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**Refined height pairing**

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For a  $d$ -dimensional regular proper variety  $X$  over the function field of a smooth variety  $B$  over a field  $k$  and for  $i \geq 0$ , we define a subgroup  $\mathrm{CH}^i(X)^{(0)}$  of  $\mathrm{CH}^i(X)$  and construct a “refined height pairing”

$$\mathrm{CH}^i(X)^{(0)} \times \mathrm{CH}^{d+1-i}(X)^{(0)} \rightarrow \mathrm{CH}^1(B)$$

in the category of abelian groups up to isogeny. For  $i = 1, d$ ,  $\mathrm{CH}^i(X)^{(0)}$  is the group of cycles numerically equivalent to 0. This pairing relates to pairings defined by P. Schneider and A. Beilinson if  $B$  is a curve, to a refined height defined by L. Moret-Bailly when  $X$  is an abelian variety, and to a pairing with values in  $H^2(B_{\bar{k}}, \mathbb{Q}_l(1))$  defined by D. Rössler and T. Szamuely in general. We study it in detail when  $i = 1$ .

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## Introduction

Let  $X$  be a regular proper (for example, smooth projective) variety of dimension  $d$  over a field  $K$ , finitely generated of transcendence degree  $\delta$  over a subfield  $k$ . Suppose given a smooth (separated)  $k$ -scheme of finite type  $B$ , with function field  $K$ . For  $i \in [0, d]$ , write  $\mathrm{CH}^i(X)$  for the  $i$ -th Chow group of  $X$ . In this paper, we define a subgroup  $\mathrm{CH}^i(X)^{[0]}$  and a “refined height pairing”

$$\mathrm{CH}^i(X)^{[0]} \times \mathrm{CH}^{d+1-i}(X)^{[0]} \rightarrow \mathrm{CH}^1(B) \tag{1}$$

in the category  $\mathbf{Ab} \otimes \mathbb{Z}[1/p]$  of abelian groups up to  $p$ -isogeny: this category is recalled in [Section 4C](#). Here  $p$  is the exponential characteristic of  $k$ , so nothing is inverted in characteristic 0; the only reason to invert it in nonzero characteristic is a lack of resolution of singularities; see [Section 4A](#).

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If  $B$  is a smooth projective curve and we compose (1) with the degree map, we get a  $\mathbb{Z}[1/p]$ -valued pairing (with values in  $p^{-s}\mathbb{Z}$  for some integer  $s \geq 0$ ), which relates to the one constructed by Beilinson [1987, §1]. Beilinson [1987, p. 5] asked what happens when  $\text{trdeg}(K/k) > 1$ : (1) gives one answer to this question.

The quotient  $\text{CH}^i(X)/\text{CH}^i(X)^{[0]}$  is finitely generated. When we vary  $(X, i)$ ,  $\text{CH}^i(X)^{[0]}$  defines an adequate equivalence relation for smooth projective  $K$ -varieties, which a priori depends on the choice of  $B$ . Its saturation  $\text{CH}^i(X)^{(0)}$  lies between the subgroups  $\text{CH}_{\text{alg}}^i(X)$  and  $\text{CH}_{\text{num}}^i(X)$  of algebraically and numerically trivial cycles, hence equals  $\text{CH}_{\text{num}}^i(X)$  when  $i = 1, d$ . We conjecture that this holds for all  $i$ , and prove it in further special cases (Theorem 5.5(ii)). One can show that it would follow in general from the Tate conjecture, or the Hodge conjecture in characteristic 0, for cycles of codimension  $< i$ , although we don't include a proof here. More generally, one might hope that Lemma 1.1 below induces pairings in  $\mathbf{Ab} \otimes \mathbb{Q}$

$$F^n \text{CH}^i(X) \times F^n \text{CH}^{d+n-i}(X) \rightarrow \text{CH}^n(B), \quad i \geq 0,$$

where  $F^* \text{CH}^*(X)$  is the conjectural Bloch–Beilinson–Murre filtration [Jannsen 1994], the case  $n = 0$  (resp. 1) being the intersection pairing (resp. (1)).

Works following Néron's seminal paper [1965] have much relied on  $l$ -adic cohomology to analyse or define height pairings (because of the cohomological definition of Hasse–Weil  $L$ -functions): for  $\delta = 1$ , this is the case in [Schneider 1982] ( $i = 1$ ,  $X$  an abelian variety), [Bloch 1984] and [Beilinson 1987]. This is also the case in the work of Damian Rössler and Tamás Szamuely [2022], which is the direct inspiration of this one: they construct a pairing

$$\text{CH}_l^i(X) \times \text{CH}_l^{d+1-i}(X) \rightarrow H_{\text{ét}}^2(B_{\bar{k}}, \mathbb{Q}_l(1)), \quad (2)$$

where  $l$  is a prime number invertible in  $k$  and  $\text{CH}_l^i(X)$  denotes cycles homologically equivalent to 0 with respect to  $l$ -adic cohomology. By contrast, our approach here is completely cycle-theoretic and very close in spirit to Moret-Bailly's geometric height [1985, chapitre III, définition 3.2]; it relies on Fulton's marvellous theory of Gysin maps [1984, Chapters 6 and 8]. This gives a different flavour to the definitions because numerical and homological equivalence have rather opposite functoriality under specialisation, as described in detail by Grothendieck in [SGA 6 1971, 7.9 and 7.13]. See Remark 2.7.

Comparing various definitions of height pairings is a highly nontrivial issue, which is solved only in a few cases: for example, as far as I know those defined by Bloch [1984] and Beilinson [1987] have still not been checked to agree. Schneider [1982] compares an  $l$ -adic height pairing [loc. cit., p. 298] with the Néron–Tate height by comparing each to an intermediate Yoneda pairing [loc. cit., p. 502]

$$H^0(B, \mathcal{A}^0) \times \text{Ext}_B^1(\mathcal{A}^0, \mathbb{G}_m) \rightarrow \text{CH}^1(B), \quad (3)$$

where  $\mathcal{A}^0$  is the connected component of the identity of the Néron model  $\mathcal{A}$  of the abelian variety  $A$  ( $= X$  here).

In [Proposition 2.11](#), I show that (1) and (2) are compatible (at least in characteristic 0) on a common subgroup  $\text{CH}_{\mathbb{B},l}^*(X)$  of  $\text{CH}_l^*(X)$  and  $\text{CH}^*(X)^{[0]}$  via the cycle class map  $\text{Pic}(B) \rightarrow H_{\text{ét}}^2(B_{\bar{k}}, \mathbb{Q}_l(1))$ : this is what Rössler and Szamuely [2022, Proposition 6.1] had checked in the special case where  $X/K$  has a smooth model, by using a variant of [Proposition 2.8](#) here. In [Theorem 5.10](#), I show that (1) is the opposite of Silverman’s refined height pairing [1994, Theorem III.9.5(b)] in the classical case of an elliptic curve  $X$  over the function field of a smooth projective curve  $B$  over an algebraically closed field  $k$ .

Another case where a compatibility should not be hard to show is that of [\[Moret-Bailly 1985\]](#).

Note that (1) is finer than (2) inasmuch as it takes homologically trivial cycles on  $B$  into account. This extra structure is presumably arithmetically significant; it is studied in [Section 6E](#) in the case  $d = 1$ ,  $B$  projective.

It may seem disturbing that (1) is essentially integral, while the classical height pairing is usually rational: this may be “explained” by (3) which is integral but takes values on the subgroup of finite index  $A^0(B) \subseteq A(K)$ . In this spirit, I show in [Remarks 5.13\(a\)](#) that in the elliptic curve case mentioned above,  $\text{CH}^1(X)^{[0]}$  contains  $\mathcal{N}^0(B)$  as a subgroup of finite index, where  $\mathcal{N}^0$  is the identity component of the Néron model of  $X$ .

The *raison d’être* of [\[Beilinson 1987; Bloch 1984\]](#) was to refine the conjectures of Tate [1965] on the orders of poles of zeta functions at integers by describing special values at these integers, when  $K$  is a global field. Thus one might like to extend (1) to the case where  $B$  is regular and flat over  $\mathbb{Z}$ . I consider this as beyond the scope of this article for two reasons:

- The present method fails in this case even if one is given a regular projective model  $f : \mathcal{X} \rightarrow B$  of  $X$ , because Fulton’s techniques do not define an intersection product on  $\mathcal{X}$ , except when  $\delta = 1$  and  $f$  is smooth [1984, p. 397]. One does get an intersection product with  $\mathbb{Q}$  coefficients, by using either  $K$ -theory as in [\[Gillet and Soulé 1987, 8.3\]](#), or alterations and deformation to the normal cone as in Andreas Weber’s thesis [2015, Corollary 4.2.3 and Theorem 4.3.3]; it is possible that the present approach may be adapted by using one of these products.
- However, the main point in characteristic 0 is to involve archimedean places to get a complete height pairing whose determinant has a chance to describe the special values as mentioned above: this is what was done successfully in [\[Bloch 1984; Beilinson 1987\]](#) when  $\delta = 1$ . In higher dimensions, one probably would have to use something like Arakelov intersection theory (see [\[Rössler and Szamuely 2022, Conjecture 7.1\]](#) for a conjectural statement).

I leave these issues to the interested readers. Rather, I hope to show here that height pairings in the style of (1) also raise interesting geometric questions. These are discussed in [Section 6](#), which is closely related to [\[Kahn 2014, Question 7.6\]](#).

**Contents.** Up to [Section 4F](#), we assume  $k$  perfect; this assumption is removed in the said subsection. In [Definition 2.2](#), we introduce subgroups  $\text{CH}^i(\mathcal{X})^0$  of *admissible cycles* in the Chow groups of a  $k$ -model  $f : \mathcal{X} \rightarrow B$  of  $f' : X \rightarrow \text{Spec } K$ , with  $\mathcal{X}$  smooth; when  $B$  is projective,  $\text{CH}^i(\mathcal{X})^0$  contains numerically trivial cycles ([Proposition 2.5](#)) and in general it contains locally homologically trivial cycles in the sense of Beilinson [1987, 1.2] ([Proposition 2.6](#)). From the intersection pairing on  $\mathcal{X}$ , pushed

forward to  $\mathrm{CH}^1(B)$ , we then get, thanks to [Proposition 2.8](#), a height pairing  $\langle \ , \ \rangle_f$  defined on the groups  $\mathrm{CH}^i(X)_f^0 := \mathrm{Im}(\mathrm{CH}^i(\mathcal{X})^0 \rightarrow \mathrm{CH}^i(X))$  (2-9). This is a pairing of genuine abelian groups. We prove in [Propositions 3.6](#) and [3.8](#) that the  $\mathrm{CH}^i(X)_f^0$  and  $\langle \ , \ \rangle_f$  are independent of  $f$  and compatible with the action of correspondences, and in [Proposition 3.9](#) that they behave well with respect to base change. The group  $\mathrm{CH}^i(X)/\mathrm{CH}^i(X)^0$  is finitely generated ([Proposition 3.11](#)).

If we are in characteristic 0, the construction is finished since  $X$  always admits a smooth model by resolution of singularities ([Proposition 4.1](#)). In characteristic  $p > 0$ , there turns out to be quite a bit of work to get a pairing in general after suitably inverting  $p$ , by using Gabber’s refinement of de Jong’s theorem: the general height pairing (4-1) is defined in [Theorem 4.14](#); as said above, it makes sense in the category  $\mathbf{Ab} \otimes \mathbb{Z}[1/p]$ . Functoriality and base change extend to this pairing ([Theorem 4.14](#)).

In [Section 5](#), we investigate [Conjecture 5.1](#):  $\mathrm{CH}^i(X)^{[0]}$  is of finite index in  $\mathrm{CH}_{\mathrm{num}}^i(X)$ , the group of cycles numerically equivalent to 0 (the inclusion is always true by [Lemma 4.3\(d\)](#)); we prove it for  $i = 1, d$  in [Theorem 5.6\(b\)](#) (see [Theorem 5.5\(ii\)](#) for other cases). In [Section 5C](#), we also relate (1) to the classical Néron–Tate height pairing in the case where  $X$  is an elliptic curve and  $B$  is a smooth projective curve.

In [Section 6](#), we study the height pairing (2-9) in the basic case  $i = 1$ . If  $B$  is projective, it leads to a coarser pairing (6-2) between the Lang–Néron groups  $\mathrm{LN}(\mathrm{Pic}_X^0, K/k)$  and  $\mathrm{LN}(\mathrm{Alb}_X, K/k)$  with values in  $N^1(B)$ , codimension 1 cycles modulo numerical equivalence ([Theorem 6.2](#)). When  $\delta = 1$ , a version of this pairing involving an ample divisor is negative definite ([Theorem 6.6](#)): one should compare this with a result of Shioda [[1999](#)] when  $d = 1$ . See also [Theorem 6.6](#) for a conjectural statement when  $\delta > 1$ . We finally get an intriguing homomorphism from  $\mathrm{LN}(\mathrm{Pic}_X^0, K/k)$  to homomorphisms between certain abelian varieties in (6-6).

**Notation and conventions.** We try and follow Fulton’s notation [[1984](#)] as much as possible. In particular, given a morphism of  $k$ -schemes  $f : X \rightarrow Y$ , we write  $\gamma_f$  for the associated graph morphism  $X \rightarrow X \times_k Y$  and  $\delta_X$  for  $\gamma_{1_X}$ ; if  $f$  admits refined Gysin morphisms as in [[loc. cit.](#), Chapters 6 and 8], we write them  $f^!$  and sometimes use the notation  $f^*$  for ordinary Gysin morphisms.

We usually abbreviate the notation  $\times_k$  (fibre product over  $k$ ) to  $\times$ , and re-establish it when it may be confused with other fibre products.

We shall encounter  $k$ -schemes essentially of finite type, being of finite type over some localisation of  $B$ . We shall sometimes commit the abuse of treating them as if they were of finite type: for example, call them smooth even if they really are essentially smooth, and take (refined) Gysin morphisms associated to morphisms between them even if these morphisms are not of finite type. This is easily justified by the fact that Chow groups commute with inverse limits of open immersions [[Bloch 2010](#), Lemma IA.1].

## 1. An elementary reduction

**1A. Intersection on regular  $K$ -schemes.** Let  $K$  be a field. If  $\mathrm{char} K = 0$ , every regular  $K$ -scheme  $X$ , separated of finite type, is smooth, so the intersection theory of [[Fulton 1984](#), Chapter 8] applies. Here we point out that this is also true in characteristic  $p > 0$ : it will be needed in and after [Section 4B](#).

We may assume  $K$  to be finitely generated over its (perfect) subfield  $k = \mathbb{F}_p$ , and  $X$  (regular) to be irreducible of dimension  $d$ . We may find a smooth connected separated  $k$ -scheme  $B$  of finite type with generic point  $\eta = \text{Spec } K$ , and a dominant morphism  $f : \mathcal{X} \rightarrow B$  with  $\mathcal{X}$   $k$ -smooth, of generic fibre  $X$ . We have the intersection pairing of [Fulton 1984, §8.1]: for  $i, r \geq 0$ ,

$$\text{CH}^i(\mathcal{X}) \times \text{CH}^{d+r-i}(\mathcal{X}) \xrightarrow{\quad} \text{CH}^{d+r}(\mathcal{X}), \tag{1-1}$$

which commutes with base change by [Fulton 1984, Proposition 6.6(c) and 8.3(a)]. Then (1-1) induces an intersection product on  $X$  by passing to the limit. If  $f_1 : \mathcal{X}_1 \rightarrow B_1$  is another choice, then  $B$  and  $B_1$  share a common open subset with isomorphic fibres, so this intersection product is independent of the choice of  $(B, f)$ .

Suppose moreover  $X$  and  $f$  proper. Composing (1-1) with  $f_*$ , we get a pairing

$$\text{CH}^i(\mathcal{X}) \times \text{CH}^{d+r-i}(\mathcal{X}) \xrightarrow{\langle \cdot, \cdot \rangle} \text{CH}^r(B). \tag{1-2}$$

For the same reason, numerical equivalence makes sense on  $X$  via (1-2), and does not depend on any choice.

**1B. The set-up.** Let now  $k$  be any perfect field; we place ourselves in the situation  $(B, \mathcal{X}, f)$  of Section 1A with  $f$  proper, and let  $f' : X \rightarrow \eta$  be the generic fibre of  $f$ . In particular, the observations of Section 1A apply to  $X$ .

For a subscheme  $Z$  of  $B$ , write  $\mathcal{X}_Z = f^{-1}(Z)$ ,  $\iota : \mathcal{X}_Z \hookrightarrow \mathcal{X}$  for the corresponding immersion and  $f_Z : \mathcal{X}_Z \rightarrow Z$  for the projection induced by  $f$ . We extend these notations to pull-backs by a morphism  $Z \rightarrow B$  when there is no ambiguity in the context.

**Lemma 1.1.** *Suppose that  $\text{codim}_B Z > r$ . Then (1-2) factors through a pairing*

$$\text{CH}^i(\mathcal{X} - \mathcal{X}_Z) \times \text{CH}^{d+r-i}(\mathcal{X} - \mathcal{X}_Z) \xrightarrow{\langle \cdot, \cdot \rangle} \text{CH}^r(B).$$

*Proof.* We have

$$\text{CH}^r(B) \xrightarrow{\sim} \text{CH}^r(B - Z). \quad \square$$

We shall use the case  $r = 1$  of this lemma in the rest of this paper.

**Remarks 1.2.** (a) Let  $\mathcal{Z}$  be the locus of nonsmoothness of  $f$ . If  $f'$  is smooth,  $f(\mathcal{Z})$  is a proper closed subset of  $B$ , hence contains only finitely many points of  $B^{(1)}$ , the set of codimension 1 points of  $B$ .

(b) If  $\delta = 1$ , any proper surjective morphism  $\varphi$  from an irreducible  $k$ -variety  $V$  to  $B$  is flat [Hartshorne 1977, Chapter II, Proposition 9.7]; in general, this is true after base-changing to the local scheme at any point  $b \in B^{(1)}$ . If  $F \subset V$  is the (closed) locus of nonflatness of  $\varphi$ , the closed subset  $\varphi(F)$  is therefore of codimension  $\geq 2$  in  $B$ . This shows that one may reduce to  $\varphi$  flat by removing a closed subset of codimension  $\geq 2$  from  $B$ . This technique may be applied to  $f$  if necessary; a variant will be used in the proof of Proposition 3.11.

Let  $\mathrm{CH}_{\mathrm{num}}^i(X)$  denote the subgroup of  $\mathrm{CH}^i(X)$  formed of cycles numerically equivalent to 0; write  $j$  for the inclusion  $X \hookrightarrow \mathcal{X}$ .

**Lemma 1.3.** *For  $\alpha \in \mathrm{CH}^i(\mathcal{X})$ , the following are equivalent:*

- (1)  $j^*\alpha \in \mathrm{CH}_{\mathrm{num}}^i(X)$ ;
- (2) for any  $\beta \in \mathrm{CH}^{d-i}(\mathcal{X})$ ,  $f_*(\alpha \cdot \beta) = 0$ .

*Proof.* We have (2)  $\Rightarrow$  (1) because of the surjectivity of  $j^*$  and the formula

$$j^* f_*(\alpha \cdot \beta) = f'_* j^*(\alpha \cdot \beta) = f'_*(j^*\alpha \cdot j^*\beta) \quad (1-3)$$

[Fulton 1984, Proposition 1.7 and 8.3(a)], where  $J : \eta \hookrightarrow B$  is the inclusion, and (1)  $\Rightarrow$  (2) because of (1-3) and the injectivity of  $j^* : \mathrm{CH}^0(B) \rightarrow \mathrm{CH}^0(\eta)$ .  $\square$

## 2. The refined height pairing

We keep the set-up of Section 1B.

**2A. Review of Fulton’s refined Gysin morphisms.** Let  $f : X \rightarrow Y$  be a morphism of algebraic  $k$ -schemes, of constant dimensions  $d_X$  and  $d_Y$  for simplicity, and let  $d = d_Y - d_X$ . In certain cases, Fulton associates to  $f$  “refined Gysin morphisms”

$$f^! : \mathrm{CH}_*(Y') \rightarrow \mathrm{CH}_{*-d}(X \times_Y Y')$$

for any  $Y$ -scheme  $Y'$ ; these morphisms are compatible with push-forward, pull-back and intersection products in the sense of [Fulton 1984, Definition 17.1]. Such collections of morphisms are called *orientations* in [loc. cit., §17.4]. Orientable morphisms are

- flat morphisms [loc. cit., Theorem 1.7],
- regular embeddings [loc. cit., §§6.2, 6.4],
- more generally, l.c.i. morphisms [loc. cit., §6.5],
- morphisms to a smooth  $Y$  [loc. cit., Definition 8.1.2].

The definitions of  $f^!$  agree when  $f$  is of several of these forms at the same time, e.g., [loc. cit., Proposition 8.1.2]. The assignment  $f \mapsto f^!$  is functorial in certain cases, many of which are summarised in [loc. cit., Example 17.4.6].

Since it is difficult to find a unified statement of all these compatibilities in [Fulton 1984], we shall strive to give precise references for all those we use; the above reminder should only be viewed as a guide to the reader.

We shall very often use the following situation, that we record as a lemma.

