Problem set 4: Beltrami differentials, quasiconformal maps and measured foliations

Exercise 1 (Beltrami differentials).

(a) Let S, X_1 and X_2 be Riemann surfaces and let

$$S \xrightarrow{f} X_1 \xrightarrow{g} X_2$$

be orientation preserving diffeomorphisms. Prove that:

$$\mu_g \circ f = \left(\frac{\partial f}{\partial z} \Big/ \overline{\left(\frac{\partial f}{\partial z}\right)} \right) \cdot \frac{\mu_{g\circ f} - \mu_f}{1 - \overline{\mu}_f \cdot \mu_{g\circ f}}.$$

<u>Solution:</u> We will write $f_z := \partial f/\partial z$ and $f_{\overline{z}} := \partial f/\partial \overline{z}$ in order to make the equations slightly shorter. The reader should however note that in general $\overline{f_z}$ does not equal $\overline{f}_z$ (but rather $\overline{f}_{\overline{z}}$). That is, attention should be paid to where the bar ends.

We compute, using the chain rule:

$$\mu_{g\circ f} = \frac{(g\circ f)_{\overline{z}}}{(g\circ f)_z} = \frac{(g_z \circ f)\cdot f_{\overline{z}} + (g_{\overline{z}} \circ f)\cdot \overline{f}_{\overline{z}}}{(g_z \circ f)\cdot f_z + (g_{\overline{z}} \circ f)\cdot \overline{f}_z}$$

So

$$\frac{f_z}{\overline{f_z}} \cdot \frac{\mu_{g\circ f} - \mu_f}{1 - \overline{\mu}_f \cdot \mu_{g\circ f}} = \frac{f_z}{\overline{f_z}} \cdot \frac{(g_z \circ f)\cdot f_{\overline{z}} + (g_{\overline{z}} \circ f)\cdot \overline{f}_{\overline{z}} - \frac{f_{\overline{z}}}{f_z}\cdot (g_z \circ f)\cdot f_z - \frac{f_{\overline{z}}}{f_z}\cdot (g_{\overline{z}} \circ f)\cdot \overline{f}_z}{(g_z \circ f)\cdot f_z + (g_{\overline{z}} \circ f)\cdot \overline{f}_z - \frac{\overline{f_{\overline{z}}}}{\overline{f_z}}\cdot (g_z \circ f)\cdot f_{\overline{z}} - \frac{\overline{f_{\overline{z}}}}{\overline{f_z}}\cdot (g_{\overline{z}} \circ f)\cdot \overline{f}_{\overline{z}}}$$

$$= \frac{(g_{\overline{z}} \circ f)\cdot (|f_z|^2 - |f_{\overline{z}}|^2)}{(g_z \circ f)\cdot (|f_z|^2 - |f_{\overline{z}}|^2)} = \frac{(g_{\overline{z}} \circ f)}{(g_z \circ f)} = \mu_g \circ f.$$

where we have used that $\overline{f}_{\overline{z}} = \overline{f_z}$ and $\overline{f}_z = \overline{f_{\overline{z}}}$.

(b) Prove the following lemma about compositions of quasiconformal maps: Suppose X, Y and Z are Riemann surfaces and $f : X \to Y$ and $g : Y \to Z$ are orientation preserving diffeomorphisms. Then the following holds:

(1) We have that

$$K_f \geq 1$$

with equality if and only if f is a biholomorphism.

(2) We have that

$$K_{g\circ f} \leq K_g \cdot K_f.$$

(3) Finally,

$$K_{f^{-1}} = K_f.$$

Hint for (2): Since $K_f(z)$ depends only on the Jacobian matrix $J_f(z)$ of f at z, this is a linear algebra question.

Solution: Recall that

$$K_f = \sup_{z \in X} K_f(z), \quad \text{where } K_f(z) = \frac{1 + |\mu_f(z)|}{1 - |\mu_f(z)|}.$$

We have seen that $\mu_f(z) = 0$ if and only if f is holomorphic at z. So $K_f(z) \geq 1$ with equality if and only if f is holomorphic at z. So this proves the inequality and also that in the equality case, f is holomorphic at all $z \in X$. Because f is invertible (and invertible holomorphic functions have holomorphic inverses), f is biholomorphic. This proves (1).

