UNIQUENESS FOR TWO DIMENSIONAL INCOMPRESSIBLE IDEAL FLOW ON SINGULAR DOMAINS

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Abstract. We prove uniqueness of the weak solution of the Euler equations for compactly supported, single signed and bounded initial vorticity in simply connected planar domains with corners forming angles larger than \( \pi/2 \). In this type of domain, the velocity is not log-lipschitz and does not belong to \( W^{1,p} \) for all \( p \), which is the standard regularity for the Yudovich’s arguments. Thanks to the explicit formula of the Biot-Savart law via a biholomorphism, we construct a Lyapunov function to prove that the vorticity never reach the boundary which is the place where the velocity is not regular. Next we adapt the proof of Yudovich although the velocity may blow up near corners. We also obtain a uniqueness result for exterior domains with large corners.

Keywords: Euler equations, domains with corners, control of trajectories, vorticity and transport equation.

MSC: 35Q31, 76B03.

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1. Introduction

The motion of a two dimensional flow can be described by the velocity \( u(t, x) = (u_1, u_2) \) and the pressure \( p \). For an incompressible ideal fluid filling an open set \( \Omega \), the pair \( (u, p) \) verifies the Euler equations:

\[
\begin{align*}
\frac{\partial}{\partial t} u + u \cdot \nabla u + \nabla p &= 0, \quad t > 0, x \in \Omega \\
\text{div} \, u &= 0, \quad t > 0, x \in \Omega 
\end{align*}
\]

endowed with an initial condition and an impermeability condition at the boundary \( \partial \Omega \):

\[
u|_{t=0} = u_0, \quad u \cdot \hat{n}|_{\partial \Omega} = 0.
\]

The well-posedness of system (1.1)-(1.2) has been the matter of many works. The case of strong solutions for smooth \( u_0 \) was solved by Wolibner [28] in smooth bounded domains (see also Kato [13] and Temam [26]). Next, McGrath [23] treated the case of the full plane, and finally Kikuchi [14] studied the exterior domains. A key quantity for these equations is the vorticity \( \omega \) defined by

\[
\omega := \text{curl} \, u = \partial_1 u_2 - \partial_2 u_1.
\]

It plays a crucial role in the study of the ideal flow due to the transport equation:

\[
\frac{\partial}{\partial t} \omega + u \cdot \nabla \omega = 0,
\]

which propagates the \( L^p \) bound of the vorticity. In smooth domains, it allows us to derive \( W^{1,p} \) estimates for the velocity globally in time. In this direction, Yudovich [29] obtained the existence and uniqueness of weak solutions when the initial vorticity is only assumed to be bounded. We quote that the well-posedness result of Yudovich applies to smooth bounded domains, and to unbounded ones under further decay assumptions.

We stress that all above studies require \( \partial \Omega \) to be at least \( C^{1,1} \). Roughly, the reason is due to the non-local character of the Euler equations. These works rely on global in space estimates of \( u \) in terms of \( \omega \). These estimates up to the boundary involve Biot and Savart type kernels, corresponding to operators such as \( \nabla \Delta^{-1} \). Unfortunately, such operators are known to behave badly in general (see e.g. [12] where the authors have constructed an example of a bounded domain with \( \partial \Omega \in C^1 \) and a function \( \omega \in C^\infty \) for which the second derivative of the solution of \( \Delta \psi = \omega \) does not belong to \( L^1(\Omega) \)).

Nevertheless, the case of singular obstacles is physically important and has practical interests: the behavior of flows around a plane wing is still a challenging question.

Without solving the question of uniqueness, Taylor established in [25] the existence of a global weak solution of (1.1)-(1.2) in a bounded lipschitz convex domain. The convexity implies that the solution \( \psi \) of the Dirichlet problem

\[
\Delta \psi = \omega \quad \text{in} \ \Omega, \quad \psi|_{\partial \Omega} = 0
\]

belongs to \( H^2(\Omega) \) when the source term \( \omega \) belongs to \( L^2(\Omega) \), irrespective of the domain regularity (see for instance [9] Chapter 3). Recently, the article [16] gave the global existence in the exterior of a \( C^2 \) Jordan arc. It is noted therein that the velocity blows up near the end-points of the arc like the inverse of the square root of the distance. In particular, this shows that the previous property on the Dirichlet problem is false in domains with some large corners.

The question of the existence of global weak solutions is now solved for a large class of singular domains in [7, 8].

Our goal here is to prove that such a solution is unique if the domain is bounded, simply connected with some obtuse corners, or if it is the complement of a simply connected compact set with some obtuse corners. We prove the uniqueness for an initial vorticity which is bounded, compactly supported in \( \Omega \) and having a definite sign.

More precisely, we consider two kinds of domains:

(D-bd) bounded domain: let \( \Omega_{bd} \) be a bounded, simply connected open set, such that \( \partial \Omega_{bd} \) is a Jordan curve with \( \partial \Omega_{bd} \in C^{1,1} \) except in a finite number of points \( z_i \) where \( \partial \Omega_{bd} \) is a corner of angles \( \alpha_i \) (i.e. locally, \( \Omega_{bd} \) coincides with the sector \( \{z_i + (r \cos \theta, r \sin \theta); r > 0, \theta_i < \theta < \theta_i + \alpha_i\} \)).
(D-ext) exterior domain: let $\Omega_{\text{ext}} := \mathbb{R}^2 \setminus C$, where $C$ is a simply connected compact set, such that $\partial \Omega_{\text{ext}}$ is a Jordan curve with $\partial \Omega_{\text{ext}} \subset C^{1,1}$ except in a finite number of points $z_i$ where $\partial \Omega_{\text{ext}}$ is a corner of angles $\alpha_i$.

The notation $\Omega$ will be used for arguments working in both cases (D-bd) and (D-ext).

To define a global weak solution to the Euler equations, let us point out that the space $L^2(\Omega_{\text{ext}})$ is not suitable in unbounded domains. Working with square integrable velocities in exterior domains is too restrictive (see (2.7) to note that $u$ behaves in general like $1/|x|$ at infinity), so we consider initial data satisfying

$$u_0 \in L^2_c(\Omega), \quad \text{curl } u_0 \in L^\infty_c(\Omega), \quad \text{div } u_0 = 0, \quad u_0 \cdot n|_{\partial \Omega} = 0,$$

$$u_0 \to 0 \quad \text{as} \quad |x| \to +\infty \quad \text{(only for exterior domains)}.$$  \hspace{1cm} (1.5)

In non-smooth domains, the divergence free and impermeability conditions have to be understood in the weak sense:

$$\int_\Omega u^0 \cdot h = 0 \quad \text{for all } h \in G_c(\Omega) := \{w \in L^2_c(\Omega) : w = \nabla p, \text{ for some } p \in H^1_{\text{loc}}(\Omega)\}. $$  \hspace{1cm} (1.6)

Let us stress that this set of initial data is large: we will show later that for any function $\omega_0 \in L^\infty_c(\Omega)$, there exists $u_0$ verifying (1.5) with $\text{curl } u_0 = \omega_0$.

Similarly, the weak form of the divergence free and tangency conditions on the Euler solution $u$ will read:

$$\forall \varphi \in D([0, +\infty) ; G_c(\Omega)), \quad \int_{\mathbb{R}^+} \int_\Omega u \cdot \nabla \varphi = 0. $$  \hspace{1cm} (1.7)

Finally, the weak form of the momentum equation on $u$ will read:

$$\forall \varphi \in D([0, +\infty \times \Omega]) \text{ with } \text{div } \varphi = 0, \quad \int_0^\infty \int_\Omega (u \cdot \partial_t \varphi + (u \otimes u) : \nabla \varphi) = -\int_\Omega u_0 \cdot \varphi(0, \cdot). $$  \hspace{1cm} (1.8)

In both cases (D-bd) and (D-ext), the existence of a global weak solution was established in [7].

**Theorem 1.1.** Assume that $u_0$ verifies (1.5). Then there exists

$$u \in L^\infty_{\text{loc}}(\mathbb{R}^+ ; L^2_{\text{loc}}(\Omega)), \quad \text{curl } u \in L^\infty(\mathbb{R}^+ ; L^1 \cap L^\infty(\Omega)), $$  \hspace{1cm} (1.9)

which is a global weak solution of (1.1)-(1.2) in the sense of (1.7) and (1.8).

In few words, this existence result follows from a compactness argument, performed on a sequence of solutions $u_n$ of the Euler equations on the sequence of approximating domains $\Omega_n$. A key ingredient of the proof is the so-called $\gamma$-convergence of $\Omega_n$ to $\Omega$ (see [7] for the details). In [7], we do not need any assumption about the regularity of the boundaries (it can be as exotic as a Koch snowflake).

The main idea is that we can perform compactness and get an existence theory only working with $H^1$ framework for the Laplace problem (1.4) (i.e. $L^2$ framework for the velocity $u = \nabla^\perp \psi$). Such a regularity is not enough to prove uniqueness. For solutions belonging in the class (1.9), the celebrated proof of Yudovich relies on two facts:

(F1) in smooth domain, the Calderón-Zygmund inequality

$$\|\nabla u\|_{L^p} \leq Cp\|\omega\|_{L^p}$$

holds for all $p \geq 2$;

(F2) $u \in L^\infty_{\text{loc}}(\mathbb{R}^+ ; L^\infty(\Omega)).$

When $\partial \Omega$ is less than $C^{1,1}$, these facts can fail (for counter-examples, see for instance [12] where $\partial \Omega \in C^1$ and [17] in the exterior of a material segment). In the bounded case (D-bd) with angles $\pi/(2n)$ ($n \in \mathbb{N}^*$), Bardos, Di Plinio and Temam [2] recovered these two facts, thanks to a new BMO estimate related to a symmetry and reflexion argument (with the assumption $\partial \Omega \subset C^2$ away from corners). This allows them to apply Yudovich’s uniqueness argument. For arbitrary acute angles (with the assumption $\partial \Omega \subset C^{2,\alpha}$ away from corners), Lacave, Miot and Wang [19] proved the uniqueness without establishing (F1): one has showed that the push forward of $u$ to the unit disk by a Riemann map is log-Lipschitz and the uniqueness followed by adapting a proof of Marchioro and Pulvirenti [22].
A few time later, Di Plinio and Temam [5] obtained (F1) for bounded domains (D-bd) with angles \( \alpha_i \leq \pi/2 \) and the uniqueness was recovered by the original Yudovich’s argument.

If the angle is greater than \( \pi/2 \), then the elliptic theory for the Laplace problem in domains with corners (see e.g. [9, 15]) implies that \( u \) does not belong to \( \cap_{p \geq 2} W^{1,p}(\Omega) \), then (F1) fails. For angles larger than \( \pi \), (F2) also fails. More precisely, if \( \partial \Omega \) admits at \( z_0 \) a corner of angle \( \alpha \), then the velocity behaves near \( z_0 \) like \( \frac{1}{|x - z_0|^{1-\alpha}} \). We recover that in the case where \( C \) is a Jordan arc (see [16]) the velocity behaves like the inverse of the square root of the distance near the end-points (\( \alpha = 2\pi \)). Hence, in a bounded domain with a cusp point (\( \alpha = 2\pi \)), the velocity belongs to \( L^p \cap W^{1,q}(\Omega) \) only for \( p < 4 \) and \( q < 4/3 \). Even without the standard regularity for the Yudovich’s arguments, the main result of this article concerns the uniqueness of global weak solutions, when the initial vorticity has a single-sign.

**Theorem 1.2.** Let \( \Omega_{bd} \) satisfy (D-bd) with angles greater than \( \pi/2 \) and let \( u_0 \) verifying (1.5). If \( \text{curl}\ u_0 \) is non-positive (respectively non-negative), then there exists a unique global weak solution of the Euler equations on \( \Omega_{bd} \) verifying

\[
    u \in L^\infty(\mathbb{R}^+; L^2_{\text{loc}}(\Omega_{bd})), \quad \text{curl}\ u \in L^\infty(\mathbb{R}^+; L^1 \cap L^\infty(\Omega_{bd})).
\]

In exterior domains, the vorticity is not sufficient to uniquely determine the velocity. As we will see in Subsection 2.3 for \( u_0 \) verifying (1.5), we can define the initial circulation:

\[
    \gamma_0 := \oint_{\partial C} u_0 \cdot \hat{t} \, ds.
\]

We will also show that for any function \( \omega_0 \in L^\infty_{\text{loc}}(\Omega_{ext}) \) and any real number \( \gamma \in \mathbb{R} \), there exists a unique \( u_0 \) verifying (1.5) with \( \text{curl}\ u_0 = \omega_0 \) and with circulation around \( C \) equal to \( \gamma \).

Adding an assumption on the value of \( \gamma_0 \), we will prove a uniqueness theorem in exterior domains.

**Theorem 1.3.** Let \( \Omega_{ext} \) satisfy (D-ext) with angles greater than \( \pi/2 \). Let \( u_0 \) verifying (1.5). If \( \text{curl}\ u_0 \) is non-positive and \( \gamma_0 \geq - \int \text{curl}\ u_0 \) (respectively \( \text{curl}\ u_0 \) non-negative and \( \gamma_0 \leq - \int \text{curl}\ u_0 \)), then there exists a unique global weak solution of the Euler equations on \( \Omega_{ext} \) verifying

\[
    u \in L^\infty(\mathbb{R}^+; L^2_{\text{loc}}(\Omega_{ext})), \quad \text{curl}\ u \in L^\infty(\mathbb{R}^+; L^1 \cap L^\infty(\Omega_{ext})).
\]

Although it is possible to show the existence of weak solutions for \( \omega_0 \in L^1 \cap L^p(\Omega) \), with some \( p > 1 \) (see [4] in smooth domains and [7, 8] in non-smooth domains), we recall that the standard proofs of uniqueness require a vorticity belonging to every \( L^p \) in order to state that the velocity belongs to \( \cap_{p \geq 2} W^{1,p} \) (see [22, 29]). There are few results with lower regularity. Among those exceptions, one can mention the case of a vorticity \( \omega_0 = \tilde{\omega}_0 + \gamma \delta_{z_0} \) in the full plane where \( \tilde{\omega}_0 \in L^1 \cap L^\infty \) is vanishing in a neighborhood of the Dirac mass [22, 18]. In that case, the authors use the explicit form of the singularity and of the Biot-Savart in \( \mathbb{R}^d \) in order to prove that the bounded part of the vorticity never meet the point vortex. In our situation, the Biot-Savart kernel is more complicated and the singularity of the velocities comes from the irregularity of the domain (and not from the initial condition). Nevertheless, the key here is to use the sign condition in Theorems 1.2 and 1.3 in order to prove that the vorticity never reach the boundary.

For a sake of clarity, we assume that \( \partial \Omega \) is locally a corner, but we can replace a corner by a singular point, where the jump of the tangent angle is equal to \( \alpha \) (see Remark 2.2 for a discussion concerning the optimal domain regularity).

The remainder of this work is organized in six sections. We introduce in Section 2 the biholomorphism \( \mathcal{T} \) and the Biot-Savart law (law giving the velocity in terms of the vorticity) in the interior or the exterior of one simply connected domain. We will recall the existence of weak solutions in this section, and derive some formulations (on vorticity and on extensions in \( \mathbb{R}^2 \)). We will take advantage of this section to show that a weak solution is a renormalized solution in the sense of DiPerna-Lions [3], which will allow us to prove that the \( L^p \) norm of the vorticity for \( p \in [1, \infty] \), the total mass \( \int_{\Omega} \omega(t, \cdot) \) and the circulation of the velocity around \( C \) are conserved quantities.
Let us mention that the explicit form of the Biot-Savart law is one of the key of this work, and it explains why \( \Omega \) is assumed to be the interior or the exterior of a simply connected domain. This law will read

\[
u(t, x) = DT(x)^T R[\omega]
\]

where \( R[\omega] \) is an integral operator. Using elliptic theory, we will obtain the exact behavior of the biholomorphism \( T \) near the corners, and then the behavior of the velocity. We note that the blow-up is stronger if the angle \( \alpha \) is bigger. We are considering here the case where all the angles \( \alpha_i \) are greater than \( \pi/2 \) (see [2] for angles equal to \( \pi/2 \) and [19, 5] for acute angles).

Section 3 is the central part of this paper: we will prove that the support of \( \omega \) never reach the boundary if we assume that the characteristics corresponding to (1.3) exist and are differentiable. Fixing point \( x_0 \) away from the boundary, the idea is to introduce a good Lyapunov function, \( L \), along the trajectory starting at \( x_0 \), and shows that it blows up if and only if the trajectory reaches the boundary in finite time. Next, we will establish that \( L \) remains bounded, so that the trajectory does not meet the boundary. It is here where the sign condition on the vorticity is used. Although we cannot say that the characteristics are regular for weak solutions, this computation gives us the main idea.

In light of this proof, we rigorously prove in Section 4 thanks to the renormalization theory, that we have the same property, even if we do not consider the characteristics.

Finally, we prove Theorems 1.2-1.3 in Section 5. We will introduce \( v := K_{R^2} \ast \omega \), where \( K_{R^2} \) is the Biot-Savart kernel in the full plane. As \( \omega \) does not meet the boundary, it means that \( \text{div} \ v = \text{curl} \ v \equiv 0 \) in a neighborhood of the boundary, i.e. \( v \) is harmonic therein. This provides in particular a control of its \( L^\infty \) norm (as well as the \( L^\infty \) norm for the gradient) by its \( L^2 \) norm. Although the total velocity is not bounded near the boundary, but just integrable, this argument allows us to yield a Gronwall-type estimate.

Therefore, the fact that the support of the vorticity stays far from the boundary will imply the uniqueness result. This idea was already used in [15], in the case of one Dirac mass in the vorticity. In that article, we consider the Euler equations in \( \mathbb{R}^2 \) when the initial vorticity is composed by a regular part \( L^\infty_c \) and a Dirac mass. The equation is called the vortex-wave system and is derived in [22]. When trajectories exist, it is proved that they do not meet the point vortex in [22] if the point vortex moves under the influence of the regular part, and in [21] if the Dirac is fixed. The method is also based on Lyapunov functions. An important issue in [18] is to generalize this result when trajectories are not regular. The Lagrangian formulation gives us a helpful feeling. For a sake of clarity, we present first the proof of uniqueness assuming the differentiability of trajectories (Section 3). Proving in Section 4 that the vorticity never meet the boundary, we state that the “weak Lagrangian flow” coming from the renormalization theory evolves in the area far from the corners. As the velocity is regular enough in this region, we can conclude that the flow is actually classical and regular.

Section 6 is devoted to the proofs of some technical lemmas.

We finish this article by Section 7 with some final comments. In the exterior of the Jordan arc (see [16]), the Euler equations are related to a special vortex sheet. We will also give some explanations about the sign assumptions in the main theorems and compare the properties of our Lyapunov function with the vortex-wave system.

We warn the reader that we write in general the proofs in the case of exterior domains. In this kind of domain, we have to take care of integrability at infinity, to control the size of the support of the vorticity, and we have to consider harmonic vector fields and circulations of velocities around \( C \). The proofs in the case of bounded domains are strictly easier, without additional arguments. We will include some remarks about that.

2. Biot-Savart law and existence

As in [11, 15, 17, 19], we work in dimension two outside (or inside) one simply connected domain. Identifying \( \mathbb{R}^2 \) with the complex plane \( \mathbb{C} \), there exists a biholomorphism \( T \) mapping \( \Omega \) to the exterior (resp. to the interior) of the unit disk. Thanks to this biholomorphism, we will obtain an explicit
formula for the Biot-Savart law: the law giving the velocity in terms of the vorticity. This explicit formula will be used to construct the Lyapunov function. We give in the following subsection the properties of this Riemann mapping.

2.1. Conformal mapping. Let $\Omega_{\text{ext}}$ satisfy (D-ext) (resp. $\Omega_{\text{bd}}$ satisfy (D-bd)), then the Riemann mapping theorem states that there exists a unique biholomorphism $T$ mapping $\Omega_{\text{ext}}$ to $B(0,1)^{+}$ (resp. $\Omega_{\text{bd}}$ to $B(0,1)$) such that $T(\infty) = \infty$ and $T'(\infty) \in \mathbb{R}_{+}^{*}$ (resp. $T(z_{0}) = 0$ and $T'(z_{0}) \in \mathbb{R}_{+}^{*}$, for a $z_{0} \in \Omega_{\text{bd}}$). We remind that the last conditions at infinity mean $T(z) = \beta z + O(1)$ at infinity, where $\beta \in \mathbb{R}_{+}^{*}$.

**Theorem 2.1.** Assume that $\partial \Omega$ is a $C^{1,1}$ Jordan curve, except in a finite number of points $z_{1}, z_{2}, \ldots, z_{n}$ where $\partial \Omega$ is a corner of angle $\alpha_{i} > \frac{\pi}{2}$ (i.e. $\Omega$ coincides locally with the sector $\{z_{i} + (r \cos \theta, r \sin \theta) ; r > 0, \theta_{i} < \theta < \theta_{i} + \alpha_{i}\}$). Then the biholomorphism $T$ defined above satisfies

- $T^{-1}$ and $T$ extend continuously up to the boundary;
- $DT^{-1}$ extends continuously up to the boundary, except at the points $T(z_{i})$ with $\alpha_{i} < \pi$ where $DT^{-1}(y)$ behaves like $1/|y - T(z_{i})|^{1 - \alpha_{i}/\pi}$;
- $DT$ extends continuously up to the boundary, except at the points $z_{i}$ with $\alpha_{i} > \pi$ where $DT(x)$ behaves like $1/|x - z_{i}|^{1 - \alpha_{i}/\pi}$;
- $D^{2}T$ belongs to $L_{\text{loc}}^{p}(\Omega)$ for any $p < 4/3$.

**Proof.** As $\partial \Omega$ is $C^{0,\alpha}$, the Kellogg-Warschawski theorem (Theorem 3.6 in [24]) states directly that $T$ and $T^{-1}$ is continuous up to the boundary. For the behavior of the derivatives, we use the classical elliptic theory: let

$$u(x) := \ln |T(x)|.$$

As $T$ is holomorphic, we have that

$$\Delta u = 0 \text{ in a neighborhood of } \partial \Omega \text{ and } u = 0 \text{ on } \partial \Omega.$$

To localize near each corners, we can introduce a smooth cutoff function $\chi$ supported in a small neighborhood of $z_{i}$. Therefore, we are exactly in the setting of elliptic studies:

$$\Delta(u\chi) = f \in C^{\infty} \text{ in } O_{i} \text{ and } u = 0 \text{ on } \partial O_{i},$$

(2.1)

where $O_{i}$ is the sector $\{z_{i} + (r \cos \theta, r \sin \theta) ; r > 0, \theta_{i} < \theta < \theta_{i} + \alpha_{i}\}$. Roughly, the idea could be to compose by $z^{\pi/\alpha_{i}}$ in order to map the sector on the half plane, where the solution of the elliptic problem $g$ is smooth. Therefore, we would have that

$$u\chi = g \circ z^{\pi/\alpha_{i}},$$

which would imply that

$$\nabla u \approx r^{\pi/\alpha_{i} - 1} \text{ and } \nabla^{2}u \approx r^{\pi/\alpha_{i} - 2}.$$  

(2.2)

This approach is developed in [19] to prove Theorem 2.1.

Let us perform here a proof based on the so-called shift theorem in non-smooth domain (see the preface of [9]): there exist some numbers $c_{k}$ such that

$$u\chi = \sum c_{k}v_{k} \in W^{m+2,p}(O_{i} \cap B(0,R)), \forall R > 0$$

where the $k$ in the summation ranges over all integers such that

$$\pi/\alpha_{i} \leq k\pi/\alpha_{i} < m + 2 - 2/p$$

and with

- $v_{k} = r^{k\pi/\alpha_{i}} \sin(k\pi\theta/\alpha_{i})$ if $k\pi/\alpha_{i}$ is not an integer;
- $v_{k} = r^{k\pi/\alpha_{i}}[\ln r \sin(k\pi\theta/\alpha_{i}) + \theta \cos(k\pi\theta/\alpha_{i})]$ if $k\pi/\alpha_{i}$ is an integer.
In this theorem, $r$ denotes the distance between $x$ and $z_i$: $r := |x - z_i|$.

