Problem set 3: Mapping class groups

Exercise 1 (Low complexity surfaces).

(a) Show that:

– any two essential (this means they run between the punctures of $\Sigma_{0,3}$ and cannot be contracted into a point or a puncture) simple proper arcs in $\Sigma_{0,3}$ with the same endpoints are isotopic and

– any two essential simple proper arcs that both start and end at the same marked point of $\Sigma_{0,3}$ are isotopic.

Solution:

- Let α and β be two arcs with the same endpoints in $\Sigma_{0,3}$. Isotope them into general position (a finite number of transverse intersections). There is one of the three punctures of $\Sigma_{0,3}$ that is not incident to α nor β. As such we can think of α and β as arcs between two points of $\mathbb{S}^2 - \{\text{pt.}\} \simeq \mathbb{R}^2$.

  Now first suppose that they're not disjoint. Then we can find a disk between α and β as follows. Parametrize α and β consistently (i.e. $\alpha(0) = \beta(0)$ and $\alpha(1) = \beta(1)$). Let $t \in (0,1)$ minimal such that $\alpha(t) = \beta(s)$ for some $s \in (0,1)$. The two segments $\alpha([0,t])$ and $\beta([0,s]$ together form a simple closed curve in $\mathbb{R}^2$ that bounds a disk. So we can isotope α in such a way that the intersection goes away. In order words, we may assume α and β don't intersect in their interior. As such their union is a simple closed curve bounding a disk in $\mathbb{R}^2$, which implies they're isotopic.

- If α and β both start and end at the same point, we argue as follows. First of all, we can apply the same argument to show that (up to isotopy) they don't intersect. Since they're both essential, they need to both separate the remaining two punctures. This means that they bound a disk and hence can be isotoped to each other.

(b) Show that $\mathrm{MCG}(\Sigma_{0,2})$ and $\mathrm{MCG}(\Sigma_{0,3})$ are trivial.

Solution: Let $k \in \{2,3\}$ and set $S = \Sigma_{0,k}$. Our goal is to show that any orientation diffeomorphism $f : S \to S$ that doesn't permute the punctures is isotopic to the identity.

Let α be a simple arc between two of the punctures of S. Then by (a), α and $f(\alpha)$ are isotopic (the same proof works when $k = 2$). We can isotope f so that is preserves α pointwise. Then we can restrict f to $S - \alpha$ which is either a disk or a once-punctured disk of which f preserves the origin. Moroever, since f preserves α pointwise, we can complete $S - \alpha$ to a closed disk and extend f with the identity on the boundary. By the Alexander trick, this map is isotopic to the identity, which implies that our original map was.

(c) Show that $\mathrm{MCG}(\Sigma_{1,1}) \simeq \mathrm{MCG}(\Sigma_1)$.

Solution: We have a forgetful homomorphism

$$\mathrm{MCG}(\Sigma_{1,1}) \longrightarrow \mathrm{MCG}(\Sigma_1),$$

thinking of $\mathrm{MCG}(\Sigma_{1,1})$ as isotopy classes of maps that fix a point on Σ_1. We need to show that this homomorphism is in fact an isomorphism. In class, we have already seen that for any pair of points $p, q \in \Sigma_1$, we can find a map $h : \Sigma_1 \to \Sigma_1$ that maps p to q and is isotopic to the identity. So in particular, in every mapping class in $\mathrm{MCG}(\Sigma_1)$ we can find a map that fixes our favorite point, which implies that the map is surjective. In reality, this doesn't use the fact that we're working with the torus and thus works for any surface.

Injectivity is however specific to the torus. Suppose $[\varphi] \in \mathrm{MCG}(\Sigma_{1,1})$ maps to the identity in $\mathrm{MCG}(\Sigma_1)$. Writing $\Sigma_1 = \mathbb{R}^2/\mathbb{Z}^2$, we may pick a lift

$$\widetilde{\varphi} : \mathbb{R}^2 \to \mathbb{R}^2$$

that fixes the origin. Since $[\varphi]$ maps to the identity in $\mathrm{MCG}(\Sigma_1)$, we have a $\mathbb{Z}^2$-equivariant homotopy $H : [0,1] \times \mathbb{R}^2 \to \mathbb{R}^2$ such that

$$H(0,\cdot) = \widetilde{\varphi} \quad \text{and} \quad H(1,\cdot) = \mathrm{Id} : \mathbb{R}^2 \to \mathbb{R}^2.$$