For (2), we use the hint. Write A for the Jacobian matrix of f and B for that of g, both with respect to some holomorphic coordinates on X, Y and Z. K_f can be computed as the ratio (major axis)/(minor axis) of the ellipse

$$\|A^{-1} \cdot z\| = 1$$

thinking of z as a real 2-dimensional vector $\begin{pmatrix} x \\ y \end{pmatrix}$. Using the standard inner product $\langle \cdot, \cdot, \rangle$ the equation for the ellipse is equivalent to

$$\langle \left(A^{-1}\right)^t \cdot A^{-1} z, z \rangle = 1$$

wher $\left(A^{-1}\right)^t$ denotes the transpose of A^{-1}. The matrix $\left(A^{-1}\right)^t \cdot A^{-1}$ is positive definite, so it has two orthogonal eigendirections (corresponding to the axes of the ellipse) and the ratio

$$K_f(z) = \frac{\text{major axis}}{\text{minor axis}} = \sqrt{\frac{\lambda_A^+}{\lambda_A^-}}$$

where λ_A^+ denotes the maximal eigenvalue of $\left(A^{-1}\right)^t \cdot A^{-1}$ and λ_A^- denotes the minimal eigenvalue of $\left(A^{-1}\right)^t \cdot A^{-1}$. Note that

$$\left(\lambda_A^+\right)^{1/2} = \|A^{-1}\|_\infty \quad \text{and} \quad \left(\lambda_A^-\right)^{-1/2} = \|A\|_\infty$$

(the latter holds because the top eigenvalue of $A \cdot A^t$ is the inverse of the bottom eigenvalue of $(A \cdot A^t)^{-1} = \left(A^{-1}\right)^t \cdot A^{-1}$. So

$$K_f(z) = \|A^{-1}\|_\infty \cdot \|A\|_\infty, \quad K_g(f(z)) = \|B^{-1}\|_\infty \cdot \|B\|_\infty$$

and

$$K_{g \circ f}(z) = \|B^{-1} A^{-1}\|_\infty \cdot \|AB\|_\infty.$$

This means that submultiplicativity of operator norms of matrices implies the inequality we're after.

Property (3) follows from the fact that $K_f(z) = \|A^{-1}\|_\infty \cdot \|A\|_\infty$.

Exercise 2 (Measured foliations and quadratic differentials using branched covers) If X and Y are closed surfaces, then a *branched covering* is a map $f : X \to Y$ such that there exists a discrete subset $S \subset X$ (called the branch points) such that $f(S) \subset Y$ is discrete and outside of S and $f(S)$, f is a covering map.

(a) Suppose $(\mathcal{F}, \mu)$ is a measured foliation of a closed surface Y and $p : X \to Y$ is a branched covering map. Explain that we can pull this back to a measured foliation $(p^*\mathcal{F}, p^*\mu)$. In particular, what are the singularities of $(p^*\mathcal{F}, p^*\mu)$?

Solution: We may pull back the decomposition of Y into leaves. Moreover, we can build charts for X using p and the charts for Y that map the leaves to horizontal lines (and k-pronged singularities at the singular points) around all the points in X that are not in the set S described above.

Around the branch points of p, we can restrict to small neighborhoods in the domain and the image that are both homeomorphic to disks. At these points, p looks like a the branched cover of the unit disk D, branched at the origin. In particular, it restricts to a covering map

$$p : D - \{0\} \to D - \{0\}.$$

Up to equivalence, these are characterized by

$$p_*(\pi_1(D - \{0\})) \simeq \mathbb{Z} \quad < \quad \pi_1(D - \{0\}) \simeq \mathbb{Z}.$$

That is, they are all equivalent to maps of the form $z \mapsto z^n$ for some $n \in \mathbb{N}^*$. This means that the pre-image of a regular point of $\mathcal{F}$ that happens to coincide with the image of a branch point of p becomes a $(n+1)$-pronged singularity. If like wise a k-pronged singularity of $\mathcal{F}$ that lies on the image of a branch point of order n becomes a $(n+1) \cdot k$-pronged singularity.

(b) Now suppose X and Y are equiped with the structure of a Riemann surface and $p : X \to Y$ is a holomorphic branched covering map, i.e. a map that is locally of the form $z \mapsto z^k$ for some $k \geq 1$. Suppose q is a quadratic differential on Y. Explain that q can be pulled back by p. Where can we find the zeroes of p^*q? And what are their orders?

