Reciprocity sheaves

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(beyond) Homotopical methods in algebraic geometry

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What is K_2?

First answer: Tate (1970es) for K_2/n:

$$K_2/n = (\mathbf{G}_m \otimes \mathbf{G}_m)/n + \text{ transfers } + \text{ projection formula.}$$

How about K_2 itself?

Two answers: Suslin, Kato (1980es).

Suslin:

$K_2 = \mathbf{G}_m \otimes \mathbf{G}_m +$ transfers + projection formula + homotopy invariance.

$\longrightarrow$ Suslin-Voevodsky motivic cohomology, Voevodsky's homotopy invariant motives.

Kato:

$$K_2 = \mathbf{G}_m \otimes \mathbf{G}_m + \text{ transfers + projection formula + Weil reciprocity.}$$

$\longrightarrow$ reciprocity sheaves (so far).

1. Rosenlicht-Serre theory (1959)

$k = \bar{k}$. Two main results:

1.1. Reciprocity property. G/k connected commutative algebraic group.

Theorem 1.1. *C/k smooth projective curve, $U \subset C$ affine open subset, $f : U \to G$ k-morphism; $\exists$ effective divisor $\mathfrak{m}$ with support $C - U$ such that*

$$f(\mathrm{div}(g)) = 0 \text{ if } g \in k(C)^*, g \equiv 1 \pmod{\mathfrak{m}}.$$

We say that f affords the modulus $\mathfrak{m}$.

Here, extend f to homomorphism $Z_0(U) \to G(k)$ by linearity; note that hypothesis on $g \Rightarrow$ support of $\mathrm{div}(g) \subset U$.

Equivalent version. $K = k(C)$: $\exists$ local symbols

$$\partial_x : G(K) \times K^* \to G(k) \quad (x \in C)$$

such that

- $\partial_x(a, g) = v_x(g)\bar{a}$ if $a \in G(\mathcal{O}_{C,x})$, $\bar{a}$ = image of a in $G(k)$;
- $\forall a \in G(K)$, $\forall g \in K^*$,

$$\sum_{x \in C} \partial_x(a, g) = 0 \tag{1}$$

(Weil reciprocity).

1.2. Representability.

Theorem 1.2. *C, U as above, $\mathfrak{m}$ effective divisor with support $C - U$. Fix $u_0 \in U$. The functor*

$$G \mapsto \{f : U \to G \mid f(u_0) = 0 \text{ and } f \text{ affords modulus } \mathfrak{m}\}$$

from commutative algebraic groups to abelian groups is corepresentable by the generalized Jacobian $J(C, \mathfrak{m})$.

$\mathfrak{m}$ reduced: get connected component of *relative Picard group*

$$J(C, \mathfrak{m}) = \mathrm{Pic}^0(C, C - U).$$

In general, extension of this by unipotent group.

2. Somekawa K-groups (1990)

k any field, $G_1, \dots, G_n$ semi-abelian varieties.

Definition 2.1 (K. Kato). $K(k, G_1, \dots, G_n)$ abelian group defined by generators and relations:

Generators: $\{g_1, \dots, g_n\}_{E/k}$, $[E : k] < \infty$, $g_i \in G_i(E)$.

Relations:

(1) Multilinearity.

(2) Projection formula (for norms on the G_i).

(3) Weil reciprocity in the style of (1).

Theorem 2.2 (Kato, Somekawa). $K(k, \mathbf{G}_m, \dots, \mathbf{G}_m) \simeq K_n^M(k)$.

Other formulas of Somekawa: $k = \bar{k}$,

$$K(k, J, \mathbf{G}_m) = V(C) \text{ (Bloch's group) if } J = J(C).$$

$$K(k, J_1, J_2) = \text{ Albanese kernel of } C_1 \times C_2 \text{ if } J_i = J(C_i).$$

Theorem 2.3 (K.-Yamazaki, 2011)**.** *k perfect, $\mathcal{F}_1, \ldots, \mathcal{F}_n$ homotopy invariant Nisnevich sheaves with transfers: can generalize Somekawa's K-groups to $K(k, \mathcal{F}_1, \ldots, \mathcal{F}_n)$ and*

$$K(k, \mathcal{F}_1, \ldots, \mathcal{F}_n) \simeq \mathrm{Hom}_{DM_-^{\mathrm{eff}}(k)}(\mathbf{Z}, \mathcal{F}_1 \otimes \cdots \otimes \mathcal{F}_n[0]).$$

3. Beyond homotopy invariance

Can one define Somekawa K-groups for more general functors than HI sheaves, e.g. unipotent groups?

