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K-duality for pseudomanifolds with isolated singularities

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Abstract

We associate to a pseudomanifold X with a conical singularity a differentiable groupoid G which plays the role of the tangent space of X. We construct a Dirac element and a dual Dirac element which induce a K-duality between the C^*-algebras $C^*(G)$ and $C(X)$. This is a first step toward an index theory for pseudomanifolds.

Keywords: Singular manifolds; Smooth groupoids; Kasparov bivariant K-theory; Poincaré duality

0. Introduction

A basic point in the Atiyah–Singer index theory for closed manifolds lies in the isomorphism:

$$K_*(V) \to K^*(T^*V), \tag{1}$$

induced by the map which assigns to the class of an elliptic pseudodifferential operator on a closed manifold V, the class of its principal symbol [2].

To prove this isomorphism, Kasparov and Connes and Skandalis [6,15], define two elements $D_V \in KK(C(V) \otimes C_0(T^*V), \mathbb{C})$ and $\lambda_V \in KK(\mathbb{C}, C(V) \otimes C_0(T^*V))$

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which induce a K-duality between $C(V)$ and $C_0(T^*V)$, i.e.

$$\lambda_V \bigotimes_{C(V)} D_V = 1_{C_0(T^*V)} \quad \text{and} \quad \lambda_V \bigotimes_{C_0(T^*V)} D_V = 1_{C(V)}.$$

Isomorphism (1) is then equal to $(\lambda_V \otimes_{C(V)} \cdot)$.

Moreover, Connes and Skandalis recover the Atiyah–Singer index theorem using this K-duality together with other tools coming from bivariant K-theory (wrong-way functoriality maps).

This notion of K-duality, also called Poincaré duality in K-theory, has a quite general meaning [14,5]: two C^*-algebras A and B are K-dual if there exist $D \in KK(A \otimes B, \mathbb{C})$ and $\lambda \in KK(\mathbb{C}, A \otimes B)$ such that

$$\lambda \bigotimes_{A} D = 1_B \in KK(B,B) \quad \text{and} \quad \lambda \bigotimes_{B} D = 1_A \in KK(A,A).$$

We usually call D a Dirac element and λ a dual Dirac element. A consequence of these equalities is that for any C^*-algebras C and E, the groups homomorphisms:

$$\left(\lambda \bigotimes_{A} \cdot\right): KK(A \otimes C, E) \to KK(C, B \otimes E),$$

$$\left(\lambda \bigotimes_{B} \cdot\right): KK(B \otimes C, E) \to KK(C, A \otimes E)$$

are isomorphisms with inverses $(\cdot \otimes_B D)$ and $(\cdot \otimes_A D)$.

It is a natural question to look for a generalization of the K-duality between a manifold and its tangent bundle for spaces less regular than smooth manifolds. Pseudomanifolds [9] offer a large class of interesting examples of such spaces. We have focused our attention on the model case of a pseudomanifold X with a conical isolated singularity c. We use bivariant K-theory, groupoids and pseudodifferential calculus on groupoids to prove a Poincaré duality in K-theory in this context.

Let us explain our choice of the algebras A and B. As in the smooth case we take $A = C(X)$. For B, we need to define an appropriate notion of tangent space for X. It should take into account the smooth structure of $X \backslash \{c\}$ and encodes the geometry of the conical singularity. This problem finds an answer no longer in the category of vector bundles but in the larger category of groupoids. Thus, we assign to X a smooth groupoid G, *the tangent space of* X, and we let B be the non-commutative C^*-algebra $C^*(G)$.

The definition of the tangent space G of X is actually motivated by the case of smooth manifolds. In particular, the concrete meaning of the isomorphism (1) was the initial source of inspiration.

