Problem set 4: Beltrami differentials, quasiconformal maps and measured foliations

Exercise 1 (Beltrami differentials and quasiconformal maps).

(a) Let S, X_1 and X_2 be Riemann surfaces and let

$$S \xrightarrow{f} X_1 \xrightarrow{g} X_2$$

be orientation preserving diffeomorphisms. Prove that:

$$\mu_g \circ f = \left(\frac{\partial f}{\partial z} \Big/ \overline{\left(\frac{\partial f}{\partial z} \right)} \right) \cdot \frac{\mu_{g \circ f} - \mu_f}{1 - \overline{\mu}_f \cdot \mu_{g \circ f}}.$$

(b) Prove the following lemma about compositions of quasiconformal maps: Suppose X, Y and Z are Riemann surfaces and $f : X \to Y$ and $g : Y \to Z$ are orientation preserving diffeomorphisms. Then the following holds:

(1) We have that

$$K_f \geq 1$$

with equality if and only if f is a biholomorphism.

(2) We have that

$$K_{g \circ f} \leq K_g \cdot K_f.$$

(3) Finally,

$$K_{f^{-1}} = K_f.$$

Hint for (2): Since $K_f(z)$ depends only on the Jacobian matrix $J_f(z)$ of f at z, this is a linear algebra question.

Exercise 2 (Measured foliations and quadratic differentials using branched covers) If X and Y are closed surfaces, then a *branched covering* is a map $f : X \to Y$ such that there exists a discrete subset $S \subset X$ such that $f(S) \subset Y$ is discrete and outside of S and $f(S)$, f is a covering map.

(a) Suppose $(\mathcal{F}, \mu)$ is a measured foliation of a closed surface Y and $p : X \to Y$ is a branched covering map. Explain that we can pull this back to a measured foliation $(p^*\mathcal{F}, p^*\mu)$. In particular, what are the singularities of $(p^*\mathcal{F}, p^*\mu)$?

(b) Now suppose X and Y are equiped with the structure of a Riemann surface and $p : X \to Y$ is a holomorphic branched covering map, i.e. a map that is locally of the form $z \mapsto z^k$ for some $k \geq 1$. Suppose q is a quadratic differential on Y. Explain that q can be pulled back by p. Where can we find the zeroes of p^*q? And what are their orders?

Exercise 3 (The Euler–Poincaré formula) Suppose V is a vector field on a compact surface S with isolated zeroes that lie in the interior of S. Recall that the *index* of a zero of V can be computed as follows. Let $x \in S$ be such that $V(x) = 0$, then take a small closed disk D around x in S that does not contain any other zeroes of V. We may identify that tangent bundle over D with the tangent bundle over some disk in the plane. We then obtain a map

$$x \in \partial D \xrightarrow{f} \frac{V(x)}{\|V(x)\|} \in \mathbb{S}^1$$

The degree (or winding number) of this map, for instance the number a such that

$$f_* : H_1(\partial D; \mathbb{Z}) \simeq \mathbb{Z} \quad \longrightarrow \quad H_1(\mathbb{S}^1; \mathbb{Z}) \simeq \mathbb{Z}$$

takes the form $n \mapsto a \cdot n$ is called the *index* of V at x and will be denoted $\mathrm{ind}_x(V)$.

(a) Let $V, W : \mathbb{R}^2 \to T\mathbb{R}^2 \simeq \mathbb{R}^2 \times \mathbb{R}^2$ denote the vector fields given by

$$V(x,y) = \frac{1}{\sqrt{2}} \cdot h(x,y) \cdot \begin{pmatrix} 1 \\ 0 \end{pmatrix} \quad \text{and} \quad W(x,y) = \begin{pmatrix} y \\ x \end{pmatrix},$$

where $h : \mathbb{R}^2 \to [0, \infty)$ is some function that satisfies $h(x,y) = 0$ if and only if $(x,y) = (0,0)$. Compute the indices of V and W at their only zero, the origin.

(b) If $\mathcal{F}_V$ and $\mathcal{F}_W$ are the foliations consisting of the integral lines of V and W respectively, how many prongs does the singularity at the origin have?

(c) Suppose a singular foliation $\mathcal{F}$ of $\mathbb{R}^2$ with smooth leaves has an even-pronged singularity at the origin as its only singularity. This means it comes from a vector field. What is the relation between the number of prongs of the singularity and the index of the vector field at the origin?

(c) The **Poincaré–Hopf theorem** states that

$$\sum_{\substack{x \in S \\ \text{zero of } V}} \mathrm{ind}_x(V) = \chi(S).$$

This was for instance treated in Julien Marché's course *Topologie algébrique des variétés I*. Use this formula to prove the Euler–Poincaré formula for singular foliations. *Hint:* Recall that a foliation is orientable if and only if its singularities are all even-pronged. Moreover, use that an orientable foliation is generated by a vector field.