We apply it for $m = 1$ and $p = 2$. As $H^3_{\text{loc}}(\mathbb{R}^2)$ embeds in $C^0$, we recover that $u$ is continuous up to the boundary.

If $\pi < \alpha_i \leq 2\pi$ then $1/2 \leq \pi/\alpha_i < 1$, which gives that $\pi/\alpha_i$ cannot be an integer. Then, the shift theorem states that $D(u(x)) - \sum c_k Dv_k$ belongs to $H^2_{\text{loc}}(\partial \Omega)$, so to $C^0$. Thanks to the formula of $v_k$, we note that $Dv$ is continuous up to the boundary, except near $z_i$ where $Dv = \mathcal{O}(r^{\pi/\alpha_i - 1})$. Next, we derive once more to obtain that $D^2(u(x)) - \sum c_k D^2 v_k$ belongs to $H^1_{\text{loc}}(\partial \Omega)$, so it belongs to $L^p_{\text{loc}}(\partial \Omega)$ for any $p$. As $\sum c_k D^2 v_k = \mathcal{O}(r^{\pi/\alpha_i - 2})$, with $2 - \pi/\alpha_i < 3/2$, then $D^2 u$ belongs to $L^p_{\text{loc}}(\partial \Omega)$ for any $p < 4/3$.

The case $\alpha_i = \pi$ is not interesting because we assume that $z_i$ is a singular point.

If $\pi/2 < \alpha_i < \pi$, then we remark that $\pi/\alpha_i$ is not an integer and that $k\pi/\alpha_i < 2$ is obtained only for $k = 1$. We apply the above argument to see that $u$ and $Dv$ is continuous up to the boundary, and $D^2 u$ belongs to $L^p_{\text{loc}}(\partial \Omega)$ for any $p < 2$.

Therefore, the shift theorem gives that $u = \mathcal{O}(r^{\pi/\alpha_i})$ and $Dv = \mathcal{O}(r^{\pi/\alpha_i - 1})$ if all the angles are greater than $\pi/2$. We show now that $Dv$ and $DT$ have the same behavior.

On the one hand, we differentiate $u$

$$\nabla u(x) = \frac{T(x)}{|T(x)|^2} DT(x)$$

hence

$$|\nabla u(x)|_{\infty} \leq 4 |DT(x)|_{\infty} \tag{2.3}$$

where $|A|_{\infty} = \max |a_{ij}|$. Indeed, by continuity of $T$, we have that $|T(x)| = \sqrt{|T_1(x)|^2 + |T_2(x)|^2} \geq 1/2$ near the boundary.

On the other hand,

$$\frac{T(x)}{|T(x)|^2} = \nabla u(x) DT(x)^{-1}.$$

By continuity of $T$, there exists a neighborhood of $\partial \Omega$ such that $|T(x)| \leq 2$. Then, we have near the boundary

$$\frac{1}{2} \leq \frac{1}{|T(x)|} \leq 2 \sqrt{2} |\nabla u(x)|_{\infty} |DT(x)^{-1}|_{\infty}.$$

Moreover, as $T$ is holomorphic, $DT$ is a $2 \times 2$ matrix on the form

$$\begin{pmatrix} a & b \\ -b & a \end{pmatrix}.$$ 

We deduce from this form that $DT(x)^{-1} = \frac{1}{\det DT(x)} DT(x)^T$. We use that $\det DT(x) = a^2 + b^2 \geq |DT(x)|_{\infty}^2$ to get

$$|DT(x)|_{\infty} \leq 4 \sqrt{2} |\nabla u(x)|_{\infty} \tag{2.4}.$$ 

Putting together (2.2), (2.3) and (2.4), we can conclude on the behavior of $DT$.

Differentiating once more, we obtain the result for $D^2 T$.

Finally, as $u = \mathcal{O}(r^{\pi/\alpha_i})$, we state that

$$|T(x)| = 1 + \mathcal{O}(r^{\pi/\alpha_i}), \quad T(z_1) = T(z_1) + \mathcal{O}(|x - z_i|^{\pi/\alpha_i}), \quad T^{-1}(y) = z_i + \mathcal{O}(|y - T(z_i)|^{\alpha_i/\pi}).$$

Next, we use the fact that $DT(x) = \mathcal{O}(|x - z_i|^{\pi/\alpha_i - 1})$ to write

$$DT^{-1}(y) = \left(DT(T^{-1}(y))\right)^{-1} = \mathcal{O}\left(\frac{1}{|y - T(z_i)|^{\alpha_i/\pi - 1}}\right) = \mathcal{O}(|y - T(z_i)|^{\alpha_i/\pi - 1})$$

which ends the proof.  

We recover the result of the exterior of the curve (see [16]): $\alpha = 2\pi$ gives that $DT$ behaves like $1/\sqrt{|x - z_i|}$. In that paper, we found the behavior of $DT$ thanks to the explicit formula of $T$. The Joukowski function $G(z) = \frac{1}{2}(z + \frac{1}{z})$ maps the exterior of the unit disk to the exterior of the segment $[(-1, 0), (1, 0)]$. Then, in the case of this segment $T = G^{-1}$ and we can compute that

$$DT(z) = z \pm \frac{z}{\sqrt{z^2 - 1}}.$$
We also note that $DT$ near a corner ($\alpha < 2\pi$) is less singular than around a cusp.

**Remark 2.2.** This kind of theorem could be useful to remark that the velocity in the exterior of a square blows-up like $1/|x|^{1/3}$ near the corner. However, the only things that we need in the sequel are:

- there exists $p_0 > 2$ such that det $DT^{-1}$ belongs to $L^p_{\text{loc}}(\Omega)$; property holding true if all the corners $z_i$ have angles $\alpha_i$ greater than $\pi/2$ (as in Theorems 1.2-1.3);
- $DT$ belongs to $L^p_{\text{loc}}(\Omega)$ for any $p < 4$ and $D^2T$ belongs to $L^p_{\text{loc}}(\Omega)$ for any $p < 4/3$.

Therefore, Theorems 1.2-1.3 can be applied for any simply connected domain (or exterior of a simply connected compact set) such that the two previous points hold true. For a sake of clarity, we express the theorems when the boundary is locally a corner at $z_i$, but we can generalize for $\Omega$ such that $\partial\Omega$ is a $C^{1,1}$ Jordan curve except in a finite number of points $z_i$. In these points, we would define

$$\alpha_i := \lim_{s \to 0} \arg(\Gamma'(s_i + s), \Gamma'(s_i - s)) + \pi,$$

where $\Gamma$ is a parametrization of $\partial\Omega$ and $z_i = \Gamma(s_i)$. Indeed, up to a smooth change of variable, the Laplace equation in $\Omega$ turns into a divergence form elliptic equation in the exterior of a corner, and we would use results related to elliptic equations in polygons, see [15].

The previous theorem gives the behavior of $T$ near the boundary. In the case of an unbounded domain (as in Theorem 1.3), we have a Laurent expansion:

$$T(z) = \beta z + \tilde{\beta} + h(z) \quad (2.5)$$

where

$$(\beta, \tilde{\beta}) \in \mathbb{R}^+ \times \mathbb{C}, \ h(z) = \mathcal{O}\left(\frac{1}{|z|}\right) \text{ and } h'(z) = \mathcal{O}\left(\frac{1}{|z|^2}\right), \text{ as } |z| \to \infty.$$  

Moreover, $T^{-1}$ admits a similar development.

### 2.2. Biot-Savart Law

Assume $\omega_0 := \text{curl} \ u_0 \in L^1 \cap L^\infty$, it will be useful to recover the velocity in terms of the vorticity.

Let $\Omega$ be the exterior (resp. the interior) of a connected, simply connected compact subset of the plane, the boundary of which is a Jordan curve. Let $T$ be a biholomorphism from $\Omega$ to $(\mathbb{B}(0,1))^c$ (resp. $B(0,1)$) such that $T(\infty) = \infty$ (resp. $T(z_0) = 0$).

We denote by $G_{\Omega} = G_{\Omega}(x,y)$ the Green’s function, whose the formula is:

$$G_{\Omega}(x,y) = \frac{1}{2\pi} \ln \frac{|T(x) - T(y)|}{|T(x) - T(y)^*||T(y)|}$$

writing $x^* = \frac{x}{|x|^2}$. The Green’s function verifies:

$$\Delta_y G_{\Omega}(x,y) = \delta(y - x) \ \forall x, y \in \Omega, \ G_{\Omega}(x,y) = 0 \ \forall (x,y) \in \Omega \times \partial\Omega, \ G_{\Omega}(x,y) = G_{\Omega}(y,x) \ \forall x, y \in \Omega.$$

The kernel of the Biot-Savart law is $K_\Omega = K_\Omega(x,y) := \nabla_z G_{\Omega}(x,y)$. With $(x_1, x_2)\perp = \left(\frac{-x_2}{x_1}\right)$, the explicit formula of $K_\Omega$ is given by

$$K_\Omega(x,y) = \frac{1}{2\pi} D^2T(x) \left(\frac{(T(x) - T(y))^\perp}{|T(x) - T(y)|^2} - \frac{(T(x) - T(y)^*)^\perp}{|T(x) - T(y)^*|^2}\right)$$

and we introduce the notation

$$K_\Omega[f] = K_\Omega[f](x) := \int_\Omega K_\Omega(x,y)f(y)dy,$$

with $f \in C^\infty_c(\Omega)$.

We will use several times the following general relation:

$$\left|\frac{a}{|a|^2} - \frac{b}{|b|^2}\right| = \frac{|a - b|}{|a||b|}, \quad (2.6)$$
which can be easily checked by squaring both sides. Using this relation with the behavior of \( T \) at infinity (see (2.5)), we obtain for large \(|x|\) that

\[
|K_{\Omega}[f]|(x) \leq \frac{C_2}{|x|^2},
\]

where \( C_2 \) depends on \( \|f\|_{L^\infty} \) and the size of the support of \( f \).

The vector field \( u = K_{\Omega}[f] \) is a solution of the elliptic system:

\[
\begin{align*}
\text{div} \ u &= 0 \quad \text{in } \Omega, \\
\text{curl} \ u &= f \quad \text{in } \Omega, \\
\ u \cdot \hat{n} &= 0 \quad \text{on } \partial\Omega, \\
\lim_{|x| \to \infty} |u| &= 0 \quad \text{(only for exterior domains)}.
\end{align*}
\]

If we consider a non-simply connected domain (as \( \Omega_{\text{ext}} \) satisfying (D-ext)), the previous system has several solutions. To uniquely determine the solution, we have to take into account the circulation. Let \( \hat{n} \) be the unit normal exterior to \( \Omega \). In what follows all contour integrals are taken in the counterclockwise sense, so that \( \oint_{\partial C} F \cdot \hat{\tau} \, ds = -\oint_{\partial C} F \cdot \hat{n} \perp \, ds \). By standard regularity arguments in domains with corners, we state that a vector field \( u \), which is divergence free and with a bounded vorticity, is continuous up to the boundary, except near the corners of angles greater than \( \pi \) where it may blow up as \( 1/|x - z_i|^{\frac{1}{2} - \pi/\alpha} \). In particular the trace on the boundary is integrable which implies that the quantity \( \oint_{\partial C} u \cdot \hat{\tau} \, ds \) is well defined.

The harmonic vector field

\[
H_{\Omega_{\text{ext}}}(x) = \frac{1}{2\pi} \nabla^\perp \ln |T(x)| = \frac{1}{2\pi} DT^T(x) \frac{T(x)^\perp}{|T(x)|^2}
\]

is the unique vector field verifying

\[
\begin{align*}
\text{div} \ H_{\Omega_{\text{ext}}} &= \text{curl} \ H_{\Omega_{\text{ext}}} = 0 \quad \text{in } \Omega_{\text{ext}}, \\
H_{\Omega_{\text{ext}}} \cdot \hat{n} &= 0 \quad \text{on } \partial C, \\
H_{\Omega_{\text{ext}}}(x) &\to 0 \quad \text{as } |x| \to \infty, \\
\oint_{\partial C} H_{\Omega_{\text{ext}}} \cdot \hat{\tau} \, ds &= 1.
\end{align*}
\]

Using (2.5), we note that

\[
H_{\Omega_{\text{ext}}}(x) = O(1/|x|) \quad \text{at infinity. (2.7)}
\]

Therefore, putting together the previous properties, we obtain the existence part of the following.

**Proposition 2.3.** Let \( \omega \in L^\infty_c(\Omega) \) and \( \gamma \in \mathbb{R} \). If \( \Omega_{\text{bd}} \) satisfies (D-bd), then there is a unique solution \( u \) of

\[
\begin{align*}
\text{div} \ u &= 0 \quad \text{in } \Omega_{\text{bd}} \\
\text{curl} \ u &= \omega \quad \text{in } \Omega_{\text{bd}} \\
\ u \cdot \hat{n} &= 0 \quad \text{on } \partial\Omega_{\text{bd}}
\end{align*}
\]

which is given by

\[
u(x) = K_{\Omega_{\text{bd}}}[\omega](x).
\]

(2.8)

If \( \Omega_{\text{ext}} \) satisfies (D-ext), then there is a unique solution \( u \) of

\[
\begin{align*}
\text{div} \ u &= 0 \quad \text{in } \Omega_{\text{ext}} \\
\text{curl} \ u &= \omega \quad \text{in } \Omega_{\text{ext}} \\
u(x) &\to 0 \quad \text{as } |x| \to \infty \\
\oint_{\partial C} u \cdot \hat{\tau} \, ds &= \gamma
\end{align*}
\]

which is given by

\[
u(x) = K_{\Omega_{\text{ext}}} [\omega](x) + (\gamma + \int \omega) H_{\Omega_{\text{ext}}}(x).
\]

(2.9)

\[\text{1} \text{see e.g. [11].}\]
Concerning the uniqueness, we can see e.g. [13 Lem 2.14] (see also [11 Prop 2.1]). We take advantage of this explicit formula to give estimates on the kernel. We introduce

\[ R[\omega](x) := \int_{\Omega} \left( \frac{(\nabla u - \nabla u_T) \cdot \nabla u}{|\nabla u - \nabla u_T|^2} \cdot \nabla u_T \right) \omega(y) dy, \]

so that (2.9) reads

\[ u(x) = \frac{1}{2\pi} DT^T(x) \left( R[\omega](x) + (\gamma + \int \omega \frac{\nabla u_T}{|\nabla u_T|^2}) \right). \tag{2.10} \]

**Proposition 2.4.** Let assume that \( \omega \) belongs to \( L^1 \cap L^\infty(\Omega) \). If all the angles of \( \Omega \) are greater than \( \pi/2 \), then there exist \((C,a) \in \mathbb{R}^+ \times (0,1/2]\) depending only on the shape of \( \Omega \) such that

\[ \|R[\omega]\|_{L^\infty(\Omega)} \leq C(\|\omega\|_{L^1}^{1/2}\|\omega\|_{L^\infty}^{1/2} + \|\omega\|_{L^1}^a\|\omega\|_{L^\infty}^{1-a} + \|\omega\|_{L^1}). \]

Moreover, \( R[\omega] \) is continuous up to the boundary.

In the case where \( \mathcal{C} \) is a Jordan arc, the uniform bound is proved in [16 Lem 4.2] and the continuity in [16 Prop 5.7]. The proof here is almost the same, except that we have to take care that \( DT^{-1} \) is not bounded if there is an angle less than \( \pi \) (see Theorem 2.1). For completeness, we write the details in Section 6. In this proof, we can understand why we have assumed that the angles are greater than \( \pi/2 \): we use that \( \det DT^{-1} \) belongs to \( L^{p_0} \) for some \( p_0 > 2 \) (see Remark 2.2).

**2.3. Existence and properties of weak solutions.** The goal of this subsection is to derive some properties about a weak solution obtained in Theorem 1.1. We will also establish similar formulations verified by extensions on the full plane.

a) **Weak solution in an unbounded domain.**

We begin by the hardest case: let \( \Omega_{ext} \) verify (D-ext) with angles greater than \( \pi/2 \). Then, there exists some pieces of the boundary which are smooth, implying that the capacity of \( \mathcal{C} \) is greater than 0 (see e.g. [7 Prop 6]). Therefore, Theorem 1.1 is a direct consequence of [7 Theo 2].

We know the existence of a global weak solution. We search now some features of such a solution. Let \( u_0 \) satisfying (1.5) and \( u \) be a global weak solution of (1.1) in the sense of (1.7) and (1.8) such that

\[ u \in L^\infty_{loc}(\mathbb{R}^+; L^2_{loc}(\Omega_{ext})), \quad \omega := \nabla \times u \in L^\infty(\mathbb{R}^+; L^1 \cap L^\infty(\Omega_{ext})). \]

As said before, by standard regularity arguments in domains with corners, we can define the real number

\[ \gamma_0 := \int_{\partial \mathcal{C}} u_0 \cdot \hat{\tau} ds. \tag{2.11} \]

We have reminded in the previous subsection that we can reconstruct the velocity in terms of the vorticity and the circulation:

\[ u_0(x) = K_{\Omega_{ext}}[\omega_0](x) + (\gamma_0 + \int \omega_0) H_{\Omega_{ext}}(x). \]

From the definition of weak solutions, we know that the quantities \( ||\omega(t,\cdot)||_{L^1 \cap L^\infty(\Omega_{ext})} \) and \( \int \omega(t,\cdot) \) are bounded in \( \mathbb{R}^+ \). Moreover, we infer that the circulation

\[ \gamma(t) := \int_{\partial \mathcal{C}} u(t,\cdot) \cdot \hat{\tau} ds \]

is bounded locally in time. Indeed, we have

\[ \gamma(t) = \oint_{J} u(t,\cdot) \cdot \hat{\tau} ds - \int_{A} \omega(t,\cdot) dx, \]

where \( J \) is a Jordan curve in \( \Omega_{ext} \) surrounding \( \mathcal{C} \) and \( A = \Omega_{ext} \cap (\text{bounded connected component of } \mathbb{R}^2 \setminus J) \). Considering a smooth neighborhood \( K \) of \( J \) we can estimate \( \oint_{J} u(t,\cdot) \cdot \hat{\tau} ds \) by \( ||u(t,\cdot)||_{W^{1,3}(K)} \leq C(||u(t,\cdot)||_{L^2(K)} + ||\omega(t,\cdot)||_{L^\infty(K)}) \) which gives:

\[ \gamma \in L^\infty_{loc}((0,\infty)). \tag{2.12} \]
Putting together this estimate, Remark 2.2 and Proposition 2.4, we obtain from (2.10) that
\[ u \in L^\infty_{\text{loc}}([0, \infty); L^p_{\text{loc}}(\Omega_{\text{ext}})), \forall p < 4. \]  
(2.13)

Let us derive a formulation verified by \( \omega \). We note that for any test function \( \varphi \in \mathcal{D}([0, \infty) \times \Omega_{\text{ext}}; \mathbb{R}) \), then \( \psi := \nabla^\perp \varphi \) belongs to the set of admissible test functions for (1.8), and an integration by parts gives
\[ \forall \varphi \in \mathcal{D}([0, \infty) \times \Omega_{\text{ext}}; \mathbb{R}), \quad \int_0^\infty \int_{\Omega_{\text{ext}}} (\omega \cdot \partial_t \varphi + \omega u \cdot \nabla \varphi) = -\int_{\Omega_{\text{ext}}} \omega_0 \varphi(0, \cdot). \]
(2.14)

Then, \((\omega, u)\) verifies the transport equation
\[ \partial_t \omega + u \cdot \nabla \omega = 0 \]
in the sense of distribution (2.14) in \( \Omega_{\text{ext}} \). We need a formulation on \( \mathbb{R}^2 \). For that, we denote by \( \tilde{\omega} \) (respectively \( \tilde{u} \)) the extension of \( \omega \) (respectively \( u \)) to \( \mathbb{R}^2 \) by zero in \( \Omega_{\text{ext}}^c \). Let us check that it verifies the transport equation for any test function \( \varphi \in C_c^\infty([0, \infty) \times \mathbb{R}^2) \).

**Proposition 2.5.** Let \((\omega, u)\) a weak solution to the Euler equations in \( \Omega_{\text{ext}} \). Then, the extensions \( \tilde{\omega}, \tilde{u} \) verify in the sense of distribution
\[
\begin{align*}
\partial_t \tilde{\omega} + \tilde{u} \cdot \nabla \tilde{\omega} &= 0, \quad \text{in } (0, \infty) \times \mathbb{R}^2, \\
\text{div } \tilde{u} &= 0 \text{ and curl } \tilde{u} = \tilde{\omega} + g_{\omega, \gamma}(s) \delta_{\partial C}, \quad \text{in } [0, \infty) \times \mathbb{R}^2, \\
|\tilde{u}| &\to 0, \quad \text{as } |x| \to \infty, \\
\tilde{\omega}(x, 0) &= \tilde{\omega}_0(x), \quad \text{in } \mathbb{R}^2,
\end{align*}
\]
(2.16)

where \( \delta_{\partial C} \) is the Dirac function along the curve and with
\[
g_{\omega, \gamma}(t, x) = u(t, x) \cdot \mathbf{T}
= \left[ \lim_{\rho \to 0^+} K_{\Omega_{\text{ext}}}[\tilde{\omega}(t, \cdot)](x - \rho \mathbf{n}) + (\gamma(t) + \int \tilde{\omega}(t, \cdot)) H_{\Omega_{\text{ext}}}(x - \rho \mathbf{n}) \right] \mathbf{T}
\]
(2.17)

**Proof.** The third and fourth points are obvious. The second point is a classical computation concerning tangent vector fields: there is no additional term on the divergence, whereas it appears on the curl the jump of the tangential velocity (see e.g. the proof of Lemma 5.8 in [10]).

Concerning the first point, we have to consider the case of a test function whose the support meets the boundary. Let \( \varphi \in C_c^\infty([0, \infty) \times \mathbb{R}^2) \). We introduce \( \Phi \) a non-decreasing function which is equal to 0 if \( s \leq 1 \) and to 1 if \( s \geq 2 \). Let
\[
\Phi^\varepsilon(x) := \Phi(\frac{|T(x)| - 1}{\varepsilon}).
\]

We note that
- it is a cutoff function of an \( \varepsilon \)-neighborhood of \( \mathcal{C} \), because \( T \) is continuous up to the boundary (see Theorem 2.1),
- we have \( \nabla \Phi^\varepsilon : H_{\Omega_{\text{ext}}} \equiv 0 \), because \( H_{\Omega_{\text{ext}}}(x) = \frac{\nabla^\perp |T(x)|}{|T(x)|} \) (see Subsection 2.2),
- the Lebesgue measure of the support of \( \nabla \Phi^\varepsilon \) is \( o(\sqrt{\varepsilon}) \). Indeed the support of \( \nabla \Phi^\varepsilon \) is contained in the subset \( \{ x \in \Omega_\varepsilon^c | 1 + \varepsilon \leq |T(x)| \leq 1 + 2\varepsilon \} \). The Lebesgue measure can be estimated thanks to Remark 2.2
\[
\int_{1 + \varepsilon \leq |T(x)| \leq 1 + 2\varepsilon} dx = \int_{1 + \varepsilon \leq |z| \leq 1 + 2\varepsilon} |\det(DT^{-1})|(z) dz \leq \sqrt{\varepsilon} \| \det(DT^{-1}) \|_{L^2(B(1 + 2\varepsilon) \setminus B(1, 1))},
\]

where the norm in the right hand side term tends to zero as \( \varepsilon \to 0 \) (by the dominated convergence theorem).