The regular part $X \backslash \{c\}$ identifies to an open subset of $G^{(0)}$ and the restriction of G to this subset is the ordinary tangent space of the manifold $X \backslash \{c\}$. The tangent space "over" the singular point is given by a pair groupoid. Furthermore the orbits space $G^{(0)}/G$ of G is topologically equivalent to X, that is $C(X) \simeq C(G^{(0)}/G)$. Thus $C(X)$

maps to the multiplier algebra of $C^*(G)$. The Dirac element is defined as the Kasparov product $D = [\Psi] \otimes \partial$ where $[\Psi]$ is the element of $KK(C(X) \otimes C^*(G), C^*(G))$ coming from the multiplication morphism Ψ and ∂ is an element of $KK(C^*(G), \mathbb{C})$ coming from a deformation groupoid $\mathscr{G}$ of G in a pair groupoid. This auxiliary groupoid $\mathscr{G}$ is the analogue of the tangent groupoid defined by Connes for a smooth manifold [5].

The construction of the dual Dirac element λ is more difficult. We let X_b be the bounded manifold with boundary L which identifies with the closure of $X \backslash \{c\}$ in $G^{(0)}$, it satisfies $X \simeq X_b/L$ and we denote by $\mathscr{A}G$ the Lie algebroid of the tangent space G. We first consider a suitable K-oriented map $X_b \to \mathscr{A}G \times X_b$. This map leads to an element λ' of $KK(\mathbb{C}, C^*(\mathscr{A}G) \otimes C(X_b))$. The adiabatic groupoid of G (see [5,16,17]), provides an element Θ of $KK(C^*(\mathscr{A}G), C^*(G))$ and we define $\lambda'' = \lambda' \otimes_{C^*(\mathscr{A}G)} \Theta$. The element λ'' can be seen as a continuous family $(\lambda''_x)_{x \in X_b}$ where $\lambda''_x \in K_0(C^*(G))$. An explicit description of λ'' shows that its restriction to L is the class of a constant family: that means λ'' determines an element $\lambda \in KK(\mathbb{C}, C^*(G) \otimes C(X))$.

An alternative and naïve description of λ is the following. To each point y of X is assigned an appropriate open subset $\hat{O}_y$ of $G^{(0)}$ satisfying $K(C^*(G|_{\hat{O}_y})) \simeq \mathbb{Z}$. We construct a continuous family $(\beta_y)_{y \in X}$, where β_y is a generator of $K(C^*(G|_{\hat{O}_y}))$. This family gives rise to an element of $K(C^*(G \times X|_{\hat{O}}))$, where $\hat{O}$ is an open subset of $G^{(0)} \times X$. We obtain λ by pushing forward this element in $K(C^*(G \times X))$ with the help of the inclusion morphism of $C^*(G \times X|_{\hat{O}})$ in $C^*(G \times X)$.

The dual Dirac element has the two following important properties:

(i) The set $\hat{O} \cap X_1 \times X_1$ is in the range of the exponential map. Here X_1 is the complement of a conical open neighborhood of c.
(ii) The equality $\lambda \otimes_{C^*(G)} \partial = 1 \in K^0(X)$ holds.

These two properties of λ are crucial to obtain our main result:

Theorem. *The Dirac element D and the dual-Dirac element λ induce a Poincaré duality between $C^*(G)$ and $C(X)$.*

All our constructions are obviously equivariant under the action of a group of automorphisms of X, that is homeomorphisms of X which are smooth diffeomorphisms of $X \backslash \{c\}$. In previous works, Julg and Kasparov and Skandalis [12,13] investigated the K-duality for simplicial complexes. Our approach is more in the spirit of [15] since it avoids the use of a simplicial decomposition of the pseudomanifold. We hope that it is better suited for applications to index theory. Indeed, the K-duality gives an isomorphism between $KK(C(X), \mathbb{C})$ and $KK(\mathbb{C}, C^*(G))$ which is, as in the smooth case, the map which assigns to the class of an elliptic pseudodifferential operator the class of its *symbol*. This point, among connections with the analysis on manifolds with boundary or conical manifolds, will be discussed in a forthcoming paper.

