Available online at www.sciencedirect.com
SCIENCE DIRECT
ELSEVIER
C. R. Acad. Sci. Paris, Ser. I 336 (2003) 577–580
COMPTES RENDUS
MATHEMATIQUE

Functional Analysis

K-duality for pseudomanifolds with an isolated singularity

K-dualité pour les pseudo-variétés à singularité isolée

Claire Debord [a], Jean-Marie Lescure [b]

[a] *Laboratoire de mathématiques et applications, Université de Haute-Alsace, 4, rue des Frères Lumière, 68093 Mulhouse cedex, France*
[b] *Laboratoire de mathématiques pures, Université Blaise Pascal, Complexe universitaire des Cézeaux, 124, av. des Landais, 63177 Aubière cedex, France*

Received 18 December 2002; accepted after revision 3 March 2003

Presented by Alain Connes

Abstract

We associate to a pseudomanifold X with an isolated singularity a differentiable groupoid G which plays the role of the tangent space of X. We construct a Dirac element D and a Dual Dirac element λ which induce a Poincaré duality in K-theory between the C^*-algebras $C(X)$ and $C^*(G)$. ***To cite this article: C. Debord, J.-M. Lescure, C. R. Acad. Sci. Paris, Ser. I 336 (2003).***

Résumé

Etant donnée une pseudo-variété X ayant une singularité isolée, nous lui associons un groupoïde différentiable G qui joue le rôle d'espace tangent à X. Nous construisons un élément Dirac D ainsi qu'un élément dual-Dirac λ qui induisent une dualité de Poincaré en K-théorie entre les C^*-algèbres $C(X)$ et $C^*(G)$. ***Pour citer cet article : C. Debord, J.-M. Lescure, C. R. Acad. Sci. Paris, Ser. I 336 (2003).***

1. The tangent bundle of a singular manifold

Let X be a pseudomanifold with an isolated singularity c, that is $X = cL \cup X_1$, where X_1 is a smooth compact manifold with boundary L glued along its boundary with the cone over L: $cL = L \times [0,1]/L \times \{0\}$. The singularity c is then the image of $L \times \{0\}$ in cL. We denote by $M = L \times]-1,1] \cup X_1$ the manifold obtained by gluing X_1 with $L \times]-1,1]$ along the boundary.

If y is a point of M or of $X \setminus \{c\}$ which is in $L \times]-1,1[$ we note $y_L \in L$ its tangential componant and $k_y \in]-1,1[$ its radial coordinate; the function k_y is smoothly extended to X_1 in such a way that $k_y \geqslant 1$ on X_1. We set $M^+ = \{y \in M \mid k_y > 0\}$, $M^- = \{y \in M \mid k_y < 0\}$ and $\overline{M^+} = \{y \in M \mid k_y \geqslant 0\}$.

E-mail addresses: C.Debord@uha.fr (C. Debord), Jean-Marie.Lescure@math.univ-bpclermont.fr (J.-M. Lescure).

1631-073X/03/$ – see front matter © 2003 Académie des sciences/Éditions scientifiques et médicales Elsevier SAS. Tous droits réservés.
doi:10.1016/S1631-073X(03)00124-9

We suppose that the manifold M is equipped with a Riemannian metric whose injectivity radius is bigger than 1 and which is a product metric on $L\times]-1,1[$.

We define the groupoid G, with source map s, range map r and space of units $G^{(0)}=M$:

$$G=M^-\times M^-\cup T\overline{M^+}\underset{r}{\overset{s}{\rightrightarrows}}M.$$

It is the union of the bundle $T\overline{M^+}=\{(z,v)\in TM\mid z\in\overline{M^+}\}$ equipped with its standard groupoid structure over $\overline{M^+}$ and the pair groupoid $M^-\times M^-\rightrightarrows M^-$.

In order to equip G with a smooth structure, we take the usual structure of manifold on $M^-\times M^-$ and on TM^+. We take a smooth positive function $\tau :]-1,+\infty[\,\to\mathbb{R}$ which satisfies $\tau^{-1}(\{0\})=[0,+\infty[$. A local chart around boundary points of $T\overline{M^+}$ is provided by the following map:

$$E_G:\mathcal{V}(TM)\to\mathcal{V}(G),\qquad \begin{cases}(y,V)\mapsto(y,V)\ \text{if}\ y\in\overline{M^+},\\ (y,V)\mapsto\big(y,\exp_y(-\tau(k_y)V)\big)\ \text{if}\ y\in M^-,\end{cases}$$

where exp is the exponential map of the Riemannian manifold M, $\mathcal{V}(TM)=\{(y,V)\in TM;\ -\tau(k_y)V\in\operatorname{dom}(\exp_y)\}$ and $\mathcal{V}(G)$ is a neighborhood of $G^{(0)}$ in G. The groupoid G is called the *tangent bundle* of X.

