of quivers, de la Peña and finally Crawley-Boevey and Schröer developed a theory of “indecomposable” irreducible components and of “decomposition” of irreducible components into indecomposable components for mod_A^d . Elementary deformation theory plays central part in the proof of this important results. We will sketch this here.

Let $d = d_1 + \cdots + d_t$ and $C_i \subset \text{mod}_A^{d_i}(\mathbf{k})$ subsets which are $\text{GL}_{d_i}(\mathbf{k})$ -stable. We consider all d -dimensional modules which are of the form $m = m_1 \oplus \cdots \oplus m_t$ with $m_i \in C_i$. Thus we may think of $m^{(i)}$ having (up to simultaneous conjugation) the shape

$$\begin{pmatrix} m_1^{(i)} & & 0 \\ & \ddots & \\ 0 & & m_t^{(i)} \end{pmatrix}.$$

We write $C_1 \oplus \cdots \oplus C_t$ for the corresponding GL_d -stable subset of $\text{mod}_A^d(\mathbf{k})$, and $\overline{C_1 \oplus \cdots \oplus C_t}$ for its Zariski closure.

The following basic result is almost folklore, see for example [3] for a proof.

Theorem 1. *If $C \subset \text{mod}_A^d$ is an irreducible component of mod_A^d , then we have $C = \overline{C_1 \oplus \cdots \oplus C_t}$ for some irreducible components C_i of $\text{mod}_A^{d_i}$ such that $C_i(\mathbf{k})$ contains an open dense subset of indecomposable modules. Moreover, the C_i are unique up to reordering.*

Thus we have a kind of Krull-Schmidt theorem for irreducible components. However, direct sums of irreducible components are not always irreducible components. This is solved by the following result:

Theorem 2 (Crawley-Boevey, Schröer). *If C_i is an irreducible component of $\text{mod}_A^{d_i}$ for $1 \leq i \leq t$ and $d = d_1 + \cdots + d_t$, then $\overline{C_1 \oplus \cdots \oplus C_t}$ is an irreducible component of mod_A^d if and only if $\text{ext}_A^1(C_i, C_j) = 0$ for all $i \neq j$.*

Here, $\text{ext}_A^1(C_1, C_2) := \min\{\dim \text{Ext}_A^1(m_1, m_2) \mid (m_1, m_2) \in C_1 \times C_2\}$. Since the map $C_1 \times C_2 \rightarrow \mathbb{N}, (m_1, m_2) \mapsto \dim \text{Ext}_A^1(m_1, m_2)$ is upper semi-continuous, $\text{ext}_A^1(C_1, C_2)$ can be considered as the “general” dimension of extensions between modules from C_1 and C_2 .

6.6. On the proof of Theorem 2.

6.6.1. Lemma. *Suppose $\text{Ext}_A^1(m_2, m_1) = 0$ and let $m \in \text{mod}_A^d(\mathbf{k})$ such that*

$$m^{(i)} = \begin{pmatrix} m_1^{(i)} & d^{(i)} \\ 0 & m_2^{(i)} \end{pmatrix} \text{ for } i = 1, 2, \dots, n.$$

Then each formal deformation of m is equivalent to a deformation m_t such that

$$m_t^{(i)} = \begin{pmatrix} \tilde{m}_1^{(i)} & \tilde{d}^{(i)} \\ 0 & \tilde{m}_2^{(i)} \end{pmatrix} \text{ for } 1 \leq i \leq n.$$

Let $d = d_1 + d_2$. For a subset

$$S \subset \text{mod}_A^{d_1}(\mathbf{k}) \times \text{mod}_A^{d_2}(\mathbf{k})$$

define

$$\mathcal{E}(S) := \{m \in \text{mod}_A^d(k) \mid \exists \text{ s.e.s. } 0 \rightarrow m_2 \rightarrow m \rightarrow m_1 \rightarrow 0 \text{ with } (m_1, m_2) \in S\}$$

Now, if $\text{ext}_A^1(C_1, C_2) \neq 0$ it is not hard to see that $\overline{\mathcal{E}(C_1 \times C_2)}$ contains properly $\overline{C_1 \oplus C_2}$. This shows the “only if” part of Theorem 2. The other direction follows basically from the following result which is interesting on its own.