I tried in 1991 and failed. Used pre-Voevodsky set-up of "Mackey functors with reciprocity" (modelled on Rosenlicht-Serre). Guessed "empirical formulas"

$$K(k, \mathbf{G}_a, \mathbf{G}_m \dots, \mathbf{G}_m) \simeq \Omega^n_{k/\mathbf{Z}} \tag{2}$$

$$K(k, \mathbf{G}_a, \mathbf{G}_a, \dots) = 0 \tag{3}$$

$$K(k, \mathbf{G}_a, A, \dots) = 0 \quad (A \text{ abelian variety}). \tag{4}$$

Taken up by Ivorra-Rülling (2012) in a framework of "reciprocity functors": k perfect field,

- Category of models: $\mathcal{C} = \{$regular curves over function fields over $k\}$.
- Reciprocity functors: presheaves with transfers restricted to $\mathcal{C}$, with reciprocity condition à la Rosenlicht-Serre.

Their main result: $\mathcal{F}_1, \ldots, \mathcal{F}_n$ reciprocity functors $\mapsto$ new reciprocity functor $T(\mathcal{F}_1, \ldots, \mathcal{F}_n)$ (a kind of tensor product).

Definition 3.1. E function field over k: $K(E, \mathcal{F}_1, \ldots, \mathcal{F}_n) = T(\mathcal{F}_1, \ldots, \mathcal{F}_n)(E)$.

- $\mathcal{F}_i \in \mathrm{HI}$: recover Somekawa K-group.
- They prove (2) and (3) above.

Problem: their tensor product T is not a priori associative.

4. Reciprocity sheaves (with S. Saito and T. Yamazaki)

In exploratory mode.

k perfect field.

- Category of models: $\mathrm{Sm}(k)$ (smooth separated k-schemes of finite type).
- Reciprocity sheaves: presheaves with transfers, with reciprocity condition à la Rosenlicht-Serre.

Makes sense more generally for a *pretheory*:

Definition 4.1. a) A pretheory $\mathcal{F}$ *has reciprocity* if, for any relative curve X/S with good compactification $\bar{X}/S$ and any $a \in \mathcal{F}(X)$, there exists an effective divisor Y with support $\bar{X} - X$ such that

$$(\mathrm{div}(g), a) = 0 \text{ if } g \in G(\bar{X}, Y) :$$

- $(,) : c(X/S) \times \mathcal{F}(X) \to \mathcal{F}(S)$ pairing given by the pretheory structure ($c(X/S)$: relative cycles)
- $G(\bar{X}, Y) \subset k(X)^*$ subgroup defined by Suslin-Voevodsky.

We say that Y is a *modulus* for a.

b) A presheaf with transfers *has reciprocity* if the associated pretheory has reciprocity.

Proposition 4.2. *a) A Nisnevich sheaf with transfers is homotopy invariant if and only if it has reciprocity “with reduced moduli” (better: “with uniformly bounded moduli”).*
b) Presheaves with transfers defined by connected commutative algebraic groups have reciprocity.

Theorem 4.3 (Global injectivity)**.** *$\mathcal{F}$ reciprocity Nisnevich sheaf with transfers. Then for any $X \in \mathrm{Sm}(k)$ and $U \subset X$ dense open subset, $\mathcal{F}(X) \to \mathcal{F}(U)$ is injective.*

(Generalizes a theorem of Voevodsky for HI Nisnevich sheaves with transfers.)

Sketch. (1) $\mathcal{F}$ is $\mathbf{P}^1$-*rigid*: $\forall X \in \mathrm{Sm}(k)$, $i_0^* = i_\infty^* : \mathcal{F}(\mathbf{P}^1_X) \to \mathcal{F}(X)$.

(2) $\mathbf{P}^1$-rigidity implies $\mathbf{P}^1$-*invariance*: $\mathcal{F}(X) \xrightarrow{\sim} \mathcal{F}(\mathbf{P}^1_X)$.

(3) $\mathbf{P}^1$-invariance implies global injectivity (an argument of Gabber).