**Lemma 2.1.** *Let*

$$\begin{array}{ccc} S' & \xrightarrow{f'} & T' \\ g' \downarrow & & g \downarrow \\ S & \xrightarrow{f} & T \end{array}$$

*be a Cartesian square of  $k$ -schemes, where  $g$  is proper and  $f$  is an l.c.i. morphism. Then:*

(a) *One has*

$$f^! g_* = g'_* f^!$$

*as homomorphisms from  $\mathrm{CH}_*(T')$  to  $\mathrm{CH}_*(S)$ .*

(b) *If  $f'$  is also an l.c.i. morphism, of same codimension, then  $f^! = f'^!$ .*

(c) *If  $f$  and  $g$  are two composable l.c.i. morphisms, then  $(g \circ f)^! = f^! \circ g^!$ .*

*Proof.* This follows from [Fulton 1984, Theorem 6.6(c)]. □

**2B. Admissible cycles.** Let  $b \in B^{(1)}$ ; write  $Z = \{\overline{b}\}$ . Recall the cap-product [Fulton 1984, p. 131]

$$\cdot_\iota : \mathrm{CH}^i(\mathcal{X}) \times \mathrm{CH}_l(\mathcal{X}_Z) \rightarrow \mathrm{CH}_{l-i}(\mathcal{X}_Z), \quad (\alpha, \beta) \mapsto \gamma_\iota^!(\beta \times \alpha),$$

where  $\iota$  is the closed immersion  $\mathcal{X}_Z \hookrightarrow \mathcal{X}$ .

Take  $l = \delta + i - 1$ . Composing with  $(f_Z)_*$ , we get a pairing

$$\langle \cdot, \cdot \rangle_b : \mathrm{CH}^i(\mathcal{X}) \times \mathrm{CH}_{\delta+i-1}(\mathcal{X}_Z) \rightarrow \mathrm{CH}_{\delta-1}(Z) = \mathrm{CH}^0(Z) = \mathbb{Z}, \quad \langle \alpha, \beta \rangle_b = (f_Z)_*(\alpha \cdot_\iota \beta). \quad (2-1)$$

We record two useful formulas:

$$\alpha \cdot \iota_* \beta = \iota_*(\alpha \cdot_\iota \beta) \in \mathrm{CH}_{l-i}(\mathcal{X}), \quad (2-2)$$

which follows from Lemma 2.1 applied to the Cartesian diagram

$$\begin{array}{ccc} \mathcal{X}_Z & \xrightarrow{\gamma_\iota} & \mathcal{X}_Z \times \mathcal{X} \\ \iota \downarrow & & \iota \times 1 \downarrow \\ \mathcal{X} & \xrightarrow{\Delta_{\mathcal{X}}} & \mathcal{X} \times \mathcal{X} \end{array}$$

of regular embeddings of codimension  $d + \delta$ . Hence

$$f_*(\alpha \cdot \iota_* \beta) = f_* \iota_*(\alpha \cdot_\iota \beta) = \iota'_*(\langle \alpha, \beta \rangle_b), \quad (2-3)$$

where  $\iota'$  is the closed immersion  $Z \hookrightarrow B$ .

**Definition 2.2.** With the above notation, we set

$$\mathrm{CH}^i(\mathcal{X})_b^0 = \{\alpha \in \mathrm{CH}^i(\mathcal{X}) \mid j^* \alpha \in \mathrm{CH}_{\mathrm{num}}^i(X) \text{ and } \langle \alpha, \beta \rangle_b = 0 \text{ for all } \beta \in \mathrm{CH}_{\delta+i-1}(\mathcal{X}_Z)\}$$



for  $b \in B^{(1)}$ , and

$$\mathrm{CH}^i(\mathcal{X})^0 = \bigcap_{b \in B^{(1)}} \mathrm{CH}^i(\mathcal{X})_b^0.$$

We call the cycles in  $\mathrm{CH}^i(\mathcal{X})^0$  *admissible*.

Even if it is not apparent anymore, this definition was inspired by [Bloch 1984, Assumption 2; Beilinson 1987, 1.2].

**Remarks 2.3.** (a) One should be careful that  $\mathrm{CH}^i(\mathcal{X})^0$  does not contain  $\mathrm{Ker} j^*$  in general. For example, let  $B = A^1 = \mathrm{Spec} k[t]$  and let  $\mathcal{X}$  be the hypersurface in  $B \times \mathbb{P}^2$  with (partly) homogeneous equation  $tX_0^2 = X_1X_2$ . Then the pull-back of the curve  $(t = X_1 = 0)$ , viewed as a codimension 1 cycle on  $\mathcal{X}$ , to the curve  $(t = X_2 = 0)$ , is the point  $(0, (1 : 0 : 0))$  which is not numerically equivalent to 0. On the other hand, if  $f$  is smooth above  $\mathrm{Spec} \mathcal{O}_{B,b}$  for a  $b \in B^{(1)}$ , then any element of  $\mathrm{Ker} j^*$  vanishes when restricted to  $\mathcal{X}_b$  thanks to [Fulton 1984, §20.3]. So this caveat only involves finitely many exceptional  $b$ .

(b) The pairing (2-1) makes sense for any  $b \in B$  (replacing  $\mathrm{CH}_{\delta+i-1}(Z)$  by  $\mathrm{CH}_{\delta+i-r}(Z)$  if  $b \in B^{(r)}$ ), and defines an equivalence relation  $\alpha \equiv_b 0$  if  $\langle \alpha, \beta \rangle_b = 0$  for any  $\beta \in \mathrm{CH}_{\delta+i-r}(Z)$ . One can show that  $\alpha \equiv_{b'} 0 \Rightarrow \alpha \equiv_b 0$  if  $b'$  is a specialisation of  $b$ ; in particular, the condition  $j^*\alpha \in \mathrm{CH}_{\mathrm{num}}^i(X)$  is superfluous in the definition of  $\mathrm{CH}^i(\mathcal{X})_b^0$ , thanks to Lemma 1.3. We shall not use these facts in the present paper, so the rather long proof is omitted (see [Kahn 2023]).

(c) Let  $b \in B^{(1)}$ . Suppose that all the irreducible components  $\mathcal{X}_b^\lambda$  of  $\mathcal{X}_b$  are of dimension  $d$  and smooth over  $k(b)$ . Then it is easy to see that  $\alpha \equiv_b 0$  if and only if  $\kappa_\lambda^! \alpha \in \mathrm{CH}_{\mathrm{num}}^i(\mathcal{X}_b^\lambda)$  for all  $\lambda$ , where  $\kappa_\lambda : \mathcal{X}_b^\lambda \hookrightarrow \mathcal{X}$  is the inclusion. Our initial approach to the refined height pairing was based on such models; they are not necessary anymore.

We obviously have

**Lemma 2.4.** *The quotient  $\mathrm{CH}^i(\mathcal{X}) / \mathrm{CH}^i(\mathcal{X})^0$  is torsion-free.* □

**2C. Comparison with numerical and homological equivalence.**

**Proposition 2.5.** *If  $B$  is projective (hence  $\mathcal{X}$  is  $k$ -proper), we have  $\mathrm{CH}_{\mathrm{num}}^i(\mathcal{X}) \subseteq \mathrm{CH}^i(\mathcal{X})^0$ .*

*Proof.* Let  $\alpha \in \mathrm{CH}_{\mathrm{num}}^i(\mathcal{X})$ : we want to show that  $\alpha \in \mathrm{CH}^i(\mathcal{X})^0$ . Let first  $\beta \in \mathrm{CH}^{d-i}(\mathcal{X}) = \mathrm{CH}_{\delta+i}(\mathcal{X})$ . Choose a 0-cycle  $z \in \mathrm{CH}_0(B)$  of nonzero degree. Then

$$0 = \mathrm{deg}(\alpha \cdot \beta \cdot f^*z) = \mathrm{deg}(f_*(\alpha \cdot \beta) \cdot z) = f_*(\alpha \cdot \beta) \mathrm{deg}(z);$$

hence  $f_*(\alpha \cdot \beta) = 0$ , and we conclude that  $j^*\alpha \in \mathrm{CH}_{\mathrm{num}}^i(X)$  by Lemma 1.3.

Let now  $b \in B^{(1)}$ , and  $Z = \overline{\{b\}}$  as above. Let  $\beta \in \mathrm{CH}_{\delta+i-1}(\mathcal{X}_Z)$ . We have this time

$$0 = f_*(\alpha \cdot \iota_*\beta \cdot f^*z) = f_*(\alpha \cdot \iota_*\beta) \cdot z$$

for any  $z \in \mathrm{CH}_1(B) = \mathrm{CH}^{\delta-1}(B)$ , i.e.,  $f_*(\alpha \cdot \iota_*\beta) = \iota'_*\langle \alpha, \beta \rangle_b \in \mathrm{CH}_{\mathrm{num}}^1(B)$  (see (2-3)). But

$$\iota'_* : \mathbb{Z} = \mathrm{CH}^0(Z) \rightarrow \mathrm{CH}^1(B) / \mathrm{CH}_{\mathrm{num}}^1(B)$$

is injective since  $Z$ , as an irreducible divisor on a smooth projective variety, is not numerically equivalent to 0 (compare [Debarre 2001, Chapter I, Theorem 1.21]). Therefore  $\langle \alpha, \beta \rangle_b = 0$ , as requested.  $\square$

Let now  $l$  be a prime number invertible in  $k$ . We have a composition

$$\mathrm{CH}^i(\mathcal{X}) \rightarrow H^{2i}(\mathcal{X}_{\bar{k}}, \mathbb{Q}_l(i)) \rightarrow H^0(B_{\bar{k}}, R^{2i} f_* \mathbb{Q}_l(i)), \tag{2-4}$$

where the first map is the (geometric) cycle class map. Write  $\mathrm{CH}_l^i(\mathcal{X})$  (resp.  $\mathrm{CH}_{\mathbb{B},l}^i(\mathcal{X})^0$ ) for the kernel of the first map (resp. of their composition): the latter group is introduced by analogy to [Beilinson 1987, 1.2], which is the special case  $\delta = 1$ ,  $k$  algebraically closed. We obviously have  $\mathrm{CH}_l^i(\mathcal{X}) \subseteq \mathrm{CH}_{\mathbb{B},l}^i(\mathcal{X})^0$ .

The following is parallel to Proposition 2.5, without assuming  $B$  projective. It will be used in Proposition 2.11 and in Remarks 5.4(a) and 3.12.

**Proposition 2.6.** *At least in characteristic 0,  $\mathrm{CH}_{\mathbb{B},l}^i(\mathcal{X})^0 \subseteq \mathrm{CH}^i(\mathcal{X})^0$ .*

*Proof.* Let  $\alpha \in \mathrm{CH}_{\mathbb{B},l}^i(\mathcal{X})^0$ . Then  $\alpha$  vanishes in  $H^0(K\bar{k}, R^{2i} f_* \mathbb{Q}_l(i)) = H^{2i}(X \otimes_k \bar{k}, \mathbb{Q}_l(i))$ , hence a fortiori in  $H^{2i}(X \otimes_K \bar{K}, \mathbb{Q}_l(i))$ : this means that  $j^* \alpha$  is  $l$ -adically homologically equivalent to 0, hence also numerically equivalent to 0. This part of the proof works in all characteristics.

We now give the sequel of the proof in characteristic 0: to oversimplify, it follows by functoriality from the fact that the cycle class map is injective in codimension 0 (sic). (So this argument is geometrically cheaper than the one for Proposition 2.5.)

We may assume  $k$  finitely generated and choose an embedding of  $k$  in  $\mathbb{C}$ . By Artin’s comparison theorem,

$$\mathrm{CH}_{\mathbb{B},l}^i(\mathcal{X})^0 = \mathrm{Ker}(\mathrm{CH}^i(\mathcal{X}) \rightarrow H_B^0(B_{\mathbb{C}}, R^{2i} f_* \mathbb{Q}(i))),$$

where  $H_B$  denotes Betti (or analytic) cohomology. Let  $b \in B^{(1)}$ , and let  $Z, \iota, \beta$  be as in Definition 2.2. To show that  $\langle \alpha, \beta \rangle_b = 0$  in  $\mathrm{CH}^0(Z) \xrightarrow{\sim} \mathrm{CH}^0(Z_{\mathbb{C}})$ , we may assume  $k = \mathbb{C}$  and drop all Tate twists.

In [Fulton 1984, Chapter 19], a cycle class map  $\mathrm{cl}$  is defined for Chow groups of complex, possibly singular, varieties, with values in their Borel–Moore homology and we have the formula

$$\mathrm{cl}(\alpha \cdot_{\iota} \beta) = \iota'^*(\mathrm{cl}(\alpha)) \cap \mathrm{cl}(\beta) \in H_{2\delta-2}(\mathcal{X}_Z) \tag{2-5}$$

[Fulton 1984, Proposition 19.2], where  $\iota'$  is the closed immersion  $Z \hookrightarrow B$  as in the previous proof, hence

$$\mathrm{cl}(\langle \alpha, \beta \rangle_b) = (f_Z)_*(\iota'^*(\mathrm{cl}(\alpha)) \cap \mathrm{cl}(\beta)) \in H_{2\delta-2}(Z) \tag{2-6}$$

since  $\mathrm{cl}$  commutes with push-forwards, by definition and [Fulton 1984, Lemma 19.1.2].

It now suffices to show that the right hand side of (2-6) vanishes since  $\mathrm{CH}_{\delta-1}(Z) \rightarrow H_{2\delta-2}(Z)$  is injective, as one sees by reducing to  $Z$  smooth by removing from it a proper closed subset. For this, it suffices to show that the pairing

$$H^{2i}(\mathcal{X}) \times H_{2\delta-2+2i}(\mathcal{X}_Z) \rightarrow H_{2\delta-2}(Z), \tag{2-7}$$

given by  $(x, y) \mapsto (f_Z)_*(\iota'^* x \cap y)$ , factors through  $H^0(B, R^{2i} f_* \mathbb{Q}) \times H_{2\delta-2+2i}(\mathcal{X}_Z)$ .

We switch by Poincaré duality from the Borel–Moore homology of  $\mathcal{X}_Z$  (resp.  $Z$ ) to the cohomology of the smooth variety  $\mathcal{X}$  (resp.  $B$ ) with supports in  $\mathcal{X}_Z$  (resp. in  $Z$ ). Then (2-7) becomes the composition

$$H^{2i}(\mathcal{X}) \times H_{\mathcal{X}_Z}^{2d+2-2i}(\mathcal{X}) \xrightarrow{\cap} H_{\mathcal{X}_Z}^{2d+2}(\mathcal{X}) \xrightarrow{f_*} H_Z^2(B), \tag{2-8}$$

where  $\cap$  is the usual cap-product. The (global) trace map  $f_*$  factors as a composition

$$H_{\mathcal{X}_Z}^{2d+2}(\mathcal{X}) \rightarrow H_Z^0(B, R^{2d+2} f_* \mathbb{Q}) \xrightarrow{(\text{Tr}_f)_*} H_Z^2(B)$$

where  $\text{Tr}_f$  is the local trace map in étale cohomology for the proper morphism  $f$ . Thus, (2-8) factor through the map

$$H^{2i}(\mathcal{X}) \times H_{\mathcal{X}_Z}^{2d+2-2i}(\mathcal{X}) \rightarrow H^0(B, R^{2i} f_* \mathbb{Q}) \times H_Z^0(B, R^{2d+2-i} f_* \mathbb{Q})$$

as requested.

In positive characteristic, the leap of faith is that (2-5) and (2-6) hold for the cycle class maps defined in  $l$ -adic Borel–Moore homology [Laumon 1976, §6]. The commutation with push-forwards causes no problem, and (2-5) indeed appears in [Laumon 1976, Theorem (7.2)], except that the extraordinary cap-product  $\cdot_l$  (defined in [Verdier 1976, 2.1.1] using intersection multiplicities) should be shown to agree with Fulton’s. (This is suggested in the notes and references of [Fulton 1984, Chapter 19]; see also [loc. cit., p. 382].)<sup>1</sup>

This being accepted, the same argument goes through. □

**Remark 2.7.** As a referee pointed out, there is an important conceptual difference between  $\text{CH}^i(\mathcal{X})_{\mathbb{B},l}^0$  and  $\text{CH}^i(\mathcal{X})^0$ : by the smooth and proper base change, we have the equality

$$\text{Ker}(H^{2i}(\mathcal{X}_{\bar{k}}, \mathbb{Q}_l(i)) \rightarrow H^{2i}(X_{\bar{k}}, \mathbb{Q}_l(i))) = \text{Ker}(H^{2i}(\mathcal{X}_{\bar{k}}, \mathbb{Q}_l(i)) \rightarrow H^0(U_{\bar{k}}, R^{2i} f_* \mathbb{Q}_l(i)))$$

for any open subset  $U \subseteq B$  over which  $f$  is smooth. Thus, the condition  $\alpha \in \text{CH}^i(\mathcal{X})_{\mathbb{B},l}^0$  for  $\alpha \in \text{CH}^i(\mathcal{X})$  only has to be checked at the generic fibre and at the “bad fibres” of  $f$ . This contrasts with the case of  $\text{CH}^i(\mathcal{X})^0$ , see Remarks 2.3(a). See also Remarks 5.4 further down.

**2D. Global height pairing.** The following proposition is the key point of this paper.

**Proposition 2.8.** *Let  $\alpha \in \text{CH}^i(\mathcal{X})^0$ . If  $\beta \in \text{CH}^{d+1-i}(\mathcal{X})$  and  $j^* \beta = 0$ , then  $f_*(\alpha \cdot \beta) = 0$  in  $\text{CH}^1(B)$ .*

*Proof.* By [Fulton 1984, Proposition 1.8], write  $\beta = \iota_* \beta'$  with  $\beta' \in \text{CH}_{\delta+i-1}(\mathcal{X}_Z)$  for some proper closed subset  $Z \subset B$ , where  $\iota : \mathcal{X}_Z \hookrightarrow \mathcal{X}$  is the inclusion. We may assume that  $\beta'$  is the class of an irreducible cycle, hence take  $Z$  irreducible. If  $\text{codim}_B Z > 1$ , the result follows from Lemma 1.1. If  $Z = \{\bar{b}\}$  for  $b \in B^{(1)}$ , the conclusion follows from (2-3). □

The proof of the following lemma is in the same spirit, so we include it here. It will be used in the proof of Proposition 3.9(ii).

<sup>1</sup>Using Olsson’s theorem [2015, Theorem 2.34] that the cycle class maps commute with refined Gysin homomorphisms, it would suffice to show the identity  $\gamma_l^1(x \times y) = \iota^* y \cap x$  in Borel–Moore  $l$ -adic homology.

**Lemma 2.9.** *Let  $b_1, \dots, b_n$  be a finite set of points on  $B^{(1)}$  and let  $Z = \overline{\{b_1, \dots, b_n\}}$ . Then one has  $(f_Z)_*(\alpha \cdot \iota_* \beta) = 0$  for any  $\alpha \in \text{CH}^i(\mathcal{X})^0$  and any  $\beta \in \text{CH}_{\delta+i-1}(\mathcal{X}_Z)$ , where  $\iota$  is the closed immersion  $\mathcal{X}_Z \hookrightarrow \mathcal{X}$ .*

*Proof.* We may assume that  $\beta$  is the class of an irreducible cycle  $\beta'$ ; then  $\beta'$  is supported on  $\mathcal{X}_{Z_r}$  for some  $r$ , where  $Z_r = \overline{\{b_r\}}$ . Let  $\kappa : \mathcal{X}_{Z_r} \hookrightarrow \mathcal{X}_Z$  be the corresponding closed immersion, and let  $\iota_r = \iota\kappa$ : by applying again Lemma 2.1 to the obvious Cartesian square involving  $\kappa$ , we get the identity

$$\alpha \cdot \iota_* \kappa_* \beta' = \kappa_* (\alpha \cdot \iota_r_* \beta')$$

etc. □

**Definition 2.10.** Let  $\text{CH}^i(X)_f^0$  be the image of  $\text{CH}^i(\mathcal{X})^0$  in  $\text{CH}^i(X)$ . By Proposition 2.8, (1-2) induces a pairing

$$\text{CH}^i(\mathcal{X})^0 \times \text{CH}^{d+1-i}(X) \rightarrow \text{CH}^1(B)$$

hence, swapping  $i$  with  $d + 1 - i$ , a “height” pairing

$$\langle \ , \ \rangle_f : \text{CH}^i(X)_f^0 \times \text{CH}^{d+1-i}(X)_f^0 \rightarrow \text{CH}^1(B). \tag{2-9}$$

We shall see in the next section (Propositions 3.6 and 3.8) that neither  $\text{CH}^i(X)_f^0$  nor  $\langle \ , \ \rangle_f$  depends in the choice of  $f$ .

