On the injectivity and non-injectivity of the l-adic cycle class maps

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Conférence inaugurale du Laboratoire de Mathématiques franco-japonais

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April 5, 2024

1. Chow groups

X algebraic variety over a field k

$n \geq 0$: $Z_n(X)$ free abelian group on closed integral subvarieties of X

Too coarse, e.g. for intersection theory

Solution: impose rational equivalence ($\equiv$ homotopy)

Get *Chow group* $CH_n(X)$.

X non singular (= smooth): can number by codimension $\mapsto CH^n(X)$.

e.g. $CH^1(X) = \mathrm{Pic}(X)$.

2. Cycle class maps

To understand algebraic cycles, use *cycle class maps*

Most classical: $k = \mathbf{C}$, singular cohomology:

$$CH^n(X) \to H^{2n}(X(\mathbf{C}), \mathbf{Z})$$

Here: use *étale cohomology*

SGA 4 1/2: l prime number invertible in k, $\nu \geq 1$

$$\mathrm{cl}^n_\nu : CH^n(X)/l^\nu \to H^{2n}_{\text{ét}}(X, \mu_{l^\nu}^{\otimes n})$$

Surjectivity? $\leftrightarrow$ Hodge conjecture (over $\mathbf{C}$), Tate conjecture (over finitely generated k).

To express the Tate conjecture, pass from k to $\bar{k}$ (algebraic closure) and replace $H^{2n}_{\text{ét}}(X, \mu_{l^\nu}^{\otimes n})$ by

$$H^{2n}(\bar{X}, \mathbf{Z}_l(n)) := \varprojlim_{\nu} H^{2n}_{\text{ét}}(\bar{X}, \mu_{l^\nu}^{\otimes n})$$

finitely generated $\mathbf{Z}_l$-module.
Over k, can do the same, but get an awkward theory: refined by U. Jannsen to *continuous étale cohomology* $H^{2n}_{\text{cont}}(X, \mathbf{Z}_l(n))$: Milnor exact sequences

$$0 \to \varprojlim{}^1 H^{2n-1}_{\text{ét}}(\bar{X}, \mu_{l^\nu}^{\otimes n}) \to H^{2n}_{\text{cont}}(X, \mathbf{Z}_l(n)) \to \varprojlim H^{2n}_{\text{ét}}(\bar{X}, \mu_{l^\nu}^{\otimes n}) \to 0$$

and *Jannsen's l-adic cycle class map*

$$\text{cl}^n_X : CH^n(X) \otimes \mathbf{Z}_l \to H^{2n}_{\text{cont}}(X, \mathbf{Z}_l(n)).$$

3. The image of cl_X^n

Theorem 1 (S. Saito, essentially)**.** *X smooth over k finitely generated. For any $n \geq 0$, the image of cl_X^n is a finitely generated $\mathbf{Z}_l$-module.*

4. The kernel of cl_X^n

Question 1. When is cl_X^n injective?

Completely false in general: e.g. $k = \bar{k}$, $n = 1$, X smooth projective curve: $0 \to \mathrm{Pic}^0(X) \to CH^1(X) \to \mathrm{NS}(X) \to 0$ but $\mathrm{Pic}^0(X) = J(X)(k)$ is divisible hence killed by cl_X^1.

Theorem 2 (Jannsen). *If k is finitely generated, cl_X^1 is injective.*

(N.B. $\mathrm{Pic}(X)$ f.g. abelian group in this case.)

Important conjecture on Chow groups: Bloch-Beilinson–Murre (BBM): for X smooth projective, $\exists$ filtration on $CH^n(X) \otimes \mathbf{Q}$ with very good properties.

Theorem 3 (Jannsen). *If $\mathrm{cl}_X^n \otimes \mathbf{Q}$ is injective for X smooth projective over finitely generated k, the BBM conjecture is true.*

People have studied the injectivity of cl^n_X restricted to $CH^n(X)_{\mathrm{tors}}$: S. Saito (positive cases), Scavia-Suzuki, Alexandrou-Schreieder, Colliot-Thélène-Scavia (negative cases). But what about $\otimes\mathbf{Q}$?

Question 2. Is the converse to Jannsen's theorem 3 true?

I don't know!

Question 3. Are there known conjectures which would imply the injectivity of $\mathrm{cl}^n_X \otimes \mathbf{Q}$ for X smooth projective over f.g. k's?

Theorem 4. *Yes in positive characteristic.*

(I don't know in characteristic 0.)