Following the construction of A. Connes for smooth manifolds [3], we define the *tangent groupoid* of X in the following way:

$$\mathcal{G}=M\times M\times]0,1]\cup G\times\{0\}\rightrightarrows M\times[0,1].$$

The groupoid $\mathcal{G}$ is the union of the groupoid $G\times\{0\}\rightrightarrows M\times\{0\}$ and the pair groupoid over M parametrized by $]0,1]$. We equip $\mathcal{G}$ with a structure of smooth groupoid similarly as we did for G.

The groupoid G is amenable [1] so that its reduced C^*-algebra coincides with the maximal one and it is nuclear. The same occurs for $\mathcal{G}$. We denote respectively $C^*(G)$ and $C^*(\mathcal{G})$ these C^*-algebras. Moreover, up to isomorphisms, these C^*-algebras do not depend on the map τ used to define the smooth structure of G.

2. The Dirac element

The partition $M\times\{0\}\cup M\times]0,1]$ of $\mathcal{G}^{(0)}$ into a saturated open subset and a saturated closed subset induces the following exact sequence of C^*-algebras [5]:

$$0\longrightarrow C^*(\mathcal{G}|_{M\times]0,1]})\longrightarrow C^*(\mathcal{G})\xrightarrow{e_0}C^*(\mathcal{G}|_{M\times\{0\}})=C^*(G)\longrightarrow 0,$$

where the first map is the inclusion and e_0 is the evaluation map at 0. The C^*-algebra $C^*(\mathcal{G}|_{M\times]0,1]})$ is isomorphic to $\mathcal{K}\otimes C_0(]0,1])$ which is contractible. So, since $C^*(\mathcal{G})$ is nuclear, the element $[e_0]$ of $KK(C^*(\mathcal{G}),C^*(G))$ corresponding to e_0, is invertible. We denote $[e_0]^{-1}\in KK(C^*(G),C^*(\mathcal{G}))$ its inverse. Let $e_1 : C^*(\mathcal{G})\to C^*(\mathcal{G}|_{M\times\{1\}})=\mathcal{K}$ be the evaluation map at 1. We let b be the generator of $KK(\mathcal{K},\mathbb{C})$. We set:

$$\partial=[e_0]^{-1}\underset{C^*(\mathcal{G})}{\otimes}[e_1]\underset{\mathcal{K}}{\otimes}b\in KK\big(C^*(G),\mathbb{C}\big).$$

The algebra $C(X)$ maps to the center of the multiplier algebra of $C^*(G)$. Let $\Psi : C^*(G)\otimes C(X)\to C^*(G)$ be the morphism induced by multiplication and $[\Psi]$ be the corresponding element in $KK(C^*(G)\otimes C(X),C^*(G))$.

The *Dirac element* is then defined as

$$D=[\Psi]\underset{C^*(G)}{\otimes}\partial\in KK\big(C^*(G)\otimes C(X),\mathbb{C}\big).$$

3. The dual Dirac element

We are looking for an element λ in $KK(\mathbb{C}, C^*(G) \otimes C(X))$, that is a continuous family $(\lambda_y)_{y \in X}$ of elements of $KK(\mathbb{C}, C^*(G))$. In order to be dual to the Dirac element D, λ is constructed so that:

(i) λ is in the image of $(i_{\mathcal{O}})_* : KK(\mathbb{C}, C^*(G \times X|_{\mathcal{O}})) \to KK(\mathbb{C}, C^*(G \times X))$ where $\mathcal{O}$ is an open subset of $\mathcal{U} = \{(x, y) \in M \times X \mid k_x < 1,\ k_y < 1\} \cup \{(x, y) \in M \times X \mid d(x, y) < 1\}$.
(ii) The equality $\lambda \otimes_{C^*(G)} \partial = 1 \in K^0(X)$ holds.

We first assign to each point y of X an open subset O_y of M which is a ball centered on y when $k_y \geqslant 1$, which is contained in M^- when $k_y \leqslant 1/2$ and equal to M^- when $k_y \leqslant \varepsilon$ (where $0 < \varepsilon < 1/2$). Furthermore the set $\mathcal{O} = \bigcup_{y \in X} O_y \times \{y\}$ is an open subset of $M \times X$ contained in $\mathcal{U}$.

Notice that $K(C^*(G|_{O_y})) \simeq \mathbb{Z}$ for each $y \in X$.

We pull back the vector bundle of differential forms on M to G and $\mathcal{G}$ using their range maps and then to $G \times X$ and $\mathcal{G} \times X$ using the first projection. We denote all these bundles by the same letter Λ. The following step is the construction of a continuous family $(\beta_y)_{y \in X}$, where $\beta_y = (F_y, C^*(G|_{O_y}, \Lambda))$ is an element of $E(\mathbb{C}, C^*(G|_{O_y}))$ which class is a generator of $K(C^*(G|_{O_y})) \simeq \mathbb{Z}$.