The issue is that $H(s,\cdot)$ might not fix the origin for all (or in fact any) $s \in (0,1)$. However, we can modify H by setting

$$\widetilde{H}(t,x) = H(t,x) - H(t,0), \quad t \in [0,1],\ x \in \mathbb{R}^2.$$

This is a $\mathbb{Z}^2$-equivariant modification, so it passes to Σ_1. It is still a homotopy between $\widetilde{\varphi}$ and the identity, because it's still continuous and the modification does not affect $H(0,\cdot)$ and $H(1,\cdot)$. Finally, $\widetilde{H}(t,\cdot)$ fixes the origin for all $t \in [0,1]$, so $[\varphi]$ is trivial in $\mathrm{MCG}(\Sigma_{1,1})$ as well.

Exercise 2 (Curve graphs). The goal of this exercise is to prove that, under a suitable complexity condition, two types of curve graphs are connected. For more on these graphs and their relation to surface homeomorphisms and their dynamics, we refer to the course called *Dynamique des homéomorphismes du tore et graphe fin des courbes* by Pierre-Antoine Guihéneuf and Frédéric Le Roux.

(a) Given a surface $\Sigma_{g,n}$ of genus g with n punctures, its *curve graph* $\mathcal{C}(\Sigma_{g,n})$ is the graph whose vertices are the isotopy classes of essential simple closed curves on $\Sigma_{g,n}$, two if which are joined by an edge if and only if they admit disjoint representatives. Show that, when $g \geq 2$, every vertex in $\mathcal{C}(\Sigma_{g,n})$ has infinite valence.

Solution: If we remove a simple closed curve from $\Sigma_{g,n}$ then the remaining surface still supports (countably) infinitely many isotopy classes of simple closed curves. All of these share an edge with the given curve in the curve graph.

(b) Show that when $g \geq 2$, $\mathcal{C}(\Sigma_{g,n})$ is connected. *Hint:* perform an induction on the number of self-intersections and use surgeries on the curves to reduce intersections.

Solution: Given two distinct vertices (i.e. isotopy classes of curves) α and β of $\mathcal{C}(\Sigma_{g,n})$, we need to find vertices $\gamma_1, \ldots, \gamma_n$ that form a path from α to β. That is, if we write $i(\gamma, \gamma')$ for the minimal number of intersections between representatives of γ and γ', then we need to find curves $\gamma_1, \ldots, \gamma_n$ such that

$$i(\alpha, \gamma_1) = 0,\ i(\gamma_1, \gamma_2) = 0,\ \ldots,\ i(\gamma_n, \beta) = 0.$$

We prove that his is possible by induction on $i(\alpha, \beta)$. If $i(\alpha, \beta) = 0$ then we're good. If $i(\alpha, \beta) = 1$, then the minimial representatives admit a regular neighborhood that is homeomorphis to a 1-holed torus in $\Sigma_{g,n}$. Since $g \geq 2$, the complement of this subsurface still supports a simple closed curve, which is connected to both α and β in the curve graph.

So, we suppose that $i(\alpha, \beta) \geq 2$. We now perform a surgery argument. Parametrize the minimal representatives α and β and take any two intersection points between the curves that are consecutive on β. The intersections might either have opposite orientations or equal orientations. In both cases, Figure 1 shows a surgery applied to α and β.

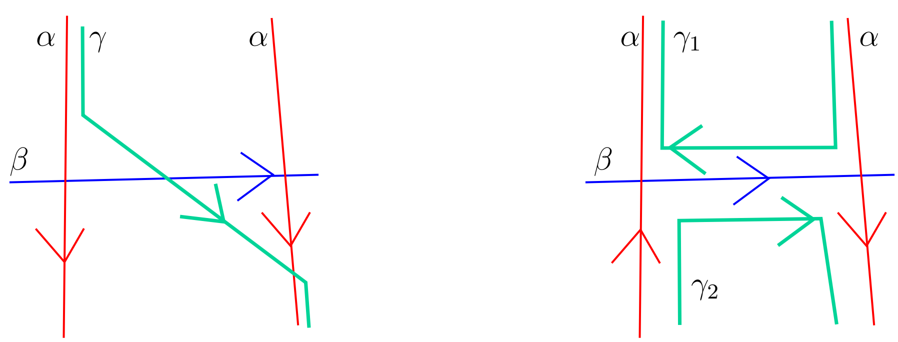

Figure 1: Two possible surgeries

If the orientations of the intersections are the same, we push α off of itself to the right and perform the surgery as shown in the picture on the left so as to obtain a curve γ that intersects α exactly once. This in particular implies that the algebraic intersection of α and γ is ± 1 and hence that γ is essential. Moreover, γ can be connected to α in the curve graph. It can also be connected to β in the curve graph, because it has one fewer intersection with β than α does.