6.6.2. Proposition. *Let $S \subset \text{mod}_A^{d_1}(k) \times \text{mod}_A^{d_2}(k)$ be $\text{GL}_{d_1}(k) \times \text{GL}_{d_2}(k)$ -stable. Then the following holds:*

- (i) *If S is a closed subset, then $\mathcal{E}(S)$ is a closed subset of $\text{mod}_A^d(k)$.*
- (ii) *If S is irreducible, locally closed and $\dim \text{Ext}_A^1(m_1, m_2) = c$ for all $(m_1, m_2) \in S$ (and some $c \in \mathbb{N}$), then $\mathcal{E}(S)$ is irreducible.*
- (iii) *If S is open and $\text{Ext}_A^1(m_2, m_1) = 0$ for all $(m_1, m_2) \in S$ then $\mathcal{E}(S)$ is open.*

Part (iii) follows from the following consideration: Let Z be the closed subscheme of mod_A^d , such that the elements $m = (m^{(1)}, \dots, m^{(n)})$ of $Z(R)$ are all of upper triangular form with respect to the decomposition $d = d_1 + d_2$. Then we have a natural map $\Delta: Z \rightarrow \text{mod}_A^{d_1} \times \text{mod}_A^{d_2}$ which sends an element of Z to the pair of its diagonal blocks. We may consider S as an open subscheme of $\text{mod}_A^{d_1} \times \text{mod}_A^{d_2}$, thus $\Delta^{-1}(S)$ is an open subscheme of Z . On the other hand, we have a morphism

$$f: \text{GL}_d \times \Delta^{-1}(S) \rightarrow \text{mod}_A^d$$

obtained from the inclusion of $\Delta^{-1}(S)$ into mod_A^d , followed by the conjugation action with GL_d . Clearly $\mathcal{E}(S)$ is the image of f . Let $m = f(g, b) \in \mathcal{E}(S)(k)$ and m_t a formal deformation of m . Then $m'_t = m_t^{g_t^{-1}}$ is a formal deformation of $b \in \Delta^{-1}(S)(k) \subset \text{mod}_A^d(k)$. By the lemma below, m'_t is equivalent to a formal deformation $m''_t \in Z(k[[t]])$. In particular, $m''_t = (m'_t)^{g_t}$ for some $g_t \in \text{E}_d(k[[t]])$. Since $\Delta^{-1}(S)$ is open in Z we get even $m''_t \in (\Delta^{-1}(S))(k[[t]])$, Lemma 4.4.2, so $m_t = f(gg_t^{-1}, m''_t)$. This implies by Lemma 4.4.1 that $\mathcal{E}(S)$ is open in mod_A^d .

7. APPENDIX: SCHEMES

For the convenience of the reader we present here some definitions for the “functorial point of view” of schemes as far as it is used in this text. Our exposition is based Part I, Section 1,2 and 5 of [10], though a few things become easier since our base k is a field (even algebraically closed). The original reference for this kind of material is a part of [5].

7.1. Basic Definitions. For an affine scheme $\text{Hom}_{k\text{-alg}}(R, -)$ and an ideal $I \subset R$ one defines the following subfunctors:

$$\begin{aligned} V_I(A) &:= \{x \in \text{Hom}_{k\text{-alg}}(R, A) \mid x(I) = 0\} \\ D_I(A) &:= \{x \in \text{Hom}_{k\text{-alg}}(R, A) \mid A \cdot x(I) = A\} \end{aligned}$$

The subfunctors of the form D_I are called *open subfunctors*, the functors of the form V_I are called *closed subfunctors* of $\text{Hom}_{k\text{-alg}}(R, -)$.

We have for example $V_I(-) \cong \text{Hom}_{k\text{-alg}}(R/I, -)$ and $D_I = D_{\sqrt{I}}$. For $f \in R$ we get $D_{(f)}(-) = \text{Hom}_{k\text{-alg}}(R_f, -)$, where R_f denotes the localization of R with respect to the multiplicative system $\{1, f, f^2, \dots\}$.

Note that

$$\mathrm{Hom}_{\mathbf{k}\text{-alg}}(R, K) = D_I(K) \cup V_I(K)$$

if K is a field, otherwise this equality is false in general. Moreover, for ideals I, J of R we have $D_I = D_J$ if and only if $D_I(\mathbf{k}) = D_J(\mathbf{k})$ since \mathbf{k} is algebraically closed.

For the rest of this section a functor

$$\{\text{commutative } \mathbf{k}\text{-algebras}\} \rightarrow \text{Sets}$$

will be called a *k-functor*.

Natural transformations are the *morphisms* between *k-functors*. A subfunctor Y of an arbitrary *k-functor* is *open*, if for every morphism $\phi: X' \rightarrow X$ with X' affine $\phi^{-1}(Y)$ is an open subfunctor of X' . A family $(Y_i)_{i \in I}$ of open subfunctors of X is called an *open covering* if $\cup_{i \in I} Y_i(K) = X(K)$ whenever K is a field.

X is called *local*, if for every *k-functor* Y with an open covering $(Y_i)_{i \in I}$ the natural map

$$\mathrm{Mor}(Y, X) \rightarrow \prod_{i \in I} \mathrm{Mor}(Y_i, X), f \mapsto (f|_{Y_i})_{i \in I}$$

induces a bijection between the morphisms from Y to X and families of morphisms $(f_i)_{i \in I} \in \prod_{i \in I} \mathrm{Mor}(Y_i, X)$ such that $f_i|_{Y_j \cap Y_i} = f_j|_{Y_i \cap Y_j} \in \mathrm{Mor}(Y_i \cap Y_j, X)$ for all $i, j \in I$. In other words, X is local if morphisms to X can be defined locally.