**2E. Comparison with the pairing of Rössler–Szamuely.**

**Proposition 2.11.** *The pairing (2-9) is compatible with the pairing (2) of the introduction on the subgroups  $\text{CH}^i(\mathcal{X})_{\mathbb{E},l}^0$  and  $\text{CH}^{d+1-i}(\mathcal{X})_{\mathbb{E},l}^0$  of Proposition 2.6.*

*Proof.* Using cup-product and push-forward in  $l$ -adic cohomology,

$$H^{2i}(\mathcal{X}_{\bar{k}}, \mathbb{Q}_l(i)) \otimes H^{2(d+1-i)}(\mathcal{X}_{\bar{k}}, \mathbb{Q}_l(d+1-i)) \xrightarrow{\cup} H^{2(d+1)}(\mathcal{X}_{\bar{k}}, \mathbb{Q}_l(d+1)) \xrightarrow{f_*} H^2(B_{\bar{k}}, \mathbb{Q}_l(1)), \tag{2-10}$$

we get from (2-4) a pairing

$$\text{CH}^i(\mathcal{X}) \otimes \text{CH}^{d+1-i}(\mathcal{X}) \rightarrow H^2(B_{\bar{k}}, \mathbb{Q}_l(1)) \tag{2-11}$$

which is evidently compatible with (1-2) (for  $r = 1$ ). On the other hand, the Leray spectral sequence

$$H^r(B_{\bar{k}}, R^s f_* \mathbb{Q}_l(i)) \Rightarrow H^{r+s}(\mathcal{X}_{\bar{k}}, \mathbb{Q}_l(i)) \tag{2-12}$$

yields Abel–Jacobi maps

$$\text{AJ}_B^i : \text{CH}^i(\mathcal{X})_{\mathbb{E},l}^0 \rightarrow H^1(B_{\bar{k}}, R^{2i-1} f_* \mathbb{Q}_l(i)). \tag{2-13}$$

We have a pairing parallel to (2-10),

$$H^1(B_{\bar{k}}, R^{2i-1} f_* \mathbb{Q}_l(i)) \otimes H^1(B_{\bar{k}}, R^{2(d-i+1)-1} f_* \mathbb{Q}_l(d-i+1)) \xrightarrow{\cup} H^2(B_{\bar{k}}, R^{2d} f_* \mathbb{Q}_l(i)) \xrightarrow{\text{Tr}_f} H^2(B_{\bar{k}}, \mathbb{Q}_l(1)), \tag{2-14}$$

which is compatible with the former via (2-12). This implies that the restriction of (2-11) to

$$\mathrm{CH}^i(\mathcal{X})_{\mathbb{B},l}^0 \otimes \mathrm{CH}^{d+1-i}(\mathcal{X})_{\mathbb{B},l}^0$$

is compatible with (2-9) via Proposition 2.6, i.e., that the diagram

$$\begin{array}{ccc} \mathrm{CH}^i(\mathcal{X})_{\mathbb{B},l}^0 \otimes \mathrm{CH}^{d+1-i}(\mathcal{X})_{\mathbb{B},l}^0 & \longrightarrow & H^2(B_{\bar{k}}, \mathbb{Q}_l(1)) \\ \downarrow & & \uparrow \\ \mathrm{CH}^i(\mathcal{X})^0 \otimes \mathrm{CH}^{d+1-i}(\mathcal{X})^0 & \longrightarrow & \mathrm{CH}^1(B) \end{array} \tag{2-15}$$

commutes.

On the other hand, the height pairing of [Rössler and Szamuely 2022] is defined on

$$\mathrm{CH}_l^i(X) \otimes \mathrm{CH}_l^{d+1-i}(X),$$

also with values in  $H^2(B_{\bar{k}}, \mathbb{Q}_l(1))$ . More precisely, by [loc. cit., Proposition 2.3], if  $\alpha \in \mathrm{CH}_l^i(X)$ ,  $j : U \hookrightarrow B$  is an open subset over which  $f$  is smooth and  $\alpha_U$  is a lift of  $\alpha$  to  $\mathrm{CH}^i(\mathcal{X}_U)$ , then  $\mathrm{AJ}_U^i(\alpha_U) \in H^1(U_{\bar{k}}, R^{2i-1}(f_U)_* \mathbb{Q}_l(i))$  lies in the subgroup  $H^{1-\delta}(B_{\bar{k}}, j_{!*} R^{2i-1}(f_U)_* \mathbb{Q}_l(i))$  [loc. cit., Proposition 2.1], and the height pairing of Rössler and Szamuely is defined by (2-14) on these subgroups. Let  $\mathcal{F} = R^{2i-1}(f_U)_* \mathbb{Q}_l(i) = j^* R^{2i-1} f_* \mathbb{Q}_l(i)$ . Since  $j^* j_{!*} \mathcal{F} = \mathcal{F}$  [Beilinson et al. 1982, remarque 1.4.14.1], the image of  $H^1(B_{\bar{k}}, R^{2i-1} f_* \mathbb{Q}_l(i))$  in  $H^1(U_{\bar{k}}, R^{2i-1}(f_U)_* \mathbb{Q}_l(i))$  is contained in  $H^{1-\delta}(B_{\bar{k}}, j_{!*} R^{2i-1}(f_U)_* \mathbb{Q}_l(i))$ . □

### 3. Independence from the (smooth) model

**3A. Review of the Corti–Hanamura category.** A morphism  $f : \mathcal{X} \rightarrow B$  as in Section 1 defines an object in the Corti–Hanamura category  $\mathrm{CHC}(B)$  of [Corti and Hanamura 2000, Definition 2.8].<sup>2</sup> Given two such objects  $f_i : \mathcal{X}_i \rightarrow B$  ( $i = 1, 2$ ), morphisms in  $\mathrm{CHC}(B)$  are defined by relative correspondences

$$\mathrm{CHC}(B)(\mathcal{X}_1, \mathcal{X}_2) = \mathrm{CH}_{\dim \mathcal{X}_2}(\mathcal{X}_1 \times_B \mathcal{X}_2) = \mathrm{CH}^{\dim X_1}(\mathcal{X}_1 \times_B \mathcal{X}_2),$$

where  $X_1$  is the generic fibre of  $\mathcal{X}_1$ .

If  $f_3 : \mathcal{X}_3 \rightarrow B$  is a third object, the composition of two such correspondences  $u : \mathcal{X}_1 \rightarrow \mathcal{X}_2$  and  $v : \mathcal{X}_2 \rightarrow \mathcal{X}_3$  is defined as

$$v \bullet u = (p_{1,3}^{1,2,3})_* \delta_2^1(u \times_k v), \tag{3-1}$$

where  $\delta^1$  is the refined Gysin morphism from [Fulton 1984, §6.2] associated to the (regular immersion) diagonal  $\delta_2 : \mathcal{X}_2 \rightarrow \mathcal{X}_2 \times_k \mathcal{X}_2$  in the (augmented) Cartesian square

$$\begin{array}{ccc} (\mathcal{X}_1 \times_B \mathcal{X}_3) & \xleftarrow{p_{1,3}^{1,2,3}} & \mathcal{X}_1 \times_B \mathcal{X}_2 \times_B \mathcal{X}_3 & \xrightarrow{\Delta} & (\mathcal{X}_1 \times_B \mathcal{X}_2) \times_k (\mathcal{X}_2 \times_B \mathcal{X}_3) \\ & & \downarrow p_2^{1,2,3} & & \downarrow p_2^{1,2} \times p_1^{2,3} \\ & & \mathcal{X}_2 & \xrightarrow{\delta_2} & \mathcal{X}_2 \times_k \mathcal{X}_2 \end{array} \tag{3-2}$$

<sup>2</sup>Except that  $f$  is assumed projective in [Corti and Hanamura 2000]; proper is sufficient to apply its formalism.

and the notation for the projections is self-evident.

As usual, one can generalise this to “graded correspondences”

$$\text{CHC}(B)_r(\mathcal{X}_1, \mathcal{X}_2) = \text{CH}_{\dim \mathcal{X}_2 - r}(\mathcal{X}_1 \times_B \mathcal{X}_2) = \text{CH}^{\dim \mathcal{X}_1 + r}(\mathcal{X}_1 \times_B \mathcal{X}_2)$$

and reduce these graded correspondences to ordinary ones if one wishes, by using the projective bundle formula [Fulton 1984, Theorem 3.3(b)].

Since  $\Delta$  is also a regular immersion of the same codimension as  $\delta$  (namely,  $\dim \mathcal{X}_2$ ), we may apply Lemma 2.1(b) which gives

$$\delta_2^!(v \times_k u) = \Delta^!(v \times_k u). \tag{3-3}$$

If the  $f_i$  are smooth, we also have a “classical” composition of correspondences à la Deninger–Murre [1991]:

$$v \circ u = (p_{1,3}^{1,2,3})_*((p_{2,3}^{1,2,3})^* v \cdot (p_{1,2}^{1,2,3})^* u).$$

**Lemma 3.1.** (a) *In the above case,  $v \circ u = v \bullet u$ .*

(b) *The category  $\text{CHC}(B)$  is contravariant for smooth  $k$ -morphisms  $\varphi : C \rightarrow B$ .*

(c) *The pro-open immersion  $j$  defines a functor to the category of Chow correspondences over  $K$  from the full subcategory of  $\text{CHC}(B)$  consisting of those  $f : \mathcal{X} \rightarrow B$  whose generic fibre is smooth.*

*Proof.* (a) We use (3-3). We have the Cartesian square

$$\begin{array}{ccc} \mathcal{X}_1 \times_B \mathcal{X}_2 \times_B \mathcal{X}_3 & \xrightarrow{\Delta_1} & (\mathcal{X}_1 \times_B \mathcal{X}_2 \times_B \mathcal{X}_3) \times_k (\mathcal{X}_1 \times_B \mathcal{X}_2 \times_B \mathcal{X}_3) \\ \parallel \downarrow & & \downarrow p_{2,3}^{1,2,3} \times p_{1,2}^{1,2,3} \\ \mathcal{X}_1 \times_B \mathcal{X}_2 \times_B \mathcal{X}_3 & \xrightarrow{\Delta} & (\mathcal{X}_2 \times_B \mathcal{X}_3) \times_k (\mathcal{X}_1 \times_B \mathcal{X}_2) \end{array}$$

in which all morphisms are l.c.i. morphisms, hence

$$\Delta^!(v \times_k u) = \Delta_1^!(p_{2,3}^{1,2,3} \times p_{1,2}^{1,2,3})^!(v \times_k u)$$

by Lemma 2.1(c),

$$(p_{2,3}^{1,2,3} \times p_{1,2}^{1,2,3})^!(v \times_k u) = (p_{2,3}^{1,2,3} \times p_{1,2}^{1,2,3})^*(v \times_k u) = (p_{2,3}^{1,2,3})^* v \times_k (p_{1,2}^{1,2,3})^* u$$

by [Fulton 1984, Proposition 6.6(b)], and finally

$$\Delta_1^!((p_{2,3}^{1,2,3})^* v \times_k (p_{1,2}^{1,2,3})^* u) = (p_{2,3}^{1,2,3})^* v \cdot (p_{1,2}^{1,2,3})^* u$$

by definition of the intersection product on smooth varieties [Fulton 1984, p. 131].

(b) The statement means that  $\varphi$  defines a functor  $\varphi^* : \text{CHC}(B) \rightarrow \text{CHC}(C)$ , given by fibre product. It is defined on objects by the smoothness of  $\varphi$ , and on morphisms because smooth morphisms are flat. To check that it respects composition involves chasing in the Cartesian cube obtained by pulling back the square of  $B$ -schemes (3-2) along the morphism  $C \times_B C \rightarrow B$ , and then further pulling back along the diagonal  $\delta' : C \rightarrow C \times_B C$ ; this latter operation is unnecessary if  $C$  is an open subset of  $B$ . The first

step involves [Fulton 1984, Proposition 6.6] as in the proof of (a), to take care of the flat l.c.i morphisms  $C \times_B (\mathcal{X}_i \times_B \mathcal{X}_j) \rightarrow \mathcal{X}_i \times_B \mathcal{X}_j$ ; the second step uses the fact that  $\delta'$  is a regular immersion.

(c) This follows from (a), (b) and [Bloch 2010, Lemma IA.1], since  $U \times_B \mathcal{X}$  is smooth over  $U$  for a suitable open subset  $U$  of  $B$  for  $\mathcal{X}$  as in the statement. □

**Remark 3.2.** The associativity of the composition  $\bullet$  is not proven in [Corti and Hanamura 2000]. It will not be used here and is left to the reader. See nevertheless Remark 3.4.

As a special case of (3-1), take  $\mathcal{X}_3 = B$ : we get pairings

$$\mathrm{CH}^{\dim X_2+r}(\mathcal{X}_1 \times_B \mathcal{X}_2) \otimes \mathrm{CH}^i(\mathcal{X}_2) \rightarrow \mathrm{CH}^{i+r}(\mathcal{X}_1), \quad (\psi, \alpha) \mapsto \psi^* \alpha := (p_1)_* \delta_2^!(\psi \times_k \alpha)$$

compatible via  $j^*$  with the usual action of correspondences over  $K$ , by Lemma 3.1(c). For clarity, we repeat (3-1) in this special case:

$$\begin{array}{ccc} \mathcal{X}_1 & \xleftarrow{p_1} & \mathcal{X}_1 \times_B \mathcal{X}_2 & \xrightarrow{\gamma_{p_2}} & (\mathcal{X}_1 \times_B \mathcal{X}_2) \times_k \mathcal{X}_2 \\ & & p_2 \downarrow & & p_2 \times 1 \downarrow \\ & & \mathcal{X}_2 & \xrightarrow{\delta_2} & \mathcal{X}_2 \times_k \mathcal{X}_2 \end{array} \tag{3-4}$$

where  $\gamma_{p_2}$  is the graph of  $p_2 := p_2^{1,2}$ .

We also write  $\psi_*$  for  $({}^t \psi)^*$ .

As an even more special case, when  $\mathcal{X}_1 = B$ : writing  $\beta$  rather than  $\psi$ , we recover the pairing (1-2)

$$\langle \alpha, \beta \rangle = (f_2)_*(\alpha \cdot \beta) = (f_2)_* \delta_2^!(\alpha \times_k \beta) = \beta^* \alpha \in \mathrm{CH}^*(B). \tag{3-5}$$

**Lemma 3.3.** *Let  $(\alpha, \beta) \in \mathrm{CH}^i(\mathcal{X}_1) \times \mathrm{CH}^{d_1-i+1}(\mathcal{X}_2)$  and  $\psi \in \mathrm{CH}^{d_2}(\mathcal{X}_1 \times_B \mathcal{X}_2)$ . Then*

$$\langle \psi^* \alpha, \beta \rangle = \langle \alpha, \psi_* \beta \rangle.$$

*Proof.* For clarity, write  $\delta_i$  for the diagonal map  $\mathcal{X}_i \rightarrow \mathcal{X}_i \times_k \mathcal{X}_i$ . As in the proof of Proposition 3.6, let  $p_i$  be the projection  $\mathcal{X}_1 \times_B \mathcal{X}_2 \rightarrow \mathcal{X}_i$ . Developing, the identity to be proven is

$$(f_1)_*((p_1)_* \delta_2^!(\psi \times \alpha) \cdot \beta) = (f_2)_*(\alpha \cdot (p_2)_* \delta_1^!({}^t \psi \times \beta)). \tag{3-6}$$

Let  $\lambda = \delta_2^!(\psi \times \alpha)$ . We have

$$(p_1)_* \lambda \cdot \beta = \delta_1^!((p_1)_* \lambda \times \beta) = \delta_1^!(p_1 \times 1)_*(\lambda \times \beta) = (p_1)_* \delta_1^!(\lambda \times \beta)$$

by Lemma 2.1(a). Similarly, if  $\lambda' = \delta_1^!({}^t \psi \times \beta)$  and  $\lambda'' := \delta_1^!(\psi \times \beta)$ :

$$\alpha \cdot (p_2)_* \lambda' = (p_2)_* \delta_2^!(\alpha \times \lambda') = (p_2)_* \delta_2^!(\lambda'' \times \alpha).$$

Since  $f_1 p_1 = f_2 p_2$ , to show (3-6) it suffices to show that

$$\delta_1^!(\lambda \times \beta) = \delta_2^!(\lambda'' \times \alpha).$$

We now observe that since  $\mathcal{X}_2$  is smooth,  $\gamma_{p_2}$  is also a regular embedding in (3-4), hence  $\delta_2^! = \gamma_{p_2}^*$  (nonrefined Gysin map) by Lemma 2.1(b) (see also (3-3)); similarly,  $\delta_1^! = \gamma_{p_1}^*$ . The expression  $\gamma_{p_i}^*(x \times y)$  is also written  $x \cdot_{p_i} y$  in [Fulton 1984, Definition 8.1.1] (cf. proof of Proposition 2.8). The formula to be proven therefore becomes

$$(\psi \cdot_{p_2} \alpha) \cdot_{p_1} \beta = (\psi \cdot_{p_1} \beta) \cdot_{p_2} \alpha$$

which is [Fulton 1984, Proposition 8.1.1(b)]. □

**Remark 3.4.** There is a much more conceptual proof by interpreting both sides as compositions of correspondences: we then have

$$\langle \psi^* \alpha, \beta \rangle = (\alpha \bullet \psi) \bullet \beta = \alpha \bullet (\psi \bullet \beta) = \langle \alpha, \psi_* \beta \rangle$$

by the associativity of  $\bullet$ .

**3B. Independence from the model and functoriality.**

**Lemma 3.5.** *Let  $b \in B^{(1)}$  and  $Z = \{\bar{b}\}$  as usual. For  $\psi \in \text{CH}^*(\mathcal{X}_1 \times_B \mathcal{X}_2)$  and  $\beta \in \text{CH}_*(\mathcal{X}_{1,Z})$ , let*

$$\psi_! \beta = (p_{2,Z})_* \delta_1^! (\psi \times \beta) \in \text{CH}_*(\mathcal{X}_{2,Z}).$$

*Then:*

(a)  $(\iota_2)_* \psi_! \beta = \psi_* ((\iota_1)_* \beta)$ .

(b) *For any  $\alpha \in \text{CH}^i(\mathcal{X}_2)$ ,  $\beta \in \text{CH}_{\delta+i-1}(\mathcal{X}_Z)$  and  $\psi \in \text{CH}^{d_2}(\mathcal{X}_1 \times_B \mathcal{X}_2)$ , we have  $\langle \alpha, \psi_! \beta \rangle_b = \langle \psi^* \alpha, \beta \rangle_b$ .*

*Proof.* (a) Let us first draw the diagram of Cartesian squares underlying the coming computation:

$$\begin{array}{ccccc}
 \mathcal{X}_{2,Z} & \xleftarrow{p_{2,Z}} & \mathcal{X}_{1,Z} \times_Z \mathcal{X}_{2,Z} & \xrightarrow{(\kappa, p_{1,Z})} & (\mathcal{X}_1 \times_B \mathcal{X}_2) \times \mathcal{X}_{1,Z} \\
 \iota_2 \downarrow & & \kappa \downarrow & & 1 \times \iota_1 \downarrow \\
 \mathcal{X}_2 & \xleftarrow{p_2} & \mathcal{X}_1 \times_B \mathcal{X}_2 & \xrightarrow{\gamma_{p_1}} & (\mathcal{X}_1 \times_B \mathcal{X}_2) \times \mathcal{X}_1 \\
 & & p_1 \downarrow & & p_1 \times 1 \downarrow \\
 & & \mathcal{X}_1 & \xrightarrow{\delta_1} & \mathcal{X}_1 \times \mathcal{X}_1
 \end{array} \tag{3-7}$$

It already explains the use of  $\delta_1^!$  in the definition of  $\psi_!$ . Now

$$\begin{aligned}
 (\iota_2)_* \psi_! \beta &= (\iota_2)_* (p_{2,Z})_* \delta_1^! (\psi \times \beta) = (p_2)_* \kappa_* \delta_1^! (\psi \times \beta) \\
 &= (p_2)_* \delta_1^! (1 \times \iota_1)_* (\psi \times \beta) = (p_2)_* \delta_1^! (\psi \times (\iota_1)_* \beta) \\
 &=: (\iota^t \psi)^* ((\iota_1)_* \beta) =: \psi_* ((\iota_1)_* \beta),
 \end{aligned}$$

where the third equality follows as usual from Lemma 2.1(a).



(b) First

$$\begin{aligned} \alpha \cdot_{i_2} \psi! \beta &:= \gamma_{i_2}^!((p_{2,Z})_* \delta_1^!(\psi \times \beta) \times \alpha) \stackrel{(a)}{=} \gamma_{i_2}^!((p_{2,Z})_* \gamma_{p_1}^!(\psi \times \beta) \times \alpha) \\ &\stackrel{(b)}{=} (p_{2,Z})_* \gamma_{i_2}^!(\gamma_{p_1} \times 1)^!(\psi \times \beta \times \alpha) \\ &\stackrel{(c)}{=} (p_{2,Z})_* \gamma_{p_2}^!(\gamma_{p_1} \times 1)^!(\psi \times \beta \times \alpha) \stackrel{(d)}{=} (p_{2,Z})_* \gamma_\lambda^!(\psi \times \beta \times \alpha), \end{aligned}$$

where  $\lambda$  is the regular embedding  $\mathcal{X}_1 \times_B \mathcal{X}_2 \hookrightarrow \mathcal{X}_1 \times \mathcal{X}_2$ , so that  $\gamma_\lambda$  is the composition of the bottom row in the diagram of Cartesian squares

$$\begin{array}{ccccc} \mathcal{X}_{2,Z} & \xrightarrow{\gamma_{i_2}} & \mathcal{X}_{2,Z} \times \mathcal{X}_2 & & \\ p_{2,Z} \uparrow & & p_{2,Z} \times 1 \uparrow & & \\ \mathcal{X}_{1,Z} \times_Z \mathcal{X}_{2,Z} & \xrightarrow{\gamma_{p_2, Z \times i_2}} & (\mathcal{X}_{1,Z} \times_Z \mathcal{X}_{2,Z}) \times \mathcal{X}_2 & \xrightarrow{(\kappa, p_{1,Z}) \times 1} & (\mathcal{X}_1 \times_B \mathcal{X}_2) \times \mathcal{X}_{1,Z} \times \mathcal{X}_2 & (3-8) \\ \kappa \downarrow & & \kappa \times 1 \downarrow & & 1 \times i_1 \times 1 \downarrow \\ \mathcal{X}_1 \times_B \mathcal{X}_2 & \xrightarrow{\gamma_{p_2}} & (\mathcal{X}_1 \times_B \mathcal{X}_2) \times \mathcal{X}_2 & \xrightarrow{\gamma_{p_1} \times 1} & (\mathcal{X}_1 \times_B \mathcal{X}_2) \times \mathcal{X}_1 \times \mathcal{X}_2 \end{array}$$

Here (a) follows from Lemma 2.1(b) applied to (3-7), (b) from Lemma 2.1(a), (c) from Lemma 2.1(b) again (applied twice), and (d) from Lemma 2.1(c).