This paper is organized as follows:

Section 1 is devoted to some preliminaries around C^*-algebras of groupoids and special KK-elements.

In Section 2, we define the *tangent space* G of a conical pseudomanifold X as well as the *tangent groupoid* $\mathcal{G}$ of X.

In Section 3, we define the Dirac element and in Section 4, we construct the dual Dirac element.

The Section 5 is devoted to the proof of the Poincaré duality.

We want to address special thanks to Georges Skandalis for his always relevant suggestions.

1. Preliminaries

1.1. C^*-algebras of a groupoid

We recall in this section some useful results about C^*-algebras of groupoids [18,5].

Let $G \overset{s}{\underset{r}{\rightrightarrows}} G^{(0)}$ be a smooth Hausdorff groupoid with source s and range r. If U is any subset of $G^{(0)}$, we let:

$$G_U := s^{-1}(U), \quad G^U := r^{-1}(U) \quad \text{and} \quad G_U^U = G|_U := G_U \cap G^U.$$

We denote by $C_c^\infty(G)$ the space of complex valued smooth and compactly supported functions on G. It is provided with a structure of involutive algebra as follows. If f and g belong to $C_c^\infty(G)$ we define

the *involution* by

$$\text{for } \gamma \in G, \quad f^*(\gamma) = \overline{f(\gamma^{-1})},$$

the *convolution product* by

$$\text{for } \gamma \in G, \quad f * g(\gamma) = \int_{\eta \in G^{r(\gamma)}} f(\eta) g(\eta^{-1}\gamma).$$

To give a sense to the integral above, we fix a Haar system for G, that is, a smooth family $\{\lambda^x, x \in G^{(0)}\}$ of left invariant measures on G indexed by $x \in G^{(0)}$ such that the support of λ^x is G^x.

Alternatively, one could replace $C_c^\infty(G)$ by the space $C_c^\infty(G, \mathcal{L}^{\frac{1}{2}})$ of compactly supported smooth sections of the line bundle of half densities $\mathcal{L}^{\frac{1}{2}}$ over G. If k denotes the dimension of the s (or r) fibers of G, the fiber $\mathcal{L}_\gamma^{\frac{1}{2}}$ over $\gamma \in G$ is defined to be the linear space of maps:

$$\rho : \Lambda^k(T_\gamma(G^{r(\gamma)})) \otimes \Lambda^k(T_\gamma(G_{s(\gamma)})) \to \mathbb{C},$$

such that $\rho(\lambda v) = |\lambda|^{\frac{1}{2}} \rho(v)$ for all λ in $\mathbb{R}$ and v in $\Lambda^k(T_\gamma(G^{r(\gamma)})) \otimes \Lambda^k(T_\gamma(G_{s(\gamma)}))$.

Then, the convolution product makes sense as the integral of a 1-density on the manifold $G^{r(\gamma)}$. Both constructions lead to the same C^*-algebra.

For each x in $G^{(0)}$, we define a $*$-representation π_x of $C_c^\infty(G)$ on the Hilbert space $L^2(G_x)$ by

$$\pi_x(f)(\xi)(\gamma) = \int_{\eta \in G^{r(\gamma)}} f(\eta)\xi(\eta^{-1}\gamma),$$

where $\xi \in L^2(G_x), f \in C_c^\infty(G)$ and $\gamma \in G_x$.

The completion of $C_c^\infty(G)$ for the norm $||f||_r = \sup_{x\in G^{(0)}} ||\pi_x(f)||$ is a C^*-algebra, called the reduced C^*-algebra of G and denoted by $C_r^*(G)$.

The maximal C^*-algebra $C^*(G)$ is the completion of $C_c^\infty(G)$ for the norm:

$$||f|| = \sup\{||\pi(f)|| \mid \pi \text{ Hilbert space } * -\text{representation of } C_c^\infty(G)\}.$$

The previous constructions still hold when the groupoid G is smooth only in the orbit direction, which means that $G|_{O_x}$ is smooth for any orbit $O_x = r(s^{-1}(x))$, $x \in G^{(0)}$. In this situation one can replace $C_c^\infty(G)$ by $C_c(G)$.