Another interesting property is the fact that the velocity is tangent to the boundary whereas \( \nabla \Phi^\varepsilon \) is normal. Indeed, we claim the following.
Lemma 2.6. If $\omega$ belongs to $L^\infty(\mathbb{R}^+; L^1 \cap L^\infty(\Omega_{ext}))$ then
\[ u \cdot \nabla \Phi^\varepsilon \to 0 \text{ strongly in } L^1(\mathbb{R}^2), \]
uniformly in time, when $\varepsilon \to 0$.

This property is not so obvious, because $|u \cdot \nabla \Phi^\varepsilon| \approx \frac{|DT|^2}{\varepsilon} R[\omega] \Phi^\varepsilon\left(\frac{|T(x)|-1}{\varepsilon}\right)$ with $\Phi^\varepsilon\left(\frac{|T(x)|-1}{\varepsilon}\right) \to O(\varepsilon)$ (in the case where $DT^{-1}$ is bounded) and $DT$ blowing up. The perpendicular argument is crucial here and we use the explicit formula (2.9) to show the cancellation effect. This lemma is proved in the case where $C$ is a Jordan arc in [16, Lem 4.6]. For a sake of completeness, we give the general proof in Section 6.

As $\Phi^\varepsilon \varphi$ belongs to $C^\infty_c(0, \infty) \times \Omega_{ext}$ for any $\varepsilon > 0$, we can write that $(\omega, u)$ is a weak solution in $\Omega_{ext}$:
\[ \int_0^\infty \int_{\mathbb{R}^2} (\Phi^\varepsilon \varphi) \omega \, dx \, dt + \int_0^\infty \int_{\mathbb{R}^2} \nabla (\Phi^\varepsilon \varphi) \cdot u \omega \, dx \, dt + \int_{\mathbb{R}^2} (\Phi^\varepsilon \varphi)(0, x) \omega_0(x) \, dx = 0. \]

As $\omega \in L^\infty(L^1 \cap L^\infty)$, it is obvious that the first and the third integrals converge to
\[ \int_0^\infty \int_{\mathbb{R}^2} \varphi \tilde{\omega} \, dx \, dt \text{ and } \int_{\mathbb{R}^2} \varphi(0, x) \tilde{\omega}_0(x) \, dx \]
as $\varepsilon \to 0$. Concerning the second integral, we have
\[ \int_0^\infty \int_{\mathbb{R}^2} \nabla (\Phi^\varepsilon \varphi) \cdot u \omega \, dx \, dt = \int_0^\infty \int_{\mathbb{R}^2} \varphi(\nabla \Phi^\varepsilon \cdot u) \omega \, dx \, dt + \int_{\mathbb{R}^2} \Phi^\varepsilon \nabla \varphi \cdot u \omega \, dx \, dt. \]
The first right hand side term tends to zero because $\nabla \Phi^\varepsilon \cdot u \to 0$ in $L^1(\mathbb{R}^2)$ and $\omega \in L^\infty(\mathbb{R}^+ \times \mathbb{R}^2)$. The second right hand side term converges to
\[ \int_0^\infty \int_{\mathbb{R}^2} \nabla \varphi \cdot \tilde{u} \tilde{\omega} \, dx \, dt \]
because $u$ belongs to $L^2(\text{supp } \varphi \cap (\mathbb{R}^+ \times \overline{\Omega}_{ext}))$ (see (2.13)). Putting together these limits, we obtain that:
\[ \int_0^\infty \int_{\mathbb{R}^2} \varphi \tilde{\omega} \, dx \, dt + \int_0^\infty \int_{\mathbb{R}^2} \nabla \varphi \cdot \tilde{u} \tilde{\omega} \, dx \, dt + \int_{\mathbb{R}^2} \varphi(0, x) \tilde{\omega}_0(x) \, dx = 0, \]
which ends the proof. \(\square\)

The goal of the following is to prove that the $L^p$ norm, the total mass of the vorticity and the circulation are conserved quantities.

In a domain with smooth boundaries and $\omega_0 \in C^\infty_c$, $\omega$ is a solution of a linear transport equation with a regular velocity, and the conservation of the previous quantities is classical. The main point is to remark that in our case $\omega$ is a renormalized solution in the sense of DiPerna and Lions (see [3]). Namely, our purpose is to state that if $\tilde{\omega}$ solves the linear transport equation (2.16) with a given velocity field $\tilde{u}$, then so does $\beta(\tilde{\omega})$ for a suitable smooth function $\beta$. This follows from the theory developed in [3] where they need that the velocity field belongs to $L^1_{loc}(\mathbb{R}^+, W^{1,1}_{loc}(\mathbb{R}^2) \cap L^1_{loc}(\mathbb{R}^+, L^1(\mathbb{R}^2) + L^\infty(\mathbb{R}^2)))$ and that $\text{div } u$ is bounded. Let us check that we are in this setting.

Lemma 2.7. Let $(\omega, u)$ be a global weak solution in $\Omega_{ext}$. Then we have that
\[ u \in L^\infty_{loc}(\mathbb{R}^+, W^{1,p}_{loc}(\Omega_{ext})) \cap L^\infty_{loc}(\mathbb{R}^+, L^1(\Omega_{ext}) + L^\infty(\Omega_{ext})), \]
for any $p \in [1, 4/3)$.

We use the explicit form of the velocity (2.9): $u(x) = DT(x)h(T(x))$, where $h$ looks like the Biot-Savart operator in $\mathbb{R}^2$. Therefore, Lemma 2.7 follows from the fact that $DT$ belongs to $W^{1,p}_{loc}(\Omega_{ext})$ for any $p < 4/3$ (see Theorem 2.1), and thanks to Proposition 2.4 and the Calderón-Zygmund inequality. The proof is written in [17] in the case where $C$ is a Jordan arc. We generalize it in Section 6.

In Proposition 2.5, we have proved that $\tilde{\omega}$ verifies the transport equation with velocity $\tilde{u}$, but actually it verifies the transport equation with any extension $\tilde{u}$ of $u$ (indeed, $\tilde{\omega} \equiv 0$ outside $\Omega_{ext}$). Let us introduce a relevant extension of $u$ in order to apply the renormalized theory. We fix $p \in (1, 4/3)$.
and \( R > 0 \) such that \( \partial \Omega_{\text{ext}} \subset B(0,R) \). We readily check that \( \tilde{\Omega}_{\text{ext}} := \Omega_{\text{ext}} \cap B(0,R + 1) \) verifies the Uniform Cone Condition (see [1], Par. 4.8 for the precise definition). Therefore by Theorem [II] Theo. 5.28] for any \( p \in (1,4/3) \) there exists a simple \((2,p)\)-extension operator \( E(p) \) from \( W^{2,p}(\tilde{\Omega}_{\text{ext}}) \) to \( W^{2,p}(\mathbb{R}^2) \), namely there exists \( K(p) > 0 \) such that for any \( v \in W^{2,p}(\tilde{\Omega}_{\text{ext}}) \) we have
\[
E(p)v = v \text{ a.e. in } \tilde{\Omega}_{\text{ext}}, \quad \|E(p)v\|_{W^{2,p}(\mathbb{R}^2)} \leq K(p)\|v\|_{W^{2,p}(\tilde{\Omega}_{\text{ext}})}.
\]

Let us consider the stream function \( \psi \) of \( u \), namely the function verifying:
\[
u = \nabla \perp \psi \text{ in } \Omega_{\text{ext}}, \quad \psi = 0 \text{ on } \partial \Omega_{\text{ext}}.
\]

By the Poincaré inequality, we have that
\[
\psi \in L^{\infty}(\mathbb{R}^+, W^{2,p}(\tilde{\Omega}_{\text{ext}})).
\]

Then we define \( \chi \) a smooth cutoff function such that \( \chi \equiv 1 \) on \( B(0,R) \) and \( \chi \equiv 0 \) on \( B(0,R + 1) \), and we put
\[
\tilde{\psi} := E(p)(\chi \psi), \quad \tilde{u} := \nabla \perp \tilde{\psi}, \quad \tilde{u}|_{B(0,R)} = u.
\]

Obviously, we have:
\[
\tilde{u} = u \text{ a.e. in } \Omega_{\text{ext}}, \quad \text{div} \tilde{u} = 0 \text{ on } \mathbb{R}^2,
\]

and
\[
\tilde{u} \in L^{\infty}(\mathbb{R}^+, W^{1,p}(B(0,R)));
\]

hence
\[
\tilde{u} \in L^{\infty}(\mathbb{R}^+, W^{1,1}(\mathbb{R}^2)) \cap L^{\infty}(\mathbb{R}^+, L^1(\mathbb{R}^2) + L^\infty(\mathbb{R}^2)).
\]

Therefore, \( \tilde{\omega} \) is a renormalized solution, namely:

**Lemma 2.8.** Let \( \tilde{\omega} \) be a solution of the linear transport equation in \( \mathbb{R}^2 \) with a velocity \( \tilde{u} \) and initial datum \( \omega_0 \). Let \( \beta : \mathbb{R} \rightarrow \mathbb{R} \) be a smooth function such that
\[
|\beta'(t)| \leq C(1 + |t|^p), \quad \forall t \in \mathbb{R},
\]

for some \( p \geq 0 \). Then \( \beta(\tilde{\omega}) \) is a solution of the transport equation in \( \mathbb{R}^2 \) (in the sense of distribution) with the velocity \( \tilde{u} \) and initial datum \( \beta(\omega_0) \).

We recall that \( \tilde{\omega} \) denotes the extension of \( \omega \) by zero in \( C \), and the previous lemma means that, for any \( \Phi \in C_c^\infty([0,\infty) \times \mathbb{R}^2) \), we have
\[
\frac{d}{dt} \int_{\mathbb{R}^2} \beta(\omega)\Phi(t,x) \, dx = \int_{\mathbb{R}^2} \beta(\omega)(\partial_t \Phi + u \cdot \nabla \Phi) \, dx \tag{2.18}
\]
in the sense of distributions on \( \mathbb{R}^+ \). Now, we write a remark from [18] in order to establish some desired properties for \( \omega \).

**Remark 2.9.** (1) Since the right-hand side in (2.18) belongs to \( L^1_{\text{loc}}(\mathbb{R}^+) \), the equality holds in \( L^1_{\text{loc}}(\mathbb{R}^+) \). With this sense, (2.18) actually still holds when \( \Phi \) is smooth, bounded and has bounded first derivatives in time and space. In this case, we have to consider smooth functions \( \beta \) which in addition satisfy \( \beta(0) = 0 \), so that \( \beta(\omega) \) is integrable. This may be proved by approximating \( \Phi \) by smooth and compactly supported functions \( \Phi_n \) for which (2.18) applies, and by letting then \( n \) go to \( +\infty \).

(2) We apply the point (1) for \( \beta(t) = t \) and \( \Phi \equiv 1 \), which gives
\[
\int_{\Omega_{\text{ext}}} \omega(t,x) \, dx = \int_{\Omega_{\text{ext}}} \omega_0(x) \, dx \text{ for all } t > 0. \tag{2.19}
\]

(3) We let \( 1 \leq p < +\infty \). Approximating \( \beta(t) = |t|^p \) by smooth functions and choosing \( \Phi \equiv 1 \) in (2.18), we deduce that for a solution \( \omega \) to (2.15), the maps \( t \mapsto \|\omega(t)\|_{L^p(\Omega_{\text{ext}})} \) are continuous and constant. In particular, we have
\[
\|\omega(t)\|_{L^1(\Omega_{\text{ext}})} \equiv \|\omega_0\|_{L^1(\Omega_{\text{ext}})} \text{ and } \|\omega(t)\|_{L^\infty(\Omega_{\text{ext}})} \equiv \|\omega_0\|_{L^\infty(\Omega_{\text{ext}})}. \tag{2.20}
\]

As the domain is unbounded, it will be useful to note that \( \omega \) stays compactly supported. Specifying our choice for \( \Phi \) in (2.18), we are led to the following.
\textbf{Proposition 2.10.} Let \((\omega, u)\) be a global weak solution in \(\Omega_{\text{ext}}\) such that
\[
\omega_0 \text{ is compactly supported in } B(0, R_0)
\]
for some positive \(R_0\). For any \(T^*\) fixed, there exists \(C > 0\) such that
\[
\omega(t, \cdot) \text{ is compactly supported in } B(0, R_0 + Ct),
\]
for any \(t \in [0, T^*]\).

The proof in different settings can be found in \cite{13} or in \cite{17}. For a sake of completeness we write the details in Section 6.

Therefore, for \(T^*\) fixed, there exists \(R_1\) such that the support of the vorticity is included in \(B(0, R_1)\) for all \(t \in [0, T^*]\). It implies that \(u\) is harmonic in \(B(0, R_1)^c\) (\(\text{div} \ u = \text{curl} \ u = 0\)), and \((1.1)\) is verified in the strong way on this set. With strong solution, the Kelvin’s circulation theorem can be used, which states that the circulation at infinity is conserved:
\[
\gamma(t) + \int_{\Omega} \omega(t, \cdot) = \gamma^\infty(t) \equiv \gamma_0^\infty = \gamma_0 + \int_{\Omega} \omega_0.
\]

Using the conservation of the total mass \((2.19)\), we obtain that the circulation of the velocity around the obstacle is conserved:
\[
\gamma(t) \equiv \gamma_0, \quad \forall t \in [0, T^*].
\] (2.21)

\textit{b) Weak solution in a bounded domain.}

The previous part can be adapted easily to the bounded case \(\Omega_{\text{bd}}\) verifying (D-bd). In simply connected domain, we do not consider the circulation:
\[
u_0(x) = K_{\Omega_{\text{bd}}} [\omega_0].
\]

As Proposition 2.5 is about the behavior near the boundary, we can check that we obtain exactly the same.

\textbf{Proposition 2.11.} Let \((\omega, u)\) a weak solution to the Euler equations in \(\Omega_{\text{bd}}\). Then, the extensions \(\bar{\omega}, \bar{u}\) verify in the sense of distribution
\[
\begin{aligned}
\partial_t \bar{\omega} + \bar{u} \cdot \nabla \bar{\omega} &= 0, & \quad & \text{in } \mathbb{R}^2 \times (0, \infty) \\
\text{div} \bar{u} &= 0 \text{ and } \text{curl} \bar{u} = \bar{\omega} + g_\omega(s) \delta_{\partial \Omega_{\text{bd}}}, & \quad & \text{in } \mathbb{R}^2 \times [0, \infty) \\
\bar{\omega}(x, 0) &= \bar{\omega}_0(x), & \quad & \text{in } \mathbb{R}^2.
\end{aligned}
\] (2.22)

where \(\delta_{\partial \Omega_{\text{bd}}}\) is the Dirac function along the curve and \(g_\omega\) is :
\[
\begin{aligned}
g_\omega(x) &= -u \cdot \bar{\tau} \\
&= - \left[ \lim_{\rho \to 0^+} K_{\Omega_{\text{bd}}} (\bar{\omega})(x - \rho \hat{n}) \right] \cdot \bar{\tau}. \quad (2.23)
\end{aligned}
\]

Moreover, we can also check that
\[
u \in L^\infty_{\text{loc}} (\mathbb{R}^+, W_1^p(\Omega_{\text{bd}})) \cap L^\infty_{\text{loc}} (\mathbb{R}^+, L^1(\Omega_{\text{bd}})),
\]
for any \(p \in [1, 4/3)\). As in the unbounded case, we fix \(p \in (1, 4/3)\), \(R > 0\) such that \(\partial \Omega_{\text{bd}} \subset B(0, R)\), and \(\chi\) a smooth cutoff function such that \(\chi \equiv 1\) on \(B(0, R)\) and \(\chi \equiv 0\) on \(B(0, R+1)\). We also consider the stream function \(\psi\) of \(\nu\):
\[
u = \nabla^\perp \psi \text{ in } \Omega_{\text{bd}}, \quad \psi = 0 \text{ on } \partial \Omega_{\text{bd}}.
\]

which verifies by the Poincaré inequality
\[
\psi \in L^\infty_{\text{loc}}(\mathbb{R}^+, W_2^p(\Omega_{\text{bd}})).
\]

We put
\[
\tilde{\psi} := \chi E(p) \psi, \quad \tilde{u} = \nabla^\perp \tilde{\psi}.
\]

Obviously, we have:
\[
\tilde{u} = u \text{ a.e. in } \Omega_{\text{bd}}, \quad \text{div} \tilde{u} = 0 \text{ on } \mathbb{R}^2.
\]
As \( \tilde{u} \) is compactly supported in \( B(0, R + 1) \) we infer that 
\[
\tilde{u} \in L_{\text{loc}}^\infty(\mathbb{R}^+; W^{1,p}(\mathbb{R}^2)),
\]
hence 
\[
\tilde{u} \in L_{\text{loc}}^\infty(\mathbb{R}^+, W^{1,1}_{\text{loc}}(\mathbb{R}^2)) \cap L_{\text{loc}}^\infty(\mathbb{R}^+, L^1(\mathbb{R}^2)).
\]

Therefore, \( \omega \) is a renormalized solution which implies that 
\[
\int_{\Omega_{bd}} \omega(t,x) \, dx = \int_{\Omega_{bd}} \omega_0(x) \, dx \text{ for all } t > 0 \tag{2.24}
\]
and 
\[
\|\omega(t)\|_{L^1(\Omega_{bd})} \equiv \|\omega_0\|_{L^1(\Omega_{bd})} \text{ and } \|\omega(t)\|_{L^\infty(\Omega_{bd})} \equiv \|\omega_0\|_{L^\infty(\Omega_{bd})}. \tag{2.25}
\]

3. Lyapunov Method

When the velocity \( u \) is smooth, it gives rise to a flow \( \phi_x(t) \) defined by 
\[
\begin{aligned}
\frac{d}{dt} \phi_x(t) &= u(t, \phi_x(t)) \\
\phi_x(0) &= x \in \mathbb{R}^2.
\end{aligned}
\tag{3.1}
\]
In view of the transport equation (1.3), we have 
\[
\frac{d}{dt} \omega(t, \phi_x(t)) \equiv 0, \tag{3.2}
\]
which gives that \( \omega \) is constant along the characteristics. We assume in this section that these trajectories exist and are differentiable, and we prove by the Lyapunov method that the support of the vorticity never meet the boundary \( \partial \Omega \). Although we do not know that the flow is smooth, the following computation is the main idea of this article, and it will be rigorously applied in Section 4.

The Lyapunov method to prove this kind of result was introduced by Marchioro and Pulvirenti [22] in the case of a point vortex which moves under the influence of the regular part of the vorticity, and by Marchioro [21] when the dirac mass is fixed. In both articles, the authors use the explicit formula of the velocity associated to the dirac centered at \( z(t) \) in the full plane: 
\[
H(x) = (x - z)^+/(2\pi|x - z|^2).
\]
The geometrical structure is the key of their analysis. Indeed, choosing \( L(t) = -\ln |\phi_x(t) - z(t)| \) they have that

a) \( L(t) \to \infty \) if and only if the trajectory meets the dirac. Then, it is sufficient to prove that \( L'(t) \) stays bounded in order to prove the result.

b) \( H(\phi_x(t)) \cdot (\phi_x(t) - z(t)) \equiv 0 \), which implies that the singular term in the velocity does not appear.

Therefore, the explicit blow up in the case of the dirac point in the full plane is crucial in two points of view: for the symmetry cancelation (point b) and for the fact that the primitive of \( 1/x \) is \( \ln x \) which blows up near the origin (point a). In our case, we do not have such an explicit form of the blow up near the corners and the primitive of \( 1/\sqrt{z} \) is \( \sqrt{z} \) which is bounded near \( 0 \). The idea is to add a logarithm. When \( C \) is a Jordan arc, \( |T| \approx 1 + \sqrt{z^2 - 1} \) and we note that \( \ln \ln(1 + \sqrt{z^2 - 1}) \) blows up near the end-points \( \pm 1 \).

However, the problem with Lyapunov function is that it is very specific on the case studied. For example, this function is different if the dirac point is fixed or if it moves with the fluid (for more details and explanations, see the discussion on Lyapunov functions in Section 7).

We fix \( x_0 \in \Omega \) and we consider \( \phi = \phi_{x_0}(t) \) the trajectory which comes from \( x_0 \) (see (3.1)). We denote 
\[
L(t) := -\ln |L_1(t, \phi(t))|
\]
where the definition of \( L_1 \) depends on the geometric property of \( \Omega \):

(1) if \( \Omega_{bd} \) satisfies (D-bd) with angles greater than \( \pi/2 \), then we choose 
\[
L_1(t, x) := \frac{1}{2\pi} \int_{\Omega_{bd}} \ln \left( \frac{|T(x) - T(y)|}{|T(x) - T(y)|^* ||T(y)||} \right) \omega(t,y) \, dy; \tag{3.3}
\]
(2) if \( \Omega_{\text{ext}} \) satisfies (D-ext) with angles greater than \( \pi/2 \), then we choose

\[
L_1(t, x) := \frac{1}{2\pi} \int_{\Omega_{\text{ext}}} \ln \left( \frac{|\mathcal{T}(x) - \mathcal{T}(y)|}{|\mathcal{T}(x) - \mathcal{T}(y)^*||\mathcal{T}(y)|} \right) \omega(t, y) \, dy + \frac{\alpha}{2\pi} \ln |\mathcal{T}(x)|,
\]

where \( \alpha := \gamma_0 + \int \omega_0 \).

These functions are related to the stream functions. In Subsection 7.5, we compare these functions with the Lyapunov functions introduced for the vortex-wave system.

When trajectories exist, it is obvious (without renormalization) that (3.1)–(3.2) imply that

\[
\int_{\Omega} \omega(t, \cdot) = \int_{\Omega} \omega_0 \quad \text{and} \quad \|\omega(t, \cdot)\|_{L^p} = \|\omega_0\|_{L^p}, \quad \forall t > 0, \forall p \in [1, \infty].
\]

We assume that \( \omega_0 \) is compactly supported, then included in \( B(0, R_0) \) for some \( R_0 > 0 \). Thanks to (2.5) and Proposition 2.4, we see that the velocity \( u \) is bounded outside this ball by a constant \( C_0 \), and (3.1)–(3.2) give

\[
\text{supp} \, \omega(t, \cdot) \subset B(0, R_0 + C_0 \cdot t), \quad \forall t \geq 0.
\]

We also have that the circulation is conserved.

If we assume that \( \omega_0 \) is non positive, then it follows from (3.2) that

\[
\omega(t, x) \leq 0, \quad \forall t \geq 0, \quad \forall x \in \Omega.
\]

3.1. Blow up of the Lyapunov function near the boundary. The first required property is that \( L \) goes to infinity iff the trajectory meets the boundary. Next, if we prove that \( L \) is bounded, then it will follow that the trajectory stays far away the boundary. We fix \( T^* > 0 \), using (3.6) we denote by \( RT^* := R_0 + C_0 \cdot T^* \), such that \( \text{supp} \, \omega(t, \cdot) \subset B(0, R T^*) \) for all \( t \in [0, T^*] \).