□

Corollary 4.4. *$\mathcal{F}$ as above, X smooth projective: the pairing $Z_0(X) \times \mathcal{F}(X) \to \mathcal{F}(k)$ factors through $CH_0(X) \times F(X)$.*

Hope: in the end will have a full Gersten conjecture (with Cousin complexes), but not there yet. At the moment also have semi-local injectivity (for reciprocity presheaves with transfers) and other theorems of Voevodsky on comparing Zariski and Nisnevich sheafifications.

5. Functorial properties

Reciprocity stable under

- subobjects
- quotients
- direct sums (possibly infinite)

Not stable under

- infinite products

(Big) open question:

- by extensions?

Consequences: $\mathrm{PST}_{\mathrm{rec}} = \{\mathcal{F} \in \mathrm{PST} \mid \mathcal{F} \text{ has reciprocity}\}$ cocomplete abelian subcategory of PST; don't know if it is thick.

$$\mathrm{HI} \subset \mathrm{PST}_{\mathrm{rec}} \subset \mathrm{PST}$$

Voevodsky: the inclusion $\mathrm{HI} \subset \mathrm{PST}$ has the left adjoint $\mathcal{F} \mapsto h_0(\mathcal{F})$. Also true (but less well-known): this inclusion has a right adjoint.

Proposition 5.1. *The inclusion* $\mathrm{PST}_{\mathrm{rec}} \subset \mathrm{PST}$ *has a right adjoint, but presumably no left adjoint [because* $\mathrm{PST}_{\mathrm{rec}}$ *not complete].*

Definition 5.2. $\mathcal{G} \in \mathrm{PST}$ is *finitely generated* if quotient of a finite direct sum of representable presheaves (equivalently: of one).

Proposition 5.3. $\mathcal{F} \in \mathrm{PST}_{\mathrm{rec}}$, $\mathcal{G} \in \mathrm{PST}$. *Then* $\underline{\mathrm{Hom}}(\mathcal{G}, \mathcal{F}) \in \mathrm{PST}_{\mathrm{rec}}$ *if* $\mathcal{G}$ *is finitely generated, but not in general.*

(Simple example: $\mathcal{G} = \mathbf{Z}^{(\mathbf{N})}$, $\mathcal{F} = \mathbf{G}_a \Rightarrow \underline{\mathrm{Hom}}(\mathcal{G}, \mathcal{F}) = \mathbf{G}_a^{\mathbf{N}}$ which does not have reciprocity.)

6. Towards "generalized Jacobians"

X/S relative curve with good compactification $\bar{X}/S$, Y effective divisor with support $\bar{X} - X$.

Definition 6.1. $\mathcal{F} \in \mathrm{PST}$:

$$\mathcal{F}_0^Y(X) = \{a \in \mathcal{F}(X) \mid a \text{ has modulus } Y\}.$$

Theorem 6.2. *$\mathcal{F} \mapsto \mathcal{F}_0^Y(X)$ is corepresented by a quotient $\mathbf{Z}_{\mathrm{tr}}^0(X/S, Y)$ of $\mathbf{Z}_{\mathrm{tr}}(X)$.*

(Proof not difficult.)

Take $S = \operatorname{Spec} k$. First guess: $\mathbf{Z}_{\mathrm{tr}}(X/k, Y) = \operatorname{Pic}(\bar{X} \times -, Y \times -)$. Is it true?

(1) $U \mapsto \operatorname{Pic}(\bar{X} \times U, Y \times U)$ defines $\mathcal{F} \in \mathrm{PST}$ (not quite immediate).

(2) By Yoneda, class of diagonal $[\Delta_X] \in \operatorname{Pic}(\bar{X} \times X, Y \times X) = \mathcal{F}(X)$ yields epimorphism $\mathbf{Z}_{\mathrm{tr}}(X) \twoheadrightarrow \mathcal{F}$.

(3) This factors through $\mathbf{Z}^0_{\mathrm{tr}}(X/k, Y) \twoheadrightarrow \mathcal{F}$.

(4) BUT not iso: kernel generated by all $G(\bar{X} \times U, Y \times U)$ ($G(\bar{X}, Y)$ not sufficient).