Now, a *scheme* is a local *k-functor* which admits an open covering by affine schemes. In particular, affine schemes are schemes in this sense. A scheme is *algebraic* over \mathbf{k} if it admits a finite open covering by affine schemes whose coordinate rings are finitely presented *k-algebras*. Since we work over a field \mathbf{k} the notion of algebraic schemes and (the otherwise weaker) notion of finite type schemes are equivalent.

Note, that for each scheme X the set of morphisms $\mathrm{Mor}(X, \mathbb{A}^1)$ is naturally a ring. In fact, $f \in \mathrm{Mor}(X, \mathbb{A}^1)$ is given by a family of maps $f_S: X(S) \rightarrow \mathbb{A}^1(S)$. Since $\mathbb{A}^1(S) = S$ is ring we can define addition and multiplication in $\mathrm{Mor}(X, \mathbb{A}^1)$ component wise. We may equip $X(\mathbf{k})$ with the Zarisky topology by defining the open subsets as $U(\mathbf{k})$ for U an open subfunctor of X , and obtain on $X(\mathbf{k})$ a sheaf of rings \mathcal{O}_X by taking $\mathcal{O}_X(U)((\mathbf{k})) = \mathrm{Mor}(U, \mathbb{A}^1)$ for each open subfunctor $U \subset X$. In particular, we may define for each $x \in X(\mathbf{k})$ the local ring $\mathcal{O}_{X,x}$ in the usual way. We say the x is *regular* if $\mathcal{O}_{X,x}$ is regular.

For a scheme X and $\mu \in X(\mathbf{k})$ we define the *tangential space*

$$T_{X,\mu} := X(p_{\mathbf{k}[\epsilon]})^{-1}(\mu) \subset X(\mathbf{k}[\epsilon])$$

where $\mathbf{k}[\epsilon] = \mathbf{k}[t]/(t^2)$ are the dual numbers and $p: \mathbf{k}[\epsilon] \rightarrow \mathbf{k}$ the canonical projection. This is indeed a finite dimensional vector space if X is algebraic. Recall, that in this case we have $\dim_{\mathbf{k}} T_{X,x} \geq \dim_{\mathrm{Krull}} \mathcal{O}_{X,x}$ and x is regular iff equality holds.

If $f: X \rightarrow Y$ is a morphism of schemes, $\mu \in X(\mathbf{k})$ and $\nu = f_{\mathbf{k}}(\mu) \in Y(\mathbf{k})$ we obtain a linear map

$$d_{f,\mu}: T_{X,\mu} \rightarrow T_{Y,\nu}, \tau \mapsto f_{\mathbf{k}[\epsilon]}(\tau),$$

the *differential* of f at x . It is a useful exercise to translate these concepts for (reduced affine schemes) into the usual language of varieties.

7.2. Group schemes. By definition, a *group scheme* is a functor

$$G: \{\text{commutative } k\text{-algebras}\} \rightarrow \{\text{groups}\}$$

which is a scheme if considered as a k -functor. It is elementary to see that the category of affine group schemes is anti-equivalent to the category of commutative Hopf-algebras. All group schemes considered in these notes will be affine. A typical examples is GL_n it is defined by

$$\text{GL}_n(S) = \{M \in \text{Mat}_{n \times n}(S) \mid \det(M) \in S^\times\}$$

and is represented by the commutative ring

$$A := k[(X_{i,j})_{1 \leq i,j \leq n}, d] / (\det(X) \cdot \det(d) - 1)$$

where X is the $n \times n$ matrix with entries $X_{i,j}$. The comultiplication is given by

$$X_{i,j} \mapsto \sum_{k=1}^n X_{i,k} \otimes X_{k,j}, \quad d \mapsto d \otimes d$$

while the antipode is determined by

$$X_{i,j} \mapsto \det(\tilde{X}_{j,i}) \cdot d, \quad d \mapsto \det X.$$

The *action* of a group scheme G on a scheme X is given by a morphism of schemes $\sigma: G \times X \rightarrow X$ such that $\sigma_S: G(S) \otimes X(S) \rightarrow X(S)$ is a group action for each commutative k -algebra S .

7.3. Orbits. In this situation, the G -orbit $G \cdot x$ of $x \in X(k)$ is defined by

$$(G \cdot x)(S) := \{y \in X(S) \mid \text{there exists a fppf-algebra } S \xrightarrow{\phi} T \text{ and } g \in G(T) \\ \text{such that } X(\phi)(y) = g \cdot x\},$$

where *fppf* means “faithfully flat finitely presented” (for its initials in French). This is in fact a scheme since k is a field, and it has the usual universal property of an orbit in the category of schemes. Moreover, $G \cdot x$ is reduced if G and the stabilizer of x are reduced.

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