Lemma 3.1. For any case (1)–(2), there exists \( C_1 = C_1(T^*, \omega_0, \gamma_0) \) such that

\[
|L_1(t, x)| \leq C_1 |||\mathcal{T}(x)| - 1||/2, \quad \forall x \in B(0, R T^*), \quad \forall t \in [0, T^*].
\]

Proof. For a sake of shortness, we write the proof in the hardest case: case (2). The other case follows easily. Recalling the notation \( z^* = z/|z|^2 \), we can compute

\[
\frac{|\mathcal{T}(x) - \mathcal{T}(y)|^2}{|\mathcal{T}(x) - \mathcal{T}(y)^*||\mathcal{T}(y)|^2} = 1 - \frac{|\mathcal{T}(x) - \mathcal{T}(y)^*|^2||\mathcal{T}(y)|^2 - |\mathcal{T}(x) - \mathcal{T}(y)|^2}{|\mathcal{T}(x) - \mathcal{T}(y)^*|^2||\mathcal{T}(y)|^2}
\]

\[
= 1 - \frac{|(\mathcal{T}(x))^2|\langle \mathcal{T}(y) \rangle^2 - 2T(x) \cdot \mathcal{T}(y) + 1 - |(\mathcal{T}(x))^2 - 2T(x) \cdot \mathcal{T}(y) + |\mathcal{T}(y)|^2}{|\mathcal{T}(x) - \mathcal{T}(y)^*|^2||\mathcal{T}(y)|^2}
\]

\[
= 1 - \frac{|(\mathcal{T}(x))^2 - 1| |\mathcal{T}(y)|^2 - 1}{|\mathcal{T}(x) - \mathcal{T}(y)|^2 |\mathcal{T}(y)|^2}.
\]

Therefore, we have

\[
\ln \left( \frac{|\mathcal{T}(x) - \mathcal{T}(y)|}{|\mathcal{T}(x) - \mathcal{T}(y)^*||\mathcal{T}(y)|} \right) = \frac{1}{2} \ln \left( \frac{|\mathcal{T}(x) - \mathcal{T}(y)|^2}{|(\mathcal{T}(x))^2 - 1||\mathcal{T}(y)|^2} \right)
\]

\[
= \frac{1}{2} \ln \left( 1 - \frac{|(\mathcal{T}(x))^2 - 1| |\mathcal{T}(y)|^2 - 1}{|\mathcal{T}(x) - \mathcal{T}(y)|^2 |\mathcal{T}(y)|^2} \right),
\]

and we need an estimate of \( \ln(1 - r) \) when \( r \in (0, 1) \), because we recall that \( |\mathcal{T}(z)| > 1 \) for any \( z \in \Omega_{\text{ext}} \). It is easy to note (studying the difference of the functions) that

\[
|\ln(1 - r)| = - \ln(1 - r) \leq \left( \frac{r}{1 - r} \right)^{1/2}, \quad \forall r \in [0, 1).
\]

Applying this inequality, we have for any \( y \neq x \) that

\[
\left| \ln \left( \frac{|\mathcal{T}(x) - \mathcal{T}(y)|}{|\mathcal{T}(x) - \mathcal{T}(y)^*||\mathcal{T}(y)|} \right) \right| \leq \frac{1}{2} \left( \frac{|(\mathcal{T}(x))^2 - 1| |\mathcal{T}(y)|^2 - 1}{|\mathcal{T}(x) - \mathcal{T}(y)|^2 |\mathcal{T}(y)|^2} \right)^{1/2}
\]

\[
\leq \frac{1}{2} \sqrt{\frac{|(\mathcal{T}(x))^2 - 1| |\mathcal{T}(y)|^2 - 1}{|\mathcal{T}(x) - \mathcal{T}(y)|^2}}.
\]
By continuity of $T$, we denote by $C_{T^*}$ a constant such that $T(B(0, R_{T^*})) \subset B(0, C_{T^*})$. Finally, we apply the previous inequality to $L_1$ and we find for all $x \in B(0, R_{T^*})$ and $t \in [0, T^*]$: 

$$|L_1(t, x)| \leq \frac{C_{T^*}(C_{T^*} + 1)_{1/2}}{4\pi}(|T(x)| - 1)^{1/2} \int_{\Omega_{\text{ext}}} \frac{|\omega(y)|}{|T(x) - T(y)|} dy + \frac{|\alpha|}{2\pi} \ln |T(x)|$$

$$\leq \frac{\sqrt{2}C_{T^*}}{4\pi}(|T(x)| - 1)^{1/2}C(||\omega||_{L^1}^2 ||\omega||_{L^\infty}^{1/2} + ||\omega||_{L^1}^{1/2} ||\omega||_{L^\infty}^{1/2}) + \frac{|\alpha|}{2\pi}(|T(x)| - 1).$$

For the last inequality, we have used a part of Proposition 2.4. As $(|T(x)| - 1) \leq C_{T^*}^{1/2}(|T(x)| - 1)^{1/2}$, the conclusion follows from (3.3)

Concerning the lower bound for the case (1)-(2), we need some conditions on the sign.

**Lemma 3.2.** If $\omega_0$ is non-positive and $\gamma_0 \geq -\int \omega_0$ (only in case (2)), then there exists $C_2 = C_2(T^*, \omega_0)$ such that

$$L_1(t, x) \geq C_2 ||T(x)| - 1|, \forall x \in B(0, R_{T^*}), \forall t \in [0, T^*].$$

**Proof.** Again, we write the details in the case (2). We denote by $r_\infty := ||\omega_0||_{L^\infty}$ and $r_1 := ||\omega_0||_{L^1}$. For $\rho > 0$, we denote by

$$V_1 := (C + B(0, \rho)) \cap \Omega_{\text{ext}} = \{x \in \Omega_{\text{ext}}; \text{dist}(x, C) < \rho\}, \ V_2 := \Omega_{\text{ext}} \setminus V_1.$$ 

We fix $\rho$ such that the lebesgue measure of $V_1$ is equal to $r_1/(2r_\infty)$.

We deduce from (3.5) that

$$r_1 = ||\omega(t, \cdot)||_{L^1(V_1)} + ||\omega(t, \cdot)||_{L^1(V_2)}$$

with $||\omega(t, \cdot)||_{L^1(V_1)} \leq r_\infty r_1/(2r_\infty) = r_1/2$, which implies that $||\omega(t, \cdot)||_{L^1(V_2)} \geq r_1/2$.

As the logarithm of the fraction is negative (see the proof of Lemma 3.1), we have, with the sign condition, that:

$$L_1(t, x) \geq \frac{1}{2\pi} \int_{V_2} \ln \left(\frac{|T(x) - T(y)|}{|T(x) - T(y)|^*} \right) \omega(y) dy.$$

Moreover, thanks to the computation made in the proof of Lemma 3.1, we have

$$\ln \left(\frac{|T(x) - T(y)|}{|T(x) - T(y)|^*} \right) = \frac{1}{2} \ln \left(\frac{|T(x) - T(y)|^2}{|T(x) - T(y)|^* |T(y)|^2} \right)$$

$$= \frac{1}{2} \ln \left(1 - \frac{(|T(x)|^2 - 1)(|T(y)|^2 - 1)}{|T(x) - T(y)|^2 |T(y)|^2} \right)$$

$$\leq -\frac{1}{2} \frac{(|T(x)|^2 - 1)(|T(y)|^2 - 1)}{|T(x) - T(y)|^2 |T(y)|^2}$$

because $\ln(1 + x) \leq x$ for any $x > -1$.

As $\rho > 0$ and $T$ is continuous, there exists $C_\rho > 0$ such that $|T(y)| \geq 1 + C_\rho$, for all $y \in V_2$. Moreover, there exists also $\bar{R}_{T^*} > 1$ such that $T(B(0, \bar{R}_{T^*})) \subset B(0, \bar{R}_{T^*})$. Adding the fact that $\omega$ is non positive, we have for all $y \in V_2 \cap \text{supp} \omega$ and $x \in B(0, \bar{R}_{T^*})$

$$\ln \left(\frac{|T(x) - T(y)|}{|T(x) - T(y)|^*} \right) \omega(y) \geq \frac{1}{2} \frac{(|T(x)|^2 - 1)(|T(y)|^2 - 1)}{|T(x) - T(y)|^2 |T(y)|^2} |\omega(y)|$$

$$\geq \frac{1}{2} \frac{(|T(x)| - 1)(|T(x)| + 1)(|T(y)| - 1)(|T(y)| + 1)}{|T(x)|^2 |T(y)|^2} |\omega(y)|$$

$$\geq \frac{1}{2} \frac{(|T(x)| - 1)C_\rho}{(\bar{R}_{T^*} + 1)\bar{R}_{T^*}} |\omega(y)|.$$
Integrating this last inequality over $V_2$, we obtain that
\[
L_1(t, x) \geq \frac{1}{2\pi} \int_{V_2} \ln\left(\frac{|T(x) - T(y)|}{|T(x) - T(y)|}\right) \omega(y) \, dy \geq \frac{(|T(x)| - 1)C_\rho}{4\pi(R_{T^*} + 1)R_{T^*}^2} \|\omega\|_{L^1(V_2)}
\]
\[
\geq \frac{C_\rho}{8\pi(R_{T^*} + 1)R_{T^*}^2}\ln(|T(x)| - 1),
\]
which ends the proof.

Multiplying by $-1$ the expression of $L_1$, we can establish the same result with the opposite sign condition:

\[\text{Remark 3.3. If } \omega_0 \text{ is non-negative and } \gamma_0 \leq -\int \omega_0, \text{ then there exists } C_2 \text{ such that}
\]
\[-L_1(t, x) \geq C_2||T(x)| - 1|, \forall x \in B(0, R_{T^*}), \forall t \in [0, T^*].\]

From these two lemmas, it follows obviously the following.

\[\text{Corollary 3.4. If } \omega_0 \text{ is non-positive and } \gamma_0 \geq -\int \omega_0 \text{ (only for (2))}, \text{ then we have that}
\]
\[\text{• } L_1(x) > 0 \text{ for all } x \in \Omega;
\]
\[\text{• } L_1(x) \to 0 \text{ if and only if } x \to \partial\Omega.
\]

If $\omega_0$ is non-negative and $\gamma_0 \leq -\int \omega_0$ (only for (2)), then we have that

\[\text{• } L_1(x) < 0 \text{ for all } x \in \Omega;
\]
\[\text{• } L_1(x) \to 0 \text{ if and only if } x \to \partial\Omega.
\]

Indeed, $|T(x)| \to 1$ iff $x \to \partial\Omega$.

\[\text{3.2. Estimate of the Lyapunov. The issue of this part is to prove that the trajectory does not reach the boundary in finite time. In other word, let } x_0 \in \text{ supp } \omega_0 \text{ (then } L_1(0, x_0) \neq 0) \text{ and } T^* > 0,
\]
we will prove that $L(t)$ stays bounded in $[0, T^*]$. Then, we differentiate $L$:

\[L'(t) = -\left(\partial_t L_1(t, \phi(t)) + \phi'(t) \cdot \nabla L_1(t, \phi(t))\right)/L_1(t, \phi(t))\]

and we want to estimate the right hand side term.

As usual, we write the details for the case (2).

On the one hand, we have that

\[u(t, x) \cdot \nabla L_1(t, x) = u(t, x) \cdot \left[\frac{1}{2\pi} \int_{\Omega_{\text{ext}}} \left(\frac{T(x) - T(y)}{|T(x) - T(y)|^2} - \frac{T(x) - T(y)^*}{|T(x) - T(y)^*|^2}\right) \omega(y) \, dy DT(x)
\]
\[+ \frac{\alpha}{2\pi} \frac{T(x)}{|T(x)|^2} DT(x)\]
\[= 0\]
thanks to the explicit formula of $u$ (see (2.9)).

On the other hand, we use the equation\footnote{to justify that it works even for a weak solution, the reader can read the first lines of the proof of Proposition 2.10} verified by $\omega$ to have

\[\partial_t L_1(t, x) = \frac{1}{2\pi} \int_{\Omega_{\text{ext}}} \ln\left(\frac{|T(x) - T(y)|}{|T(x) - T(y)^*||T(y)|}\right) \partial_t \omega(y) \, dy
\]
\[= -\frac{1}{2\pi} \int_{\Omega_{\text{ext}}} \ln\left(\frac{|T(x) - T(y)|}{|T(x) - T(y)^*||T(y)|}\right) \text{div}(u(y) \omega(y)) \, dy
\]
\[= \frac{1}{2\pi} \int_{\Omega_{\text{ext}}} \nabla_y \left[\ln\left(\frac{|T(x) - T(y)|}{|T(x) - T(y)^*||T(y)|}\right)\right] u(y) \omega(y) \, dy.
\]

Now, we use the symmetry of the Green kernel (see Subsection 2.2)

\[\nabla_y \ln\left(\frac{|T(x) - T(y)|}{|T(x) - T(y)^*||T(y)|}\right) = \nabla_y \ln\left(\frac{|T(y) - T(x)|}{|T(y) - T(x)^*||T(x)|}\right).
\]
and the explicit formula of \( u(y) \) to write
\[
\partial_t L_1(t, x) = \frac{1}{2\pi} \int_{\Omega_{\text{ext}}} \left( \frac{T(y) - T(x)}{|T(y) - T(x)|^2} - \frac{T(y) - T(x)^*}{|T(y) - T(x)^*|^2} \right) DT(y) \frac{1}{2\pi} DT^T(y) \left( R[\omega](y) + \alpha \frac{T(y)}{|T(y)|^2} \right) \omega(y) dy.
\]
As \( T \) is holomorphic, \( DT \) is of the form \( \begin{pmatrix} a & b \\ -b & a \end{pmatrix} \) and we can check that \( DT(y) DT^T(y) = (a^2 + b^2) I_d = |\det(DT)(y)| I_d \). Hence
\[
\partial_t L_1(t, x) = \frac{1}{(2\pi)^2} \int_{\Omega_{\text{ext}}} \left( \frac{T(y) - T(x)}{|T(y) - T(x)|^2} - \frac{T(y) - T(x)^*}{|T(y) - T(x)^*|^2} \right) \left( R[\omega](y) + \alpha \frac{T(y)}{|T(y)|^2} \right) |\det(DT)(y)| \omega(y) dy.
\]

The goal is to estimate \( \partial_t L_1 / L_1 \). However, Corollary 3.4 states that \( L_1 \) goes to zero if and only if \( x \to \partial \Omega_{\text{ext}} \). Then it is important to show that \( \partial_t L_1 \) tends to zero as \( x \to \partial \Omega_{\text{ext}} \), and to compare the rate with \( L_1 \). We will need the following general lemma.

**Lemma 3.5.** Let \( h \) be a bounded function, compactly supported in \( B(0, R_h) \) for some \( R_h > 1 \). Then, there exists \( C_h = C(\|h\|_{L^\infty}, R_h) \) such that
\[
\int_{D^c} \frac{|h(y)|}{|y-x||y-x^*|} dy \leq C_h \left( |\ln(|x| - 1)| + |x| \right), \quad \forall x \in D^c
\]
with the notation \( x^* = x/|x|^2 \) and \( D = B(0, 1) \).

*Proof.* We fix \( x \in D^c \) and we denote \( \rho = |x| - 1 \) and \( \rho^* = 1 - |x^*| = 1 - \frac{1}{1 + \rho} = \frac{\rho}{1 + \rho} \).

We compute
\[
\int_{D^c} \frac{|h(y)|}{|y-x||y-x^*|} dy = \int_{D^c \cap B(x, 4\rho)} \frac{|h(y)|}{|y-x||y-x^*|} dy + \int_{D^c \cap B(x, 4\rho)^c} \frac{|h(y)|}{|y-x||y-x^*|} dy =: I_1 + I_2.
\]

For \( I_1 \), we know that \( |y-x^*| \geq |y| - |x^*| \geq \rho^* \), hence
\[
I_1 \leq \frac{\rho^*}{\rho} \int_{D^c \cap B(x, 4\rho)} \frac{|h(y)|}{|y-x|} dy \leq \frac{\|h\|_{L^\infty}}{\rho} \int_{B(x, 4\rho)} \frac{1}{|y-x|} dy \leq \frac{(1 + \rho)\|h\|_{L^\infty}}{2\pi \rho^2}
\]
which gives that \( I_1 \leq C_1 |x| \).

Concerning \( I_2 \), we note that
\[
|x - x^*| = \rho + \rho^* = \rho + \frac{\rho}{1 + \rho} \leq 2\rho \leq \frac{1}{2} |y - x|
\]
for any \( y \in B(x, 4\rho)^c \). Hence,
\[
|y - x^*| \geq |y - x| - |x - x^*| \geq \frac{1}{2} |y - x|
\]
and we have
\[
I_2 \leq \int_{D^c \cap B(x, 4\rho)} \frac{2|h(y)|}{|y-x|^2} dy \leq 2\|h\|_{L^\infty} \int_{4\rho}^{2\pi} \int_0^{\rho + R_h} \frac{1}{r} dr d\theta \leq 4\pi \|h\|_{L^\infty} \ln \frac{|x| + R_h}{4\rho}.
\]
This implies that \( I_2 \leq C_2 \left( |\ln(|x| - 1)| + \ln \frac{|x| + R^c}{4} \right). \)

We conclude because there exists \( C_3 = C_3(R^c) \) such that \( \ln \frac{|x| + R^c}{4} \leq C_3 |x| \) for any \( x \in D^c. \)

We recall that we have fixed \( T^* > 0, x_0 \in \text{supp} \, \omega_0 \) and \( R_{T^*} = R_0 + C_0 T^* \), such that \( \text{supp} \, \omega(t, \cdot) \subset B(0, R_{T^*}) \) for all \( t \in [0, T^*]. \) Finally, we estimate \( \partial_t L_1 \) without sign conditions.

**Lemma 3.6.** There exists \( C_3 = C_3(T^*) \) such that

\[
|\partial_t L_1(t, x)| \leq C_3 ||T(x)||^{-1} \left( 1 + |\ln ||T(x)|| - 1| \right), \forall x \in B(0, R_{T^*}), \forall t \in [0, T^*].
\]

**Proof.** Using (2.6), we know that

\[
\frac{\lvert T(y) - T(x) \rvert}{\lvert T(y) - T(x) \rvert^2} - \frac{\lvert T(y) - T(x)^* \rvert}{\lvert T(y) - T(x)^* \rvert^2} = \frac{\lvert T(x) - T(x)^* \rvert}{\lvert T(y) - T(x) \rvert \lvert T(y) - T(x)^* \rvert^2}.
\]

Then, Proposition 2.4 and (3.5) allow us to estimate (3.8)

\[
|\partial_t L_1(t, x)| \leq C||T(x) - T(x)^*|| \int_{\Omega_{\text{ext}}} \frac{|\omega(y)|}{\lvert T(y) - T(x) \rvert \lvert T(y) - T(x)^* \rvert} |\det(\partial T)(y)| \, dy.
\]

On the one hand, we have for all \( x \in B(0, R_{T^*}) \)

\[
|T(x) - T(x)^*| = \frac{|T(x)|^2 - T(x)|^2}{T(x)^2} = \frac{|T(x)|^2 - 1}{|T(x)|} = \frac{(|T(x)| - 1)(|T(x)| + 1)}{|T(x)|} \leq 2(|T(x)| - 1).
\]

On the other hand, we change variables \( \eta = T(y) \) and we compute

\[
\int_{\Omega_{\text{ext}}} \frac{|\omega(y)|}{\lvert T(y) - T(x) \rvert \lvert T(y) - T(x)^* \rvert} |\det(\partial T)(y)| \, dy = \int_{\partial D^c} \frac{|\omega(T^{-1}(\eta))|}{|\eta - T(x)| \lvert \eta - T(x)^* \rvert} \, d\eta.
\]

As \( \|\omega \circ T^{-1}\|_{L^\infty} = \|\omega_0\|_{L^\infty} \) and as

\[
\text{supp} \omega \circ T^{-1} = T(\text{supp} \omega) \subset T(B(0, R_{T^*})) \subset B(0, \tilde{R}_{T^*}),
\]

we apply Lemma 3.5 to establish that

\[
\int_{\Omega_{\text{ext}}} \frac{|\omega(y)|}{\lvert T(y) - T(x) \rvert \lvert T(y) - T(x)^* \rvert} |\det(\partial T)(y)| \, dy \leq C \left( |\ln(||T(x)|| - 1)| + \tilde{R}_{T^*} \right), \forall x \in B(0, R_{T^*}).
\]

This finishes the proof. \( \square \)

**Remark 3.7.** In the bounded case \( \Omega_{\text{bd}}, \) there is a subtle difference. We note that

\[
|T(x) - T(x)^*| = \frac{(1 - |T(x)|)(|T(x)| + 1)}{|T(x)|} \leq 2 \frac{(1 - |T(x)|)}{|T(x)|}
\]

with \( |T(x)| \) which can go to zero. To fix this problem, we can prove a similar result to Lemma 3.5 there exists \( C_h = C(||h||_{L^\infty}) \) such that

\[
\frac{1}{|x|} \int_D \frac{|h(y)|}{y - x||y - x^*|} \, dy \leq C_h \left( |\ln(1 - |x|)| + 1 \right), \forall x \in D.
\]

Indeed, we can write \(|x||y - x^*| = |(y - x^*) - x||x| - \frac{x}{|x|} \) and we deduce putting \( \rho := 1 - |x| \) that:

- for \( y \in B(x, 4\rho) \cap D, \ |y - \frac{x}{|x|}| \geq |\frac{x}{|x|} - |y - x^*| \geq 1 - |x| = \rho; \)
- for \( y \in B(x, 4\rho^c) \cap D, \ |y - \frac{x}{|x|}| \geq |y - x^*| = (1 - |y|^2)(1 - |x|^2) \geq 0. \)

Using these two inequalities, we follow exactly the proof of Lemma 3.5. The inequality (3.9) allows us to establish Lemma 3.6 in the bounded case.
In light of Lemmas 3.2 and 3.6, we see that we have an additional logarithm which implies that \( \frac{\partial_t L_1}{L_1} \to \infty \) if \( x \to C \). However, the logarithm is exactly what we can estimate by Gronwall inequality: 
\[ L'(t) = \frac{\partial_t L_1}{L_1} \approx \ln L_1 = L(t) \]
This is the general idea to establish the main result of this section.

**Proposition 3.8.** We assume that \( \omega_0 \) is non-positive, compactly supported in \( \Omega_{ext} \) and \( \gamma_0 \geq -\int \omega_0 \).

Then, for any \( T^* > 0 \), there exists \( C_{T^*} \) such that

\[ L(t) \leq C_{T^*}, \forall x_0 \in \text{supp} \omega_0, \forall t \in [0, T^*]. \]

**Proof.** As the support of \( \omega_0 \) does not intersect \( \partial \Omega_{ext} \), we have by continuity of \( T \) and by Lemma 3.2 that

\[ L(0) = -\ln L_1(0, x_0) \leq -\ln C_2(|T(x_0)| - 1) \]
is bounded uniformly in \( x_0 \in \text{supp} \omega_0 \).