Next

$$\psi^* \alpha \cdot_{i_1} \beta := \gamma_{i_1}^!(\beta \times (p_1)_* \delta_2^!(\psi \times \alpha)) \stackrel{(a)}{=} \gamma_{i_1}^!(\beta \times (p_1)_* \gamma_{p_2}^!(\psi \times \alpha)) \stackrel{(b)}{=} (p_{1,Z})_* \gamma_{i_1}^!(1 \times \gamma_{p_2})^!(\beta \times \psi \times \alpha),$$

where (a) follows from Lemma 2.1(b) applied to (3-4) and (b) follows from Lemma 2.1(a) applied to the Cartesian square

$$\begin{array}{ccc} \mathcal{X}_{1,Z} \times_Z \mathcal{X}_{2,Z} & \xrightarrow{(p_{1,Z}, \kappa)} & \mathcal{X}_{1,Z} \times (\mathcal{X}_1 \times_B \mathcal{X}_2) \\ p_{1,Z} \downarrow & & 1 \times p_1 \downarrow \\ \mathcal{X}_{1,Z} & \xrightarrow{\gamma_{i_1}} & \mathcal{X}_{1,Z} \times \mathcal{X}_1 \end{array}$$

Since  $f_{1,Z} p_{1,Z} = f_{2,Z} p_{2,Z}$ , we are left to prove the equality

$$\gamma_\lambda^!(\psi \times \beta \times \alpha) = \gamma_{i_1}^!(1 \times \gamma_{p_2})^!(\beta \times \psi \times \alpha).$$

For this we draw the diagram of Cartesian squares, similar to (3-8):

$$\begin{array}{ccccc} \mathcal{X}_{1,Z} & \xrightarrow{t \gamma_{i_1}} & \mathcal{X}_1 \times \mathcal{X}_{1,Z} & & \\ p_{1,Z} \uparrow & & p_1 \times 1 \uparrow & & \\ \mathcal{X}_{1,Z} \times_Z \mathcal{X}_{2,Z} & \xrightarrow{(\kappa, p_{1,Z})} & (\mathcal{X}_1 \times_B \mathcal{X}_2) \times \mathcal{X}_{1,Z} & \xrightarrow{\gamma_{p_2} \times 1_{\mathcal{X}_{1,Z}}} & (\mathcal{X}_1 \times_B \mathcal{X}_2) \times \mathcal{X}_2 \times \mathcal{X}_{1,Z} \\ \kappa \downarrow & & 1 \times i_1 \downarrow & & 1 \times 1 \times i_1 \downarrow \\ \mathcal{X}_1 \times_B \mathcal{X}_2 & \xrightarrow{\gamma_{p_1}} & (\mathcal{X}_1 \times_B \mathcal{X}_2) \times \mathcal{X}_1 & \xrightarrow{\gamma_{p_2} \times 1_{\mathcal{X}_1}} & (\mathcal{X}_1 \times_B \mathcal{X}_2) \times \mathcal{X}_2 \times \mathcal{X}_1 \end{array}$$

Here the composition of the bottom row is  $\gamma_\lambda$ , up to permuting  $\mathcal{X}_1$  and  $\mathcal{X}_2$ . By Lemma 2.1(b),  $({}^t\gamma_1)^\dagger$  and  $\gamma_{p_1}^\dagger$  both compute the refined Gysin map corresponding to the arrow  $(\kappa, p_{1,Z})$ , and also  $(\gamma_{p_2} \times 1_{\mathcal{X}_{1,Z}})^\dagger = (\gamma_{p_2} \times 1_{\mathcal{X}_1})^\dagger$ ; we conclude by applying Lemma 2.1(c) to the bottom row once again.  $\square$

**Proposition 3.6.** *Let  $f_1 : \mathcal{X}_1 \rightarrow B$ ,  $f_2 : \mathcal{X}_2 \rightarrow B$  be two proper morphisms with generic fibres  $X_1, X_2$  of dimensions  $d_1, d_2$ , where  $\mathcal{X}_1$  and  $\mathcal{X}_2$  are smooth; let  $r \in \mathbb{Z}$  and let  $\gamma \in \text{CH}^{d_2+r}(X_1 \times_K X_2)$  be a Chow correspondence of degree  $r$ . Then*

$$\gamma^* \text{CH}^i(X_2)_{f_2}^0 \subseteq \text{CH}^{i+r}(X_1)_{f_1}^0 \tag{3-9}$$

for any  $i \geq 0$ . In particular,

- (i) if  $r = 0$ , we also have  $\gamma_* \text{CH}_i(X_1)_{f_2}^0 \subseteq \text{CH}_i(X_2)_{f_1}^0$ ;
- (ii) the group  $\text{CH}^i(X)_f^0$  does not depend on  $f$ .

*Proof.* First, (i) (resp. (ii)) follows from (3-9) by considering  ${}^t\gamma$  (resp. by taking  $X_1 = X_2 = X$ ,  $\gamma = \Delta_X$ ). To prove (3-9), we may assume that  $\gamma$  is the class of an integral cycle  $\Gamma \subset X_1 \times_K X_2$ .

Let  $j_i : X_i \hookrightarrow \mathcal{X}_i$  be the corresponding immersions, and  $\psi$  be the closure of  $\Gamma$  in  $\mathcal{X}_1 \times_B \mathcal{X}_2$ . By Lemma 3.1(c),

$$\gamma^* \circ j_2^* = j_1^* \circ \psi^*, \tag{3-10}$$

and it suffices to show that  $\psi^*\alpha \in \text{CH}^{i+r}(\mathcal{X}_1)^0$  for any  $\alpha \in \text{CH}^i(\mathcal{X}_2)^0$ . Formula (3-10) shows that  $j_1^*(\psi^*\alpha) \in \text{CH}_{\text{num}}^{i+r}(X_1)$ ; the other condition follows from Lemma 3.5(b).  $\square$

**Remark 3.7.** If  $B$  is projective, Lemma 3.5(a) is sufficient for the proof of Proposition 3.6 by using (2-3), as in the proof of Proposition 2.5.

**Proposition 3.8.** *The pairing (2-9) does not depend on the choice of  $f$  (we drop  $f$  from its notation from now on). Moreover, in the situation of Proposition 3.6 with  $r = 0$ , we have the identity*

$$\langle \gamma^*\alpha, \beta \rangle = \langle \alpha, \gamma_*\beta \rangle \tag{3-11}$$

for  $(\alpha, \beta) \in \text{CH}^i(X_2)^0 \times \text{CH}_{i-1}(X_1)^0$ .

*Proof.* As in the proof of Proposition 3.6, the first claim follows from the second by taking  $X_1 = X_2 = X$ ,  $\gamma = \Delta_X$ . For the second claim, we take  $\gamma$  and  $\psi$  as in the proof of Proposition 3.6. Then (3-11) follows from Lemma 3.3 applied to lifts  $\tilde{\alpha}$  and  $\tilde{\beta}$  of  $\alpha$  and  $\beta$  in  $\text{CH}^i(\mathcal{X}_2)^0$  and  $\text{CH}^{d_1-i+1}(\mathcal{X}_1)^0$ , respectively.  $\square$

### 3C. Base change.

**Proposition 3.9.** *Consider a commutative diagram*

$$\begin{array}{ccc} \mathcal{X}_1 & \xrightarrow{g} & \mathcal{X}_2 \\ f_1 \downarrow & & f_2 \downarrow \\ B_1 & \xrightarrow{\bar{g}} & B_2 \end{array} \tag{3-12}$$

where  $f_1, f_2$  satisfy the hypotheses of [Section 1](#),  $\bar{g}$  is finite surjective and  $g$  proper; we assume that the diagram of generic fibres,

$$\begin{array}{ccc} X_1 & \xrightarrow{g'} & X_2 \\ f_1 \downarrow & & f_2 \downarrow \\ \eta_1 & \xrightarrow{\bar{g}'} & \eta_2 \end{array}$$

is Cartesian (in particular,  $g$  is generically finite). Then, for all  $i \geq 0$ , one has:

- (i)  $g^* \text{CH}^i(\mathcal{X}_2)^0 \subseteq \text{CH}^i(\mathcal{X}_1)^0$ , hence  $g'^* \text{CH}^i(X_2)^0 \subseteq \text{CH}^i(X_1)^0$ .
- (ii)  $g_* \text{CH}^i(\mathcal{X}_1)^0 \subseteq \text{CH}^i(\mathcal{X}_2)^0$ , hence  $g'_* \text{CH}^i(X_1)^0 \subseteq \text{CH}^i(X_2)^0$ .
- (iii)  $(g^*)^{-1} \text{CH}^i(\mathcal{X}_1)^0 = \text{CH}^i(\mathcal{X}_2)^0$ .
- (iv) One has the identities

$$\bar{g}_* \langle g'^* \alpha, \beta' \rangle = \langle \alpha, g'_* \beta' \rangle, \tag{3-13}$$

$$\langle g'^* \alpha, g'^* \beta \rangle = \bar{g}^* \langle \alpha, \beta \rangle, \tag{3-14}$$

for any  $i \geq 0$  and any  $(\alpha, \beta, \beta') \in \text{CH}^i(X_2)^0 \times \text{CH}^{d+1-i}(X_2)^0 \times \text{CH}^{d+1-i}(X_1)^0$ .

*Proof.* (i) Write  $j_i : X_i \hookrightarrow \mathcal{X}_i$  for the inclusions. Let  $\alpha \in \text{CH}^i(\mathcal{X}_2)^0$ : then  $j_1^* g^* \alpha = g'^* j_2^* \alpha \in \text{CH}_{\text{num}}^i(X_1)$ . Next, let  $b \in B_1^{(1)}$  and  $Z = \{\bar{b}\}$ . Let  $\beta \in \text{CH}_{\delta+i-1}(\mathcal{X}_{1,Z})$ ,  $f_{1,Z} : \mathcal{X}_{1,Z} \rightarrow Z$  be the restriction of  $f_1$  and  $\iota_1 : \mathcal{X}_{1,Z} \hookrightarrow \mathcal{X}_1$  be the closed immersion: we need to prove that  $(f_{1,Z})_*(g^* \alpha \cdot \iota_1 \beta) = 0$ . Let  $T = \bar{g}(Z)$  and  $\bar{h} : Z \rightarrow T$  be the (finite surjective) projection: it suffices to show that  $\bar{h}_*(f_{1,Z})_*(g^* \alpha \cdot \iota_1 \beta) = 0 \in \text{CH}^0(T)$ . This follows from the computation

$$\begin{aligned} 0 &\stackrel{(a)}{=} (f_{2,T})_*(\alpha \cdot \iota_2 h_* \beta) = (f_{2,T})_* \gamma_{\iota_2}^! (h_* \beta \times \alpha) \\ &\stackrel{(b)}{=} (f_{2,T})_* h_* \gamma_{g \iota_1}^! (\beta \times \alpha) = \bar{h}_*(f_{1,Z})_* \gamma_{g \iota_1}^! (\beta \times \alpha) \\ &\stackrel{(c)}{=} \bar{h}_*(f_{1,Z})_* \gamma_{\iota_1}^! (1 \times g)^! (\beta \times \alpha) = \bar{h}_*(f_{1,Z})_*(g^* \alpha \cdot \iota_2 \beta), \end{aligned}$$

where  $h : \mathcal{X}_{1,Z} \rightarrow \mathcal{X}_{2,T}$  is the restriction of  $g$  and  $\iota_2$  is the inclusion  $\mathcal{X}_{2,T} \hookrightarrow \mathcal{X}_2$ , in which (a) is by hypothesis, (b) follows from [Lemma 2.1](#), and (c) follows from [\[Fulton 1984, Proposition 8.1.1\(a\)\]](#) (see comment in [\[op. cit., mid p. 134\]](#)).

(ii) The inclusion  $j_2^* g_* \text{CH}^i(\mathcal{X}_1)^0 \subseteq \text{CH}_{\text{num}}^i(X_2)$  is obtained this time from the identity  $j_2^* g_* = g'_* j_1^*$ . Next, let  $b \in B_2^{(1)}$ ,  $Z = \{\bar{b}\}$  and  $\iota_2 : \mathcal{X}_{2,Z} \hookrightarrow \mathcal{X}_2$ ,  $f_{2,Z} : \mathcal{X}_{2,Z} \rightarrow Z$  be the inclusion and the projection. Let  $\alpha \in \text{CH}^i(\mathcal{X}_1)^0$  and  $\beta \in \text{CH}_{\delta+i-1}(\mathcal{X}_{2,Z})$ : we need to prove that  $(f_{2,Z})_*(g_* \alpha \cdot \iota_2 \beta) = 0 \in \text{CH}^0(Z)$ .

Let  $T = \bar{g}^{-1}(Z)$ . Then  $\mathcal{X}_{1,T} \xrightarrow{\sim} \mathcal{X}_1 \times_{\mathcal{X}_2} \mathcal{X}_{2,Z}$ ; hence refined Gysin morphisms

$$g^! : \text{CH}_j(\mathcal{X}_{2,Z}) \rightarrow \text{CH}_j(\mathcal{X}_{1,T}).$$

By [Lemma 2.9](#), we have  $(f_{1,T})_*(\alpha \cdot_{\iota_1} g^! \beta) = 0$ , where  $\iota_1$  is the inclusion  $\mathcal{X}_{1,T} \hookrightarrow \mathcal{X}_1$ . The commutative square

$$\begin{array}{ccc} \mathcal{X}_{1,T} & \xrightarrow{h} & \mathcal{X}_{2,Z} \\ f_{1,T} \downarrow & & f_{2,Z} \downarrow \\ T & \xrightarrow{\bar{h}} & Z \end{array}$$

where  $h$  and  $\bar{h}$  are the restrictions of  $g$  and  $\bar{g}$ , gives the identity of push-forwards

$$\bar{h}_*(f_{1,T})_* = (f_{2,Z})_* h_*$$

Therefore, it suffices to prove the identity (projection formula)

$$g_* \alpha \cdot_{\iota_2} \beta = h_*(\alpha \cdot_{\iota_1} g^! \beta). \tag{3-15}$$

For this, consider the commutative diagram of Cartesian squares

$$\begin{array}{ccccc} \mathcal{X}_1 & \xrightarrow{\delta_{\mathcal{X}_1}} & \mathcal{X}_1 \times \mathcal{X}_1 & \xrightarrow{g \times 1} & \mathcal{X}_2 \times \mathcal{X}_1 \\ \iota_1 \uparrow & & \iota_1 \times 1 \uparrow & & \iota_2 \times 1 \uparrow \\ \mathcal{X}_{1,T} & \xrightarrow{\gamma_{\iota_1}} & \mathcal{X}_{1,T} \times \mathcal{X}_1 & \xrightarrow{h \times 1} & \mathcal{X}_{2,Z} \times \mathcal{X}_1 \\ h \downarrow & & & & 1 \times g \downarrow \\ \mathcal{X}_{2,Z} & & \xrightarrow{\gamma_{\iota_2}} & & \mathcal{X}_{2,Z} \times \mathcal{X}_2 \\ \iota_2 \downarrow & & & & \iota_2 \times 1 \downarrow \\ \mathcal{X}_2 & & \xrightarrow{\delta_{\mathcal{X}_2}} & & \mathcal{X}_2 \times \mathcal{X}_2 \end{array}$$

Applying [Lemma 2.1](#) to the two bottom squares yields first

$$g_* \alpha \cdot_{\iota_2} \beta := \gamma_{\iota_2}^! (1 \times g)_*(\beta \times \alpha) = h_* \gamma_{\iota_2}^! (\beta \times \alpha) = h_* \delta_{\mathcal{X}_2}^! (\beta \times \alpha).$$

We are now left to show the identity

$$\delta_{\mathcal{X}_2}^! (\beta \times \alpha) = \alpha \cdot_{\iota_1} g^! \beta := \gamma_{\iota_1}^! (g \times 1)^! (\beta \times \alpha),$$

where the right hand side stems from the top part of the diagram (with vertical arrows pointing upwards). But  $\gamma_{\iota_1}^! (g \times 1)^! (\beta \times \alpha) = \delta_{\mathcal{X}_1}^! (g \times 1)^! (\beta \times \alpha)$  and  $\delta_{\mathcal{X}_1}^! (g \times 1)^! (\beta \times \alpha) = [(g \times 1) \delta_{\mathcal{X}_1}]^! (\beta \times \alpha) = {}^t \gamma_g^! (\beta \times \alpha)$ , both by [Lemma 2.1](#). Here,  ${}^t \gamma$  denotes the transpose of a graph (graph composed with the switch of factors). Finally,  ${}^t \gamma_g^! (\beta \times \alpha) = \delta_{\mathcal{X}_2}^! (\beta \times \alpha)$  by applying once again [Lemma 2.1\(b\)](#) to the diagram of

Cartesian squares

$$\begin{array}{ccc}
 \mathcal{X}_{1,T} & \xrightarrow{(h, t_1)} & \mathcal{X}_{2,Z} \times \mathcal{X}_1 \\
 t_1 \downarrow & & t_2 \times 1 \downarrow \\
 \mathcal{X}_1 & \xrightarrow{t \gamma_g} & \mathcal{X}_2 \times \mathcal{X}_1 \\
 g \downarrow & & 1 \times g \downarrow \\
 \mathcal{X}_2 & \xrightarrow{\delta_{\mathcal{X}_2}} & \mathcal{X}_2 \times \mathcal{X}_2
 \end{array}$$

(iii) This follows from (i) and (ii) by the projection formula  $g_* g^* = \text{deg}(g)$  (generic degree), and [Lemma 2.4](#).

(iv) This follows from the special case  $B_1 = B_2, \bar{g} = 1_B$  in (i) or (ii).

In (iv), the identities can be checked on the level of  $\mathcal{X}_1$  and  $\mathcal{X}_2$ . The first, (3-13), is an easy consequence of the projection formula. Let us prove (3-14). The diagram of Cartesian squares

$$\begin{array}{ccccc}
 \mathcal{X}_1 & \xrightarrow{(g, f_1)^B} & \mathcal{X}_2 \times_{B_2} B_1 & \xrightarrow{1 \times_{B_2} \bar{g}} & \mathcal{X}_2 \\
 \parallel \downarrow & & \text{inj} \downarrow & & (1, f_2) \downarrow \\
 \mathcal{X}_1 & \xrightarrow{(g, f_1)} & \mathcal{X}_2 \times_k B_1 & \xrightarrow{1 \times \bar{g}} & \mathcal{X}_2 \times_k B_2
 \end{array}$$

together with [Lemma 2.1](#) (b), and (c) gives a factorisation of  $g^*$  into a composition of refined Gysin morphisms

$$g^* = (g, f_1)^!(1 \times \bar{g})^! \tag{3-16}$$

Next, [\[Fulton 1984, Example 8.1.7\]](#) applied to the left square with  $x = [\mathcal{X}_1]$  and  $y = (1 \times \bar{g})^! z$  for some  $z \in \text{CH}^*(\mathcal{X}_2)$  yields via [\[Fulton 1984, Proposition 8.1.2\(b\)\]](#) the identity

$$(g, f_1)_*^B (g, f_1)^! y = (g, f_1)_*^B [\mathcal{X}_1] \cdot y = y; \tag{3-17}$$

indeed,  $(g, f_1)^B$  maps  $\mathcal{X}_1$  birationally onto an irreducible component of  $\mathcal{X}_2 \times_{B_2} B_1$ , and the other irreducible components have support away from  $\eta_2$ , hence have smaller dimensions. Taking  $z = \alpha \cdot \beta$  for  $(\alpha, \beta) \in \text{CH}^i(\mathcal{X}_2)^0 \times \text{CH}^{d+1-i}(\mathcal{X}_2)^0$ , we get

$$\begin{aligned}
 \langle g^* \alpha, g^* \beta \rangle &= (f_1)_* (g^* \alpha \cdot g^* \beta) = (f_1)_* g^* (\alpha \cdot \beta) \\
 &\stackrel{(3-16)}{=} (f_2 \times 1)_* (g, f_1)_*^B (g, f_1)^! (1 \times \bar{g})^! (\alpha \cdot \beta) \\
 &\stackrel{(3-17)}{=} (f_2 \times 1)_* (1 \times \bar{g})^! (\alpha \cdot \beta) \\
 &= \bar{g}^* (f_2)_* (\alpha \cdot \beta) = \bar{g}^* \langle \alpha, \beta \rangle,
 \end{aligned}$$

where the last but one equality follows once again from [Lemma 2.1](#). This readily implies (3-14). □

**Remark 3.10.** In [Proposition 3.9](#), suppose that  $\bar{g}$  is only an alteration. I cannot prove (i). On the other hand, (ii) holds with the same proof, as well as (iv) for  $(\alpha, \beta) \in \text{CH}^i(X_2)^0 \times \text{CH}^{d+1-i}(X_2)^0$  such that

$(g^*\alpha, g^*\beta) \in \text{CH}^i(X_1)^0 \times \text{CH}^{d+1-i}(X_1)^0$ . This is not very important, in view of [Remarks 1.2\(b\)](#) (see proof of [Proposition 3.11](#)).