5. With $\mathbf{Q}$ coefficients

k finite field, X smooth projective: Tate-Beilinson conjecture

Conjecture 1. $\mathrm{cl}_X^n \otimes \mathbf{Q}$ *is bijective for all* $n \geq 0$.

Theorem 5. *Conjecture 1 implies that* $\mathrm{cl}_X^n \otimes \mathbf{Q}$ *is injective for any smooth projective* X *over any* k *finitely generated over* $\mathbf{F}_p$ *(*$p \neq l$*).*

Sketch. Extend Conjecture 1 to a statement on general smooth $\mathbf{F}_q$-varieties; if $k = \mathbf{F}_q(U)$ with U smooth small enough, spread X to smooth projective morphism $\mathcal{X} \to U$, apply Deligne's degeneration criterion for Leray spectral sequence and pass to the limit. □

In characteristic 0 I don't know any similar conjecture implying the injectivity of $\mathrm{cl}_X^* \otimes \mathbf{Q}$.

6. RESTRICTION TO TORSION

6.1. Positive results: decomposition of the diagonal.

Theorem 6. *k finitely generated of char. 0, X smooth projective. Assume that $CH_0(X_{k(X)}) \otimes \mathbf{Q} \xrightarrow{\deg \otimes \mathbf{Q}} \mathbf{Q}$ is an isomorphism. Let $\delta = |\operatorname{Coker} \deg|$. Then $\delta \operatorname{Ker} \mathrm{cl}^2_X = 0$, and $\operatorname{Ker} \mathrm{cl}^2_X$ is finite if k is of Kronecker dimension ≤ 2. If $\operatorname{char} k = p > 0$, this is true up to p-primary groups of finite exponent.*

(Kronecker dimension: in characteristic p, transcendence degree over $\mathbf{F}_p$; in characteristic 0, transcendence degree over $\mathbf{Q}$ $+1$.)

Sketch. All is due to S. Saito (in char. 0), except the case of Kronecker dimension 2. In this case, use finite generation of CH_0 of arithmetic surfaces (Bloch, Kato-Saito). (Tricky!) □

6.2. Counterexamples. Of two types: 1) for CH^2; for CH^n, $n > 2$. To understand these examples, refine cycle class map using étale Chow groups:

$$CH^n(X) \otimes \mathbf{Z}_l \xrightarrow{\alpha_X^n} CH^n_{\text{ét}}(X) \otimes \mathbf{Z}_l \xrightarrow{\beta_X^n} H^{2n}_{\text{cont}}(X, \mathbf{Z}_l(n))$$

where $CH^n_{\text{ét}}(X) = H^{2n}_{\text{ét}}(X, \mathbf{Z}(n))$ is étale motivic cohomology ($\text{cl}_X^n = \beta_X^n \circ \alpha_X^n$).

For $n = 2$, short exact sequence

$$0 \to CH^2(X) \xrightarrow{\alpha_X^2} CH^2_{\text{ét}}(X) \to H^3_{\text{nr}}(X, \mathbf{Q}/\mathbf{Z}(2)) \to 0$$

but α_X^n not injective for $n > 2$ in general. In fact:

Theorem 7. *In all examples of Scavia-Suzuki and Alexandrou-Schreieder with* $n > 2$*, the cycle in* $\operatorname{Ker}\operatorname{cl}_X^n$ *is already killed by* α_X^n.

Trivialises their proofs, e.g. no need of refined Bloch maps in Alexandrou-Schreieder.

For $n = 2$, two examples: Scavia-Suzuki and Colliot-Thélène-Scavia. First one uses a norm variety (used in proof of Bloch-Kato conjecture) and corresponding Rost motive $\mathcal{R}$, with $CH^2(\mathcal{R}) = \mathbf{Z}/l$. One way to understand it is

Proposition 1. *a) The canonical map*

$$CH^2_{\text{ét}}(k) \otimes \mathbf{Z}_{(l)} \to CH^2_{\text{ét}}(\mathcal{R})$$

is an isomorphism.
b) $\beta^2_{\mathcal{R}} = 0$.

a) is basically because $\mathcal{R}$ becomes trivial over $\bar{k}$. In view of a), the reason for b) is that $H^4_{\text{cont}}(k, \mathbf{Z}_l(2)) = 0$ because k is chosen with cohomological dimension 3!

Second counterexample much more delicate to "explain" in this way...

The End