Problem set 3: Mapping class groups

Exercise 1 (Low complexity surfaces).

(a) Show that:

- any two essential (this means they run between the punctures of $\Sigma_{0,3}$ and cannot be contracted into a point or a puncture) simple proper arcs in $\Sigma_{0,3}$ with the same endpoints are isotopic and
- any two essential simple proper arcs that both start and end at the same marked point of $\Sigma_{0,3}$ are isotopic.

(b) Show that $\mathrm{MCG}(\Sigma_{0,2})$ and $\mathrm{MCG}(\Sigma_{0,3})$ are trivial.

(c) Show that $\mathrm{MCG}(\Sigma_{1,1}) \simeq \mathrm{MCG}(\Sigma_1)$.

Exercise 2 (Curve graphs). The goal of this exercise is to prove that, under a suitable complexity condition, two types of curve graphs are connected. For more on these graphs and their relation to surface homeomorphisms and their dynamics, we refer to the course called *Dynamique des homéomorphismes du tore et graphe fin des courbes* by Pierre-Antoine Guihéneuf and Frédéric Le Roux.

(a) Given a surface $\Sigma_{g,n}$ of genus g with n punctures, its *curve graph* $\mathcal{C}(\Sigma_{g,n})$ is the graph whose vertices are the isotopy classes of essential simple closed curves on $\Sigma_{g,n}$, two if which are joined by an edge if and only if they admit disjoint representatives. Show that, when $g \geq 2$, every vertex in $\mathcal{C}(\Sigma_{g,n})$ has infinite valence.

(b) Show that when $g \geq 2$, $\mathcal{C}(\Sigma_{g,n})$ is connected. *Hint:* perform an induction on the number of self-intersections and use surgeries on the curves to reduce intersections.

(c) The *non-separating curve graph* $\mathcal{C}^{\mathrm{ns}}(\Sigma_{g,n})$ is the subgraph of $\mathcal{C}(\Sigma_{g,n})$ whose vertices are all isotopy classes of non-separating curves. The edge relation remains the same as before. Show that $\mathcal{C}(\Sigma_{g,n})$ is connected. *Hint:* use the path that you found for the previous question and find a way to throw out separating curves.

(d) Let $\mathcal{C}^*(\Sigma_{g,n})$ denote the graph whose vertices are all isotopy classes of non-separating curves on $\Sigma_{g,n}$ that share an edge whenever their intersection number (minimized over the isotopy classes) equals 1. Show that $\mathcal{C}^*(\Sigma_{g,n})$ is connected.

Exercise 3 (The Birman exact sequence). Let S be an orientable surface without boundary with $\chi(S) < 0$. Let $\mathrm{Homeo}^+(S)$ denote the group of orientation preserving self homeomorphisms of S.

(a) Fix $x \in S$, define the map $e_x : \mathrm{Homeo}^+(S) \to S$ defined by

$$e_x(f) = f(x), \quad f \in \mathrm{Homeo}^+(S).$$

What is the fiber of this map?

(b) Show that this defines a fiber bundle $e_x : \mathcal{F} \to S$ with fibers homeomorphic to the group $\mathrm{Homeo}^+(S, x)$ of orientation preserving homeomorphisms that fix x.

(c) Recall that if $F \to E \to B$ is a fiber bundle, then there is a long exact sequence of homotopy groups

$$\cdots \to \pi_n(F) \to \pi_n(E) \to \pi_n(B) \to \pi_{n-1}(F) \to \cdots$$

Since $\chi(S) < 0$, $\pi_1(\mathrm{Homeo}^+(S)) = \{e\}$. Prove that there exists an exact sequence

$$1 \to \pi_1(S, x) \to \mathrm{MCG}(S, x) \to \mathrm{MCG}(S) \to 1.$$

This is called the *Birman exact sequence*. It turns out the the image of the loop $\alpha \in \pi_1(S, x)$ is $T_{\alpha_1} T_{\alpha_2}^{-1}$, where α_1 and α_2 are the boundary curves of a regular neighborhood of α in S and T_{α_1} and T_{α_2} denote the Dehn twists in these curves.

Exercise 4 (The Dehn–Lickorish theorem). The goal of this exercise is to combine the results of the previous three exercises into the Dehn–Lickorish theorem: the fact that the mapping class group of $\Sigma_{g,n}$ can be generated by Dehn twists in non-separating simple closed curves.

(a) Suppose that G is a group that acts by graph automorphisms on a connected graph Γ such that

- G acts transitively on the vertices of Γ and
- G acts transitively on ordered pairs of vertices of Γ that share an edge.

Suppose v and w are two vertices of Γ that are connected by an edge and let $h \in G$ be such that $h(w) = v$. Then

$$G = \langle h, \mathrm{stab}_G(v) \rangle.$$

(b) Let $\overrightarrow{\alpha}$ be an oriented non-separating simple closed curve on $\Sigma_{g,n}$. Write $\mathrm{MCG}(\Sigma_{g,n}, \overrightarrow{\alpha})$ for the subgroup of $\mathrm{MCG}(\Sigma_{g,n})$ consisting of mapping classes that preserve $\overrightarrow{\alpha}$ and its orientation and $\mathrm{MCG}(\Sigma_{g,n}, \alpha)$ for those mapping classes that preserve α but not necessarily its orientation. Show there exists a short exact sequence

$$1 \to \mathrm{MCG}(\Sigma_{g,n}. \overrightarrow{\alpha}) \to \mathrm{MCG}(\Sigma_{g,n}, \alpha) \to \mathbb{Z}/2\mathbb{Z} \to 1.$$

Hint: Let β be another non-separating simple curve that intersects α exactly once and consider the element $T_\beta T_\alpha^2 T_\beta$, where T_α and T_β denote the Dehn twists in α and β respectively.

(c) Prove by induction on the pair (g, n), with base cases $(g, n) = (1, 1)$ and $(g, n) = (1, 0)$ that the mapping class group $\mathrm{MCG}(\Sigma_{g,n})$ is generated by Dehn twists in non-separating curves. *Hint:* Use the action on $\mathcal{C}^*(\Sigma_{g,n})$ for the induction on genus. Along the way it will be useful to know that there is a short exact sequence

$$1 \to \langle T_\alpha \rangle \to \mathrm{MCG}(\Sigma_{g,n}, \overrightarrow{\alpha}) \to \mathrm{MCG}(\Sigma_{g,n} - \alpha) \to 1,$$

where the map between the two mapping class groups is the restriction to the complement of α