For any \( x_0 \in \text{supp} \omega_0 \), (3.6) gives that \( \phi(t) \in B(0, R_{T^*}) \), for all \( t \in [0, T^*] \). Therefore, the computation made in the begin of this subsection gives

\[ L'(t) = -\partial_t L_1(t, \phi(t))/L_1(t, \phi(t)). \]

Lemma 3.2 states that there exists \( C_2 \) such that

\[ L_1(t, \phi(t)) \geq C_2(|T(\phi(t))| - 1). \] (3.10)

Moreover, thanks to Lemma 3.1, it is easy to find \( C_4 \) such that

\[ L_1(t, x) \leq C_4, \forall x \in B(0, R_{T^*} \cap \Omega_{ext}), \forall t \in [0, T^*]. \] (3.11)

Finally, we have proved in Lemma 3.6 that there exists \( C_3 \) such that

\[ |\partial_t L_1(t, \phi(t))| \leq C_3(|T(\phi(t))| - 1)
\[ \left(1 + |\ln |T(\phi(t))|| - 1\right)\right). \]

(3.12)

We can easily check that in the interval \((0, e^{-1})\) the function \( x \mapsto x|\ln x| \) is equal to the map \( x \mapsto -x \ln x \), which is increasing. By (3.10) and (3.11), we use the fact that

\[ 0 \leq \frac{C_2(|T(\phi(t))| - 1)}{eC_4} \leq \frac{L_1(t, \phi(t))}{eC_4} \leq e^{-1} \]
to apply this remark on (3.12):

\[ |\partial_t L_1(t, \phi(t))| \leq C_3(|T(\phi(t))| - 1)
\[ \left(1 + |\ln \frac{eC_4}{C_2}\right)\left(1 + \frac{C_2C_4}{eC_4}\right)\ln \frac{C_2(|T(\phi(t))| - 1)}{eC_4} \]
\[ \leq \frac{C_3}{C_2} \left(1 + \ln \frac{eC_4}{C_2}\right) L_1(t, \phi(t)) - \frac{C_3C_4}{eC_4} L_1(t, \phi(t)) \ln \frac{L_1(t, \phi(t))}{eC_4} \]
\[ \leq L_1(t, \phi(t))(C_5 - C_6 \ln L_1(t, \phi(t))). \]

As \( L_1 \) is positive, we finally obtain that

\[ L'(t) = -\partial_t L_1(t, \phi(t))/L_1(t, \phi(t)) \leq \frac{|\partial_t L_1(t, \phi(t))|}{L_1(t, \phi(t))} \leq C_5 - C_6 \ln L_1(t, \phi(t)) = C_5 + C_6 L(t). \]

The constants \( C_5 \) and \( C_6 \) are uniform for \( x_0 \in \text{supp} \omega_0 \) and \( t \in [0, T^*] \). Gronwall’s lemma implies that

\[ L(t) \leq (L(0) + \frac{C_6}{C_5})e^{C_6 T^*}, \forall x_0 \in \text{supp} \omega_0, \forall t \in [0, T^*]. \]

□

By Corollary 3.4, the corollary of this proposition is that the support of \( \omega(t, \cdot) \) never reach the boundary. As before, we have the same proposition with the opposite sign condition:
Remark 3.9. We assume that the support of \( \omega_0 \) is outside a neighborhood of \( \partial \Omega_{\text{ext}} \), that \( \omega_0 \) is non-negative and \( \gamma_0 \leq -\int \omega_0 \). Then, for any \( T^* > 0 \), there exists \( C_{T^*} \) such that
\[
L(t) \leq C_{T^*}, \quad \forall x_0 \in \text{supp} \omega_0, \quad \forall t \in [0, T^*].
\]
Indeed, replacing everywhere \( L_1 \) by \(-L_1\), the last inequality in the proof would be
\[
L'(t) = \frac{\partial_t L_1(t, \phi(t))}{-L_1(t, \phi(t))} \leq \frac{\left| \partial_t L_1(t, \phi(t)) \right|}{-L_1(t, \phi(t))} \leq C_5 - C_6 \ln -L_1(t, \phi(t)) = C_5 + C_6 L(t),
\]
which allows us to conclude in the same way.

4. Vorticity far from the boundary

The role of this section is to prove rigorously that the vorticity never reach the boundary. In Section 3, we have assumed that the flows exist and are regular enough to compute derivatives. However, the solution considered in Theorems 1.2 and 1.3 are weak, and such a property is not established in the existence proofs (see [16, 7]).

Without considering trajectories, we have proved in Subsection 2.3 that the weak solutions verify the classical estimates:

- conservation of the total mass of the vorticity (2.19);
- conservation of the \( L^p \) norm of the vorticity for \( p \in [1, \infty] \) (2.20);
- conservation of the circulation (2.21) (only for exterior domains);
- compact support for the vorticity: Proposition 2.10 (only for exterior domains).

We can easily prove that the conservations of the total mass and the \( L^1 \) norm of the vorticity imply that
\[
\omega_0 \geq 0 \quad \text{a.e. in } \Omega \implies \omega(t, x) \geq 0, \quad \forall t \geq 0, \quad \text{a.e. in } \Omega.
\]

Thinking of the Lyapunov function used in Section 3, we can construct a good test function in order to use the renormalization theory. We establish now the key result for proving the uniqueness.

**Proposition 4.1.** Let \( \omega \) be a global weak solution of (2.15) such that \( \omega_0 \) is compactly supported in \( \Omega \).

If \( \omega_0 \) is non-positive and \( \gamma_0 \geq -\int \omega_0 \) (only for exterior domains), then, for any \( T^* > 0 \), there exists a neighborhood \( U_{T^*} \) of \( \partial \Omega \) such that
\[
\omega(t) \equiv 0 \quad \text{on } U_{T^*}, \quad \forall t \in [0, T^*].
\]

**Proof.** Let us perform the proof in the unbounded domain \( \Omega_{\text{ext}} \) verifying (D-ext). According to Proposition 2.10, we have
\[
\text{supp } \omega(t) \subset B(0, R_0 + C_6 t), \quad \forall t \geq 0.
\]

We note \( R_{T^*} := R_0 + C_6 T^* \).

Thanks to Lemma 3.1, it is easy to find \( C_4 \) such that
\[
L_1(t, x) \leq C_4, \quad \forall x \in B(0, R_{T^*} \cap \Omega_{\text{ext}}), \quad \forall t \in [0, T^*].
\]

We also deduce from the conservation of the vorticity sign that Corollary 3.4 holds true.

We aim to apply (2.18) with the choice \( \beta(t) = t^2 \) and we set
\[
\Phi(t, x) = \chi_0 \left( -\ln L_1(t, x) + \ln C_4 \right), \quad R(t) = \left( -\ln L_1(t, x) + \ln C_4 \right),
\]
where \( \chi_0 \) is a smooth function: \( \mathbb{R} \to \mathbb{R}^+ \) which is identically zero for \( |x| \leq 1/2 \) and identically one for \( |x| \geq 1 \) and increasing on \( \mathbb{R}^+ \), \( L_1 \) is defined in (3.4) and \( R(t) \) is an increasing continuous function to be determined later on.

As \( L_1(t, x) \leq C_4 \), we have that \( -\ln L_1(t, x) + \ln C_4 \) is positive \( \forall x \in B(0, R_{T^*} \cap \Omega_{\text{ext}}), \quad \forall t \in [0, T^*] \).

On the one hand, Lemma 3.2 states that there exists \( C_2 \) such that
\[
L_1(t, x) \geq C_2(|\mathcal{T}(x)| - 1), \quad \forall x \in B(0, R_{T^*} \cap \Omega_{\text{ext}}), \quad \forall t \in [0, T^*].
\]
We have proved in Lemma 3.6 that there exists $C_3$ such that
\[ |\partial_t L_1(t, x)| \leq C_3(|T(x)| - 1) \left(1 + |\ln(|T(x)| - 1)|\right), \quad \forall x \in B(0, R_{T*} \cap \Omega_{\text{ext}}), \quad \forall t \in [0, T^*]. \]

Then, using the fact that $x \mapsto -x \ln x$ is increasing in $[0, e^{-1}]$ (see the proof of Proposition 3.8) we have that
\[ |\partial_t L_1(t, x)| \leq L_1(t, x)(C_5 - C_6 \ln \frac{L_1(t, x)}{C_4}), \quad \forall x \in B(0, R_{T*} \cap \Omega_{\text{ext}}), \quad \forall t \in [0, T^*]. \tag{4.1} \]

On the other hand, we have
\[ \nabla_x L_1(t, x) = -u^\perp(t, x), \]
therefore
\[ u \cdot \nabla \Phi = u \cdot u^\perp \frac{\chi_0}{RL_1} \equiv 0. \]

Besides,
\[ \partial_t \Phi(t, x) = \left(\frac{R'(t)}{R^2(t)} \ln \frac{L_1(t, x)}{C_4} - \frac{1}{R} \frac{\partial_t L_1(t, x)}{L_1(t, x)}\right) \chi_0 \left(\frac{-\ln L_1(t, x) + \ln C_4}{R(t)}\right). \]

In view of (2.18), this yields for any $T \in [0, T^*]$ \( \int_{\mathbb{R}^2} \Phi(T, x)\omega^2(T, x) \, dx - \int_{\mathbb{R}^2} \Phi(0, x)\omega_0^2(x) \, dx \]
\[ = \int_0^T \int_{\mathbb{R}^2} \omega^2(t, x) \chi_0 \left(\frac{-\ln L_1(t, x) + \ln C_4}{R} \right) \left(\frac{R'(t)}{R} \ln \frac{L_1(t, x)}{C_4} - \frac{\partial_t L_1(t, x)}{L_1(t, x)}\right) \, dx \, dt. \]

Since $-\ln \frac{L_1(t, x)}{C_4} \geq 0$, the term $\chi_0 \left(\frac{-\ln L_1(t, x) + \ln C_4}{R}\right)$ is non-negative and non-zero provided $\frac{1}{2} \leq -\ln \frac{L_1(t, x)}{C_4} \leq 1$, so we obtain
\[ \int_{\mathbb{R}^2} \Phi(T, x)\omega^2(T, x) \, dx - \int_{\mathbb{R}^2} \Phi(0, x)\omega_0^2(x) \, dx \leq \int_0^T \int_{\mathbb{R}^2} \omega^2 \chi_0 \left(\frac{R'}{2} + C_5 + C_6 R\right) \, dx \, dt. \]

In the last inequality, we have used (4.1), which is allowed because $\text{supp } \omega \subset B(0, R_{T*} \cap \Omega_{\text{ext}})$ for all $t \in [0, T^*]$. We now choose
\[ R(t) = \lambda_0 e^{2C_6 t} - \frac{C_5}{C_6}, \]
with $\lambda_0$ to be determined later on, so that
\[ \int_{\mathbb{R}^2} \Phi(T, x)\omega^2(T, x) \, dx \leq \int_{\mathbb{R}^2} \Phi(0, x)\omega_0^2(x) \, dx. \]

Since the support of $\omega_0$ does not intersect some neighborhood of $\mathcal{C}$, the continuity of $T$ implies that there exists $\mu_0 > 0$ such that $\text{supp } \omega_0 \subset B(0, \mu_0 + 1)^c$. Then,
\[ 0 \leq -\ln L_1(0, x) + \ln C_4 \leq -\ln \left(C_2(|T(x)| - 1)\right) + \ln C_4 \leq -\ln (C_2 \mu_0) + \ln C_4 \]
for all $x$ in the support of $\omega_0$. We finally choose $\lambda_0$ so that
\[ 0 < \frac{-\ln (C_2 \mu_0) + \ln C_4}{\lambda_0 - \frac{C_5}{C_6}} \leq \frac{1}{2}. \]

For this choice, we have
\[ \Phi(0, x)\omega_0^2(x) = \chi_0 \left(\frac{-\ln L_1(0, x) + \ln C_4}{\lambda_0 - \frac{C_5}{C_6}}\right) \omega_0^2(x) \equiv 0. \]

\(^{3}\text{see the proof of Proposition 2.10 to check that this equality holds for all } T.\)
We deduce that for all $T \in [0, T^*]$, $\Phi(T, x)\omega^2(T, x) \equiv 0$. Thanks to Lemma 3.1, we know that there exists $C_1$ such that

$$L_1(T, x) \leq C_1(|T(x)| - 1)^{1/2}, \forall x \in B(0, R_{T^*}), \forall T \in [0, T^*].$$

Therefore, for any $x \in T^{-1}\left(B(0, 1 + e^{-\frac{x^2}{4}(R(T^*)^2 - \ln C_1)} \setminus B(0, 1)\right)$ and any $T \in [0, T^*]$, we have that

$$|T(x)| \leq 1 + e^{-\frac{x^2}{4}(R(T^*)^2 - \ln C_1)}$$

$$\ln(|T(x)| - 1) \leq -\frac{2}{C_1}(R(T^*) - \ln C_1)$$

$$-\frac{C_1}{2}\ln(|T(x)| - 1) \geq (R(T^*) - \ln C_1)$$

which implies that

$$-\frac{C_1}{2}\ln(|T(x)| - 1) + \ln C_1 \geq 1. \quad (4.2)$$

Moreover, for any $x \in B(0, R_{T^*})$ and $T \in [0, T^*]$ we have that

$$\ln L_1(T, x) \leq \frac{C_1}{2}\ln(|T(x)| - 1)$$

$$-\ln L_1(T, x) + \ln C_4 \geq -\frac{C_1}{2}\ln(|T(x)| - 1) + \ln C_4$$

which gives (using that $R$ is an increasing function and that $-\ln L_1(T, x) + \ln C_4 \geq 0$):

$$-\frac{\ln L_1(T, x) + \ln C_4}{R(T)} \geq -\ln L_1(T, x) + \ln C_4 \geq -\frac{C_1}{2}\ln(|T(x)| - 1) + \ln C_4.$$

Putting together the last inequality and $(4.2)$, $\Phi(T, x)\omega^2(T, x) \equiv 0$ for any $T \in [0, T^*]$ implies that

$$\omega(T, x) \equiv 0, \forall x \in T^{-1}\left(B(0, 1 + e^{-\frac{x^2}{4}(R(T^*)^2 - \ln C_1)} \setminus B(0, 1)\right), \forall T \in [0, T^*]$$

and the conclusion follows.

\[\square\]

**Remark 4.2.** Of course, as in Remarks 3.3 and 3.9 the previous proposition holds true for the opposite sign condition:

$\omega_0$ non negative and $\gamma_0 \leq -\int \omega_0$.

Actually, we can prove Propositions 2.10 and 3.1 without the renormalized solutions. Indeed, as we have proved in Remark 2.9 that $\omega$ stays definite sign (thanks to the renormalization theory), then we can use $\omega$ instead of $\omega^2$ in the proofs. In this case, we just need that $\omega$ is a weak solution for test functions possibly supported up to the boundary. However, we have presented here the proofs with $\beta(\omega) = \omega^2$ in order to extend the theorems in the case where $\omega_0$ is constant near the boundary (see Section 7).

5. **Uniqueness of Eulerian solutions**

5.1. **Velocity formulation.** In order to follow the proof of Yudovich, we derive a velocity formulation in the full plane from [2,16]. We begin by introducing

$$v(x) := \int_{\mathbb{R}^2} K_{\mathbb{R}^2}(x - y)\tilde{\omega}(y)dy$$

with $K_{\mathbb{R}^2}(x) = \frac{1}{2\pi} \frac{x^1}{|x|^2}$, the solution in the full plane of

$$\text{div } v = 0 \text{ on } \mathbb{R}^2, \quad \text{curl } v = \tilde{\omega} \text{ on } \mathbb{R}^2, \quad \lim_{|x| \to \infty} |v| = 0.$$

This vector field is bounded, and we denote the difference by $w = \tilde{u} - v$, which belongs to $L^\infty_{loc}(\mathbb{R}^+; L^p_{loc}(\mathbb{R}^2))$ for $p < 4$, and verifies

$$\text{div } w = 0 \text{ on } \mathbb{R}^2, \quad \text{curl } w = g_{\omega, \gamma}(s)\delta_{B1} \text{ on } \mathbb{R}^2, \quad \lim_{|x| \to \infty} |w| = 0.$$
We infer that $v$ verifies the following equation:

$$\begin{align*}
\begin{cases}
v_t + v \cdot \nabla v + v \cdot \nabla w + w \cdot \nabla v - v(s)^+ \dot{g}_{v,\gamma}(s) \cdot \delta_{\partial \Omega} &= -\nabla p, &\text{in } \mathbb{R}^2 \times (0, \infty) \\
\text{div } v &= 0, &\text{in } \mathbb{R}^2 \times (0, \infty) \\
w(x) &= \frac{1}{2\pi} \int_{\partial \Omega} \left( \frac{x-s}{|x-s|^2} \right) \dot{g}_{v,\gamma}(s) \, ds, &\text{in } \mathbb{R}^2 \times (0, \infty) \\
v(x, 0) &= K_{\mathbb{R}^2}[\tilde{w}], &\text{in } \mathbb{R}^2.
\end{cases}
\end{align*}$$

(5.1)

with $\dot{g}_{v,\gamma} := \text{curl } g_{v,\gamma,\gamma}$ (see (2.17)).

In order to prove the equivalence of (2.16) and (5.1) it is sufficient to show that

$$\text{curl } [v \cdot \nabla w + w \cdot \nabla v - v(s)^+ \dot{g}_{v,\gamma}(s) \cdot \delta_{\partial \Omega}] = \text{div } (\tilde{\omega} w)$$

(5.2)

for all divergence free fields $v \in W^{1,p}_{\text{loc}}$, with some $p > 2$. Indeed, if (5.2) holds, then we get for $\tilde{\omega} = \text{curl } v$

$$0 = -\text{curl } \nabla p = \text{curl } [v_t + v \cdot \nabla v + v \cdot \nabla w + w \cdot \nabla v - v(s)^+ \dot{g}_{v,\gamma}(s) \cdot \delta_{\partial \Omega}]$$

$$\partial_t \tilde{\omega} + v \cdot \nabla \tilde{\omega} + w \cdot \nabla \tilde{\omega} = 0$$

so relation (2.16) holds true. And vice versa, if (2.16) holds then we deduce that the left hand side of (5.1) has zero curl so it must be a gradient.

We now prove (5.2). As $W^{1,p}_{\text{loc}} \subset C^0$, $v(s)$ is well defined. Next, it suffices to prove the equality for smooth $v$, since we can pass to the limit on a subsequence of smooth approximations of $v$ which converges strongly in $W^{1,p}_{\text{loc}}$ and $C^0$. Now, it is trivial to check that, for a $2 \times 2$ matrix $A$ with distribution coefficients, we have

$$\text{curl div } A = \text{div } \left( \frac{\text{curl } C_1}{\text{curl } C_2} \right)$$

where $C_i$ denotes the $i$-th column of $A$. For smooth $v$, we deduce

$$\text{curl } [v \cdot \nabla w + w \cdot \nabla v] = \text{curl div } (v \otimes w + w \otimes v)$$

$$\quad = \text{div } \left( \frac{\text{curl } (vw_1)}{\text{curl } (vw_2)} \right)$$

$$\quad = \text{div } (w \text{ curl } v + v \cdot \nabla w + v \text{ curl } w + w \cdot \nabla v).$$

It is a simple computation to check that

$$\text{div } (v \cdot \nabla w + w \cdot \nabla v) = v \cdot \nabla \text{div } w + w \cdot \nabla \text{div } v + \text{curl } v \text{ div } w + \text{curl } w \text{ div } v.$$

Taking into account that we have free divergence fields, we can finish by writing

$$\text{curl } [v \cdot \nabla w + w \cdot \nabla v] = \text{div } (w \text{ curl } v + v \dot{g}_{v,\gamma}(s) \cdot \delta_{\partial \Omega}) = \text{div } (w \text{ curl } v) + \text{curl } [v(s)^+ \dot{g}_{v,\gamma}(s) \cdot \delta_{\partial \Omega}],$$

which proves (5.2).

5.2. Proof of Theorems 1.2-1.3. The goal is to adapt the proof of Yudovich: let $u_1$ and $u_2$ be two weak solutions of (1.1) (Theorem 1.1) from the same initial datum $u_0$ verifying (1.5)-(1.6). We define as above $v_1, w_1$ (resp. $v_2, w_2$) associated to $\omega_1 := u_1$ (resp. $\omega_2 := u_2$) and $\gamma_0$ (see (2.11) and (2.21)). We denote

$$\bar{\omega} := \omega_1 - \omega_2$$

where the bar means that we extend by zero outside $\Omega$ and

$$\bar{v} := v_1 - v_2,$$

which verifies

$$\partial_t \bar{v} + \bar{v} \cdot \nabla v_1 + v_2 \cdot \nabla \bar{v} + \text{div } (\bar{v} \otimes w_1 + v_2 \otimes \bar{\omega} + w_1 \otimes \bar{v} + \bar{\omega} \otimes v_2)$$

$$\quad - (v_1(s)^+ \dot{g}_{\bar{v},\gamma}(s) - \bar{v}(s)^+ \dot{g}_{v_2,\gamma_0}(s)) \cdot \delta_{\partial \Omega} = -\nabla p.$$  

(5.3)

Next, we will multiply by $\bar{v}$ and integrate. The difficulty compared with the Yudovich’s original proof is that we have some terms as $\int_{\mathbb{R}^2} |w_1| |\bar{v}| |\nabla \bar{v}|$ with $w_1$ blowing up near the corners. The general idea

---

4The original proof comes from [11] and we copy it for a sake of clarity.
is to divide such an integral in two parts: on $U$ a small neighborhood of the boundary where the vorticity vanishes (see Proposition 4.1) and on $\mathbb{R}^2 \setminus U$ where the velocity $w_1$ is regular. Far from the boundary, we follow what Yudovich did, and near the boundary we compute

$$
\int_U |w_1| |\tilde{v}| |\nabla \tilde{v}| \leq \|w_1\|_{L^1(U)} \|\tilde{v}\|_{L^\infty(U)} \|\nabla \tilde{v}\|_{L^\infty(U)}.
$$

Indeed $w_1$ is integrable near the boundary, and as $\tilde{v}$ is harmonic in $U$ ($\text{div} \tilde{v} = \text{curl} \tilde{v} = 0$), then we have

$$
\int_U |w_1| |\tilde{v}| |\nabla \tilde{v}| \leq C \|\tilde{v}\|^2_{L^2}
$$

which will allow us to conclude by the Gronwall’s lemma. We note here why Proposition 4.1 is the main key of the uniqueness proof.

This idea was used in [15] in order to prove the uniqueness of the vortex-wave system, and we follow the same plan.

We denote by $W^{1,4}_\sigma(\mathbb{R}^2)$ the set of functions belonging to $W^{1,4}(\mathbb{R}^2)$ and which are divergence-free in the sense of distributions, and by $W^{-1,4/3}_\sigma(\mathbb{R}^2)$ its dual space.

First, we prove that we can multiply by $\tilde{v}$ and integrate. As a consequence of (5.1) and (5.3), we obtain the following properties for $\tilde{v}$.

**Proposition 5.1.** Let $u_0$ verifying (1.5), $u_1, u_2$ be two weak solutions of (1.1) with initial condition $u_0$. Let $\tilde{v} = v_1 - v_2$. Then we have

$$
\tilde{v} \in L^2_{\text{loc}}(\mathbb{R}^+, W^{1,4}_\sigma(\mathbb{R}^2)), \quad \partial_t \tilde{v} \in L^2_{\text{loc}}(\mathbb{R}^+, W^{-1,4/3}_\sigma(\mathbb{R}^2)).
$$

In addition, we have $\tilde{v} \in C(\mathbb{R}^+, L^2(\mathbb{R}^2))$ and for all $T \in \mathbb{R}^+$,

$$
\|\tilde{v}(T)\|_{L^2(\mathbb{R}^2)}^2 = 2 \int_0^T \langle \partial_t \tilde{v}, \tilde{v} \rangle_{W^{-1,4/3}_\sigma, W^{1,4}_\sigma} \, ds, \quad \forall T \in \mathbb{R}^+.
$$

The proof follows easily from the estimates established in Section 2. The reader can find the details in Section 6.

Now, we take advantage of the fact that $\omega_i$ is equal to zero near $\partial \Omega$ (Proposition 4.1) to give harmonic regularity estimates on $\tilde{v}(t)$.

**Lemma 5.2.** Let $T^* > 0$. We assume that $\omega_0$ is compactly supported in $\Omega$ and has the sign conditions of Proposition 4.2 (or of Remark 4.3). Then, there exists a neighborhood $U_{T^*}$ of $\partial \Omega$ such that for all $t \leq T^*$, $\tilde{v}(t, \cdot)$ is harmonic on $U_{T^*}$. In particular, for $O_{T^*}$ an open set such that $\partial \Omega \subset O_{T^*} \subset U_{T^*}$, we have the following estimates:

1. $\|\tilde{v}(t, \cdot)\|_{L^\infty(O_{T^*})} \leq C \|\tilde{v}(t, \cdot)\|_{L^2(\mathbb{R}^2)}$,

2. $\|\nabla \tilde{v}(t, \cdot)\|_{L^\infty(O_{T^*})} \leq C \|\tilde{v}(t, \cdot)\|_{L^2(\mathbb{R}^2)},$

where $C$ only depends on $O_{T^*}$.

The proof is a direct consequence of the mean-value formula (see e.g. the proof of Lemma 3.9 in [15]). In order to prepare the Gronwall estimate, we establish the following estimates on $w_1 - w_2$.