**3D. Structure of  $\text{CH}^i(X)/\text{CH}^i(X)^0$ .**

**Proposition 3.11.** *The groups  $\text{CH}^i(\mathcal{X})/\text{CH}^i(\mathcal{X})^0$  and  $\text{CH}^i(X)/\text{CH}^i(X)^0$  are finitely generated.*

*Proof.* It suffices to show the first claim. We proceed in several steps.

(1) Suppose  $B'$  is an open subset of  $B$ , let  $\mathcal{X}' = \mathcal{X} \times_B B'$  and let  $\lambda : \mathcal{X}' \rightarrow \mathcal{X}$  be the corresponding open immersion. Then  $\text{CH}^i(\mathcal{X})^0 \subseteq (\lambda^*)^{-1} \text{CH}^i(\mathcal{X}')^0$ , with equality if  $B - B'$  has codimension  $\geq 2$ . Therefore the claim for  $\mathcal{X}$  implies the claim for  $\mathcal{X}'$ , and conversely in the latter case.

(2)  $B$  is projective: this follows from [Proposition 2.5](#).

(3) In general, let  $\bar{B}$  be a compactification of  $B$  and  $\bar{\mathcal{X}} \xrightarrow{\bar{f}} \bar{B}$  a projective morphism extending  $f$  (in the sense that  $\mathcal{X} = \bar{\mathcal{X}} \times_{\bar{B}} B$ ).

- By [\[de Jong 1996, Theorem 4.1\]](#), alter  $\bar{B}$  into a smooth projective  $k$ -variety  $\bar{B}_1$ .
- Let  $K_1 = k(\bar{B}_1)$  (a finite extension of  $K$ ), and let  $\bar{\mathcal{X}}'$  be the closure of  $X \otimes_K K_1$  in  $\bar{\mathcal{X}} \times_{\bar{B}} \bar{B}_1$ . Again by [\[de Jong 1996, Theorem 4.1\]](#), alter  $\bar{\mathcal{X}}'$  into a smooth projective  $k$ -variety  $\bar{\mathcal{X}}_1$ . We are now in the situation of (2).
- Let  $B_1 = B \times_{\bar{B}} \bar{B}_1$  and  $\mathcal{X}_1 = B_1 \times_{\bar{B}_1} \bar{\mathcal{X}}_1$ .
- By [Remarks 1.2\(b\)](#), the alteration  $B_1 \rightarrow B$  becomes flat, hence finite, after removing from  $B$  a closed subset  $F$  of codimension  $\geq 2$ . Let  $B' = B - F$  and  $\mathcal{X}', B'_1, \mathcal{X}'_1$  be the corresponding base changes of  $\mathcal{X}, B_1$  and  $\mathcal{X}_1$ .

By (2), the claim is true for  $\bar{\mathcal{X}}_1$ ; therefore it is also true for  $\mathcal{X}'_1$  by (1). By [Proposition 3.9](#) (i), (ii), the projection  $\mathcal{X}'_1 \rightarrow \mathcal{X}'$  induces maps between  $\text{CH}^i(\mathcal{X}')/\text{CH}^i(\mathcal{X}')^0$  and  $\text{CH}^i(\mathcal{X}'_1)/\text{CH}^i(\mathcal{X}'_1)^0$ , whose composition is multiplication by  $[K_1 : K]$ . Since  $\text{CH}^i(\mathcal{X}')/\text{CH}^i(\mathcal{X}')^0$  is torsion-free by [Lemma 2.4](#), it is finitely generated, and so is  $\text{CH}^i(\mathcal{X})/\text{CH}^i(\mathcal{X})^0$  by reapplying (1). □

**Remark 3.12.** [Proposition 2.6](#) gives a more direct proof of [Proposition 3.11](#) in characteristic 0, by the comparison theorem between Betti and  $l$ -adic cohomology.

**3E. A vanishing result.** Let  $l$  be a prime number invertible in  $k$ . For any smooth  $k$ -variety  $V$ , there are cycle class maps with values in Jannsen’s continuous étale cohomology

$$\text{cl}^i : \text{CH}^i(V) \rightarrow H_{\text{cont}}^{2i}(V, \mathbb{Z}_l(i))$$

which are compatible with pull-backs, push-forwards and products [\[Jannsen 1988, \(3.25\) and \(6.14\)\]](#).<sup>3</sup>

<sup>3</sup>Strangely, [\[Jannsen 1988, \(3.25\)\]](#) only mentions push-forwards for closed immersions, but the case of a general proper morphism is proven in the same way.

**Lemma 3.13.** *Suppose  $k$  finitely generated. Then the composition of  $\text{cl}^1$  with the projection*

$$H_{\text{cont}}^2(V, \mathbb{Z}_l(1)) \rightarrow H_{\text{cont}}^2(V, \mathbb{Z}_l(1))/H_{\text{cont}}^2(k, \mathbb{Z}_l(1))$$

*has finite kernel.*

*Proof.* By construction of  $\text{cl}^i$ , there is a commutative diagram

$$\begin{array}{ccc} \text{CH}^1(V) & \xrightarrow{\alpha} & \text{CH}^1(V)^\wedge \\ \text{cl}^1 \downarrow & & (\text{cl}^1)^\wedge \downarrow \\ H_{\text{cont}}^2(V, \mathbb{Z}_l(1)) & \longrightarrow & \varprojlim H^2(V, \mu_l^n) \end{array}$$

where the bottom map is part of the Milnor exact sequence of [Jannsen 1988, (3.16)] and  $\text{CH}^1(V)^\wedge$  is the  $l$ -adic completion of  $\text{CH}^1(V)$ . The Kummer exact sequences imply the injectivity of  $(\text{cl}^1)^\wedge$ . Since  $k$  is finitely generated,  $\text{CH}^1(V)$  is a finitely generated abelian group, which implies that  $\alpha$  has finite kernel of order prime to  $l$ . Hence the same holds for  $\text{cl}^1$ .

On the other hand, the choice of a 0-cycle of nonzero degree on  $V$  (e.g., a closed point), plus transfer, provide a map  $\rho : H_{\text{cont}}^2(V, \mathbb{Z}_l(1)) \rightarrow H_{\text{cont}}^2(k, \mathbb{Z}_l(1))$  such that the composition

$$H_{\text{cont}}^2(k, \mathbb{Z}_l(1)) \rightarrow H_{\text{cont}}^2(V, \mathbb{Z}_l(1)) \xrightarrow{\rho} H_{\text{cont}}^2(k, \mathbb{Z}_l(1))$$

is multiplication by some integer  $m > 0$ . Since  $\text{CH}^1(k) = 0$ , the naturality of the cycle class map implies that  $\rho \circ \text{cl}^1 = 0$ . Hence the lemma. □

The following proposition will be used in the proof of Proposition 6.8.

**Proposition 3.14.** *Let  $(\alpha, \beta) \in \text{CH}^i(\mathcal{X}) \times \text{CH}^{d+1-i}(\mathcal{X})$ . Consider the pairing (1-2). As in Section 2C, let  $\text{CH}_l^i(\mathcal{X})$  be the kernel of the geometric cycle class map. If  $(\alpha, \beta) \in \text{CH}_l^i(\mathcal{X}) \times \text{CH}_l^{d+1-i}(\mathcal{X})$ , then  $\langle \alpha, \beta \rangle$  is torsion.*

*Proof.* We may assume  $k$  to be the perfect closure of a finitely generated field. We use the spectral sequences of [Jannsen 1988, Theorem (3.3)]

$$E_2^{p,q} = H_{\text{cont}}^p(k, H^q(V_{\bar{k}}, \mathbb{Z}_l(n))) \Rightarrow H_{\text{cont}}^{p+q}(V, \mathbb{Z}_l(n)).$$

They are compatible with the action of correspondences, in particular with products and push-forwards. Thus, if  $F^\bullet H_{\text{cont}}$  is the filtration on  $H_{\text{cont}}$  induced by the spectral sequence, we have

$$\begin{aligned} \text{cl}^1(f_*(\alpha \cdot \beta)) &= f_* \text{cl}^{d+1}(\alpha \cdot \beta) \\ &= f_*(\text{cl}^i(\alpha) \cup \text{cl}^{d+1-i}(\beta)) \in F^2 H_{\text{cont}}^2(B, \mathbb{Z}_l(1)) = \text{Im}(H_{\text{cont}}^2(k, \mathbb{Z}_l(1)) \rightarrow H_{\text{cont}}^2(B, \mathbb{Z}_l(1))) \end{aligned}$$

if  $(\alpha, \beta) \in \text{CH}_l^i(\mathcal{X}) \times \text{CH}_l^{d+1-i}(\mathcal{X})$ . We conclude by Lemma 3.13. □

**Question 3.15.** When  $B$  is projective, can one prove Proposition 3.14 with  $\text{CH}_l$  replaced by  $\text{CH}_{\text{num}}$ , without assuming the standard conjectures?

**3F. Local height pairing.** In this context, there is not much to say. Let  $f$  be as in Section 1. Let  $C_1 \in \mathcal{Z}^i(X)$ ,  $C_2 \in \mathcal{Z}^{d+1-i}(X)$  be two integral cycles with disjoint supports. Let  $\mathcal{C}_i$  be the closure of  $C_i$  in  $\mathcal{X}$ ; then  $\mathcal{C}_1 \times_{\mathcal{X}} \mathcal{C}_2$  has support in  $\mathcal{X}_Z$  for some proper closed subset  $Z$  of  $B$ , whence a refined intersection product [Fulton 1984, §8.1],

$$\mathcal{C}_1 \cdot \mathcal{C}_2 \in \text{CH}_{\delta-1}(\mathcal{X}_Z).$$

Given the isomorphism

$$\text{CH}_{\delta-1}(Z) \xrightarrow{\sim} \bigoplus_{b \in Z \cap B^{(1)}} \mathbb{Z},$$

the class  $(f_Z)_*(\mathcal{C}_1 \cdot \mathcal{C}_2)$  defines a divisor on  $B$ , whose class in  $\text{Pic}(B) = \text{CH}^1(B)$  is obviously  $\langle \mathcal{C}_1, \mathcal{C}_2 \rangle$  (cf. [Beilinson 1987, Lemma 2.0.1]). One may then extend by bilinearity and get an expression of  $\langle \cdot, \cdot \rangle$  as the class of a divisor.

We leave it to the interested reader to refine Lemma 3.3 to this local height pairing in the style of [Bloch 1984, (A.2)].

#### 4. Extension to the general case

Let  $X$  be regular, connected and proper over  $K$ . In the previous section, we defined subgroups  $\text{CH}^i(X)^0 \subset \text{CH}^i(X)$  and pairings (1) assuming the existence of a  $k$ -smooth model  $\mathcal{X}$  of  $X$ , proper over  $B$ .

##### 4A. Characteristic 0.

**Proposition 4.1.** *Assuming resolution of singularities à la Hironaka, a smooth model always exists. This is the case in particular if  $\text{char } k = 0$ , or if  $d + \delta \leq 3$  [Cossart and Piltant 2009].*

*Proof.* Start from an integral proper model  $f : \mathcal{X} \rightarrow B$  of  $X/K$ . Let  $U \subseteq \mathcal{X}$  be the regular locus of  $\mathcal{X}/k$ : it is open [EGA IV<sub>2</sub> 1965, corollaire 6.12.6] and since  $X$  is regular, we have  $X \subset U$ . By hypothesis, we may find  $\mathcal{X}_1$  regular over  $k$  and a projective morphism  $\pi : \mathcal{X}_1 \rightarrow \mathcal{X}$  such that  $\pi|_{\pi^{-1}(U)} : \pi^{-1}(U) \rightarrow U$  is an isomorphism. Then the immersion  $X \hookrightarrow \mathcal{X}$  lifts to  $X \hookrightarrow \mathcal{X}_1$ , and  $\mathcal{X}_1$  is the desired smooth model of  $X$  (since  $k$  is assumed to be perfect). □

**4B. Positive characteristic.** Here we cannot directly use de Jong’s theorem [1996] to replace Hironaka resolution, because there is no control in this theorem on the centre of the alteration. Instead we must proceed more carefully.

**Definition 4.2.** Let  $X$  be an integral proper  $K$ -scheme.

- (a)  $X$  is *good* (relatively to  $B$ ) if it admits a  $k$ -regular proper model  $\mathcal{X} \xrightarrow{f} B$ . (In particular,  $X$  is then regular.)
- (b) A  $K$ -morphism  $\pi : X_1 \rightarrow X$  is *admissible* if  $X_1$  is good.
- (c) We set

$$\text{CH}^i(X)^0 = \{\alpha \in \text{CH}^i(X) \mid \pi^* \alpha \in \text{CH}^i(X_1)^0 \ \forall \pi \text{ admissible}\}.$$



- Lemma 4.3.** (a) For any  $X$  as in [Definition 4.2](#), admissible alterations exist; in particular  $\mathrm{CH}^i(X)^0 \neq \emptyset$ .  
 (b) If  $X$  is good,  $\mathrm{CH}^i(X)^0$  agrees with the subgroup of [Proposition 3.6\(ii\)](#).  
 (c) Given two admissible morphisms  $\pi_i : X_i \rightarrow X$ , there exists an admissible morphism  $\pi_3 : X_3 \rightarrow X$  factoring through  $\pi_1$  and  $\pi_2$ .  
 (d) If  $X$  is regular, we have  $\mathrm{CH}^i(X)^0 \subseteq \mathrm{CH}_{\mathrm{num}}^i(X)$ .

*Proof.* (a) This follows from [\[de Jong 1996, Theorem 4.1\]](#) applied to a (not necessarily smooth) model.

(b) This follows from [Proposition 3.6](#).

(c) Let  $\mathcal{X}, \mathcal{X}_1, \mathcal{X}_2$  be  $B$ -proper models of  $X, X_1$  and  $X_2$ . Taking the graphs of the rational maps  $\pi_i : \mathcal{X}_i \dashrightarrow \mathcal{X}$ , we may assume these to be  $B$ -morphisms. Applying [\[de Jong 1996, Theorem 4.1\]](#) again to an irreducible component of  $\mathcal{X}_1 \times_{\mathcal{X}} \mathcal{X}_2$  dominant over  $B$ , we get a  $k$ -smooth  $\mathcal{X}$ -scheme  $\mathcal{X}_3$ , projective over  $B$  and mapping to  $\mathcal{X}_1$  and  $\mathcal{X}_2$ , whose generic fibre  $X_3$  maps to  $X_1$  and  $X_2$  over  $X$ .

(d) Let  $\alpha \in \mathrm{CH}^i(X)^0$  and  $\beta \in \mathrm{CH}^{d-i}(X)$ . Choose an admissible  $\pi$ . Writing  $[\ , \ ]$  for the intersection product, we have  $[\pi_{\eta}^* \alpha, \pi_{\eta}^* \beta] = 0$  by definition of  $\mathrm{CH}^i(X_1)^0$ , hence  $[\alpha, \beta] = 0$ . □

To go further, we need to invert  $p$  in characteristic  $p$ ; this is the object of the next subsections.

**4C. The category  $\mathbf{Ab} \otimes R$ , where  $R$  is a subring of  $\mathbb{Q}$ .** (See also [\[Barbieri-Viale and Kahn 2016, Appendix B\]](#).) This category has two equivalent descriptions:

- It is the localisation of the category  $\mathbf{Ab}$  of abelian groups with respect to the Serre subcategory of abelian groups killed by some integer invertible in  $A$ ; in particular,  $\mathbf{Ab} \otimes R$  is abelian and the localisation functor  $\mathbf{Ab} \rightarrow \mathbf{Ab} \otimes R$  is exact.
- It has the same objects as  $\mathbf{Ab}$ , while morphisms are those of  $\mathbf{Ab}$  tensored with  $A$ .

If  $R = \mathbb{Z}[1/p]$ , we shall abbreviate  $\mathbf{Ab} \otimes R$  to  $\mathbf{Ab}[1/p]$ .

**Lemma 4.4.** The tensor product of  $\mathbf{Ab}$  induces a tensor structure on  $\mathbf{Ab} \otimes R$ , still denoted by  $\otimes$ .

(This allows us to talk of a “pairing in  $\mathbf{Ab} \otimes R$ ”.)

*Proof.* It suffices to show that, if  $f \in \mathbf{Ab}(A, B)$  becomes invertible in  $\mathbf{Ab} \otimes R$  (i.e.,  $\mathrm{Ker} f, \mathrm{Coker} f$  have  $p$ -power exponent), the same holds for  $f \otimes 1_C$  for any  $C \in \mathbf{Ab}$ . By considering the image of  $f$ , we may treat separately the cases where  $f$  is injective and  $f$  is surjective. Both hold because, if  $G \in \mathbf{Ab}$  has  $p$ -power exponent, so do  $G \otimes C$  and  $\mathrm{Tor}(G, C)$  for any  $C \in \mathbf{Ab}$ . □

**Remarks 4.5.** (a) Let  $A, B$  be two abelian groups. By definition, a morphism in  $(\mathbf{Ab} \otimes R)(A, B) = \varinjlim_{N \neq 0} \mathbf{Ab}(A, B)$  is represented by a pair  $(\varphi, N)$  with  $\varphi : A \rightarrow B$  and  $N$  an integer invertible in  $R$ ; two pairs  $(\varphi_1, N_1)$  and  $(\varphi_2, N_2)$  are equivalent if there exist two such integers  $d_1, d_2$  such that  $d_1 N_1 = d_2 N_2 =: N_3$  and  $(d_1 \varphi_1, N_3) = (d_2 \varphi_2, N_3)$ . We get a well-defined homomorphism

$$\rho : (\mathbf{Ab} \otimes R)(A, B) \rightarrow \mathbf{Ab}(A, B \otimes R)$$

by sending a pair  $(\varphi, N)$  to  $\psi; = N^{-1}\varphi$ ; its image is contained in the subgroup formed of those homomorphisms  $\psi : A \rightarrow B \otimes R$  such that  $\psi(A) \subseteq N^{-1}\bar{B}$  for some  $N \neq 0$ , with  $\bar{B} = B/\text{torsion}$ . If  $B$  is torsion-free,  $\rho$  is injective with the above image.

(b) In any category, the commutativity of a diagram (i.e., the equality of two arrows) is equivalent to the commutativity of a family of diagrams of sets, thanks to Yoneda’s lemma. In the category of modules over a ring  $R$ , one can test such commutativity on elements, because the  $R$ -module  $R$  is a generator.

In the sequel, we shall extend identities such as (3-11), (3-13) and (3-14) to  $\mathbf{Ab} \otimes R$ . However this category is not Grothendieck (note that abelian groups with finite exponent are not closed under infinite direct sums), so reasoning with “elements” is abusive. Writing out the above identities as commutative diagrams in  $\mathbf{Ab}$  is straightforward, but cumbersome. (For example, (3-11) means that two homomorphisms from  $\text{CH}^{d_2}(X_1 \times_K X_2) \otimes \text{CH}^i(X_2)^0 \otimes \text{CH}_{i-1}(X_1)^0$  to  $\text{CH}^1(B)$  agree.) We shall therefore sometimes make the abuse of talking of such identities in  $\mathbf{Ab} \otimes R$  when we mean the corresponding commutative diagrams.

In Theorem 4.14, we shall use a local-to-global result for these localisations (Corollary 4.8 below).

**Theorem 4.6.** *Let  $H$  be a module over an integral domain  $R$  with quotient field  $Q$ . Suppose given, for each maximal ideal  $\mathfrak{m} \subset R$ , an element  $f_{\mathfrak{m}} \in H_{\mathfrak{m}}$ , all of which become equal in  $Q \otimes_R H$ . Then there exists at most one element  $f \in H$  which becomes equal to  $f_{\mathfrak{m}}$  in  $H_{\mathfrak{m}}$  for every  $\mathfrak{m}$ ;  $f$  exists provided*

- (i)  $H$  is torsion free, or
- (ii)  $R$  is Noetherian and  $S = \text{Supp}(M_{\text{tors}})$  is a finite set of maximal ideals.

(Counterexample without Hypothesis (ii):  $R = \mathbb{Z}$ ,  $H = \bigoplus_{\mathfrak{m}} \mathbb{Z}/\mathfrak{m}$ ,  $f_{\mathfrak{m}} = 1_{\mathfrak{m}}$ .)