If the intersections have opposite orientations (the picture on the right), then we perform a surgery that yields two curves γ_1 and γ_2. Both of them are non-null homotopic, because we assumed that α and β are in minimal position. If they both bound a once-punctured disk, then α bounds a twice-punctured disk. We can build similar curves γ_3 and γ_4 on the other side of α, if they also both bound once-puntured disks, we would get that our surface is $\Sigma_{0,4}$, which violates the condition on genus. In conclusion, we may assume at least one of γ_1 and γ_2 is essential, say γ_1. This curve does not intersect α and intersect β twice fewer than α does, so we're done.

(c) The *non-separating curve graph* $\mathcal{C}^{\mathrm{ns}}(\Sigma_{g,n})$ is the subgraph of $\mathcal{C}(\Sigma_{g,n})$ whose vertices are all isotopy classes of non-separating curves. The edge relation remains the same as before. Show that $\mathcal{C}(\Sigma_{g,n})$ is connected. *Hint:* use the path that you found for the previous question and find a way to throw out separating curves.

<u>Solution:</u> We start with the case that $n \leq 1$. Suppose that γ_i in the path we constructed for the previous question is separating. If γ_{i-1} and γ_{i+1} lie on different components of $\Sigma_{g,n}$, we can simply remove γ_i from the sequence. If γ_{i-1} and γ_{i+1} lie in the same component then the other component has positive genus (because $g \geq 2$ and $n \leq 1$). In particular, it supports a non-separating curve that we may replace γ_i with.

Now we perform an induction on n to deal with the remaining cases. Suppose γ_i is separating. The only issue is that one of the components of $\Sigma_{g,n} - \gamma_i = S \sqcup S'$, say S' might be a punctured disk (otherwise we reduce to the previous case). By induction, we can find a path in $\mathcal{C}^{\mathrm{ns}}(S)$ between γ_{i-1} and γ_{i+1} that we can replace γ_i with.

(d) Let $\mathcal{C}^*(\Sigma_{g,n})$ denote the graph whose vertices are all isotopy classes of non-separating curves on $\Sigma_{g,n}$ that share an edge whenever their intersection number (minimized over the isotopy classes) equals 1. Show that $\mathcal{C}^*(\Sigma_{g,n})$ is connected.

<u>Solution:</u> Given two vertices α and β of $\mathcal{C}^*(\Sigma_{g,n})$, they represent vertices of $\mathcal{C}^{\mathrm{ns}}(\Sigma_{g,n})$ as well. We can find a connected path $\alpha, \gamma_1, \dots, \gamma_n, \beta$ in $\mathcal{C}^{\mathrm{ns}}(\Sigma_{g,n})$ between these curves. Any two consecutive vertices in this path represent curves that don't intersect on $\Sigma_{g,n}$. Using the classification of surfaces, we can, for each consecutive pair, find a curve that intersects both of them once, thus yielding a path in $\mathcal{C}^*(\Sigma_{g,n})$.

Exercise 3 (The Birman exact sequence). Let S be an orientable surface without boundary with $\chi(S) < 0$. Let $\mathrm{Homeo}^+(S)$ denote the group of orientation preserving self homeomorphisms of S.

(a) Fix $x \in S$, define the map $e_x : \mathrm{Homeo}^+(S) \to S$ defined by

$$e_x(f) = f(x), \quad f \in \mathrm{Homeo}^+(S).$$

What is the fiber of this map?

<u>Solution:</u> Above $p \in S$, this is the space of orientation preserving homeomorphisms that map x to p.

(b) Show that this defines a fiber bundle $e_x : \mathcal{F} \to S$ with fibers homeomorphic to the group $\mathrm{Homeo}^+(S, x)$ of orientation preserving homeomorphisms that fix x.

<u>Solution:</u> Our goal is to show that $\mathrm{Homeo}^+(S)$ is locally homeomorphic to a product of S with $\mathrm{Homeo}^+(S, x)$ in such a way that, locally, e_x corresponds to the projection on the first factor.