When O_y is a subset of M^+ it is natural to state $F_y = a_y$ where $a_y \in C_b^\infty(T^* O_y, \operatorname{End} \Lambda)$ is the symbol of a pseudo-differential operator G_y which satisfies:

- G_y belongs to $\Psi_c^0(O_y, \Lambda) + C_b^\infty(O_y, \operatorname{End} \Lambda)$ and $G_y^2 - 1$ belongs to $\Psi_c^{-1}(O_y, \Lambda)$,
- G_y is of the form $G_y = \begin{pmatrix} 0 & G_y^- \\ G_y^+ & 0 \end{pmatrix}$ with respect to the usual decomposition of $\Lambda = \Lambda^{\mathrm{ev}} \otimes \Lambda^{\mathrm{odd}}$,
- G_y^+ is surjective and $\operatorname{Ker}(G_y^+) = \mathbb{C} \cdot \epsilon_y$ where ϵ_y belongs to $C_b^\infty(O_y, \Lambda)$,
- the family $(G_y)_{\substack{y \in X \\ k_y > \varepsilon}}$ defines a continuous section of $B(H)$ where H is the bundle $\bigcup_{\substack{y \in X \\ k_y > \varepsilon}} L^2(O_y, \Lambda)$.

The existence of such an operator is a consequence of Theorem 19.2.12 of [6].

When y comes closer to the singularity c, we gradually pass from a situation where $G|_{O_y} = T O_y$ to a situation where $G|_{O_y} = O_y \times O_y$. The pseudo-differential calculus on groupoids, first introduced by A. Connes in [2] (see also [4]), enables us to construct a continuous family $(F_y)_{y \in X,\ k_y > \varepsilon}$ such that up to a compact operator $F_y = a_y$ when $G|_{O_y} = T O_y$ and $F_y = G_y$ when $G|_{O_y} = O_y \times O_y$. Thus we "replace" the symbol a_y by the operator G_y.

The last case is when O_y becomes equal to M^-. We first choose an appropriate extension of F_y which belongs to $B(L^2(O_y, \Lambda))$ into an element of $B(L^2(M^-, \Lambda))$. Afterwards, thanks to the properties of the operator G_y listed above, we can replace F_y by a constant operator F_c.

We construct in this way a continuous family β_y of elements of $E(\mathbb{C}, C^*(G|_{O_y}))$. This family induces an element β of $KK(\mathbb{C}, C^*(G \times X|_{\mathcal{O}}))$. We set $\lambda = (i_{\mathcal{O}})_*(\beta)$. The dual-Dirac element λ satisfies the properties (i) and (ii) mentionned above.

4. The Poincaré duality

Theorem 4.1. *The Dirac element D and the dual-Dirac element λ induce a Poincaré duality between the C^*-algebras $C^*(G)$ and $C(X)$, that is*:

$$\lambda \underset{C^*(G)}{\otimes} D = 1_{C(X)} \in KK\big(C(X), C(X)\big) \quad \textit{and} \quad \lambda \underset{C(X)}{\otimes} D = 1_{C^*(G)} \in KK\big(C^*(G), C^*(G)\big).$$

Idea of the proof. Let us consider the morphisms $\Psi', \Delta' : C^*(G \times X) \otimes C(X) \to C^*(G) \otimes C(X)$ given by $\Psi'(f \otimes g \otimes h) = \Psi(f \otimes h) \otimes g$ and $\Delta'(f \otimes g \otimes h) = f \otimes \Delta(h \otimes g)$ where $\Delta : C(X) \otimes C(X) \to C(X)$ is the multiplication map. Their restrictions $C^*(G \times X|_{\mathcal{O}}) \otimes C(X) \to C^*(G) \otimes C(X)$ are homotopic, hence, since

$\lambda = (i_{\mathcal{O}})_*(\beta)$, the following equality holds:

$$\lambda \underset{C^*(G)}{\otimes} [\Psi] = \lambda \underset{C(X)}{\otimes} [\Delta].$$

This ensures that

$$\lambda \underset{C^*(G)}{\otimes} D = (\lambda \underset{C^*(G)}{\otimes} \partial) \underset{C(X)}{\otimes} [\Delta] = 1_{C(X)}.$$