*Proof. Uniqueness.* Let  $f, f'$  verifying the condition. Then  $f$  and  $f'$  become equal in  $H_{\mathfrak{m}}$  for all  $\mathfrak{m}$ . This means that, for every  $\mathfrak{m}$ , there exists  $M_{\mathfrak{m}} \in R - \mathfrak{m}$  such that  $M_{\mathfrak{m}}(f - f') = 0$ . Since the  $M_{\mathfrak{m}}$  generate  $R$  as an ideal, we get  $f = f'$ .

*Existence.* We may write  $f_{\mathfrak{m}} = r_{\mathfrak{m}}^{-1}\tilde{f}_{\mathfrak{m}}$  with  $\tilde{f}_{\mathfrak{m}} \in H$  and  $r_{\mathfrak{m}} \in R - \mathfrak{m}$ ; again, the  $r_{\mathfrak{m}}$  generate the unit ideal of  $R$ . In case (i), if  $g \in Q \otimes_R H$  is the common value of the  $f_{\mathfrak{m}}$ , then  $r_{\mathfrak{m}}g \in H$  for all  $\mathfrak{m}$ ; if  $(a_{\mathfrak{m}})$  is a family of elements of  $R$  with finite support such that  $\sum a_{\mathfrak{m}}r_{\mathfrak{m}} = 1$ , then  $g = \sum a_{\mathfrak{m}}r_{\mathfrak{m}}g \in H$ .

In case (ii), write  $T = H_{\text{tors}}$  for notational simplicity. Considering  $H/T$ , we find  $f_0$  such that  $1_{\mathfrak{m}} \otimes f_0 - f_{\mathfrak{m}}$  is torsion for all  $\mathfrak{m}$ , hence is 0 for  $\mathfrak{m} \notin S$ .

**Claim 4.7.** *The monomorphism  $T \mapsto \prod_{\mathfrak{m} \in S} T_{\mathfrak{m}}$  is surjective.*

*Proof.* For each  $\mathfrak{m} \in S$ , let  $T^{\mathfrak{m}} = \text{Ker}(T \rightarrow \prod_{\mathfrak{m}' \neq \mathfrak{m}} T_{\mathfrak{m}'})$ : we must show that  $T = \sum T^{\mathfrak{m}}$ . Let  $t \in T$ ; by assumption, the radical of  $\text{Ann}(t)$  (the annihilator of  $t$ ) is of the form  $\prod_{\mathfrak{m} \in S'} \mathfrak{m}$  for a subset  $S'$  of  $S$ . By [Bourbaki 1985, IV.2.5, proposition 9],  $R(t) = R/\text{Ann}(t)$  is Artinian, hence  $R(t) \xrightarrow{\sim} \prod_{\mathfrak{m} \in S'} R(t)_{\mathfrak{m}}$  [loc. cit., corollaire 1]; equivalently,  $Rt \xrightarrow{\sim} \prod_{\mathfrak{m} \in S'} (Rt)_{\mathfrak{m}}$ , which shows that  $t \in \sum T^{\mathfrak{m}}$ . □

Coming back to the proof of case (ii), the claim yields an element  $t \in T$  such that  $t_{\mathfrak{m}} = 1_{\mathfrak{m}} \otimes f_0 - f_{\mathfrak{m}}$  for all  $\mathfrak{m} \in S$ ; then  $f = f_0 - t$  yields the desired element. □

**Corollary 4.8.** *Let  $A, B \in \mathbf{Ab}$  and  $R$  be a subring of  $\mathbb{Q}$ . Suppose given, for each prime number  $l$  not invertible in  $R$ , a morphism  $f_l : A \rightarrow B$  in  $\mathbf{Ab} \otimes \mathbb{Z}_{(l)}$ , all of which become equal in  $\mathbf{Ab} \otimes \mathbb{Q}$ . Then there exists at most one morphism  $f : A \rightarrow B$  in  $\mathbf{Ab} \otimes R$  which becomes equal to  $f_l$  in  $\mathbf{Ab} \otimes \mathbb{Z}_{(l)}$  for every  $l$ ;  $f$  exists provided  $B$  is  $l$ -torsion free for almost all  $l$  not invertible in  $R$ .*

*Proof.* Apply [Theorem 4.6](#) to  $H = \text{Hom}(A, B) \otimes R$ , noting that the hypothesis on  $B$  implies the hypothesis on  $H$ . □

**4D.  $p$ -covers.**

**Definition 4.9.** Let  $X$  be an integral proper  $K$ -scheme. A  $p$ -cover of  $X$  is a finite family  $(\pi_l : X_l \rightarrow X)$ , indexed by prime numbers  $l \neq p$  and such that

- (i) for each  $l$ ,  $\pi_l$  is an admissible alteration of generic degree  $d_l$  prime to  $l$ ;
- (ii)  $\text{gcd}_l(d_l)$  is a power of  $p$ .

**Proposition 4.10.** (a)  $p$ -covers exist.

(b) *Given two  $p$ -covers  $(\pi_l), (\pi'_l)$ , there exists a third  $p$ -cover  $(\pi''_l)$  such that, for each  $l$ ,  $\pi''_l$  factors through  $\pi_l$  and  $\pi'_l$ .*

(c) *Given a  $p$ -cover  $(\pi_l)$  and an admissible morphism  $f_1 : X_1 \rightarrow X$ , there exists a  $p$ -cover  $(\pi_{1,l})$  of  $X_1$  such that the composition  $X_{1,l} \rightarrow X_1 \rightarrow X$  factors through  $X_l$  for each  $l$ .*

*Proof.* (a) We use Gabber’s refinement of de Jong’s alteration theorem [[Illusie and Temkin 2014](#), Theorem 2.1]: given a model  $\mathcal{X}$  of  $X$  and a prime number  $l \neq p$ , we may find an alteration  $\mathcal{X}_l \rightarrow \mathcal{X}$  with  $\mathcal{X}_l$  regular (hence smooth over  $k$ ) and of generic degree  $d_l$  prime to  $l$ ; the induced alteration  $\pi_l : X_l \rightarrow X$  is then admissible of generic degree  $d_l$ . Considering the other prime divisors of  $d_l$  different from  $p$ , we may find a finite number of  $l$  and  $\pi_l$  such that the gcd of the  $d_l$  is a power of  $p$ .

(b) and (c) These are proven similarly to (a). □

**4E. The refined height pairing (characteristic  $p$ ).**

**Definition 4.11.** We set

$$\text{CH}^i(X)^{[0]} = \{\alpha \in \text{CH}^i(X) \mid \exists s \geq 0 : p^s \alpha \in \text{CH}^i(X)^0\}.$$

**Proposition 4.12.** (a) *If  $X$  is regular,  $\text{CH}^i(X)/\text{CH}^i(X)^0$  is an extension of a finitely generated abelian group by a torsion group of  $p$ -power exponent, and  $\text{CH}^i(X)/\text{CH}^i(X)^{[0]}$  is finitely generated with prime-to- $p$  torsion.*

(b) *Let  $(\pi_l)$  be a  $p$ -cover of  $X$ , and let  $\alpha \in \text{CH}^i(X)$ . Then  $\alpha \in \text{CH}^i(X)^{[0]}$  if and only if  $\pi_l^* \alpha \in \text{CH}^i(X_l)^{[0]}$  for each  $l$ .*

(c) *Propositions 3.6 and 3.9 (i), (ii), (iii) extend to all regular  $X$  after replacing  $\text{CH}^i(X)^0$  by  $\text{CH}^i(X)^{[0]}$ .*

*Proof.* (a) Given a  $p$ -cover  $(\pi_l)$ , since  $(\pi_l)_*\pi_l^*$  is multiplication by  $d_l$  for each  $l$ ,  $\text{Ker}(\text{CH}^i(X)/\text{CH}^i(X)^0 \rightarrow \prod_l \text{CH}^i(X_l)/\text{CH}^i(X_l)^0)$  is killed by a power of  $p$ , say  $p^s$ , and the first claim follows from [Proposition 3.11](#). The second follows by definition of  $\text{CH}^i(X)^{[0]}$ .

(b) The condition is necessary by definition; the converse follows from [Proposition 4.10](#) (c), as in (a).

(c) Let  $X_1, X_2, \gamma$  be as in [Proposition 3.6](#). To prove (3-9), we must show that  $\pi^*\gamma^*\text{CH}^i(X_2)^0 \subseteq \text{CH}^{i+r}(X_1)^0$  for any admissible  $\pi : X_1' \rightarrow X_1$ ; replacing  $\gamma$  by  $\gamma \circ \pi$ , we may assume that  $X_1$  is good and  $\pi = 1_{X_1}$ . Choose a  $p$ -cover  $(\pi_l)$  of  $X_2$ . For  $\alpha \in \text{CH}^i(X_2)^{[0]}$ , we have  $\pi_l^*\alpha \in \text{CH}^i(X_l)^{[0]}$ , hence

$$d_l\gamma^*\alpha = (\gamma^*(\pi_l)_*)\pi_l^*\alpha \in \text{CH}^{i+r}(X_1)^{[0]}$$

for all  $l$  thanks to [Proposition 3.6](#), hence  $p^s\gamma^*\alpha \in \text{CH}^{i+r}(X_1)^{[0]}$  and finally  $\gamma^*\alpha \in \text{CH}^{i+r}(X_1)^{[0]}$  as desired. The cases in [Proposition 3.9](#) are treated similarly. □

**Lemma 4.13.** *Let  $\pi : X_1 \rightarrow X$  be an admissible alteration, of generic degree  $d$  prime to  $l$ , where  $l \neq p$ . Then the morphism in  $\mathbf{Ab} \otimes \mathbb{Z}_{(l)}$*

$$\langle \cdot, \cdot \rangle_{(l)} : \text{CH}^i(X)^{[0]} \otimes \text{CH}^{d+1-i}(X)^{[0]} \xrightarrow{(\pi^* \otimes \pi^*)} \text{CH}^i(X_1)^{[0]} \otimes \text{CH}^{d+1-i}(X_1)^{[0]} \xrightarrow{d^{-1}\langle \cdot, \cdot \rangle} \text{CH}^1(B)$$

*does not depend on the choice of  $\pi$ , and coincides with  $\langle \cdot, \cdot \rangle$  if  $X$  is good. For two prime numbers  $l, l' \neq p$ , we have  $\langle \cdot, \cdot \rangle_{(l)} = \langle \cdot, \cdot \rangle_{(l')}$  in  $\mathbf{Ab} \otimes \mathbb{Q}$ .*

*Proof.* Let  $\pi' : X_1' \rightarrow X$  another such alteration, with generic degree  $d'$ . By [Proposition 4.10\(c\)](#) applied to an irreducible component of  $X_1 \times X X_1'$  dominating  $X$ , we can find admissible alterations  $X_1'' \xrightarrow{\rho} X_1, X_1'' \xrightarrow{\rho'} X_1'$  of generic degrees  $\delta, \delta'$  such that  $\pi\rho = \pi'\rho'$ , hence  $\delta d = \delta' d'$ . Using elements to clarify the argument, we have for  $(\alpha, \beta) \in \text{CH}^i(X)^{[0]} \times \text{CH}^{d+1-i}(X)^{[0]}$ ,

$$d^{-1}\langle \pi^*\alpha, \pi^*\beta \rangle = d^{-1}\delta^{-1}\langle \rho^*\pi^*\alpha, \rho^*\pi^*\beta \rangle = d'^{-1}\delta'^{-1}\langle \rho'^*\pi'^*\alpha, \rho'^*\pi'^*\beta \rangle = d'^{-1}\langle \pi'^*\alpha, \pi'^*\beta \rangle,$$

where we used (3-11) and the identities  $\rho_*\rho^* = \delta, \rho'_*\rho'^* = \delta'$ . The second claim follows by taking  $\pi = 1_X$ . For the third claim, we argue similarly by using an admissible alteration covering two admissible alterations of generic degrees prime to  $l$  and  $l'$ . □

**Theorem 4.14.** (a) *There exists a unique pairing*

$$\langle \cdot, \cdot \rangle : \text{CH}^i(X)^{[0]} \otimes \text{CH}^{d+1-i}(X)^{[0]} \rightarrow \text{CH}^1(B) \tag{4-1}$$

*in  $\mathbf{Ab}[1/p]$  which coincides with  $\langle \cdot, \cdot \rangle_{(l)}$  in  $\mathbf{Ab} \otimes \mathbb{Z}_{(l)}$  for each  $l$ .*

(b) *The identities of Propositions 3.8 (see [Remarks 4.5\(b\)](#)) and 3.9(iv) extend to these pairings.*

*Proof.* (a) Suppose first that  $k$  is the perfect closure of a field  $k_0$  finitely generated over  $\mathbb{F}_p$ , and that  $B = B_0 \otimes_{k_0} k$  for some smooth  $k_0$ -variety  $B_0$ . Then  $\text{CH}^1(B_0)$  is a finitely generated abelian group [[Kahn 2006](#)], and  $\text{CH}^1(B_0) \otimes \mathbb{Z}[1/p]$  does not change under purely inseparable extensions; in particular,  $\text{CH}^1(B) \otimes \mathbb{Z}[1/p]$  has finite torsion and a fortiori verifies the hypothesis of [Corollary 4.8](#). The result then follows from this theorem and [Lemma 4.13](#).

In general, the situation is defined over such a subfield of  $k$ , so reduces to the first case.

(b) Let  $X_1, X_2$  be (proper) regular, and let  $\gamma \in \text{CH}^{\dim X_2}(X_1 \times_K X_2)$ . We need to prove the analogue of (3-11),

$$\langle \cdot, \cdot \rangle_1 \circ \gamma^* \otimes 1 = \langle \cdot, \cdot \rangle_2 \circ 1 \otimes \gamma_*$$

where  $\langle \cdot, \cdot \rangle_i$  is the height pairing of  $X_i$ . By the uniqueness statement of Corollary 4.8, it suffices to prove this identity after localising at  $l$  for all  $l \neq p$ . Let  $\pi_i : X_{i,l} \rightarrow X_i$  ( $i = 1, 2$ ) be two admissible alterations of generic degrees  $d_i$  prime to  $l$ , and let  $\gamma_l = \pi_2^* \circ \gamma \circ (\pi_1)_* \in \text{CH}^{\dim X_2}(X_{1,l} \times_K X_{2,l})$ , so that  $d_2 \gamma \circ (\pi_1)_* = (\pi_2)_* \gamma_l$  and  $\gamma_l \circ \pi_1^* = d_1 \pi_2^* \circ \gamma$ . By Lemma 4.13, we have, with obvious notation,

$$\begin{aligned} \langle \cdot, \cdot \rangle_1 \circ \gamma^* \otimes 1 &= d_1^{-1} \langle \cdot, \cdot \rangle_{1,l} \circ \pi_1^* \gamma^* \otimes \pi_1^* = d_1^{-1} d_2^{-1} \langle \cdot, \cdot \rangle_{1,l} \circ \gamma_l^* \pi_2^* \otimes \pi_1^* \\ &\stackrel{(a)}{=} d_1^{-1} d_2^{-1} \langle \cdot, \cdot \rangle_{2,l} \circ \pi_2^* \otimes (\gamma_l)_* \pi_1^* = d_2^{-1} \langle \cdot, \cdot \rangle_{2,l} \circ \pi_2^* \otimes \pi_2^* \gamma_* \\ &= \langle \cdot, \cdot \rangle_2 \circ 1 \otimes \gamma_*, \end{aligned}$$

where (a) used (3-11) for  $\gamma_l$ .

The identity of Proposition 3.9(iv) is extended in similar fashion. □

We shall use the following fact in the proof of Theorem 6.2:

**Example 4.15.** Suppose that  $X$  is an abelian variety. For  $a \in X(K)$ , write  $\tau_a$  for the translation by  $a$ . It yields a self-correspondence of degree 0 still denoted by  $\tau_a$ , and we have the obvious formula  ${}^t \tau_a = \tau_{-a}$ . This yields the identity (see Remarks 4.5(b))

$$\langle \tau_a^* \alpha, \beta \rangle = \langle \alpha, \tau_{-a}^* \beta \rangle$$

for  $(\alpha, \beta) \in \text{CH}^i(X)^{[0]} \times \text{CH}^{d+1-i}(X)^{[0]}$ .

**Remark 4.16.** The functoriality of Proposition 4.12(c) means that the subgroups  $\text{CH}^i(X)^{[0]}$ , for varying  $X$  and  $i$ , define an *adequate equivalence relation* on algebraic cycles with integral coefficients on smooth projective  $K$ -varieties. This adequate relation a priori depends on the choice of  $B$ , but see Conjecture 5.1 and Remarks 5.4 below.

**4F. Extension to imperfect fields.** Let  $X, K, B$  be as in the introduction, but relax the assumption that  $k$  is perfect; specifically, we assume  $k$  imperfect of characteristic  $p$ . Write  $k^p$  (resp.  $K^p, B^p, X^p$  for the perfect closure of  $k$  (resp. for  $K \otimes_k k^p, B \otimes_k k^p, X \otimes_K K^p$ ).

We define  $\text{CH}^i(X)^{[0]}$  as the inverse image of  $\text{CH}^i(X^p)^{[0]}$  under the pull-back morphism  $\text{CH}^i(X) \rightarrow \text{CH}^i(X^p)$ . We claim that the pairing (4-1) for  $X^p$  induces a similar pairing for  $X$ , with the same properties.

Since the homomorphism  $\lambda : \text{CH}^1(B) \rightarrow \text{CH}^1(B^p)$  has  $p$ -primary torsion kernel and cokernel, this is trivial if we accept to replace  $\text{CH}^1(B)$  by  $\text{CH}^1(B) \otimes \mathbb{Z}[1/p]$  (note that  $\text{Ker } \lambda$  and  $\text{Coker } \lambda$  do not have finite exponent, so  $\lambda$  is not an isomorphism in  $\mathbf{Ab}[1/p]$ ). We can avoid this, however, by observing that all constructions involved in constructing (4-1) for  $X^p$  and proving its properties are defined over some finite subextension of  $k^p/k$ .

### 5. Homologically and algebraically trivial cycles

From now on, we write

$$\mathrm{CH}^i(X)^{(0)} = \{\alpha \in \mathrm{CH}^i(X) \mid \exists n \neq 0 : n\alpha \in \mathrm{CH}^i(X)^0\}$$

for the saturation of  $\mathrm{CH}^i(X)^0$ . We have the inclusion

$$\mathrm{CH}^i(X)^{(0)} \subseteq \mathrm{CH}_{\mathrm{num}}^i(X) \tag{5-1}$$

by [Lemma 4.3\(d\)](#) and the fact that  $\mathrm{CH}^i(X)/\mathrm{CH}_{\mathrm{num}}^i(X)$  is torsion-free.

**5A. Conjectures.** The following is a numerical analogue to [\[Beilinson 1987, Conjecture 2.2.5\]](#).

**Conjecture 5.1.** *The inclusion (5-1) is an equality.*

Let the index  $l$  denote homological equivalence for  $l$ -adic cohomology,  $l \neq \mathrm{char} k$ . [Conjecture 5.1](#) implies

**Conjecture 5.2.** *One has the inclusion  $\mathrm{CH}_l^i(X) \subseteq \mathrm{CH}^i(X)^{(0)}$ .*

Conversely, [Conjecture 5.2](#) implies [Conjecture 5.1](#) under Grothendieck’s standard Conjecture D, by [Proposition 3.11](#) (and [Proposition 4.12\(a\)](#) in characteristic  $p$ ).

**Proposition 5.3.** *Conjecture 5.2 is true if  $X$  admits a model  $f : \mathcal{X} \rightarrow B$  with  $f$  smooth.*

*Proof.* This follows from the smooth and proper base change theorem (see [Remark 2.7](#)). □

**Remarks 5.4.** (a) More generally, [Proposition 2.6](#) shows that  $\mathrm{CH}^i(X)^0$  contains the image of  $\mathrm{CH}^i(\mathcal{X})_{B,l}^0$  for any model  $f : \mathcal{X} \rightarrow B$  of  $X$  with  $\mathcal{X}$  smooth.

(b) Suppose  $X$  smooth (not just regular). For clarity, let us write  $\mathrm{CH}^i(X)_B^{(0)}$  to mark the dependence of  $\mathrm{CH}^i(X)^{(0)}$  on the model  $B$ . If  $U$  is an open subset of  $B$ , we obviously have  $\mathrm{CH}^i(X)_B^{(0)} \subseteq \mathrm{CH}^i(X)_U^{(0)}$ , and this direct system is *essentially constant* by [Proposition 4.12\(b\)](#). For  $U$  small enough, [Proposition 5.3](#) thus gives inclusions

$$\mathrm{CH}_l^i(X) \subseteq \mathrm{CH}^i(X)_U^{(0)} \subseteq \mathrm{CH}_{\mathrm{num}}^i(X),$$

where the middle group does not change when  $U$  gets smaller (note that equality on the right is *not* clear: see [Remark 2.7](#)). In view of [Remark 4.16](#), this defines a new adequate equivalence on smooth projective  $K$ -varieties, this time independent of the choice of  $B$  (and which conjecturally agrees with numerical equivalence).

**Theorem 5.5.** *Conjecture 5.1 is true in the following cases:*

- (i)  $i = 1, d$ .
- (ii)  $\mathrm{char} K = 0$ ,  $f$  is smooth and
  - either  $i \in \{2, d - 1\}$ ,

- or  $X$  is “of abelian type” (i.e., its homological motive is isomorphic to a direct summand of the motive of an abelian variety).