Let $U \subset S$ be a neighborhood of $x \in S$ that is homeomorphic to a disk. We define a map $U \to \text{Homeo}^+(S)$ that assigns ϕ_u to $u \in U$, where

$$\phi_u(x) = u$$

and ϕ_u varies continuously as a function of u. The map

$$(u, \psi) \in U \times \text{Homeo}^+(S, x) \quad \mapsto \quad \phi_u \circ \psi \in e_x^{-1}(U) \subset \text{Homeo}^+(S)$$

is a homeomorphism with inverse

$$\psi \in e_x^{-1}(U) \mapsto (\psi(x), \phi_{\psi(x)}^{-1} \circ \psi) \in U \times \text{Homeo}^+(S, x).$$

Now suppose $y \in S$ is some other point. Then let $\xi : S \to S$ be some homomorphism with $\xi(x) = y$. then we obtain a homeomorphism $e_x^{-1}(U) \to e_x^{-1}(\xi(U))$ given by $\psi \mapsto \xi \circ \psi$. So this indeed has the structure of a fiber bundle.

(c) Recall that if $F \to E \to B$ is a fiber bundle, then there is a long exact sequence of homotopy groups

$$\cdots \to \pi_n(F) \to \pi_n(E) \to \pi_n(B) \to \pi_{n-1}(F) \to \cdots$$

Since $\chi(S) < 0$, $\pi_1(\text{Homeo}^+(S)) = \{e\}$. Prove that there exists an exact sequence

$$1 \to \pi_1(S) \to \text{MCG}(S, x) \to \text{MCG}(S) \to 1.$$

This is called the *Birman exact sequence*. It turns out the the image of the loop $\alpha \in \pi_1(S, x)$ is $T_{\alpha_1} T_{\alpha_2}^{-1}$, where α_1 and α_2 are the boundary curves of a regular neighborhood of α in S and T_{α_1} and T_{α_2} denote the Dehn twists in these curves.

Solution: We get that the sequence

$$\pi_1(\text{Homeo}^+(S)) \to \pi_1(S) \to \pi_0(\text{Homeo}^+(S, x)) \to pi_0(\text{Homeo}^+(S)) \to \pi_0(S)$$

is exact. Now using that $\pi_1(\text{Homeo}^+(S)) = \{e\}$, $\pi_0(\text{Homeo}^+(S, x)) = \text{MCG}(S, x)$, $\pi_0(\text{Homeo}^+(S)) = \text{MCG}(S)$ and $\pi_0(S) = \{e\}$, we get that there is a short exact sequence

$$1 \to \pi_1(S) \to \text{MCG}(S, x) \to \text{MCG}(S) \to 1$$

Exercise 4 (The Dehn–Lickorish theorem). The goal of this exercise is to combine the results of the previous three exercises into the Dehn–Lickorish theorem: the fact that the mapping class group of $\Sigma_{g,n}$ can be generated by Dehn twists in non-separating simple closed curves.

(a) Suppose that G is a group that acts by graph automorphisms on a connected graph Γ such that

- G acts transitively on the vertices of Γ and

* G acts transitively on ordered pairs of vertices of Γ that share an edge.

Suppose v and w are two vertices of Γ that are connected by an edge and let $h \in G$ be such that $h(w) = v$. Then

$$G = \langle h, \mathrm{stab}_G(v) \rangle.$$

Solution: Given $g \in G$, our goal is to show that it can be written as a word in elements of $\mathrm{stab}_G(v)$ and copies of h.

Because Γ is connected, there is a sequence of vertices

$$v_0 = v,\ v_1,\ \ldots, v_n = g(v)$$

such that v_i is connected to v_{i-1} by an edge for all $i = 1, \ldots, n$. Now choose $g_i \in G$ such that

$$g_i(v) = v_i.$$

By the transitivity of the action, these elements exist. Moreover, we can take $g_0 = e$ and $g_n = g$. We will prove by induction that

$$g_i \in \langle h, \mathrm{stab}_G(v) \rangle.$$

For g_0 we're good, which settles the base case.

Now suppose $g_i \in \langle h, \mathrm{stab}_G(v) \rangle$. Apply g_i^{-1} to the ordered pair (v_i, v_{i+1}). We obtain the ordered $(v, g_i^{-1} \circ g_{i+1}(v))$ that share an edge (because v_i and v_{i+1} do and the action is simplicial). By transitivity on ordered pairs of vertices sharing an edge, we can find some element $f \in G$ that maps $(v, g_i^{-1} \circ g_{i+1}(v))$ to (v, w). Observe that

$$f \in \mathrm{stab}_G(v) \quad \text{and} \quad h \circ g_i^{-1} \circ g_{i+1}(v) = v,$$

so $hg_i^{-1}g_{i+1} \in \langle h, \mathrm{stab}_G(v) \rangle$ and hence $g_{i+1} \in \langle h, \mathrm{stab}_G(v) \rangle$. This means that we're done.