**Lemma 5.3.** Let $T^* > 0$ and $\partial \Omega \subset O_{T^*} \subset U_{T^*}$ as in Lemma 5.2. Then $\tilde{w} := w_1 - w_2$ verifies the following estimates for any $t \in [0, T^*]$

1. $\|\tilde{w}(t, \cdot)\|_{L^2(\mathbb{R}^2)} \leq 2 \|\tilde{v}(t, \cdot)\|_{L^2(\mathbb{R}^2)}$,

2. $\|\nabla \tilde{w}(t, \cdot)\|_{L^\infty(O_{T^*})} \leq C \|\tilde{v}(t, \cdot)\|_{L^2(\mathbb{R}^2)}$,

3. $\|\nabla \tilde{w}(t, \cdot)\|_{L^2(O_{T^*})} \leq C \|\tilde{v}(t, \cdot)\|_{L^2(\mathbb{R}^2)},$

where $C$ only depends on $O_{T^*}$.
Proof. We fix $t \in [0, T^*)$ and we denote $\tilde{u} := \tilde{u}_1 - \tilde{u}_2$. From the explicit formula and the conservation law, we have that

$$\begin{align*}
\text{div} \tilde{u} &= 0 \quad \text{on } \Omega, \\
curl \tilde{u} &= \tilde{\omega} \quad \text{on } \Omega, \\
\tilde{u} \cdot \tilde{n} &= 0 \quad \text{on } \partial \Omega,
\end{align*}$$

and

$$\begin{align*}
\int_{\partial \Omega} \tilde{u} \cdot \tilde{\tau} &= 0 \quad \text{(only for exterior domains)}, \\
\lim_{|x| \to \infty} |\tilde{u}| &= 0 \quad \text{(only for exterior domains)},
\end{align*}$$

Indeed, in the case of exterior domains, $\tilde{\omega} \equiv 0$ on $\mathcal{C}$ which implies that the circulation of $\tilde{v}$ around $\mathcal{C}$ is equal to zero. Therefore, $\tilde{u}$ is the orthogonal projection of $\tilde{v}$ on the set of the vector fields which are divergence free, tangent to the boundary and belonging to $L^2(\Omega)$. In arbitrary domains, this projection is orthogonal (see [6, Theo 1.1 in Chap III.1.]):

$$\|\tilde{u}(t, \cdot)\|_{L^2(\Omega)} \leq \|\tilde{v}(t, \cdot)\|_{L^2(\Omega)}.$$ 

Then the first point follows directly:

$$\|\tilde{w}(t, \cdot)\|_{L^2(\mathbb{R}^2)} \leq \|\tilde{u}(t, \cdot)\|_{L^2(\Omega)} + \|\tilde{v}(t, \cdot)\|_{L^2(\mathbb{R}^2)} \leq \|\tilde{v}(t, \cdot)\|_{L^2(\Omega)} + \|\tilde{v}(t, \cdot)\|_{L^2(\mathbb{R}^2)} \leq 2\|\tilde{v}(t, \cdot)\|_{L^2(\mathbb{R}^2)}.$$ 

The second point is exactly the same thing as in Lemma 5.2. $\tilde{w}$ is harmonic in $\Omega$ then there exists $C$ depending on $O_{T^*}$ such that

$$\|\tilde{w}(t, \cdot)\|_{L^\infty(O_{T^*})} \leq C\|\tilde{w}(t, \cdot)\|_{L^2(\Omega)} \leq 2C\|\tilde{v}(t, \cdot)\|_{L^2(\mathbb{R}^2)}.$$ 

Another consequence of the mean-value Theorem is that

$$\|\nabla \tilde{w}(t, \cdot)\|_{L^2(O_{T^*})} \leq C\|\tilde{w}(t, \cdot)\|_{L^2(\Omega)} \leq 2C\|\tilde{v}(t, \cdot)\|_{L^2(\mathbb{R}^2)}.$$ 

Indeed, there is $R_1$ such that $\text{dist}(\partial \Omega, \partial O_{T^*}) > R_1$, then

$$\|\nabla \tilde{w}(t, x)\|_{L^2(O_{T^*})} = \left\| \frac{1}{\pi R_1^2} \int_{B(x, R_1)} \nabla \tilde{w}(t, y) dy \right\|_{L^2(O_{T^*})} \leq \frac{1}{\pi R_1^2} \int_0^{2\pi} \|\tilde{w}(t, x + R_1 e^{i\theta})\|_{L^2(\Omega)} d\theta \leq \frac{2\|\tilde{w}(t, \cdot)\|_{L^2(\Omega)}}{R_1}.$$

□

Remark 5.4. We remark that the result from Galdi’s book does not require any regularity of $\partial \Omega$ when we consider the $L^2$ norm (thanks to the Hilbert structure). In contrast for $p \neq 2$, the author states that the Leray projector is continuous from $L^p$ to $L^p$ if the boundary $\partial \Omega$ is $C^2$. Indeed, in our case we see that $\tilde{v}$ belongs to $L^p$ for any $p > 1$, whereas $\tilde{u} = \Pi \tilde{v}$ does not belong in $L^p(\Omega)$ for some $p > 4$ (if there is an angle greater than $\pi$, see Remark 2.2).

We can adapt now the Yudovich proof, as it is done in [18]. We fix $T^* > 0$ in order to fix $O_{T^*}$ in Lemmas 5.2 and 5.3. We consider smooth and divergence-free functions $\Phi_n \in C_c^\infty(\mathbb{R}^+ \times \mathbb{R}^2)$ converging to $\tilde{v}$ in $L^2_{\text{loc}}(\mathbb{R}^+ \times W^{1,4}(\mathbb{R}^2))$ as test functions in (5.3), and let $n$ go to $+\infty$. First, we have for all $T \in [0, T^*)$

$$\int_0^T \langle \partial_t \tilde{v}, \Phi_n \rangle_{W^{-1,4/3}_{\sigma}, W^{1/4}_{\sigma}} ds \to \int_0^T \langle \partial_t \tilde{v}, \tilde{v} \rangle_{W^{-1,4/3}_{\sigma}, W^{1/4}_{\sigma}} ds,$$
and we deduce the limit of the other terms from the several bounds for \( v_1 \) stated in the proof of Proposition 5.1. This yields
\[
\frac{1}{2} \| \tilde{v}(T, \cdot) \|_{L^2}^2 = I + J + K,
\]
where
\[
I = - \int_0^T \int_{\mathbb{R}^2} \tilde{v} \cdot (\tilde{v} \cdot \nabla v_1 + v_2 \cdot \nabla \tilde{v}) \, dx \, dt,
\]
\[
J = \int_0^T \int_{\mathbb{R}^2} (\tilde{v} \otimes w_1 + v_2 \otimes \tilde{w} + w_1 \otimes \tilde{v} + \tilde{w} \otimes v_2) : \nabla \tilde{v} \, dx \, dt,
\]
\[
K = \int_0^T \int_{\partial \Omega} v_1(s) \tilde{g}_{E,0}(s) \cdot \tilde{v}(s) \, ds.
\]
The goal is to estimate all the terms in the right-hand side in order to obtain a Gronwall-type inequality. For the first term \( I \) in (5.4), we begin by noticing that
\[
\int_{\mathbb{R}^2} (v_2 \cdot \nabla \tilde{v}) \cdot \tilde{v} \, dx = \frac{1}{2} \int_{\mathbb{R}^2} v_2 \cdot \nabla |\tilde{v}|^2 \, dx = -\frac{1}{2} \int_{\mathbb{R}^2} |\tilde{v}|^2 \text{div} v_2 \, dx = 0,
\]
where we have used that \( v_2 = O(1/|x|) \) and \( \tilde{v} = O(1/|x|^2) \) at infinity. Moreover, Hölder’s inequality gives
\[
\left| \int_{\mathbb{R}^2} (\tilde{v} \cdot \nabla v_1) \cdot \tilde{v} \, dx \right| \leq \| \tilde{v} \|_{L^q} \| \nabla v_1 \|_{L^p},
\]
with \( \frac{1}{p} + \frac{1}{q} = \frac{1}{2} \). On the one hand, the Calderón-Zygmung inequality states that \( \| \nabla v_1 \|_{L^p} \leq C \| \omega_1 \|_{L^p} \) for \( p \geq 2 \). On the other hand, we write by interpolation \( \| \tilde{v} \|_{L^p} \leq \| \tilde{v} \|_{L^2}^{1-a} \| \tilde{v} \|_{L^\infty}^{a} \) with \( \frac{1}{q} = \frac{a}{2} + \frac{1-a}{\infty} \). We have that \( a = 1 - \frac{2}{p} \), so we are led to
\[
|I| \leq C \int_0^T \| \tilde{v} \|_{L^2}^{2-2/p} \, dt. \tag{5.5}
\]
We now estimate \( J \). We have
\[
\int_{\mathbb{R}^2} (\tilde{v} \otimes w_1) : \nabla \tilde{v} \, dx = \int_{\mathbb{R}^2} \sum_{i,j} \tilde{v}_i w_{1,j} \partial_j \tilde{v}_i \, dx = \frac{1}{2} \sum_{i} \int_{\mathbb{R}^2} \sum_{j} w_{1,j} \partial_j \tilde{v}_i^2 \, dx
\]
\[
= \frac{1}{2} \sum_{i} \int_{\mathbb{R}^2} \tilde{v}_i^2 \text{div} w_1 \, dx = 0,
\]
since \( w_1 \) is divergence-free, and
\[
\left| \int_0^T \int_{\mathbb{R}^2} (w_1 \otimes \tilde{v}) : \nabla \tilde{v} \, dx \, dt \right| \leq \left| \int_0^T \int_{\Omega_T} (w_1 \otimes \tilde{v}) : \nabla \tilde{v} \, dx \, dt \right| + \left| \int_0^T \int_{\partial \Omega_T} (w_1 \otimes \tilde{v}) : \nabla \tilde{v} \, dx \, dt \right|. \tag{5.6}
\]
We perform an integration by part for the second term in the right-hand side of (5.6). Arguing that \( \text{div} \tilde{v} = 0 \), we obtain
\[
\left| \int_0^T \int_{\mathbb{R}^2} (w_1 \otimes \tilde{v}) : \nabla \tilde{v} \, dx \, dt \right| \leq \left| \int_0^T \int_{\Omega_T} (w_1 \otimes \tilde{v}) : \nabla \tilde{v} \, dx \, dt \right|
\]
\[
+ \left| \int_0^T \left( \int_{\Omega_T} (\tilde{v} \cdot \nabla w_1) \cdot \tilde{v} \, dx + \int_{\partial \Omega_T} (w_1 \cdot \tilde{v})(\tilde{v} \cdot \nu) \, ds \right) \, dt \right|
\]
\[
\leq \int_0^T \| w_1 \|_{L^1(\Omega_T)} \| \tilde{v} \|_{L^\infty(\Omega_T)} \| \nabla \tilde{v} \|_{L^\infty(\Omega_T)} \, dt
\]
\[
+ \int_0^T \| \nabla w_1 \|_{L^\infty(\Omega_T)} \| \tilde{v} \|_{L^2}^2 \, dt
\]
\[
+ \int_0^T \| w_1 \|_{L^\infty(\partial \Omega_T)} \| \tilde{v} \|_{L^\infty(\partial \Omega_T)} \| \partial \Omega_T \| \, dt.
\]
As remarked when we have introduced $w$: $\|w_1\|_{L^1(\Omega^+)} \leq C$ with $C$ depending only on $\Omega$, $T^*$ and $u_0$. Moreover, using the harmonicity of $w_1$, we know that $\|\nabla w_1\|_{L^\infty(\partial \Omega^+)}$ is bounded by a constant times $\|w_1\|_{L^\infty(V^+)}$, with $\partial \Omega \subset V^+ \subset \Omega^+$. Using the behavior of $DT$ at infinity (2.5), Proposition 2.4, conservation laws (2.19), (2.20), (2.21), then (2.9) allows us to state that $\|u_1\|_{L^\infty((0,T^*) \times V^+)} \leq C_0$ with $C_0$ depending only on $\Omega$, $T^*$ and $u_0$. As $v_1$ is uniformly bounded, we obtain that $\|\nabla w_1\|_{L^\infty(\partial \Omega^+)}$ and $\|w_1\|_{L^\infty(\partial \Omega^+)}$ is bounded uniformly in $(0,T^*)$. Then, according to Lemma 5.2, this gives

$$\left| \int_0^T \int_{\mathbb{R}^2} (w_1 \otimes \tilde{v}) : \nabla \tilde{v} \, dx \, dt \right| \leq C \int_0^T \|\tilde{v}\|_{L^2}^2 \, dt.$$

In the same way, we integrate by part to get

$$\left| \int_0^T \int_{\mathbb{R}^2} (v_2 \otimes \tilde{w}) : \nabla \tilde{v} \, dx \, dt \right| \leq \left| \int_0^T \int_{\Omega^+} (v_2 \otimes \tilde{w}) : \nabla \tilde{v} \, dx \, dt \right|$$

$$+ \left| \int_0^T \left( \int_{\partial \Omega^+} (\tilde{w} \cdot \nabla v_2) \cdot \tilde{v} \, ds + \int_{\partial \Omega^+} (v_2 \cdot \tilde{v})(\tilde{w} \cdot v) \, ds \right) \, dt \right|.$$

Therefore,

$$\left| \int_0^T \int_{\mathbb{R}^2} (v_2 \otimes \tilde{w}) : \nabla \tilde{v} \, dx \, dt \right| \leq \int_0^T \|\tilde{w}\|_{L^2(\Omega^+)} \|v_2\|_{L^2(\Omega^+)} \|\nabla \tilde{v}\|_{L^\infty(\partial \Omega^+)} \, dt$$

$$+ \int_0^T \|\tilde{w}\|_{L^\infty(\partial \Omega^+)} \|\tilde{v}\|_{L^2} \|v_2\|_{L^2} \, dt$$

$$+ \int_0^T \|\tilde{w}\|_{L^\infty(\partial \Omega^+)} \|\tilde{v}\|_{L^\infty(\partial \Omega^+)} \|v_2\|_{L^\infty(\partial \Omega^+)} \, \|\tilde{w}\|_{L^2} \|v_2\|_{L^\infty(\partial \Omega^+)} \|\tilde{v}\|_{L^\infty(\partial \Omega^+)} \, |\partial H| \, dt.$$

Using again Calderón-Zygmund inequality for $v_2$ and Lemmata 5.2 and 5.3 we get

$$\left| \int_0^T \int_{\mathbb{R}^2} (v_2 \otimes \tilde{w}) : \nabla \tilde{v} \, dx \, dt \right| \leq C \int_0^T \|\tilde{v}\|_{L^2}^2 \, dt.$$

A very similar computation yields

$$\left| \int_0^T \int_{\mathbb{R}^2} (\tilde{w} \otimes v_2) : \nabla \tilde{v} \, dx \, dt \right| \leq C \int_0^T \left( \|\tilde{v}\|_{L^2(\Omega^+)} + \|\nabla \tilde{w}\|_{L^2(\Omega^+)} + \|\tilde{w}\|_{L^\infty(\partial \Omega^+)} \right) \|\tilde{v}\|_{L^2} \, dt$$

$$\leq C \int_0^T \|\tilde{v}\|_{L^2}^2 \, dt.$$

Therefore, we arrive at

$$|J| \leq 3C \int_0^T \|\tilde{v}\|_{L^2}^2 \, dt. \quad \text{(5.7)}$$

Finally, using (2.17) we write the third term $K$ in (5.4) as follows:

$$K = \pm \int_0^T \int_{\partial \Omega} (\tilde{u} \cdot \tilde{v})(v_1^+ \cdot \tilde{v}) \, ds$$

$$= \pm \int_0^T \int_{\Omega} \text{curl} \, \tilde{u}(v_1^+ \cdot \tilde{v}) \, dx \pm \int_0^T \int_{\Omega} \tilde{u} \cdot \nabla^+(v_1^+ \cdot \tilde{v}) \, dx,$$
where ± depends if we treat exterior or interior domains. Using that \( \mathbf{curl} \bar{u} = \mathbf{curl} \bar{v} \) in \( \Omega \), \( \mathbf{div} \bar{v} = 0 \) and the behaviors at infinity, we obtain by several integrations by parts:

\[
\int_{\Omega} \mathbf{curl} \bar{u}(v_{11}^+ \cdot \bar{v}) \, dx = \int_{\Omega} \mathbf{curl} \bar{v}(v_{11}^+ \cdot \bar{v}) \, dx = \int_{\mathbb{R}^2} \mathbf{curl} \bar{v}(v_{11}^+ \cdot \bar{v}) \, dx
\]

\[
= \int_{\mathbb{R}^2} \left( v_{12}^+ \bar{v}_2 + v_{11} \frac{\partial \bar{v}_1}{\partial x_1} \right) \, dx
\]

\[
= \int_{\mathbb{R}^2} \left( \bar{v}_2 \frac{\partial \bar{v}_1}{\partial x_1} - \bar{v}_1 \frac{\partial \bar{v}_2}{\partial x_1} \right) \, dx
\]

Hence,

\[
\left| \int_{0}^{T} \int_{\Omega} \mathbf{curl} \bar{u}(v_{11}^+ \cdot \bar{v}) \, dx \, dt \right| \leq 4 \int_{0}^{T} \int_{\mathbb{R}^2} |\nabla v_1| |\bar{v}|^2 \, dx \, dt
\]

which gives by Calderón-Zygmund inequality (as for 1):

\[
\left| \int_{0}^{T} \int_{\Omega} \mathbf{curl} \bar{u}(v_{11}^+ \cdot \bar{v}) \, dx \, dt \right| \leq C \left| \frac{\bar{v}}{L_2} \right|^{2-2/p} \, dt.
\]

With similar computations, and using Lemmata 5.2 and 5.3, we can prove that the second term of \( K \) can be treated thanks to:

\[
\int_{0}^{T} \int_{\Omega} |\bar{u}| |\nabla v_1||\bar{v}| \, dx \, dt \leq C \int_{0}^{T} \left| \frac{\bar{v}}{L_2} \right|^{2-2/p} \, dt
\]

\[
\int_{0}^{T} \int_{\partial \Omega_T} |\bar{u}| |\nabla \bar{v}| \, dx \, dt \leq C \int_{0}^{T} \left| \frac{\bar{v}}{L_2} \right| \, dt
\]

\[
\int_{0}^{T} \int_{\partial \Omega_T} |\bar{v}| |\nabla v_1||\bar{v}| \, dx \, dt \leq C \int_{0}^{T} \left| \frac{\bar{v}}{L_2} \right|^{2-2/p} \, dt
\]

\[
\int_{0}^{T} \int_{\partial \Omega_T} |\nabla \bar{v}| |\nabla v_1||\bar{v}| \, dx \, dt \leq C \int_{0}^{T} \left| \frac{\bar{v}}{L_2} \right| \, dt
\]

\[
\int_{0}^{T} \int_{\partial \Omega_T} (|\bar{v}| + |\bar{u}|) |\nabla v_1||\bar{v}| \, dx \, dt \leq C \int_{0}^{T} \left| \frac{\bar{v}}{L_2} \right| \, dt,
\]

which implies that

\[
|K| \leq C \int_{0}^{T} \left| \frac{\bar{v}}{L_2} \right|^{2} \, dt + C \int_{0}^{T} \left| \frac{\bar{v}}{L_2} \right|^{2-2/p} \, dt.
\]  

(5.8)

Therefore, the estimates (5.5), (5.7) and (5.8) with (5.4) imply that

\[
\left| \frac{\bar{v}}{L_2} \right|^{2} \leq C \int_{0}^{T} \left| \frac{\bar{v}}{L_2} \right|^{2} \, dt + C \int_{0}^{T} \left| \frac{\bar{v}}{L_2} \right|^{2-2/p} \, dt.
\]

As we have choose \( p > 2 \) and as \( \left| \frac{\bar{v}}{L_2} \right| \leq C_0 \) for all \( t \in [0, T^*] \) (see Proposition 5.1), we have \( \left| \frac{\bar{v}}{L_2} \right|^{2/p} \leq C_0^{2/p} \). Hence, for \( p \) large enough, the previous inequality becomes

\[
\left| \frac{\bar{v}(T, \cdot)}{L_2} \right|^{2} \leq 2Cp \int_{0}^{T} \left| \frac{\bar{v}}{L_2} \right|^{2-2/p} \, dt.
\]
Using a Gronwall-like argument, this implies
\[ \| \tilde{v}(T, \cdot) \|_{L^2} \leq (2CT)^p, \quad \forall p \geq 2. \]
Letting \( p \) tend to infinity, we conclude that \( \| \tilde{v}(T, \cdot) \|_{L^2} = 0 \) for all \( T < \min(T^*, 1/(2C)) \). Finally, we consider the maximal interval of \([0, T^*] \) on which \( \| \tilde{v}(T, \cdot) \|_{L^2} = 0 \), which is closed by continuity of \( \| \tilde{v}(T, \cdot) \|_{L^2} \). If it is not equal to the whole of \([0, T^*] \), we may repeat the proof above, which leads to a contradiction by maximality. Therefore uniqueness holds on \([0, T^*] \), and this ends the proof of Theorems 1.2 and 1.3. Indeed, Lemma 5.3 implies that \( \| u_1 - u_2 \|_{L^2} \leq \| \tilde{w} \|_{L^2} + \| \tilde{v} \|_{L^2} \leq 2 \| \tilde{v} \|_{L^2} \).