*Proof.* For (i), see [Theorem 5.6\(b\)](#) below. For (ii), homological and numerical equivalences agree in the said cases by [Lieberman \[1968\]](#). Therefore, the statement follows from [Proposition 5.3](#).  $\square$

**5B. Algebraic equivalence.**

**Theorem 5.6.** (a) One has  $\text{CH}_{\text{alg}}^i(X) \subseteq \text{CH}^i(X)^{(0)}$ .

(b) [Conjecture 5.1](#) is true for  $i = 1, d$ .

Of course, (b) follows from (a) (using Matsusaka’s theorem [\[1957\]](#) in the case  $i = 1$ ).

To prove (a), we first reduce to the case where  $X$  has a smooth model  $\mathcal{X}$  as in [Section 4](#): this is automatic if  $\text{char } k = 0$  by [Proposition 4.1](#), and if  $\text{char } k > 0$  we first reduce to  $k$  perfect as in [Section 4F](#), then we can use [Proposition 4.10\(a\)](#) and a transfer argument.

We now give ourselves a model  $f : \mathcal{X} \rightarrow B$  of  $X$  with  $\mathcal{X}$  smooth. The proof is in two steps.

*Step 1.* Assume  $d = 1$  and two sections  $\tilde{c}_0, \tilde{c}_1$  of  $f$  are given. Let  $c_0, c_1$  be their generic fibres and  $\alpha = [c_0] - [c_1]$ .

**Lemma 5.7.** *There exists an integer  $N > 0$  such that  $N\alpha \in \text{CH}^1(X)^0$ .*

*Proof.* Let  $\tilde{\alpha} = [\tilde{c}_0(B)] - [\tilde{c}_1(B)] \in \text{CH}^1(\mathcal{X})$ . Then  $j^*\tilde{\alpha} \in \text{CH}_{\text{alg}}^1(X) \subseteq \text{CH}_{\text{num}}^1(X)$ . We now need to find  $N > 0$  and  $\xi \in \text{Ker } j^*$  such that  $N\tilde{\alpha} + \xi \in \text{CH}^1(\mathcal{X})^0$ . We shall look for  $\xi$  in the form

$$\xi = \sum_{b \in B^{(1)}} (\iota_b)_* \xi_b,$$

where  $\iota_b : \mathcal{X}_{Z_b} \hookrightarrow \mathcal{X}$  is the inclusion (with  $Z_b = \overline{\{b\}}$  as usual) and each  $\xi_b$  is a linear combination of classes of irreducible  $\delta$ -dimensional components  $\mathcal{X}_{Z_b}^\lambda$  of  $\mathcal{X}_{Z_b}$  (almost all  $\xi_b$  will be 0). For this, I claim that the method of [\[Silverman 1994, III.8\]](#) extends to this case:

The first thing to check is that the hypothesis of [\[loc. cit., Proposition III.8.3\]](#) is verified, namely that  $\langle \tilde{\alpha}, [\mathcal{X}_{Z_b}] \rangle_b = 0$  for all  $b \in B^{(1)}$ . For simplicity, write  $Z$  and  $\iota$  instead of  $Z_b$  and  $\iota_b$ . Up to removing a proper closed subset from  $Z$ , we may assume it smooth. In the Cartesian square of the diagram

$$\begin{array}{ccc} Z & \xrightarrow{g_i=(d_i, \iota')} & \mathcal{X}_Z \times B \\ d_i \downarrow & & 1 \times \tilde{c}_i \downarrow \\ Z & \xleftarrow{f_Z} \mathcal{X}_Z \xrightarrow{\gamma_i} & \mathcal{X}_Z \times \mathcal{X} \end{array}$$

where  $d_i = (\tilde{c}_i)|_Z$  and  $\iota'$  is the inclusion  $Z \hookrightarrow B$ , the top horizontal map  $g_i$  is a regular embedding of codimension  $\delta + 1$  as the composite of the two regular embeddings

$$Z \xrightarrow{\delta} Z \times Z \xrightarrow{d_i \times \iota'} \mathcal{X}_Z \times B.$$

Here we use that the embedding  $d_i$  is regular [EGA IV<sub>4</sub> 1967, proposition 19.1.1]. Then

$$\begin{aligned} \langle [\tilde{c}_i(B)], [\mathcal{X}_Z] \rangle_b &= (f_Z)_* \gamma_i^1([\mathcal{X}_Z] \times (\tilde{c}_i)_*[B]) = (f_Z)_* \gamma_i^1((1 \times \tilde{c}_i)_*[\mathcal{X}_Z \times B]) \\ &\stackrel{(a)}{=} (f_Z)_*(d_i)_* \gamma_i^1[\mathcal{X}_Z \times B] = \gamma_i^1[\mathcal{X}_Z \times B] \\ &\stackrel{(b)}{=} g_i^1[\mathcal{X}_Z^\lambda \times B] \stackrel{(c)}{=} \delta^!(d_i^![\mathcal{X}_Z] \times \iota^![B]) = [Z], \end{aligned}$$

where (a) (resp. (b), (c)) is once again Lemma 2.1(a) (resp. (b), (c)).

Now

$$\text{CH}_{\delta+i-1}(\mathcal{X}_Z) = \text{CH}_\delta(\mathcal{X}_Z) \leftarrow \bigoplus_{\lambda} \mathbb{Z}[\mathcal{X}_Z^\lambda],$$

where the  $\mathcal{X}_Z^\lambda$  are the irreducible components of  $\mathcal{X}_Z$  of dimension  $\delta$ : this follows from [Fulton 1984, Example 1.8.1] by induction on the number of components. The second thing to observe is that the statement and proof of [Silverman 1994, Proposition III.8.2] apply verbatim, namely that the quadratic form  $\alpha \mapsto \langle \iota_*\alpha, \alpha \rangle_b$  on  $\text{CH}_\delta(\mathcal{X}_Z)$  is negative, with kernel generated by  $[\mathcal{X}_Z]$ . Indeed, this is a local computation so we can consider the fibre of  $\mathcal{X}$  over  $\text{Spec } \mathcal{O}_{B,b}$  and simply apply the said proposition. (The fact that  $f_*\mathcal{O}_{\mathcal{X}} = \mathcal{O}_B$ , which is used in its proof, follows from the fact that  $X$  is geometrically connected since it has rational points, and that  $B$  is normal.)

We can now find  $N$  and  $\xi$  just as in [Silverman 1994, Proposition III.8.3]. □

*Step 2.* The general case. Let  $\alpha \in \text{CH}_{\text{alg}}^i(X)$ . By [Achter et al. 2019, Lemma 3.8], there exist an integer  $s \geq 0$ , a smooth projective  $K$ -curve  $C$ , two rational points  $c_0, c_1 \in C(K)$  and an element  $y \in \text{CH}^i(C \times X)$  such that  $p^s\alpha = (c_0^* - c_1^*)y$  (recall that  $p$  is the exponential characteristic of  $k$ ).

**Lemma 5.8** (Q. Liu; cf. [Liu and Tong 2016]). *There exists a closed subset  $F \subset B$  of codimension  $> 1$  such that  $C$  lifts to a regular proper  $(B - F)$ -scheme  $\mathcal{C}$  and the  $c_i$  lift to sections  $\tilde{c}_i$  of  $\mathcal{C} \rightarrow B - F$ .*

*Proof.* See Proposition A.1 of the Appendix. □

By Step 1 and Lemma 5.8, there exists an integer  $N > 0$  such that  $N([c_1] - [c_0]) \in \text{CH}^1(C)^0$ . Then  $Np^s\alpha = N(c_0^* - c_1^*)y = y^*N([c_0] - [c_1]) \in \text{CH}^i(X)^0$  by Proposition 3.6, where  $y$  is considered as a correspondence (Weil–Bloch trick). This concludes the proof of Theorem 5.6.

**Remark 5.9.** There is a statement parallel to Theorem 5.6 in [Beilinson 1987, Lemma 2.2.2(b)], with a similar proof.

**5C. Example: elliptic curves.** In Step 1 of the proof of Theorem 5.6, suppose  $\delta = 1$ ,  $B$  projective and that  $X$  is an elliptic curve. Applying  $\text{deg} : \text{CH}^1(B) \rightarrow \mathbb{Z}$ , we get an integral pairing  $\langle \cdot, \cdot \rangle$  on the finite index subgroup  $\text{CH}^1(X)^0 \cap X(K)$  of  $\text{CH}_{\text{alg}}^1(X) = \text{Pic}^0(X) = X(K)$ ; this pairing coincides with the Néron–Tate height pairing by the description in [Silverman 1994, Theorem III.9.3].

**Theorem 5.10.** *Assume  $k$  algebraically closed. Then:*

(a)  $\text{CH}^1(X)^0 \cap X(K)$  contains the subset denoted by  $X(K)_0$  in [Silverman 1994, Remark III.9.4.2].



(b) If  $\mathcal{X}$  is a minimal model,  $X(K)_0$  is a subgroup and the pairing

$$X(K)_0 \times X(K)_0 \rightarrow \text{Pic}(B) \tag{5-2}$$

of [loc. cit., Theorem III.9.5(b)] equals  $-\langle \ , \ \rangle$ .

*Proof.* (a) Let  $P \in X(K)$ . As in Lemma 5.7, write  $\tilde{P} : B \rightarrow \mathcal{X}$  for the section of  $f$  extending  $P$  (here its existence as a morphism is automatic since  $\delta = 1$ , by the valuative criterion of properness). What is written  $[\tilde{P}(B)] = \tilde{P}_*[B]$  in its proof of is denoted by  $(P)$  in [Silverman 1994]. Since

$$(P) \cdot [\mathcal{X}_b] = \tilde{P}_*[B] \cdot f^*b = [B] \cdot \tilde{P}^* f^*b = \text{deg}(b) = 1$$

(projection formula), and the intersection numbers of  $(P)$  with the components of  $\mathcal{X}_b$  are  $\geq 0$ , this implies that  $P$  meets exactly one component of  $\mathcal{X}_b$ , with multiplicity 1.

By definition,  $X(K)_0$  is the set of  $P$  such that  $(P)$  meets the same component of  $\mathcal{X}_b$  as  $(0)$  for all  $b \in B^{(1)}$ . Equivalently,  $\text{deg}(((P) - (0)) \cdot [\mathcal{X}_b^\lambda]) = 0$  for all  $b$  and all such components. By (2-3), this degree is none else than  $\langle (P) - (0), \mathcal{X}_b^\lambda \rangle_b$ , so we get that

$$P \in X(K)_0 \Rightarrow (P) - (0) \in \text{CH}^1(\mathcal{X})^0 \Rightarrow P - 0 \in \text{CH}^1(X)^0.$$

(b) What we use here is that

$$\tilde{P} = \tau_P \circ \tilde{0} \tag{5-3}$$

for all  $P \in X(K)$ , where  $\tau_P$  is the translation by  $P$  [Silverman 1994, Proposition III.9.1]. This already implies that  $X(K)_0$  is a subgroup of  $X(K)$ .

We start with a convenient description of (5-2) by reformulating part (a) of [Silverman 1994, Theorem III.9.5]. For  $P, Q \in X(K)$ , we have  $j^*((P + Q) - (P) - (Q) + (0)) = 0$  in  $\text{Pic}^0(X)$ ; the sequence

$$0 \rightarrow \text{Pic}(B) \xrightarrow{f^*} \text{Pic}(\mathcal{X}) \xrightarrow{j^*} \text{Pic}(X) \rightarrow 0$$

is exact except at  $\text{Pic}(\mathcal{X})$  where its homology is given by  $\bigoplus_b \text{CH}_1(\mathcal{X}_b)/[\mathcal{X}_b]$  (see [Kahn 2009, 3.2(a)]). If now  $P, Q \in X(K)_0$ , then

$$(P + Q) - (P) - (Q) + (0) = ((P + Q) - (0)) - ((P) - (0)) - ((Q) - (0)) \in \text{CH}^1(\mathcal{X})^0,$$

which implies that its homology class is 0 by the nondegeneracy of the intersection pairings on  $\text{CH}_1(\mathcal{X}_b)/[\mathcal{X}_b]$ . Thus  $(P + Q) - (P) - (Q) + (0) = f^*[P, Q]$  for a unique  $[P, Q] \in \text{Pic}(B)$ . In particular,

$$[P, Q] = \tilde{R}^*((P + Q) - (P) - (Q) + (0)) \quad \text{for all } R \in X(K). \tag{5-4}$$

For convenience, we now write

$$P * Q = f_*((P) \cdot (Q)) \in \text{Pic}(B)$$

for  $P, Q \in X(K)$ .

**Lemma 5.11.** *We have the identities  $P * Q = \tilde{Q}^*(P)$  and  $P * Q = 0 * (P - Q)$ .*

*Proof.* For the first identity,

$$f_*((P) \cdot (Q)) = f_*(\tilde{P}_*[B] \cdot \tilde{Q}_*[P]) = f_*(\tilde{Q}_* \tilde{Q}^* \tilde{P}_*[B]) = \tilde{Q}^*(P)$$

by the projection formula. For the second one,

$$\tilde{Q}^*(P) = \tilde{0}^* \tau_Q^* (\tau_P)_* \tilde{0}_*[B] = \tilde{0}^* (\tau_Q)_*^{-1} (\tau_P)_* \tilde{0}_*[B] = \tilde{0}^* (\tau_{P-Q})_* \tilde{0}_*[B] = \tilde{0}^*(P - Q). \quad \square$$

**Remark 5.12.** Since  $P * Q = Q * P$ , we also get the intriguing identity  $0 * P = 0 * (-P)$ .

To prove the claim of [Theorem 5.10\(b\)](#), we now apply [\(5-4\)](#) with  $R = Q$ :

$$\begin{aligned} [P, Q] &= \tilde{Q}^*((P + Q) - (P) - (Q) + (0)) \\ &= (P + Q) * Q - P * Q - Q * Q + 0 * Q \\ &= 0 * P - P * Q - 0 * 0 + 0 * Q = P * 0 - P * Q - 0 * 0 + 0 * Q = -\langle P, Q \rangle. \quad \square \end{aligned}$$

**Remarks 5.13.** (a) We have  $X(K)_0 = \mathcal{N}^0(B)$ , where  $\mathcal{N}^0$  is the identity component of the Néron model  $\mathcal{N}$  of  $X$ . Indeed,  $\mathcal{N}$  is isomorphic to the smooth locus  $\mathcal{X}_{\text{sm}}$  of  $\mathcal{X}$  [[Artin 1986](#), Proposition 1.15] and  $\mathcal{N}^0$  contains the 0-section  $(0)$ . For any  $P \in X(K)$ ,  $(P) \subset \mathcal{X}_{\text{sm}}$  (see end of Step 2 in the proof of [Proposition A.1](#)), and  $(P) \in \mathcal{N}^0$  if and only if  $P \in X(K)_0$  since  $\mathcal{N}_b^0$  is the identity component of  $\mathcal{N}_b$  for all  $b \in B^{(1)}$  by definition of  $\mathcal{N}^0$ .

(b) Suppose that  $P - 0 \in \text{CH}^1(X)^0$ . We can find a fibral divisor  $\xi$  such that  $(P) - (0) - \xi$  is orthogonal to all fibral divisors (as in the proof of [Lemma 5.7](#), with  $N = 1$ ), and this divisor is unique modulo  $\text{Im } f^*$  by [[Silverman 1994](#), Proposition III.8.3]. By [[loc. cit.](#), Lemma III.9.4] (or by (a)),  $X(K)/X(K)_0 = X(K)/\mathcal{N}^0(B)$  is finite, so the class of  $\xi$  is torsion in each  $\text{CH}_1(\mathcal{X}_b)/[\mathcal{X}_b]$ . Thus  $\text{CH}^1(X)^0/\mathcal{N}^0(B) \hookrightarrow \bigoplus_{b \in B^{(1)}} (\text{CH}_1(\mathcal{X}_b)/[\mathcal{X}_b])_{\text{tors}}$ .

## 6. The pairing in codimension 1

In this section, we assume  $X$  projective and geometrically irreducible. Recall that  $\delta = \text{trdeg}(K/k) = \dim B$ . We shall study the height pairing [\(4-1\)](#) for  $i = 1$ , in  $\mathbf{Ab} \otimes \mathbb{Q}$ ; note that  $\text{CH}^i(X)^{(0)} = \text{CH}_{\text{num}}^i(X)$  for  $i = 1, d$  by [Theorem 5.6](#).

**6A. A general result.** We write  $T(X) \subset \text{CH}_{\text{num}}^d(X) = \text{CH}^d(X)_0$  for the Albanese kernel. For an abelian  $K$ -variety  $A$ , write  $\text{Tr}_{K/k} A$  for its  $K/k$ -trace and

$$\text{LN}(A, K/k) = A(K)/(\text{Tr}_{K/k} A)(k)$$

for its *Lang–Néron group*: it is finitely generated by the Lang–Néron theorem [[1959](#)]. We shall need the following classical fact:

**Lemma 6.1.** *The Albanese map  $a_X : \text{CH}^d(X)_0 \rightarrow \text{Alb}_X(K)$  has a cokernel of finite exponent.*

*Proof.* This could be deduced from [Kahn 2021, Proposition A.1]; here is a different and more direct proof. Choose a smooth irreducible multiple hyperplane section of dimension 1  $i : C \hookrightarrow X$ . By the usual transfer argument, we may assume that  $X$  has a rational point lying on  $C$ . Then  $a_C$  is bijective. By [Murre 1990, Lemma 2.3], the composition

$$\text{Pic}_X^0 \xrightarrow{i^*} \text{Pic}_C^0 = \text{Alb}_C \xrightarrow{i_*} \text{Alb}_X \tag{6-1}$$

is an isogeny, hence  $\text{Coker } i_*(K)$  has finite exponent and so does its quotient  $\text{Coker } a_X$ . □

**Theorem 6.2.** (a) *The pairing  $\langle \cdot, \cdot \rangle$  vanishes on  $\text{CH}_{\text{num}}^1(X) \times T(X)$ .*

(b) *This induces a pairing (in  $\mathbf{Ab} \otimes \mathbb{Q}$ )*

$$\langle \cdot, \cdot \rangle : \text{Pic}^0(X) \times \text{Alb}_X(K) \rightarrow \text{CH}^1(B).$$

(c) *Suppose  $B$  projective. Composing this pairing with the projection  $\text{CH}^1(B) \rightarrow N^1(B)$  (where  $N^1(B)$  is the group of cycles of codimension 1 modulo numerical equivalence) induces a pairing*

$$\langle \cdot, \cdot \rangle_{\text{num}} : \text{LN}(\text{Pic}_X^0, K/k) \times \text{LN}(\text{Alb}_X, K/k) \rightarrow N^1(B). \tag{6-2}$$

*Proof.* (a) Up to extending scalars to the perfect closure of  $k$ , we may assume  $k$  perfect. Let  $L/K$  be a finite extension. Let  $B_L$  be the normalisation of  $B$  in  $L$ ; up to removing from  $B$  a closed subset  $F$  of codimension  $\geq 2$  and from  $B_L$  the inverse image of  $F$  (which does not affect  $\text{CH}^1(B)$  or  $\text{CH}^1(B_L)$ ), we may assume  $B_L$  smooth. In  $\mathbf{Ab}[1/p]$ , the map  $\text{CH}^1(B) \rightarrow \text{CH}^1(B_L)$  is a monomorphism (transfer argument). In view of the functoriality in Theorem 4.14(b), to prove the vanishing we may thus increase scalars as much as we wish. In particular, we may assume that  $X(K) \neq \emptyset$ .

Let  $x \in X(K)$  and let  $a : X \rightarrow \text{Alb}_X$  be the corresponding Albanese map. Then  $a$  induces an isomorphism  $a^* : \text{Pic}^0(\text{Alb}_X) \xrightarrow{\sim} \text{Pic}^0(X)$ , and  $a_* : \text{CH}_0(X) \rightarrow \text{CH}_0(\text{Alb}_X)$  sends  $T(X)$  into  $T(\text{Alb}_X)$ . Still by functoriality, we are reduced to the case  $X = \text{Alb}_X =: A$ .

The sequel is inspired by Néron’s proof of [Néron 1965, Proposition 7]. In order to reason with elements, pick a representative of  $\langle \cdot, \cdot \rangle$  in  $\mathbf{Ab}$  as in Remarks 4.5(a). Let  $\beta \in T(A)$ , and let  $\bar{K}$  be an algebraic closure of  $K$ . In  $T(A_{\bar{K}})$ , we may write  $\beta_{\bar{K}} = \sum_i ([a_i + b_i] - [a_i] - [b_i] + [0])$ , with  $a_i, b_i \in A(\bar{K})$ . Choose  $L/K$  finite such that all the  $a_i$  are rational over  $L$ . As above, we may extend scalars from  $K$  to  $L$ , and thus reduce to  $\beta = [a + b] - [a] - [b] + [0]$  for  $a, b \in A(K)$ . The vanishing now follows from Example 4.15 and the theorem of the square [Mumford 2008, II.6, Corollary 4].

(b) This follows immediately from (a) and Lemma 6.1, which implies that  $\text{CH}^d(X)_0/T(X) \rightarrow \text{Alb}_X(K)$  is an isomorphism in  $\mathbf{Ab} \otimes \mathbb{Q}$ .