Yields stronger notion of reciprocity:

Definition 6.3 (First form)**.** X/S relative curve with good compactification $\bar{X}/S$, Y effective divisor with support $\bar{X} - X$, $\mathcal{F} \in \mathrm{PST}$: $a \in \mathcal{F}(X)$ has *universal modulus* Y if for any $f : S' \to S$, $f_X^* a$ has modulus $f_X^* Y$ $(f_X : X \times_S S' \to X)$.

$$\mathcal{F}_1^Y(X) := \{a \in \mathcal{F}(X) \mid a \text{ has universal modulus } Y\}.$$

Theorem 6.4. *$F \mapsto F_1^Y(X)$ is corepresentable by a quotient $\mathbf{Z}_{\mathrm{tr}}^1(X/S, Y)$ of $\mathbf{Z}_{\mathrm{tr}}^0(X/S, Y)$.*

Case $X = \operatorname{Spec} k$: get exactly $\mathbf{Z}_{\mathrm{tr}}^1(X/k, Y) \xrightarrow{\sim} \mathrm{Pic}(\bar{X} \times -, Y \times -)$.

Moreover: $\mathrm{Pic}(\bar{X} \times -, Y \times -) \in \mathrm{PST}_{\mathrm{rec}}$ (proof to be fu!lly checked).

How about higher dimensions?

Definition 6.5. A *modulus pair* is a pair $(\bar{X}, Y)$, $\bar{X}$ normal proper, $Y \subset \bar{X}$ effective Cartier divisor such that $X = \bar{X} - Y$ is affine and smooth over k.

Define *universal modulus* Y (second form) for $a \in \mathcal{F}(X)$ by using relative curves in $\bar{X} \times S$, S variable. This notion is probably corepresentable by a quotient $\mathbf{Z}_{\mathrm{tr}}(X, Y)$ of $\mathbf{Z}_{\mathrm{tr}}(X)$.

Guess: $\mathbf{Z}_{\mathrm{tr}}(X, Y)(U) = CH^d(\bar{X} \times U, Y \times U)$, $d = \dim X$ (Kerz-Saito Chow group with modulus)…

7. Towards tensor products

Ivorra-Rülling idea:

Definition 7.1. $\mathcal{F}_1, \dots, \mathcal{F}_n, \mathcal{G} \in \mathrm{PST}$. An element $\varphi \in \mathrm{PST}(\mathcal{F}_1 \otimes \cdots \otimes \mathcal{F}_n, \mathcal{G})$ is *continuous* if for any relative curve X/S with good compactification $\bar{X}/S$ and any effective divisor Y with support $\bar{X} - X$,

$$\varphi(\mathcal{F}_1^Y(X) \otimes \cdots \otimes \mathcal{F}_n^Y(X)) \subseteq \mathcal{G}^Y(X).$$

($n = 1$: automatic.)
Defines subgroup $\mathrm{PST}^{\mathrm{cont}}(\mathcal{F}_1 \otimes \cdots \otimes \mathcal{F}_n, \mathcal{G}) \subseteq \mathrm{PST}(\mathcal{F}_1 \otimes \cdots \otimes \mathcal{F}_n, \mathcal{G})$.

Theorem 7.2. $\mathcal{F}_1, \dots, \mathcal{F}_n \in \mathrm{PST}_{\mathrm{rec}}$*: the functor*

$$\mathcal{G} \mapsto \mathrm{PST}^{\mathrm{cont}}(\mathcal{F}_1 \otimes \cdots \otimes \mathcal{F}_n, \mathcal{G})$$

is corepresentable by $T(\mathcal{F}_1, \dots, \mathcal{F}_n) \in \mathrm{PST}$.

In Ivorra-Rülling setting: $T(\mathcal{F}_1, \dots, \mathcal{F}_n) \in \mathrm{PST}_{\mathrm{rec}}$. Here, not completely clear (to be continued).

8. What are we after?

Ultimate objective: get a "reciprocity" version of DM.

Several problems:

- $\mathrm{PST}_{\mathrm{rec}}$ perhaps not thick in PST.
- Presumably no left adjoint to the inclusion.
- Tensor structure not clear (guess: OK when restricted to *finitely generated* $\mathcal{F} \in \mathrm{PST}_{\mathrm{rec}}$).

In Voevodsky's setting: "small" category $DM^{\mathrm{eff}}_{\mathrm{gm}}$ of geometric motives, "big" category DM^{eff} of motivic complexes + "Yoneda functor"

$$DM^{\mathrm{eff}}_{\mathrm{gm}} \to DM^{\mathrm{eff}}$$

Here: seem to go towards a geometric category and a category of complexes of sheaves, but their functoriality is not clear.