6. Technical results

We will use several times the following from [10]:

**Lemma 6.1.** Let \( S \subset \mathbb{R}^2 \), \( \alpha \in (0, 2) \) and \( g : S \to \mathbb{R}^+ \) be a function belonging in \( L^1(S) \cap L^\gamma(S) \), for \( r > \frac{2}{2-\alpha} \). Then
\[ \frac{g(y)}{|x - y|^\alpha} dy \leq C\|g\|_{L^1(S)}^{\frac{2-\gamma+2\alpha}{\gamma}} \|g\|_{L^\gamma(S)}^{\frac{-\gamma}{\gamma}}. \]

6.1. Proof of Proposition 2.4. We make the proof in the unbounded case (which is the hardest case). We decompose \( R[\omega] \) in two parts:
\[ R_1(x) := \int_{\Omega_{\text{ext}}} \frac{(T(x) - T(y))^\perp}{|T(x) - T(y)|^2} \omega(y) dy \quad \text{and} \quad R_2(x) := \int_{\Omega_{\text{ext}}} \frac{(T(x) - T(y))^\perp}{|T(x) - T(y)|^2} \omega(y) dy. \]
a) Estimate and continuity of \( R_1 \).
Let \( z := T(x) \) and \( f(\eta) := \omega(T^{-1}(\eta)) \det(DT^{-1}(\eta)) \chi_E(\eta) \), with \( \chi_E \) the characteristic function of the set \( E \). Making the change of variables \( \eta = T(y) \), we find
\[ R_1(T^{-1}(z)) = \int_{\mathbb{R}^2} \frac{(z - \eta)^\perp}{|z - \eta|^2} f(\eta) d\eta. \]
Changing variables back, we get
\[ \| f \|_{L^1(\mathbb{R}^2)} = \| \omega \|_{L^1}. \]
We choose \( p_0 > 2 \) such that \( \det(DT^{-1}) \) belongs to \( L^{p_0}_{\text{loc}}(\Omega_{\text{ext}}) \) (see Remark 2.2). If all the angles are greater than \( \pi \), we can choose \( p_0 = \infty \) (thanks to Theorem 2.1 and (2.5)) and we would have \( \| f \|_{L^\infty(\mathbb{R}^2)} \leq C \| \omega \|_{L^\infty}. \) However, if there is one angle less than \( \pi \), we have to decompose the integral in two parts:
\[ R_1(T^{-1}(z)) = \int_{|\eta| \geq 2} \frac{(z - \eta)^\perp}{|z - \eta|^2} f(\eta) d\eta + \int_{|\eta| \leq 2} \frac{(z - \eta)^\perp}{|z - \eta|^2} f(\eta) d\eta \]
with
\[ \| f \|_{L^\infty(\mathbb{R}^2 \setminus B(0,2))} \leq C_1 \| \omega \|_{L^\infty} \]
by (2.5), and
\[ \| f \|_{L^{p_0}(B(0,2))} \leq C_2 \| \omega \|_{L^{p_0}}, \]
by Remark 2.2. Then we use the classical estimate for the Biot-Savart kernel in \( \mathbb{R}^2 \) (see Lemma 6.1):
\[ \left| \int_{|\eta| \geq 2} \frac{(z - \eta)^\perp}{|z - \eta|^2} f(\eta) d\eta \right| \leq C_0 \| f \|_{L^{1/2}(\mathbb{R}^2 \setminus B(0,2))} \| f \|_{L^{1/2}(\mathbb{R}^2 \setminus B(0,2))} \leq C_4 \| \omega \|_{L^1}^{1/2} \| \omega \|_{L^\infty}^{1/2} \]
and
\[ \left| \int_{|\eta| \leq 2} \frac{(z - \eta)^\perp}{|z - \eta|^2} f(\eta) d\eta \right| \leq C_0 \| f \|_{L^{1/2}(B(0,2))} \| f \|_{L^{1/2}(B(0,2))} \leq C_5 \| \omega \|_{L^{1/2}}^{p_0} \| \omega \|_{L^{1/2}}^{p_0}, \]
which give the uniform estimate
\[ \| R_1 \|_{L^\infty(\Omega_{\text{ext}})} \leq C(\| \omega \|_{L^1}^{1/2} \| \omega \|_{L^\infty}^{1/2} + \| \omega \|_{L^1} \| \omega \|_{L^\infty}^{1/2}). \]
where \( a = \frac{p_0-2}{2p_0-1} \) belongs to \((0,1/2] \). Concerning the continuity, we approximate \( f_{\mathcal{X}B(0,2)} \) by \( f_n \in C^\infty_c(B(0,2)) \) and \( f_{\mathcal{Y}B(0,2)} \) by \( g_n \in C^\infty_c(B(0,2)^c) \) such that

\[
\|f_n - f\|_{L^1(B(0,2)^c)} \to 0, \quad \|g_n - f\|_{L^1(B(0,2)^c)} \to 0, \quad \|g_n\|_{L^\infty} \leq C(f) \text{ as } n \to \infty.
\]

As \( f_n \) and \( g_n \) are smooth, we infer that the functions

\[
z \mapsto \int_{\mathbb{R}^2} \frac{\xi}{|\xi|^2} f_n(z - \xi) \, d\xi \quad \text{and} \quad t \mapsto \int_{\mathbb{R}^2} \frac{\xi}{|\xi|^2} g_n(z - \xi) \, d\xi
\]

are continuous. Moreover, we deduce from the previous estimates that

\[
\left\| R_1(T^{-1}(z)) - \int_{\mathbb{R}^2} \frac{(z - \eta)^\perp}{|z - \eta|^2} g_n(\eta) \, d\eta \right\|_{L^\infty(B(0,1)^c)} \leq C_0 \left( \|f - g_n\|_{L^1(B(0,2)^c)}^{1/2} \|g - g_n\|_{L^1(B(0,2)^c)}^{1/2} + \|f - f_n\|_{L^p(B(0,2)^c)} \|f - f_n\|_{L^p(B(0,2))}^{(p_0-1)/p_0} \right).
\]

Passing to the limit \( n \to \infty \), we prove the continuity of \( R_1 \circ T^{-1} \). Using Theorem 2.1, we conclude that \( R_1 \) is continuous up to the boundary.

**b) Estimate and continuity of \( R_2 \).**

We use, as before, the notations \( f, z \) and the change of variables \( \eta \)

\[
R_2(T^{-1}(z)) = \int_{|\eta| \geq 1} \frac{(z - \eta^*)^\perp}{|z - \eta^*|^2} f(\eta) \, d\eta
\]

\[
= \int_{|\eta| \geq 2} \frac{(z - \eta^*)^\perp}{|z - \eta^*|^2} f(\eta) \, d\eta + \int_{1 \leq |\eta| \leq 2} \frac{(z - \eta^*)^\perp}{|z - \eta^*|^2} f(\eta) \, d\eta
\]

\[
:= R_{21}(z) + R_{22}(z).
\]

If \( |\eta| \geq 2 \), then \( |z - \eta^*| \geq 1/2 \) because \( |z| \geq 1 \) (see the definition of \( T \)). Therefore, we obtain obviously that

\[
\|R_{21}\|_{L^\infty(B(0,1)^c)} \leq 2 \|f\|_{L^1(B(0,2)^c)} \leq 2 \|\omega\|_{L^1}.
\]

The continuity is easier than above:

- we approximate \( f_{\mathcal{X}B(0,2)^c} \) by \( g_n \in C^\infty_c(B(0,2)^c) \) such that \( \|g_n - f\|_{L^1(B(0,2)^c)} \to 0 \) as \( n \to \infty \);
- the functions

\[
z \mapsto \int_{|\eta| \geq 2} \frac{(z - \eta^*)^\perp}{|z - \eta^*|^2} g_n(\eta) \, d\eta
\]

is continuous up to the boundary \( \partial B(0,1) \) because \( |z - \eta^*| \geq 1/2 \);

- the previous estimates gives

\[
\left\| R_{21}(z) - \int_{|\eta| \geq 2} \frac{(z - \eta^*)^\perp}{|z - \eta^*|^2} g_n(\eta) \, d\eta \right\|_{L^\infty(B(0,1)^c)} \leq 2 \|f - g_n\|_{L^1(B(0,2)^c)};
\]

which implies the continuity of \( R_{21} \).

Concerning \( R_{22} \), we again change variables writing \( \theta = \eta^* \), to obtain:

\[
R_{22}(z) = \int_{1/2 \leq |\theta| \leq 1} \frac{(z - \theta)^\perp}{|z - \theta|^2} f(\theta^*) \, d\theta / |\theta|^4.
\]

Let \( g(\theta) := f(\theta^*) / |\theta|^4 \). As above, we deduce by changing variables back that

\[
\|g\|_{L^1(1/2 \leq |\theta| \leq 1)} \leq \|\omega\|_{L^1}.
\]

It is also easy to see that

\[
\|g\|_{L^p(1/2 \leq |\theta| \leq 1)} \leq 2 \|f\|_{L^p(B(0,2))} \leq C_6 \|\omega\|_{L^\infty}.
\]

Then, by the classical estimates of the Biot-Savart law in \( \mathbb{R}^2 \), we have

\[
\|R_{22}\|_{L^\infty(B(0,1)^c)} \leq C \|\omega\|_{L^1} \|\omega\|_{L^\infty}^{1-a}.
\]
Reasoning as for $R_1$, where we approximate $g$, we get that $R_{22}$ is continuous.

The continuity of $T$ allows us to conclude that $R_2$ is continuous up to the boundary, which ends the proof in the case of $\Omega_{\text{ext}}$ unbounded.

**Remark about the bounded case.**

Concerning $R_1$, we do not need to decompose the integral in two parts:

$$
\|f\|_{L^p_0(B(0,1))} \leq C_2 \|\omega\|_{L^\infty},
$$

where $f(\eta) := \omega(T^{-1}(\eta))|\det(DT^{-1}(\eta))|\chi_{\{|\eta|\leq 1\}}$.

Even with $R_2$, we directly have

$$
R_2(T^{-1}(z)) = \int_{|\theta|\geq 1} \frac{(z - \theta)\check{\eta}}{|z - \theta|^2} f(\theta^*) \frac{d\theta}{|\theta|^2}
$$

and we conclude following the proof concerning $R_{22}$.

**6.2 Proof of Lemma 2.6** Using the explicit formula of $\Phi$ and (2.9), we write

$$
u(x) \cdot \nabla \Phi^e(x) = \nu^\perp(x) \cdot \nabla \Phi^\perp(x)
$$

$$
= -\frac{1}{2\pi\varepsilon} \Phi'\left(\frac{|T(x)| - 1}{\varepsilon}\right) \int_{\Omega_{\text{ext}}} \left( \frac{T(y) \cdot T(x)}{|T(x) - T(y)|^2} - \frac{T(y)^* \cdot T(x)^*}{|T(x) - T(y)^*|^2} \right) \omega(t, y) \, dy
$$

As $T$ is holomorphic, $DT$ is of the form $\begin{pmatrix} a & b \\ -b & a \end{pmatrix}$ and we can check that $DT(x)DT^T(x) = (a^2 + b^2)Id = |\det(DT(x))|Id$, so

$$
\nu(x) \cdot \nabla \Phi^e(x) = \frac{\Phi'\left(\frac{|T(x)| - 1}{\varepsilon}\right) |\det(DT(x))|}{2\pi \varepsilon |T(x)|} \int_{\Omega_{\text{ext}}} \left( \frac{T(y) \cdot T(x)}{|T(x) - T(y)|^2} - \frac{T(y)^* \cdot T(x)^*}{|T(x) - T(y)^*|^2} \right) \omega(t, y) \, dy.
$$

We compute the $L^1$ norm, next we change variables twice $\eta = T(y)$ and $z = T(x)$, to have

$$
\|\nu \cdot \nabla \Phi^e\|_{L^1} = \frac{1}{2\pi \varepsilon} \int_{|z|\geq 1} \left| \Phi' \left( \frac{|z| - 1}{\varepsilon} \right) \right| \left| \int_{|\eta|\geq 1} \left( \frac{\eta \cdot z^\perp/|z| - \eta^* \cdot z^\perp/|z|^2}{|z - \eta|^2} \right) f(t, \eta) \, d\eta \right| \, dz,
$$

where $f(t, \eta) = \omega(t, T^{-1}(\eta))|\det(DT^{-1}(\eta))|$.

Thanks to the definition of $\Phi$, we know that $\left\| \frac{\Phi'\left(\frac{|z| - 1}{\varepsilon}\right)}{\varepsilon} \right\|_{L^1} \leq C$. So it is sufficient to prove that

$$
\left\| \int_{|\eta|\geq 1} \left( \frac{\eta \cdot z^\perp/|z| - \eta^* \cdot z^\perp/|z|^2}{|z - \eta|^2} \right) f(t, \eta) \, d\eta \right\|_{L^\infty(1+\varepsilon \leq |z| \leq 1+2\varepsilon)} \to 0 \quad (6.1)
$$

as $\varepsilon \to 0$, uniformly in time.

Let

$$
A := \frac{\eta \cdot z^\perp/|z| - \eta^* \cdot z^\perp/|z|}{|z - \eta|^2 - |z - \eta^*|^2}.
$$

We compute

$$
A = \left( \frac{|z|^2 - 2z \cdot \eta/|\eta|^2 + 1/|\eta|^2 - 1/|\eta|^2(|z|^2 - 2z \cdot \eta + |\eta|^2)}{|z - \eta|^2 z - \eta^*|^2} \right) \eta \cdot z^\perp/|z|
$$

$$
= \left( \frac{|z|^2 - 1(1 - 1/|\eta|^2)}{|z - \eta|^2 z - \eta^*|^2} \right) \eta \cdot z^\perp/|z|.
$$

We use that $|z| \geq 1$, to write

$$
|z - \eta^*| \geq 1 - \frac{1}{|\eta|}.
$$
Moreover, $|\eta^*| \leq 1$ allows us to have

$$|z - \eta^*| \geq |z| - 1.$$  

We can now estimate $A$ by:

$$|A| \leq \frac{(|z| + 1)(1 + 1/|\eta|)(|z| - 1)^b}{|z - \eta|^2|z - \eta^*|^b}|\eta| \cdot \frac{z^\perp}{|z|}$$

with $0 \leq b \leq 1$, to be chosen later. We remark also that $\eta \cdot \frac{z^\perp}{|z|} = (\eta - z) \cdot \frac{z^\perp}{|z|}$ and the Cauchy-Schwarz inequality gives

$$|\eta \cdot \frac{z^\perp}{|z|}| \leq |\eta - z|.$$  

We use the fact that $|z| - 1 \leq 2 \varepsilon$, to estimate (6.1):

$$\left| \int_{|\eta| \geq 1} Af(t, \eta) \, d\eta \right| \leq (2 + 2\varepsilon)\cdot 2.2(2\varepsilon)^b \int_{|\eta| \geq 1} \frac{|f(t, \eta)|}{|z - \eta||z - \eta^*|^b} \, d\eta,$$

hence, the Hölder inequality gives

$$\left| \int_{|\eta| \geq 1} Af(t, \eta) \, d\eta \right| \leq (2 + 2\varepsilon)\cdot 2.2(2\varepsilon)^b \left\| \frac{f(t, \eta)^{1/p}}{|z - \eta|} \right\|_{L^p} \left\| \frac{f(t, \eta)^{1/q}}{|z - \eta^*|^b} \right\|_{L^q}$$

with $1/p + 1/q = 1$ chosen later.

In the same way as we have estimated $R_2$ in the proof of Proposition 2.4, we obtain for $bq = 1$:

$$\left\| \frac{f(t, \eta)^{1/q}}{|z - \eta|^b} \right\|_{L^q} = \left( \int_{|\eta| \geq 1} \frac{|f(t, \eta)|}{|z - \eta|^b} \, d\eta \right)^{1/q} \leq C_q,$$

where we have used that $\omega$ belongs to $L^\infty(\mathbb{R}_+^*; L^1 \cap L^\infty(\Omega_{ext}))$.

Now we use Lemma 6.1 for $f \in L^1 \cap L^{p_0}$, with $p_0 > 2$ and for $f \in L^1 \cap L^\infty$ (see the proof of Proposition 2.4). Then, we choose $p \in (1, 2)$ such that $p_0 > \frac{2}{2 - p}$ and we follow the estimate of $R_1$ in the proof of Proposition 2.4 to obtain:

$$\left\| \frac{f(t, \eta)^{1/p}}{|z - \eta|} \right\|_{L^p} = \left( \int_{|\eta| \geq 1} \frac{|f(t, \eta)|}{|z - \eta|^p} \, d\eta \right)^{1/p} \leq C_p,$$

where we have used again that $\omega$ belongs to $L^\infty(\mathbb{R}_+^*; L^1 \cap L^\infty(\Omega_{ext}))$.

Fixing a $p \in (1, 2)$ such that $p_0 > \frac{2}{2 - p}$, it gives $q \in (2, \infty)$ and $b \in (0, 1/2)$ and it follows

$$\|u \cdot \nabla \Phi\|_{L^1} \leq C(2 + 2\varepsilon)\cdot 2.2(2\varepsilon)^b C_p C_q$$

which tends to zero when $\varepsilon$ tends to zero, uniformly in time.

6.3. Proof of Lemma 2.7 Let $T > 0$ fixed. We rewrite (2.9):

$$u(x) = \frac{1}{2\pi} DT^T(x) \left( \int_{\Omega_{ext}} \frac{\mathcal{T}(x) - \mathcal{T}(y)}{|\mathcal{T}(x) - \mathcal{T}(y)|^2} - \frac{\mathcal{T}(x) - \mathcal{T}(y)^*}{|\mathcal{T}(x) - \mathcal{T}(y)^*|^2} \right)^{1/2} \omega(y) \, dy + \alpha \mathcal{T}(x)^{1/2}$$

$$:= \frac{1}{2\pi} DT^T(x) h(T(x))$$

where $\alpha$ is bounded by $\|\gamma\|_{L^\infty([0, T])} + \|\omega\|_{L^\infty(L^1)}$ in $[0, T]$ (see (2.12)).

We start by treating $h$. We change variables $\eta = T(y)$, and we obtain

$$h(z) = \int_{B(0,1)^c} \left( \frac{z - \eta}{|z - \eta|^2} - \frac{z - \eta^*}{|z - \eta^*|^2} \right) \omega(T^{-1}(\eta)) \, d\eta + \alpha \frac{z^\perp}{|z|^2}$$

$$= \int_{B(0,2)^c} \frac{z - \eta}{|z - \eta|^2} f(t, \eta) \, d\eta + \int_{B(0,2) \setminus B(0,1)} \frac{z - \eta}{|z - \eta|^2} f(t, \eta) \, d\eta - \int_{B(0,2)^c} \frac{z - \eta^*}{|z - \eta^*|^2} f(t, \eta) \, d\eta$$

$$- \int_{B(0,2) \setminus B(0,1)} \frac{z - \eta^*}{|z - \eta^*|^2} f(t, \eta) \, d\eta + \alpha \frac{z^\perp}{|z|^2}$$

$$:= h_1(z) + h_2(z) - h_3(z) + h_4(z) + \alpha h_5(z),$$
with \(f(t, \eta) = \omega(t, T^{-1}(\eta))\) \(|\det DT^{-1}(\eta)|\) belonging to \(L^{\infty}(L^1 \cap L^{p_0}(B(0,2) \setminus B(0,1))\) for some \(p_0 > 2\) and to \(L^{\infty}(L^1 \cap L^{\infty}(B(0,2)\c))\) (see the proof of Proposition 2.4). As \(|z| = |T(x)| \geq 1\), we are looking for estimates in \(B(0,1)\). Obviously we have that

\[
|h_5| \text{ belongs to } L^{\infty}(B(0,1)\c) \text{ and } |Dh_5| \text{ belongs to } L^{\infty}(B(0,1)\c).
\]

Concerning \(h_1\), we introduce \(f_1 := f \chi_{B(0,2)\c}\) where \(\chi_S\) denotes the characteristic function on \(S\). Hence

\[
h_1(z) = \int_{\mathbb{R}^2} \frac{(z - \eta)\perp}{|z - \eta|^2} f_1(\eta) \, d\eta \quad \text{with} \quad f_1 \in L^{\infty}(\mathbb{R}^+; L^1(\mathbb{R}^2) \cap L^{\infty}(\mathbb{R}^2)).
\]

We have used the relation between the \(L^p\) norm of \(f\) and of \(\omega\) (see the proof of Proposition 2.4). The standard estimates on the Biot-Savart kernel in \(\mathbb{R}^2\) (i.e. Lemma 6.1) and the Calderón-Zygmund inequality give that

\[
h_1 \text{ belongs to } L^{\infty}(\mathbb{R}^+ \times B(0,1)\c) \text{ and } |Dh_1| \text{ belongs to } L^{\infty}(\mathbb{R}^+; L^p(B(0,1)\c)), \quad \forall p \in (1, \infty).
\]

For \(h_2\), it is almost the same argument: we introduce \(f_2 := f \chi_{B(0,2)\setminus B(0,1)}\), hence

\[
h_2(z) = \int_{\mathbb{R}^2} \frac{(z - \eta)\perp}{|z - \eta|^2} f_2(\eta) \, d\eta \quad \text{with} \quad f_2 \in L^{\infty}(\mathbb{R}^+; L^1(\mathbb{R}^2) \cap L^{p_0}(\mathbb{R}^2)).
\]

The standard estimates on the Biot-Savart kernel in \(\mathbb{R}^2\) and the Calderón-Zygmund inequality give that

\[
h_2 \text{ belongs to } L^{\infty}(\mathbb{R}^+ \times B(0,1)\c) \text{ and } |Dh_2| \text{ belongs to } L^{\infty}(\mathbb{R}^+; L^{p_0}(B(0,1)\c)).
\]

For \(h_3\), we can remark that for any \(\eta \in B(0,2)\c\) we have \(|z - \eta^\ast| \geq \frac{1}{2}\). Therefore, the function \((z, \eta) \mapsto \frac{(z-\eta^\perp)}{|z-\eta|^2}\) is smooth in \(B(0,1)\c \times B(0,2)\c\), which gives us, by a classical integration theorem, that

\[
h_3 \text{ belongs to } L^{\infty}(\mathbb{R}^+ \times B(0,1)\c) \text{ and } |Dh_3| \text{ belongs to } L^{\infty}(\mathbb{R}^+ \times B(0,1)\c).
\]

To treat the last term, we change variables \(\theta = \eta^\ast\)

\[
h_4(z) = \int_{B(0,1)\setminus B(0,1/2)} \frac{(z - \theta)\perp}{|z - \theta|^2} f(\theta^\ast) \, d\theta^\ast := \int_{\mathbb{R}^2} \frac{(z - \theta)\perp}{|z - \theta|^2} f_4(\theta) \, d\theta,
\]

with \(f_4(\theta) := \frac{f(t, \theta^\ast)}{|\theta|^4} \chi_{B(0,1)\setminus B(0,1/2)}(\theta)\) which belongs to \(L^{\infty}(\mathbb{R}^+; L^1(\mathbb{R}^2) \cap L^{p_0}(\mathbb{R}^2))\). Therefore, standard estimates on the Biot-Savart kernel and the Calderón-Zygmund inequality give that

\[
h_4 \text{ belongs to } L^{\infty}(\mathbb{R}^+ \times B(0,1)\c) \text{ and } |Dh_4| \text{ belongs to } L^{\infty}(\mathbb{R}^+; L^{p_0}(B(0,1)\c)).
\]

Now, we come back to \(u\). As \(u(x) = \frac{1}{2\pi} DT^T(x) h(T(x))\), with \(DT\) belonging to \(L^1_{\text{loc}}(\overline{\Omega_{\text{ext}}}\c)\) (see Remark 2.2) and as \(h \circ T\) is uniformly bounded, we have that

\[
u \text{ belongs to } L^{\infty}(\Omega; L^1_{\text{loc}}(\overline{\Omega_{\text{ext}}}\c))\).
\]

Adding the bounded behavior of \(DT\) at infinity, we have that

\[
u \text{ belongs to } L^{\infty}(\Omega; L^1_{\text{loc}}(\overline{\Omega_{\text{ext}}}\c) + L^{\infty}(\overline{\Omega_{\text{ext}}}\c)\).
\]

Moreover, we have

\[
\text{as } u \text{ is uniformly bounded and } DT \text{ belongs to } L^p_{\text{loc}}(\overline{\Omega_{\text{ext}}}\c) \text{ for any } p < 4/3 \text{ (see Theorem 2.1). We note that the second right hand side term belongs to } L^{\infty}(\Omega; L^{4/3}_{\text{loc}}(\overline{\Omega_{\text{ext}}}\c)) \text{ because } DT \text{ belongs to } L^{8/3}_{\text{loc}}(\overline{\Omega_{\text{ext}}}\c) \text{ and } (-Dh_3 + \alpha Dh_5)(T(x)) \text{ belongs to } L^{\infty}(\Omega; L^{8/3}_{\text{loc}}(\overline{\Omega_{\text{ext}}}\c)).
\]
Concerning the third right hand side term, we use that $T$ holomorphic implies that $DT$ is of the form $\left( \begin{array}{cc} a & b \\ -b & a \end{array} \right)$. Hence, we get easily that

$$|DT(x)|^2_{\omega,0} = (\sup(|a|,|b|))^2 \leq a^2 + b^2 = |\det DT(x)|.$$  

Therefore, changing variables, we have for $i = 1, 2, 4$ and $K$ any compact set of $\Omega_{ext}$:

$$||DT||^2|Dh_i \circ T||_{L^2(K)} \leq ||Dh_i||_{L^2(K)}$$

with $\tilde{K} := T(K)$ a compact set (by the continuity of $T$), which is bounded because $2 < p_0$. As $DT$ belongs to $L^p(K)$ for any $p < 4$, we have by the H"older inequality that $|DT|^2|Dh_i \circ T|$ is uniformly bounded in $L^p(K)$ for any $p \in [1, 4/3]$. This ends the proof.