Solution: Suppose (U, z) is a chart on X and $(p(U), w)$ a chart on Y. Moreover suppose that in the latter coordinates, q takes the form $q(w) = \varphi(w)dw^2$. Then we can set

$$(p^*q)(z) = \varphi(p(z)) \cdot d(p(z))^2$$

Since p is a branched covering map, we can cover X with such charts. If q has a zero of order m ($m = 0$ is allowed) at a point, which is a branch point of degree k, then p^*q has a zeroes of order

$$k \cdot m + 2 \cdot (k - 1)$$

at this the pre-image(s) of this point.

Exercise 3 (The Euler–Poincaré formula) Suppose V is a vector field on a compact surface S with isolated zeroes that lie in the interior of S. Recall that the *index* of a zero of V can be computed as follows. Let $x \in S$ be such that $V(x) = 0$, then take a small closed disk D around x in S that does not contain any other zeroes of V. We may identify that tangent bundle over D with the tangent bundle over some disk in the plane. We then obtain a map

$$x \in \partial D \xrightarrow{f} \frac{V(x)}{\|V(x)\|} \in \mathbb{S}^1$$

The degree (or winding number) of this map, for instance the number a such that

$$f_* : H_1(\partial D; \mathbb{Z}) \simeq \mathbb{Z} \quad \longrightarrow \quad H_1(\mathbb{S}^1; \mathbb{Z}) \simeq \mathbb{Z}$$

takes the form $n \mapsto a \cdot n$ is called the *index* of V at x and will be denoted $\mathrm{ind}_x(V)$.

(a) Let $V, W : \mathbb{R}^2 \to T\mathbb{R}^2 \simeq \mathbb{R}^2 \times \mathbb{R}^2$ denote the vector fields given by

$$V(x,y) = \frac{1}{\sqrt{2}} \cdot h(x,y) \cdot \begin{pmatrix} 1 \\ 0 \end{pmatrix} \quad \text{and} \quad W(x,y) = \begin{pmatrix} y \\ x \end{pmatrix},$$

where $h : \mathbb{R}^2 \to [0, \infty)$ is some function that satisfies $h(x,y) = 0$ if and only if $(x,y) = (0,0)$. Compute the indices of V and W at their only zero, the origin.

<u>Solution:</u> In the case of V, the map f is constant, this means it induces the 0-map in homology and hence $\mathrm{ind}_{(0,0)}(V) = 0$.

In the case of W, the map f is the reflection through the line $x = y$, which has degree -1 because it's one-to-one and reverses orientation.

(b) If $\mathcal{F}_V$ and $\mathcal{F}_W$ are the foliations consisting of the integral lines of V and W respectively, how many prongs does the singularity at the origin have?

<u>Solution:</u> The origin is not really a singularity of $\mathcal{F}_V$, that is, it has two prongs and we can replace the three integral lines $(-\infty, 0) \times \{0\}$, $\{(0,0)\}$ and $(0, \infty) \times \{0\}$ with a single horizontal leaf.

In the case of $\mathcal{F}_W$, the number of prongs is 4. Indeed, the integral lines of the vector field are the solutions $t \mapsto (x(t), y(t))$ to

$$\frac{dx(t)}{dt} = y(t), \quad \text{and} \quad \frac{dy(t)}{dt} = x(t)$$

So $x(t)$ is a solution to

$$\frac{d^2}{dt^2} x(t) = x(t),$$

i.e. $x(t) = a \cdot \cosh(t) + b \cdot \sinh(t)$ for some $a, b \in \mathbb{R}$. Which gives $y(t) = a \cdot \sinh(t) + b \cdot \cosh(t)$. These are exactly the lines of the form $x^2 + C = y^2$ (where $C = b^2 - a^2$). So the prongs are the half-lines $x = \pm y$, of which there are 4.

(c) Suppose a singular foliation $\mathcal{F}$of $\mathbb{R}^2$ with smooth leaves has an even-pronged singularity at the origin as its only singularity. This means it comes from a vector field. What is the relation between the number of prongs of the singularity and the index of the vector field at the origin?

Solution: If V denotes the vector field and $P_{(0,0)}$ then number of prongs at the origin, then

$$1 - P_{(0,0)}/2 = \mathrm{ind}_{(0,0)}(V)$$

Indeed, in the singular Euclidean metric induced by the foliation, the total angle around the singularity is $P_{(0,0)} \cdot \pi$. So the map f we use to define the index is $P_{(0,0)}/2$-to-1. This means that its degree is $\pm(P_{(0,0)}/2 - 1)$. Now, when we loop around the singular point in the clockwise direction, the orientation of the leaves rotates in the anti-clockwise direction (the reader is encouraged to draw a picture at this point). This means that the map f is orientation reversing and hence that its degree is indeed $1 - P_{(0,0)}/2$.