(c) We may assume  $k$  algebraically closed; then the claim follows from the divisibility of  $Y(k)$  for an abelian  $k$ -variety  $Y$  and the finite generation of  $N^1(B)$ . □

**6B. Another conjecture.** For the needs of Theorem 6.6 below, we introduce a new conjecture. From now on,  $B$  is projective as in Theorem 6.2(c).

Let  $R$  be a discrete valuation ring with quotient field  $K$  and residue field  $E$ . Suppose that an abelian  $K$ -variety  $A$  has good reduction with respect to  $R$ ; then its Néron model  $\mathcal{A}$  is an abelian scheme over  $\text{Spec } R$ , whose special fibre  $A_s$  is an abelian  $E$ -variety. We have a specialisation homomorphism

$$A(K) = \mathcal{A}(R) \rightarrow A_s(E). \tag{6-3}$$

Suppose now that  $R$  contains  $k$ . The notion of  $K/k$ -trace readily extends to a notion of  $R/k$ -trace for abelian  $R$ -schemes; viewing these traces as right adjoints shows that

- $\text{Tr}_{R/k} \mathcal{A}$  exists and equals  $\text{Tr}_{K/k} A$ ;
- the ‘special fibre’ functor yields a canonical morphism  $\text{Tr}_{K/k} A \rightarrow \text{Tr}_{E/k} A_s$ .

It follows that (6-3) induces a homomorphism of Lang–Néron groups

$$\text{LN}(A, K/k) \rightarrow \text{LN}(A_s, E/k). \tag{6-4}$$

**Conjecture 6.3.** *Assume that  $A$  has semistable reduction at every point of  $B^{(1)}$ , and that  $\delta > 1$ . For any projective embedding  $B \hookrightarrow \mathbb{P}^N$ , there exists a smooth, geometrically connected hyperplane section  $h$  of  $B$  such that  $A$  has good reduction at  $h$  and the kernel of (6-4) is finite, with  $E = k(h)$ .*

Suppose  $A$  constant. Then (6-4) may be rewritten as

$$\text{Hom}_k(\text{Alb}_B, A) \rightarrow \text{Hom}_k(\text{Alb}_h, A),$$

and Conjecture 6.3 follows from the surjectivity of  $\text{Alb}_h \rightarrow \text{Alb}_B$  (see (6-1)). This gives some evidence for this conjecture.

**Remark 6.4.** Perhaps the hypotheses of Conjecture 6.3 are too weak.<sup>4</sup> In any case, we only need it in the special case  $A = \text{Pic}_X^0$ , when  $X$  satisfies the conclusion of Lemma 6.5 (or any suitable variant of it); it may be easier to prove in such a case.

**6C. A technical lemma.** This lemma will be needed in the proofs of Theorem 6.6 and Proposition 6.8 below.

**Lemma 6.5.** *Suppose that  $d = 1$ . Then there exists an alteration  $\tilde{B} \rightarrow B$ , with  $\tilde{B}$  smooth, such that  $X \otimes_K k(\tilde{B})$  has a projective model  $f : \mathcal{X} \rightarrow \tilde{B}$  where  $\mathcal{X}$  is smooth over  $k$  and, for all  $b \in \tilde{B}^{(1)}$ , the irreducible components of  $\mathcal{X}_b$  are smooth over  $k(b)$ .*

*Proof.* Start from a projective embedding  $X \hookrightarrow \mathbb{P}_K^N$  and consider its closure  $\mathcal{X}_0$  in  $\mathbb{P}_B^N$ . In the following reasoning using results of [de Jong 1997], we always take the group  $G$  appearing there equal to 1. By [de Jong 1997, Theorem 5.9] (or just [de Jong 1997, Theorem 2.4 and Lemma 5.7]), we may (projectively) alter  $f_0 : \mathcal{X}_0 \rightarrow B$  into  $f_1 : \mathcal{X}_1 \rightarrow B_1$  so that  $f_1$  is a projective quasismooth semistable curve in the sense of [de Jong 1997, section after Lemma 5.6]. This condition is stable under base change, hence, by the reasoning at the end of the proof of [de Jong 1996, Theorem 5.13], we may alter  $B_1$  into  $B_2$  so that  $B_2$  is

<sup>4</sup>Added in proof: A version of this conjecture has now been proven: see Bruno Kahn and Long Liu, *A specialisation theorem for Lang–Néron groups*, in preparation.

smooth and  $f_2 : \mathcal{X}_2 := \mathcal{X}_1 \times_{B_1} B_2 \rightarrow B_2$  verifies the hypotheses of [de Jong 1997, Proposition 5.11] (note that varieties over a field verify [de Jong 1997, (5.12.1)] by [de Jong 1996, Theorem 4.1]); in particular,  $\tilde{B} := B_2$  is smooth. Next, the beginning of the proof of [de Jong 1997, Proposition 5.11] yields a modification  $\pi : \mathcal{X}_3 \rightarrow \mathcal{X}_2$  such that the singular locus  $\Sigma$  of  $\mathcal{X}_3$  is smooth of codimension  $\geq 3$  and  $f_3 : \mathcal{X}_3 \rightarrow \tilde{B}$  is still a quasisplit semistable curve. The end of this proof then yields a desingularisation  $\mathcal{X}_4$  of  $\mathcal{X}_3$  by blowing up the components of  $\Sigma$ . Since they lie over points of codimension  $\geq 2$  in  $\tilde{B}$ , this does not affect the fibres of  $f_3$  at points of codimension 1, so  $f_4 : \mathcal{X}_4 \rightarrow \tilde{B}$  is “quasisplit semistable in codimension 1”.

We are left to desingularise the singular components of  $(\mathcal{X}_4)_b$  for all  $b \in \tilde{B}^{(1)}$ . Let  $D$  be such a component, and let  $x$  be a singular point of  $D$ . Note that  $x$  does not lie on any other component, since all singular points of  $(\mathcal{X}_4)_b$  are quadratic by the “semistable” condition. By the “quasisplit” one, the completion of  $\mathcal{O}_{\mathcal{X}_4, x}$  is isomorphic to  $k[[u, v, t_1, \dots, t_\delta]]/(uv - t_1)$ , where  $t_1$  is a local equation of  $D$  (compare [de Jong 1996, 2.16]). The ideal of  $x$  is  $(u, v, t_1)$ . Blowing up this ideal retains the regularity of  $\mathcal{X}_4$ , separates the two branches of  $D$  at  $x$  (making its strict transform regular at the two corresponding points) and adds a smooth irreducible exceptional divisor. We have therefore decreased by 1 the total number of singular points of the irreducible components of  $(\mathcal{X}_4)_b$ . Since only finitely many  $b$  are involved, we end the process after a finite number of steps. □

**6D. A negativity theorem.**

**Theorem 6.6.** *Let  $L \in \text{Pic}(X)$  and  $\ell \in \text{Pic}(B) - \{0\}$ . Consider the quadratic form*

$$q = q(X, B, L, \ell) : \text{LN}(\text{Pic}_X^0, K/k) \ni \alpha \mapsto \deg(\langle \alpha, L^{d-1} \alpha \rangle_{\text{num}} \cdot \ell^{\delta-1})$$

*obtained from the pairing of Theorem 6.2(c). If  $L$  is ample and  $\delta = 1$  (hence  $\ell^{\delta-1} = 1$ ), then  $q(X, B, L, \ell)$  is negative definite (in particular, nondegenerate). If Conjecture 6.3 holds for  $\text{Pic}_X^0$  when  $d = 1$  and in the situation of Lemma 6.5, this extends to  $\delta > 1$  for  $\ell$  ample.*

**Remark 6.7.** As pointed out in Remarks 4.5(b), the notation using elements is abusive in  $\mathbf{Ab} \otimes \mathbb{Q}$ . Theorem 6.6 could be converted into an arrow-theoretic statement; similarly, the notion “negative definite” for a quadratic form with values in  $\mathbb{Z}$  is unambiguous in  $\mathbf{Ab} \otimes \mathbb{Q}$ , by using Remarks 4.5(a).

However, converting the proof below into arrow-theoretic notation would be cumbersome at best. Since the source and target of the quadratic form  $q$  are finitely generated abelian groups, we can tensor everything with  $\mathbb{Q}$  (i.e., apply the natural functor from  $\mathbf{Ab} \otimes \mathbb{Q}$  to  $\mathbb{Q}$ -vector spaces) without losing information, and reason with honest elements. This is what we do in this proof.

*Proof.* (a) We first reduce to  $d = 1$  as follows. Suppose  $d > 1$ . We may assume  $L$  very ample. Let  $i : C \hookrightarrow X$  be a smooth irreducible curve given by successive hyperplane sections from the projective embedding determined by  $L$ . By the functoriality of Theorem 4.14, we have

$$\langle i^* \alpha, i^* \alpha \rangle_{\text{num}} = \langle \alpha, i_* i^* \alpha \rangle_{\text{num}} = \langle \alpha, L^{d-1} \cdot \alpha \rangle_{\text{num}};$$

hence  $q(X, B, L, \ell)(\alpha) = q(C, B, i^* L, \ell)(i^* \alpha)$ . By the isogeny (6-1),  $\text{LN}(\text{Pic}_X^0, K/k) \rightarrow \text{LN}(\text{Pic}_C^0, K/k)$  is mono in  $\mathbf{Ab} \otimes \mathbb{Q}$ .

We now assume  $d = 1$ .

(b) We reduce to the situation of [Lemma 6.5](#). Let  $\mathcal{X} \xrightarrow{f} \tilde{B}$  be as in [Lemma 6.5](#). Since  $\pi : \tilde{B} \rightarrow B$  is projective, pick a very ample divisor  $\mathcal{L}$  relative to  $\pi$ . By [\[EGA II 1961, proposition 4.4.10\(ii\)\]](#),  $\mathcal{L} + n\pi^*\ell$  is then very ample (relative to  $\tilde{B} \rightarrow \text{Spec } k$ ) for all  $n \gg 0$ . Let  $\alpha \in \text{LN}(\text{Pic}_X^0, K/k) - \{0\}$ . Assuming the theorem true over  $\tilde{B}$ , we have

$$\deg(\langle \pi^*\alpha, \pi^*L^{d-1}\pi^*\alpha \rangle_{\text{num}} \cdot (\mathcal{L} + n\pi^*\ell)^{\delta-1}) < 0$$

for all  $n \gg 0$ . This is a polynomial in  $n$ , with dominant term

$$\deg(\langle \pi^*\alpha, \pi^*L^{d-1}\pi^*\alpha \rangle_{\text{num}} \cdot \pi^*\ell^{\delta-1}) = \deg(\langle \alpha, L^{d-1}\alpha \rangle_{\text{num}} \cdot \ell^{\delta-1})$$

by [\(3-14\)](#), which must be negative.

We now assume that we are in the situation of [Lemma 6.5](#).

(c) Assume  $\delta = 1$ . Observe that the pairing [\(1-2\)](#), composed with the degree, is then the intersection pairing. By the Hodge index theorem, this pairing has signature  $(1, \rho - 1)$  where  $\rho = \text{rk } N^1(\mathcal{X})$ . Since  $N^1(\mathcal{X})^0$  is the orthogonal of the isotropic vector  $f^*t$  for  $t \in N^1(B) - \{0\}$ , the restriction of the intersection pairing to this subspace is negative with kernel generated by  $f^*t$ . Since  $f^*t$  also generates the kernel of  $N^1(\mathcal{X})^0 \rightarrow \text{LN}(\text{Pic}_X^0, K/k)$ , the quadratic form  $q$  is negative definite, as requested.

(d) Assume finally  $\delta > 1$ . Similarly to (a), we may assume  $\ell$  very ample. We may also assume  $k$  algebraically closed (in particular, infinite). Let  $Z \subset B$  be the locus of nonsmoothness of  $f$ . In the family of hyperplane sections of  $B$  relative to the projective embedding given by  $\ell$ , only finitely many may be contained in  $Z$ , therefore we can pick a smooth hyperplane section  $h \not\subset Z$ . By induction, there exists a smooth ample curve  $i : \Gamma \subset B$  determined by  $\ell$  such that the generic fibre  $X(E)$  of  $\mathcal{X}_\Gamma = f^{-1}(\Gamma)$  is smooth over  $E = k(\Gamma)$ .

Write  $I : \mathcal{X}_\Gamma \hookrightarrow \mathcal{X}$ ,  $g : \mathcal{X}_\Gamma \rightarrow \Gamma$  for the two corresponding projections. For  $\tilde{\alpha} \in \text{CH}^1(\mathcal{X})$ , we have

$$\langle \tilde{\alpha}, \tilde{\alpha} \rangle \cdot \ell^{\delta-1} = i_*i^*f_*(\tilde{\alpha}^2) = i_*g_*I^1(\tilde{\alpha}^2).$$

Since  $\deg_B \circ i_* = \deg_\Gamma$ , it is enough to compute  $g_*I^1(\tilde{\alpha}^2)$ .

Choose a resolution of singularities  $\pi : \mathcal{Y} \rightarrow \mathcal{X}_\Gamma$  of the surface  $\mathcal{X}_\Gamma$ ; let  $\tilde{I} = I \circ \pi$  and  $\tilde{g} = g \circ \pi$ . The same reasoning as in the proof of [Proposition 3.9\(iv\)](#) yields the identity  $I^1 = \pi_*\tilde{I}^*$ , hence

$$g_*I^1(\tilde{\alpha}^2) = \tilde{g}_*\tilde{I}^*(\tilde{\alpha}^2) = \tilde{g}_*(\tilde{I}^*\tilde{\alpha})^2.$$

Now there exists a finite extension  $E'/E$  with smooth projective  $k$ -model  $\Gamma'$ , and a semistable model  $\mathcal{Y}'$  of  $X(E) \otimes_E E'$  over  $\Gamma'$  mapping to  $\mathcal{Y}$  by a morphism  $\varphi$ . If  $d = [E' : E]$ , we therefore have

$$(\tilde{g} \circ \varphi)_*(\tilde{I} \circ \varphi)^*(\tilde{\alpha})^2 = d\tilde{g}_*(\tilde{I}^*\tilde{\alpha})^2.$$

Under [Conjecture 6.3](#),  $\Gamma$  may be chosen such that the map induced by  $\tilde{I}^*$

$$\text{LN}(\text{Pic}_X^0, K/k) \rightarrow \text{LN}(\text{Pic}_{X(E)}^0, E/k)$$

has finite kernel, and our reduction to  $\delta = 1$  is complete. □

**6E. Another pairing.** Here we assume  $B$  projective; we write  $A = \mathrm{Tr}_{K/k} \mathrm{Pic}_X^0$  and  $P = \mathrm{Pic}_B^0$ .

**Proposition 6.8.** *Suppose  $d = 1$ . In the pairing of [Theorem 6.2\(b\)](#), we have  $\langle A(k), A(k) \rangle \subseteq \mathrm{Pic}^0(B)\{p\}$  in  $\mathbf{Ab} \otimes \mathbb{Q}$ , where  $p$  is the exponential characteristic of  $k$ . This induces a pairing in  $\mathbf{Ab} \otimes \mathbb{Q}$*

$$\mathrm{LN}(\mathrm{Pic}_X^0, K/k) \times A(k) \rightarrow P(k)/P(k)\{p\}. \tag{6-5}$$

*Proof.* We may first reduce to  $k$  perfect and then pass to a finite extension of  $K$ , hence reduce to the existence of a smooth model  $\mathcal{X}$  (e.g., as in [Lemma 6.5](#)). By [[Kahn 2009](#), 3.2(a)], we have

$$j^* \mathrm{Pic}^0(\mathcal{X}) = A(k).$$

By [Proposition 3.14](#),  $\langle \mathrm{Pic}^0(\mathcal{X}), \mathrm{Pic}^0(\mathcal{X}) \rangle$  is  $p$ -primary torsion, hence the claim. □

**Question 6.9.** Does (6-5) extend to arbitrary  $d$ , replacing  $A(k)$  by  $\mathrm{Tr}_{K/k} \mathrm{Alb}_X(k)$ ?

Let  $E = k(A)$ . Using [[Milne 1986](#), Theorem 3.1], we deduce from (6-5) a pairing

$$\mathrm{LN}(\mathrm{Pic}_X^0, K/k) \times \mathrm{Mor}_k(A, A) \rightarrow \mathrm{Mor}_k(A, P)/\mathrm{Mor}_k(A, P)\{p\}.$$

Evaluating on the identity  $1_A \in \mathrm{Mor}_k(A, A)$ , we get a homomorphism

$$\mathrm{LN}(\mathrm{Pic}_X^0, K/k) \rightarrow \mathrm{Mor}_k(A, P)/\mathrm{Mor}_k(A, P)\{p\}$$

and using the canonical isomorphism  $\mathrm{Mor}_k(A, P) \simeq P(k) \oplus \mathrm{Hom}_k(A, P)$ , a final homomorphism

$$\mathrm{LN}(\mathrm{Pic}_X^0, K/k) \rightarrow \mathrm{Hom}(\mathrm{Tr}_{K/k} \mathrm{Pic}_X^0, \mathrm{Pic}_B^0) \tag{6-6}$$

because the right hand group is torsion-free. It is an exercise to check that, evaluating this homomorphism on  $k$ -points, we get back (6-5) (improved).

If  $B = \mathbb{P}^1$  or  $\mathrm{Tr}_{K/k} \mathrm{Pic}_X^0 = 0$ , the right hand side is 0 while the left hand side is nonzero in general. Yet we may ask:

**Question 6.10.** When is (6-6) surjective (in  $\mathbf{Ab} \otimes \mathbb{Q}$ )?

### Appendix: Extending rational points to sections

by Qing Liu

**Proposition A.1.** *Let  $B$  be a noetherian connected regular excellent scheme. Let  $C$  be a connected projective regular curve over the function field  $K$  of  $B$ . Let  $c_1, \dots, c_n \in C(K)$ . Then there exist an open subset  $U \subseteq B$  with  $\mathrm{codim}(B \setminus U, B) \geq 2$  and a proper scheme  $\mathcal{C} \rightarrow U$ , with  $\mathcal{C}$  regular, containing the  $c_i$  such that the latter extend to sections of  $\mathcal{C} \rightarrow U$ .*

**Step 1.** We extend  $C$  to a projective regular scheme  $\mathcal{C}_0$  over some “big” open subset  $U_0$  of  $B$ .

First we extend  $C$  to an integral projective scheme  $f : \mathcal{X} \rightarrow B$  (taking for instance the scheme-theoretical closure of  $C$  in a suitable  $\mathbb{P}_B^n$ ). Let  $\mathcal{X}_{\text{sing}}$  be the closed subset of the singular points of  $\mathcal{X}$ . Then  $V := B \setminus f(\mathcal{X}_{\text{sing}})$  is a dense open subset of  $B$  such that  $\mathcal{X}_V$  is regular.

Let  $b_1, \dots, b_m$  be the codimension 1 points of  $B$  inside of  $B \setminus V$ . We are going to extend  $\mathcal{X}_V$  above an open subset  $U_0$  of  $B$  containing all the  $b_j$ . For each  $j \leq m$ , we have a relative integral curve  $\mathcal{X} \times_B \text{Spec } \mathcal{O}_{B,b_j}$  over the discrete valuation ring  $\mathcal{O}_{B,b_j}$  with regular generic fibre  $C$ . As  $B$  is excellent, there exists a resolution of singularities

$$\mathcal{X}'_j \rightarrow \mathcal{X} \times_B \text{Spec } \mathcal{O}_{B,b_j} \rightarrow \text{Spec } \mathcal{O}_{B,b_j}.$$

Each  $\mathcal{X}'_j$  is a projective regular curve over  $\text{Spec } \mathcal{O}_{B,b_j}$  and extends to a projective regular curve  $\mathcal{X}_j$  over some open neighbourhood  $V_j \ni b_j$ . Shrinking the (finitely many)  $V_j$  if necessary, we can suppose that for all  $j, \ell \leq m$ ,  $\mathcal{X}_j$  and  $\mathcal{X}_\ell$  coincide over  $V_j \cap V_\ell$  and that  $\mathcal{X}_j$  coincides with  $\mathcal{X}_V$  over  $V \cap V_j$ . Let  $U_0$  be the union of  $V$  and the  $V_j$  and let  $\mathcal{C}_0 \rightarrow U_0$  be obtained by glueing  $\mathcal{X}_V$  and the  $\mathcal{X}_j$ . Then  $\mathcal{C}_0$  is regular, proper over  $U_0$  (by the fpqc descent  $V \coprod (\coprod_{1 \leq i \leq m} V_j) \rightarrow U_0$  of properness; see [EGA IV<sub>2</sub> 1965, proposition 2.7.1(vii)]), and  $\text{codim}(B \setminus U_0, B) \geq 2$ .

**Step 2.** For all  $i \leq n$  we let  $P_i \subseteq \mathcal{C}_0$  be the scheme-theoretical closure of  $\{c_i\}$ . Then  $P_i \rightarrow U_0$  is proper birational, hence is an isomorphism away from a closed subset  $Z_i \subset U_0$  of codimension at least 2. To finish we let  $U := U_0 \setminus (\bigcup_{1 \leq i \leq n} Z_i)$  and let  $\mathcal{C} = (\mathcal{C}_0)_U \rightarrow U$ . (As  $U$  and  $\mathcal{C}$  are regular, the section  $(P_i)_U$  of  $\mathcal{C} \rightarrow U$  is contained in the smooth locus of  $\mathcal{C}$  [Bosch et al. 1990, 3.1, Proposition 2 and following paragraph].)

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