(b) Let $\overrightarrow{\alpha}$ be an oriented non-separating simple closed curve on $\Sigma_{g,n}$. Write $\mathrm{MCG}(\Sigma_{g,n}, \overrightarrow{\alpha})$ for the subgroup of $\mathrm{MCG}(\Sigma_{g,n})$ consisting of mapping classes that preserve $\overrightarrow{\alpha}$ and its orientation and $\mathrm{MCG}(\Sigma_{g,n}, \alpha)$ for those mapping classes that preserve α but not necessarily its orientation. Show there exists a short exact sequence

$$1 \to \mathrm{MCG}(\Sigma_{g,n}. \overrightarrow{\alpha}) \to \mathrm{MCG}(\Sigma_{g,n}, \alpha) \to \mathbb{Z}/2\mathbb{Z} \to 1.$$

Hint: Let β be another non-separating simple curve that intersects α exactly once and consider the element $T_\beta T_\alpha^2 T_\beta$, where T_α and T_β denote the Dehn twists in α and β respectively.

Solution: We have a map $\mathrm{MCG}(\Sigma_{g,n}, \alpha) \to \mathbb{Z}/2\mathbb{Z}$ measuring whether or not the given mapping class preserves the orientation. $\mathrm{MCG}(\Sigma_{g,n}. \overrightarrow{\alpha})$ is the kernel of this map, so the only thing to check is whether the map is surjective. This is indeed the case, because $T_\beta T_\alpha^2 T_\beta$ flips the orientation of α.

(c) Prove by induction on the pair (g, n), with base cases $(g, n) = (1, 1)$ and $(g, n) = (1, 0)$ that the mapping class group $\mathrm{MCG}(\Sigma_{g,n})$ is generated by Dehn twists in non-separating curves. *Hint:* Use the action on $\mathcal{C}^*(\Sigma_{g,n})$ for the induction on genus. Along the way it will be useful to know that there is a short exact sequence

$$1 \to \langle T_\alpha \rangle \to \mathrm{MCG}(\Sigma_{g,n}, \overrightarrow{\alpha}) \to \mathrm{MCG}(\Sigma_{g,n} - \alpha) \to 1,$$

where the map between the two mapping class groups is the restriction to the complement of α.

Solution: We have seen the base cases in the first exercise and the lectures, so we focus on the induction step.

First we perform the induction on n, so we assume that $\mathrm{MCG}(\Sigma_{g,0})$ (or $\mathrm{MCG}(\Sigma_{1,1})$ in the case of genus 1, because we need negativity of the Euler characteristic) can be generated by Dehn twists. By the Birman exact sequence

$$1 \to \pi_1(\Sigma_{g,n}) \to \mathrm{MCG}(\Sigma_{g,n+1}) \to \mathrm{MCG}(\Sigma_{g,n}) \to 1,$$

$\mathrm{MCG}(\Sigma_{g,n+1})$ can be generated by the image of $\pi_1(\Sigma_{g,n})$ (which is finitely generated by Dehn twists along the curves of a generating set of $\Sigma_{g,n}$ consisting of non-separating curves, by the previous exercise) and $\mathrm{MCG}(\Sigma_{g,n})$, which is finitely generated by Dehn twists in non-separating curves (that remain non-separating on $\Sigma_{g,n+1}$) by induction.

Next up, we deal with the induction on the genus g. We combine item (a) of this exercise with the result of Exercise 2. Indeed, the action of $\mathrm{MCG}(\Sigma_{g+1,n})$ on $\mathcal{C}^*(\Sigma_{g+1,n})$ satisfies the conditions of item (a) (essentially by the classification of surfaces) so we can generate $\mathrm{MCG}(\Sigma_{g+1,n})$ by the stabilizer of a non-separating curve α on $\Sigma_{g+1,n}$ and an element h that maps some non-separating curve β on $\Sigma_{g+1,n}$ that intersects α once to α. Denoting the Dehn twists in α and β by T_α and T_β respectively, $h = T_\beta T_\alpha$ does the job. So all we need to do is understand the stabilizer of α. This uses tbe short exact sequence from the hint and the one from (b).