(d) The **Poincaré–Hopf theorem** states that

$$\sum_{\substack{x \in S \\ \text{zero of } V}} \mathrm{ind}_x(V) = \chi(S).$$

This was for instance treated in Julien Marché's course *Topologie algébrique des variétés I*. Use this formula to prove the Euler–Poincaré formula for singular foliations. *Hint:* Recall that a foliation is orientable if and only if its singularities are all even-pronged. Moreover, use that an orientable foliation is generated by a vector field.

Solution: Let $\mathcal{F}$ be a singular foliation on a compact surface S. Looking at the hint, the first we want to do is make our foliation orientable. To this end, suppose $\Sigma \subset S$ is the set of singularities of S. If S has boundary, we can suppose all these singularities lie in the interior of S. Indeed, if they don't we can attach small cylinders to the boundary and extend the foliation on these (using the fact that we only allow four types of behavior of $\mathcal{F}$ near the boundary) in such a way that the resulting foliation no longer has singularities on the boundary and still has the same number of singularities with the same number of prongs each. This doesn't change the Euler characteristic of S, so if the Euler–Poincaré formula is true for this new foliated surface, it is true for the original one.

First suppose that $\mathcal{F}$ is orientable, which means it's defined by a 1-form or dually by a vector field the zeroes of which lie in the singularities of $\mathcal{F}$. If P_x counts the number of prongs of a singularity, then the index of this vector field is $(2 - P_x)/2$ as we've seen in the previous exercise. So the Poincaré–Hopf theorem immediately implies the Euler–Poincaré formula.

Now suppose $\mathcal{F}$ is non-orientable. We will build a branched cover of S of degree 2 on which the pullback of $\mathcal{F}$ is orientable. To this end, set $S' = S - \Sigma$. The restiction of $\mathcal{F}$ to S' is a smooth foliation and thus has a well-defined tangent bundle. This allows us to define an orientation homomorphism

$$\pi_1(S', x) \to \mathbb{Z}/2\mathbb{Z}.$$

We remind the reader that his can be done as follows. Given $[\alpha] \in \pi_1(S', x)$ we may homotope α such that it's a smooth curve transverse to $\mathcal{F}$. Since S' is orientable, we can designate one of the sides of α as its outside. We pick a parametrization $t \in \mathbb{S}^1 \mapsto \alpha(t)$ consider the map $\mathbb{S}^1 \to \mathbb{S}^1$ that assigns to t the angle between the tangent vector $\dot{\alpha}(t)$ and any outward pointing (in the direction of the outside of α) tangent vector to $\mathcal{F}$. The degree of this map is some element in $\mathbb{Z}$. The image modulo 2 of this map, yields a well defined homomorphism $\pi_1(S', x) \to \mathbb{Z}/2\mathbb{Z}$. The image of this map is non-trivial because $\mathcal{F}$ is non-orientable.

So we obtain a double cover of $S'' \to S'$ corresponding to the subgroup of the orientation homomorphism. Singularities of $\mathcal{F}$ with an odd number of prongs lift to a single singularity with double the number of prongs and singularities of $\mathcal{F}$ with an even number of prongs lift to two singularities each with the same number of prongs.

We have $\chi(S'') = 2\chi(S')$. Write $\overline{S''}$ for the surface in which we have filled the missing points back in. The construction above yields a branched cover $p : \overline{S''} \to S$. We have

$$\begin{aligned} 2\chi(\overline{S''}) &= 2\chi(S'') + 2\#\{\text{singularities on } S''\} \\ &= 4\chi(S') + 4|\Sigma| - 2\#\{\text{odd-pronged singularities on } S\} \\ &= 4\chi(S) - 2\#\{\text{odd-pronged singularities on } S\} \end{aligned}$$

On the other hand, by the Euler–Poincaré formula that we have already proved for oriented foliations (like $p^*\mathcal{F}$),

$$\begin{aligned} 2\chi(\overline{S''}) &= \sum_{\substack{x \in \overline{S''} \\ \text{sing. of } p^*\mathcal{F}}} 2 - P_x \\ &= -2 \cdot \#\{\text{odd-pronged singularities on } S\} + \sum_{\substack{x \in S \\ \text{sing. of } \mathcal{F}}} 4 - 2P_x \end{aligned}$$

Combining the two equations above, we obtain the formula we want.