In order to show the second equality, we study the invariance of the element $\lambda \otimes_{C(X)} [\Psi]$ under the *flip* automorphism f of $C^*(G \times G)$, that is the automorphism induced by $(\gamma, \eta) \in G \times G \mapsto (\eta, \gamma)$. The motivation comes from the equality

$$\Big(\Big(\lambda \underset{C(X)}{\otimes} [\Psi]\Big) \underset{C^*(G\times G)}{\otimes} [f]\Big) \underset{C^*(G)}{\otimes} \partial = \Big(\lambda \underset{C^*(G)}{\otimes} \partial\Big) \underset{C(X)}{\otimes} [\Psi] = 1_{C^*(G)}, \quad \text{which implies}$$
$$\lambda \underset{C(X)}{\otimes} D - 1_{C^*(G)} = \Big(\Big(\lambda \underset{C(X)}{\otimes} [\Psi]\Big) \underset{C^*(G\times G)}{\otimes} [\mathrm{id} - f]\Big) \underset{C^*(G)}{\otimes} \partial.$$

If B is a symmetric geodesically convex neighborhood of the diagonal of $M^+ \times M^+$ contained in the range of exp, then the flip automorphism of $C^*(TB)$ is homotopic to identity. Let $C = L \times]-1, 1[\subset M$ and $F = M \times M \setminus C \times C$. We denote by $r_* : KK(C^*(G), C^*(G \times G)) \to KK(C^*(G), C^*(G \times G|_F))$ the morphism induced by the restriction r.

Using again that λ is in the image of $(i_{\mathcal{O}})_*$ and that $\mathcal{O} \cap M^+ \times M^+$ is a small enough neigborhood of the diagonal, we show that $r_*(\lambda \otimes_{C(X)} [\Psi])$ is invariant under the flip automorphism of $C^*(G \times G|_F)$. Then, the long exact sequence in KK-theory associated to:

$$0 \longrightarrow C^*(G \times G|_{C\times C}) \xrightarrow{i_{C\times C}} C^*(G \times G) \xrightarrow{r} C^*(G \times G|_F) \longrightarrow 0$$

ensures that $(\lambda \otimes_{C(X)} [\Psi]) \otimes_{C^*(G\times G)} [\mathrm{id} - f]$ belongs to the image of $(i_{C\times C})_* : KK(C^*(G), C^*(G \times G|_{C\times C})) \to KK(C^*(G), C^*(G \times G))$. This enables us to show that $\lambda \otimes_{C(X)} D - 1_{C^*(G)}$ is in the image of $(i_C)_* : KK(C^*(G), C^*(G|_C)) \to KK(C^*(G), C^*(G))$ where i_C is the inclusion morphism of $C^*(G|_C)$ into $C^*(G)$.

On the other hand the equality $\lambda \otimes_{C^*(G)} D = 1_{C(X)}$ ensures that $\lambda \otimes_{C(X)} D - 1_{C^*(G)}$ is in the kernel of the map $(\cdot \otimes_{C^*(G)} D) : KK(C^*(G), C^*(G)) \to KK(C^*(G) \otimes C(X), \mathbb{C})$. To finish the proof we show that this map is injective in restriction to the image of $(i_C)_*$. This last point comes from the fact that the inclusion $i^{\mathcal{K}}$ of $C^*(G|_{M^-}) \simeq \mathcal{K}$ into $C^*(G|_C)$ induces a KK-equivalence between $\mathcal{K}$ and $C^*(G|_C)$; and $(\cdot \otimes_{C^*(G)} D) \circ (i_C)_* \circ (i^{\mathcal{K}})_* = e_c^*$ where $e_c^* : KK(C^*(G), \mathbb{C}) \to KK(C^*(G) \otimes C(X), \mathbb{C})$ comes from the evaluation map at c, $e_c : C(X) \to \mathbb{C}$. □

A consequence of the preceding theorem is a Poincaré duality between $C_0(T^*\overline{M^+})$ and $C_0(M^+)$. Thus we get a second Poincaré duality for manifolds with boundary, the first one has been stated by G. Kasparov in [7] for the algebras $C_0(T^*M^+)$ and $C(\overline{M^+})$.

References

[1] C. Anantharaman-Delaroche, J. Renault, Amenable Groupoids, in: Monograph. Enseign. Math., Vol. 36, L'Enseignement Mathématique, Genève, 2000.

[2] A. Connes, Sur la théorie non commutative de l'intégration, in: Lecture Notes in Math., Vol. 729, Springer, 1979, pp. 19–143.

[3] A. Connes, Noncommutative Geometry, Academic Press, 1994.

[4] A. Connes, G. Skandalis, The longitudinal index theorem for foliations, Publ. Res. Inst. Math. Sci. 20 (1984) 1139–1183.

[5] M. Hilsum, G. Skandalis, Morphismes K-orientés d'espaces de feuilles et fonctorialité en théorie de Kasparov, Ann. Sci. École Norm. Sup. 20 (4) (1987) 325–390.

[6] L. Hörmander, The Analysis of Linear Partial Differential Operators, III, in: A Series of Comprehensive Studies in Mathematics, Springer-Verlag, Berlin, 1985.

[7] G.G. Kasparov, Equivariant *KK*-theory and the Novikov conjecture, Invent. Math. 91 (1) (1988) 147–201.