### 6.4. Proof of Proposition 2.10

We set $\beta(t) = t^2$ and use (2.18) with this choice. Let $\Phi \in D(\mathbb{R}^+ \times \mathbb{R}^2)$

$$\int_{\mathbb{R}^2} \Phi(T, x)(\tilde{\omega})^2(T, x) \, dx - \int_{\mathbb{R}^2} \Phi(0, x)(\tilde{\omega})^2(0, x) \, dx = \int_0^T \int_{\mathbb{R}^2} (\tilde{\omega})^2(\partial_t \Phi + \bar{\omega} \cdot \nabla \Phi) \, dx \, dt.$$

This is actually an improvement of (2.18), in which the equality holds in $L^1_{loc}(\mathbb{R}^+)$. Indeed, we have $\partial_t \tilde{\omega} = -\text{div}(\bar{\omega}u)$ (in the sense of distributions) with $\tilde{\omega} \in L^\infty$ and $\bar{\omega} \in L^p_{loc}(\mathbb{R}^+, L^p_{loc}(\mathbb{R}^2))$ for all $p < 4$ (see [2.13]), which implies that $\partial_t \tilde{\omega}$ belongs to $L^1_{loc}(\mathbb{R}^+, W^{-1,p}_{loc}(\mathbb{R}^2))$. Hence, $\tilde{\omega}$ belongs to $C(\mathbb{R}^+, W^{-1,p}_{loc}(\mathbb{R}^2)) \cap L^\infty(\mathbb{R}^+, L^2(\mathbb{R}^2)) \subset C_w(\mathbb{R}^+, L^2_{loc}(\mathbb{R}^2))$, where $C_w L^2_{loc}$ stands for the space of maps $f$ such that for any sequence $t_n \to t$, the sequence $f(t_n)$ converges to $f(t)$ weakly in $L^2_{loc}$. Since on the other hand $t \mapsto \|\tilde{\omega}(t)\|_2$ is continuous by Remark 2.9, we have $\tilde{\omega} \in C(\mathbb{R}^+, L^2(\mathbb{R}^2))$. Therefore the previous integral equality holds for all $T$.

Now, we choose a good test function. We let $\Phi_0$ be a non-decreasing function on $\mathbb{R}$, which is equal to 1 for $s \geq 2$ and vanishes for $s \leq 1$ and we set $\Phi(t, x) = \Phi_0(|x|/R(t))$, with $R(t)$ a smooth, positive and increasing function to be determined later on, such that $R(0) = R_0$. For this choice of $\Phi$, we have $(\omega_0(x))^2 \Phi(0, x) = 0$.

We compute then

$$\nabla \Phi = \frac{x}{|x|} \frac{\Phi_0'}{R(t)}$$

and

$$\partial_t \Phi = -\frac{R'(t)}{R^2(t)} |x| \Phi_0'.$$

We obtain

$$\int_{\mathbb{R}^2} \Phi(T, x)(\tilde{\omega})^2(T, x) \, dx = \int_0^T \int_{\mathbb{R}^2} (\tilde{\omega})^2 \frac{\Phi_0(|x|)}{R} \left( u(x) \cdot \frac{x}{|x|} - \frac{R'}{R} |x| \right) \, dx \, dt$$

$$\leq \int_0^T \int_{\mathbb{R}^2} (\tilde{\omega})^2 \frac{\Phi_0(|x|)}{R} (C - R') \, dx \, dt,$$

where $C$ is independent of $t$ and $x$. Indeed, we have that

$$u(t, x) = \frac{1}{2\pi} DT^T(x) \left( R[\omega](x) + (\gamma + \int \omega_0) \frac{T(x)^{\perp}}{|T(x)|^2} \right)$$

with $|R[\omega]| \leq C_1$ (see Proposition 2.4) and $1/|T(x)| \leq 1$. Using (2.5), we know that there exists a positive $C_2$ such that

$$|DT(x)| \leq C_2|\beta|, \, \forall |x| \geq R_0.$$

Putting together all these inequalities with (2.12), we obtain

$$C = \frac{1}{2\pi} C_2|\beta| \left( C_1 + \|\gamma\|_{L^\infty(0, T^*)} + \|\omega_0\|_{L^1} \right).$$
Taking $R(t) = R_0 + Ct$, we arrive at
\[ \int_{\mathbb{R}^2} \Phi(T, x)(\tilde{\omega})^2(T, x) \, dx \leq 0, \]
which ends the proof.

6.5. **Proof of Proposition 5.1** By the conservation of the total mass of $\omega_i$ (2.19), we have that
\[ \int_{\mathbb{R}^2} \tilde{\omega}(t, \cdot) \equiv 0, \quad \forall t \geq 0. \]
Moreover, Proposition 2.10 states that there exists $C_1(\omega_0, \Omega)$ such that $\omega_1(t, \cdot)$ and $\omega_2(t, \cdot)$ are compactly supported in $B(0, R_0 + C_1 t)$. So we first infer that $\tilde{v}(t) \in L^2(\mathbb{R}^2)$ for all $t$ (see e.g. 20). Using that $\|\omega_i\|_{L^1(\Omega) \cap L^\infty(\Omega)} \in L^\infty(\mathbb{R}^+)$, we even obtain
\[ \tilde{v} \in L^\infty_{\text{loc}}(\mathbb{R}^+, L^2(\mathbb{R}^2)). \]

We now turn to the first assertion in Proposition 5.1. By Lemma 6.1 and the Calderón-Zygmund inequality we state that (2.20) implies that $v_i = K_{\mathbb{R}^2} * \tilde{\omega}$ belongs to $L^\infty(\mathbb{R}^+ \times \mathbb{R}^2)$ and its gradient $\nabla v_i$ to $L^\infty(\mathbb{R}^+, L^4(\mathbb{R}^2))$. On the other hand, since the vorticity $\omega_i$ is compactly supported, we have for large $|x|$ \[ |v_i(t, x)| \leq \frac{C}{|x|} \int_{\mathbb{R}^2} |\tilde{\omega}_i(t, y)| \, dy, \]hence $v_i$ belongs to $L^\infty_{\text{loc}}(\mathbb{R}^+, L^p(\mathbb{R}^2))$ for all $p > 2$. It follows in particular that
\[ v_i \in L^\infty_{\text{loc}}(\mathbb{R}^+, W^{1,4}(\mathbb{R}^2)) \]
and also that $v_i \otimes v_i$ belongs to $L^\infty_{\text{loc}}(L^{4/3})$. Since $v_i$ is divergence-free, we have $v_i \cdot \nabla v_i = \text{div} (v_i \otimes v_i)$, and so $v_i \cdot \nabla v_i \in L^2_{\text{loc}}(\mathbb{R}^+, W^{-1,4}(\mathbb{R}^2))$.

Thanks to (2.13), we know that $v_i(t) \otimes w_i(t)$ belongs to $L^{4/3}_{\text{loc}}$. At infinity, we use the explicit formula of $u$ (2.9), the compact support of the vorticity and the behavior of $T$ at infinity (2.5) to note that $w_i$ is bounded by $C/|x|$. $v_i$ has the same behavior at infinity, which belongs to $L^{8/3}$. This yields
\[ \text{div} (v_i \otimes w_i), \quad \text{div} (w_i \otimes v_i) \in L^2_{\text{loc}}(\mathbb{R}^+, W^{-1,4}(\mathbb{R}^2)). \]

Besides, we can infer from the behavior of $T$ on the boundary (Theorem 2.1) and Proposition 2.4 that $\tilde{g}_{v_i, \gamma_0}$, defined in (2.17), is uniformly bounded in $L^1(\partial \Omega)$. Then we deduce from the embedding of $W^{1,4}(\mathbb{R}^2)$ in $C^0_0(\mathbb{R}^2)$ that $\tilde{g}_{v_i, \gamma_0} \delta_0$ belongs to $L^2_{\text{loc}}(W^{-1,4})$. Therefore, $v_i \tilde{g}_{v_i, \gamma_0} \delta_0 \in L^2_{\text{loc}}(\mathbb{R}^+, W^{-1,4}(\mathbb{R}^2))$.

According to (5.1), we finally obtain
\[ \langle \partial_t v_i, \Phi \rangle = \langle \partial_t v_i - \nabla p_i, \Phi \rangle \leq C\|\Phi\|_{L^2(W^{1,4}_a)}, \]
for all divergence-free smooth vector field $\Phi$. This implies that
\[ \partial_t v_i \in L^2_{\text{loc}}(\mathbb{R}^+, W^{-1,4/3}_\sigma(\mathbb{R}^2)), \quad i = 1, 2, \]
and the same holds for $\partial_t \tilde{v}$. Now, since $\tilde{v}$ belongs to $L^2_{\text{loc}}(\mathbb{R}^+, W^{1,4}_\sigma)$, we deduce from (6.2) and Lemma 1.2 in Chapter III of [27] that $\tilde{v}$ is almost everywhere equal to a function continuous from $\mathbb{R}^+$ into $L^2$ and we have in the sense of distributions on $\mathbb{R}^+$:
\[ \frac{d}{dt} \|\tilde{v}\|^2_{L^2(\mathbb{R}^2)} = 2\langle \partial_t \tilde{v}, \tilde{v} \rangle_{W^{-1,4/3}_\sigma, W^{1,4}_\sigma}. \]

We finally conclude by using the fact that $\tilde{v}(0) = 0$. 


7. Final remarks and comments

7.1. No extraction in convergence results. In [25, 17, 4], the existence of a weak solution is a consequence of a compactness argument. Indeed, we consider therein the unique solution \( u_n \) of the Euler equations in the smooth domain \( \Omega_n \), which converges to \( \Omega \) in some geometrical senses. Then, we extract a subsequence such that \( u_{\varphi(n)} \to u \) and we check that \( u \) is a solution of the Euler equations in \( \Omega \). Putting together the present result with [17], we can state the following.

**Theorem 7.1.** Let \( \omega_0, \gamma_0, \Omega \) as in Theorems 1.2 or 1.3. For any sequence of smooth open simply connected domains (or exterior of simply connected domains) \( \Omega_n \) converging to \( \Omega \) in the Hausdorff sense, then the unique solution \( u_n \) of the Euler equations on \( \Omega_n \), with initial datum \( u_0^n \) such that

\[
\text{div} u_0^n = 0, \quad \text{curl} u_0^n = \omega_0, \quad u_n^0 \cdot \hat{n}|_{\partial\Omega_n} = 0, \quad \lim_{|x| \to +\infty} u_n^0 = 0, \quad \oint_{\partial\Omega_n} u_n^0 \cdot \hat{\tau} \, ds = \gamma_0 \quad \text{(only for exterior domains)},
\]

converges in \( L^2_{\text{loc}}(\mathbb{R}^+ \times \overline{\Omega}) \) to the unique solution \( u \) of the Euler equations on \( \Omega \) with initial datum \( u_0 \) such that

\[
\text{div} u_0 = 0, \quad \text{curl} u_0 = \omega_0, \quad u^0 \cdot \hat{n}|_{\partial\Omega} = 0, \quad \lim_{|x| \to +\infty} u^0 = 0, \quad \oint_{\partial\Omega} u^0 \cdot \hat{\tau} \, ds = \gamma_0 \quad \text{(only for exterior domains)}.
\]

7.2. Special vortex sheet. In [10], we consider some smooth domains \( \Omega_\varepsilon = \mathbb{R}^2 \setminus C_\varepsilon \) where the thin obstacles \( C_\varepsilon \) shrink to a \( C^2 \) Jordan arc \( \Gamma \) as \( \varepsilon \) tends to zero. For \( \omega_0 \in L^\infty(\Gamma^c) \) and \( \gamma \in \mathbb{R} \) given, we denote by \( (u_\varepsilon, \omega_\varepsilon) \) the corresponding regular solution of the Euler equations on \( \Omega_\varepsilon \). Truncating smoothly around the obstacle, it is proved therein that the resulting \( \tilde{u}_\varepsilon \) and \( \tilde{\omega}_\varepsilon \), defined on the full plane \( \mathbb{R}^2 \), converge in appropriate topologies to a solution \( u, \omega \) of the system

\[
\begin{cases}
\partial_t \omega + u \cdot \nabla \omega = 0, & t > 0, \ x \in \mathbb{R}^2, \\
\text{div} u = 0, & t > 0, \ x \in \mathbb{R}^2, \\
\text{curl} u = \omega + g_{\omega, \gamma}\delta_{\Gamma}, & t > 0, \ x \in \mathbb{R}^2.
\end{cases}
\]

This is an Euler like equation, modified by a Dirac mass along the arc. The density function \( g_{\omega, \gamma} \) is given explicitly in terms of \( \omega \) and \( \Gamma \). Moreover, it is shown that it is equal to the jump of the tangential component of the velocity across the arc. We refer to [16] for all necessary details.

Actually, the presence of this additional measure is mandatory in order that the velocity \( u \) is tangent to the curve, with circulation \( \gamma \) around it.

Therefore, in the exterior of a Jordan arc, (7.1) appears to be a special vortex sheet, “special” because the support of the dirac mass does not move (staying to be \( \Gamma \)) and because the normal component of the velocity on the curve is equal to zero. For a general vortex sheet, we can prove that the normal component is continuous, but not necessarily zero. In both case, we have a jump of the tangential component, which is equal to the density of the vortex sheet \( g \).

A consequence of the present work is the uniqueness of a solution of (7.1), with the good sign conditions for \( \omega_0 \) and \( \gamma \) (see Theorem 1.3).

For instance, if we assume that \( \Gamma \) is the segment \([-1, 0); (1, 0)]\), then we have the explicit expression of the harmonic vector field thanks to the Joukowski function, and we can find in [16, p. 1144] the following:

\[
\text{curl} H_\Gamma = \frac{1}{\pi} \frac{1}{\sqrt{1 - x_1^2}} \chi_{(-1,1)}(x_1) \delta_0(x_2).
\]

Then, choosing \( \omega_0 \equiv 0 \) and \( \gamma = 1 \), we have proven that the stationary shear flow \( u(t, x) = H_\Gamma(x) \) is the unique solution of the Euler equations with initial vorticity \( \frac{1}{\pi} \frac{1}{\sqrt{1 - x_1^2}} \chi_{(-1,1)}(x_1) \delta_0(x_2) \).

Adding a vorticity or considering other shapes for \( \Gamma \) complicate a lot the expression of \( g_{\omega, \gamma} \) (see [16]). In particular, we do not prove the uniqueness for the so-called Prandtl-Munk vortex sheet:

\[
\frac{1}{\pi} \frac{1}{\sqrt{1 - x_1^2}} \chi_{(-1,1)}(x_1) \delta_0(x_2).
\]
7.3. Extension for constant vorticity near the boundary. As it is remarked several times, the crucial point is to prove that the vorticity never meet the boundary if we consider an initial vorticity compactly supported in $\Omega$. However, we can extend easily this result to the case of an initial vorticity constant to the boundary. Indeed, for $\alpha \in \mathbb{R}$ given, choosing $\beta(t) = (t-\alpha)^2$ in the proof of Proposition 4.1 gives the following.

**Proposition 7.2.** Let $\omega$ be a global weak solution of (2.15) such that $\omega_0$ is compactly supported in $\overline{\Omega}$ and such that $\omega_0 \equiv \alpha$ in a neighborhood of the boundary. If $\omega_0$ is non-positive and $\gamma_0 \geq -\int \omega_0$ (only for exterior domains), then, for any $T^* > 0$, there exists a neighborhood $U_{T^*}$ of $\partial \Omega$ such that

$$
\omega(t) \equiv \alpha \quad \text{on} \quad U_{T^*}, \quad \forall t \in [0, T^*].
$$

Therefore, in the proof of the uniqueness, we still have on $U_{T^*}$

$$
\text{curl } \tilde{v} = \text{curl } v_1 - \text{curl } v_2 = \alpha - \alpha = 0,
$$

which implies that the velocity $\tilde{v}$ is harmonic near the boundary, allowing us to follow exactly the proof made in Section 5.

7.4. Motivation for extending equations on $\mathbb{R}^2$. We present here some reasons for considering extensions as in Proposition 2.5.

Forgetting for a while the extension in $\mathbb{R}^2$, the important point in that proposition is to prove that the vorticity equation is verified for test functions possibly supported up to the boundary. This fact is used in Section 4 to prove that the vorticity does not meet the boundary in finite time. Indeed, we considered therein a test function supported in a neighborhood of the boundary (whose the size of the support depends on the Liapounov function constructed in Section 3).

A (minor) motivation to define the extension in Section 2 is to use the renormalized theory which was mainly developed for smooth domains. But as it is noted at the end of Section 4, the renormalized theory is not necessary with our sign assumption.

However, it is interesting in Section 5 to define a formulation in both side of $\partial \Omega$. Even if we have established that the velocity $u$ is harmonic in the neighborhood of the corners ($\text{div } u = \text{curl } u = 0$), it does not imply that $u$ has the regularity for the Yudovich’s arguments: for example $x \rightarrow \ln |T(x)|$ is harmonic in the neighborhood of the boundary but does not belong to $\cap_{p \geq 2} W^{1,p}(\Omega)$ if the angles are greater than $\pi$. The key in Section 5 is to introduce $v = K_{\mathbb{R}^2} [\omega]$ and to state that $\|v\|_{L^\infty(V)} + \|\nabla v\|_{L^\infty(V)} \leq \|v\|_{L^2(U)}$ where $\partial \Omega \Subset V \Subset U \subset (\text{supp } \omega)^c$. Moreover, we can use the standard Calderón-Zygmund inequality in $\mathbb{R}^2$.

7.5. Lyapunov and sign conditions. Let us present in this subsection the different Lyapunov functions, the advantage of each, and why it is specific to the case studied.

**Vortex wave system in $\mathbb{R}^2$.** Let us consider that the initial vorticity is composed on a regular part plus a dirac mass centered at the point $z(t)$. Then Marchioro and Pulvirenti proved in [22] that there exists one solution to the following system:

\[
\begin{cases}
 v(t, \cdot) = (K_{\mathbb{R}^2} \ast \omega)(\cdot, t), \\
 \dot{z}(t) = v(t, z(t)), \\
 \dot{\phi}_x(t) = v(t, \phi_x(t)) + \frac{(\phi_x(t) - z(t))}{2\pi |\phi_x(t) - z(t)|^2}, \\
 \phi_x(0) = x, \ x \neq z_0, \\
 \omega(t, \phi_x(t)) = \omega_0(x),
\end{cases}
\]

which means that the point vortex $z(t)$ moves under the velocity field $v$ produced by the regular part $\omega$ of the vorticity, whereas the regular part and the vortex point give rise to a smooth flow $\phi$ along which $\omega$ is constant. In this case, we can prove that the trajectories never meet the point vortex considering the following Lyapunov function:

$$
L(t) := -\ln |\phi_x(t) - z(t)|,
$$
for \( x \neq z_0 \) fixed. We note that \( L \) goes to \( +\infty \) iff \( \phi_x(t) \to z(t) \), so we want to prove that \( L \) stays bounded. Next we compute:
\[
L'(t) = -\frac{(\phi_x(t) - z(t)) \cdot (\dot{\phi}_x(t) - \dot{z}(t))}{|\phi_x(t) - z(t)|^2} - \frac{(\phi_x(t) - z(t)) \cdot (v(t, \phi_x(t)) - v(t, z(t)))}{|\phi_x(t) - z(t)|^2}.
\]
Next, we use that the regular part \( v \) is log-lipschitz in order to obtain a Gronwall-type inequality. To summarize, we remark that in the case, the important points are:
\[
L(t) \to \infty \text{ iff } \phi_x(t) \to z(t) \quad \text{and} \quad (\phi_x(t) - z(t)) \cdot \frac{(\phi_x(t) - z(t))}{2\pi |\phi_x(t) - z(t)|^2} \equiv 0
\]
removing the singular part.

**Dirac mass fixed in \( \mathbb{R}^2 \).** Marchioro in [21] studied exactly the same problem as above, assuming that the vortex mass cannot move. Therefore, the previous Lyapunov does not work, because we do not have a difference of two velocities and we cannot use the log-lipschitz regularity. In that article, the author introduced a new Lyapunov:
\[
L(t) := -\int_{\mathbb{R}^2} \left( \ln |\phi_x(t) - y| \right) \omega(t, y) \, dy - \ln |\phi_x(t) - z_0|,
\]
where the first integral is the stream function associated to \( v \). Then, the first step was to prove that the integral is bounded, which implies that \( L \) goes to \( +\infty \) iff \( \phi_x(t) \to z_0 \). Next, he computed:
\[
L'(t) = -\left( \int_{\mathbb{R}^2} \frac{\phi_x(t) - y}{|\phi_x(t) - y|^2} \omega(t, y) \, dy + \frac{\phi_x(t) - z_0}{|\phi_x(t) - z_0|^2} \dot{\phi}_x(t) - \int_{\mathbb{R}^2} \left( \ln |\phi_x(t) - y| \right) \partial_t \omega(t, y) \, dy \right.
\]
\[
= -\int_{\mathbb{R}^2} \left( \ln |\phi_x(t) - y| \right) \partial_t \omega(t, y) \, dy - \int_{\mathbb{R}^2} \nabla \left( \ln |\phi_x(t) - y| \right) \cdot \left( v(t, y) + \frac{(y - z_0)}{2\pi |y - z_0|^2} \right) \omega(t, y) \, dy.
\]
The second step was to prove some good estimates for the right hand side integral in order to conclude by the Gronwall lemma. Here, we see that the singular term is now passed in a integral, which is bounded. Similarly, we note that the important points in this case are:
\[
L(t) \to \infty \text{ iff } \phi_x(t) \to z_0 \quad \text{and} \quad \left( \int_{\mathbb{R}^2} \frac{\phi_x(t) - y}{|\phi_x(t) - y|^2} \omega(t, y) \, dy + \frac{\phi_x(t) - z_0}{|\phi_x(t) - z_0|^2} \dot{\phi}_x(t) \right) \equiv 0.
\]

**Interior or exterior of simply connected domains.** In our case, we have again an explicit formula of the velocity by the Biot-Savart law (see (2.8) and (2.9)). As the velocity near the boundary blows up, we have to make appear a cancellation as Marchioro did, in order that the singular part goes in an integral. To do that, we introduce the stream function associated to the velocity:
\[
L_1(t, x) := \frac{1}{2\pi} \int_{\partial \Omega} \ln \left( \frac{|T(x) - T(y)|}{|T(x) - T(y)|^3} \right) \omega(y) \, dy + \frac{\alpha}{2\pi} \ln |T(x)|
\]
with \( \alpha = 0 \) in the bounded case. However, as this function tends to zero (instead to \( \infty \)) when \( x \to \partial \Omega \) (see Lemma 3.1), we add a logarithm:
\[
L(t) := -\ln |L_1(t, \phi_x(t))|,
\]
and the goal is to prove that \( L \) stays bounded. Then, we computed in Section 3
\[
L'(t) = -\frac{\partial_t L_1(t, \phi_x(t))}{|L_1(t, \phi_x(t))|},
\]
and we have proved that \( \partial_t L_1 \) tends to zero as \( \phi_x(t) \to \partial \Omega \), comparing the rate with \( L_1 \). Then, we note here that it is important that \( \partial_t L_1 \) goes to zero where \( L_1 \) tends to zero. We have managed to prove that \( \partial_t L_1 \) tends to zero near the boundary, and the sign condition allows us to state that the boundary is the only set where \( L_1 \) vanishes (see Lemma 3.2). For instance, in bounded domain (i.e. \( \alpha = 0 \)) we remark that a vorticity with both signs can imply that \( L_1 = 0 \) somewhere else than on \( \partial \Omega \). This last remark is the main reason of the sign condition of the vorticity. Next, the sign condition on the circulation follows from the fact that we want the same sign for both terms in \( L_1 \).
Remark 7.3. In Section 5, we have proved the uniqueness up to the time $T$, only using that the vorticity does not meet the boundary between $[0,T]$. Therefore, without any sign condition about the initial vorticity, it is an easy consequence of the uniform estimate of the velocity far away the boundary that we have local uniqueness for any $\omega_0 \in L^\infty_c(\Omega)$. The main part of this paper is to prove the global uniqueness.

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