The identity map of $C_c^\infty(G)$ induces a surjective morphism from $C^*(G)$ onto $C_r^*(G)$. The injectivity of this morphism is related to amenability of groupoids [1]. When G is an amenable groupoid, its reduced and maximal C^*-algebras are equal and, moreover, this common C^*-algebra is nuclear.

1.2. Subalgebras and exact sequences of groupoid C^*-algebras

To an open subset O of $G^{(0)}$ corresponds an inclusion i_O of $C_c^\infty(G|_O)$ into $C_c^\infty(G)$ which induces an injective morphism, again denoted by i_O, from $C^*(G|_O)$ into $C^*(G)$.

When O is saturated, $C^*(G|_O)$ is an ideal of $C^*(G)$. In this case, $F := G^{(0)} \setminus O$ is a saturated closed subset of $G^{(0)}$ and the restriction of functions induces a surjective morphism r_F from $C^*(G)$ to $C^*(G|_F)$. Moreover, according to [10], the following sequence of C^*-algebras is exact:

$$0 \to C^*(G|_O) \xrightarrow{i_O} C^*(G) \xrightarrow{r_F} C^*(G|_F) \to 0.$$

1.3. C^*-modules arising from bundles and groupoids

Let us now consider an hermitian bundle E on $G^{(0)}$. We equip the space $C_c^\infty(G, r^*E)$ with the $C^*(G)$-valued product:

$$\langle f, g\rangle(\gamma) = \int_{\eta \in G^{r(\gamma)}} \langle f(\eta^{-1}), g(\eta^{-1}\gamma)\rangle_{s(\eta)}.$$

This endows $C_c^\infty(G, r^*E)$ with a structure of $C^*(G)$-pre-Hilbert module and we denote by $C^*(G, E)$ the corresponding $C^*(G)$-Hilbert module. As usual, we note

$\mathscr{L}(\mathscr{E})$ and $\mathscr{K}(\mathscr{E})$ the C^*-algebras of (adjointable) endomorphisms and compact endomorphisms of any Hilbert module $\mathscr{E}$.

1.4. KK-tools

This paper makes an intensive use of Kasparov's bivariant K-theory. The unfamiliar reader may consult [3,14,19]. In this section, we recall some basic constructions and fix the notations.

When A is a C^*-algebra, the element $1_A \in KK(A,A)$ is the class of the triple $(A, i_A, 0)$, where A is graded by $A^{(1)} = 0$ and $i_A : A \to \mathscr{L}(A)$ is given by $i(a)b = ab,\ a,b \in A$.

If B and C are additional C^*-algebras, $\tau_C : KK(A,B) \to KK(A \otimes C, B \otimes C)$ is the group homomorphism defined by $\tau_C[(E,\rho,F)] = [(E \otimes C, \rho \otimes i_C, F \otimes 1)]$.

The heart of Kasparov theory is the existence of a product which generalizes various functorial operations in K-theory. Recall that the Kasparov product is a well defined bilinear coupling $KK(A,B) \times KK(B,C) \to KK(A,C)$ denoted $(x,y) \mapsto x \otimes_B y$ which is associative, covariant in C, contravariant in A and satisfies:

- $f_*(x) \otimes_E y = x \otimes_B f^*(y)$ for any $*$-homomorphism $f : B \to E$, $x \in KK(A,B)$ and $y \in KK(E,C)$.
- $x \otimes_B 1_B = 1_A \otimes_A x = x$, for $x \in KK(A,B)$.
- $\tau_D(x \otimes_B y) = \tau_D(x) \otimes_{B \otimes D} \tau_D(y)$, when $x \in KK(A,B)$ and $y \in KK(B,C)$.

In the sequel, we will denote simply $x \otimes y$ the product $x \otimes_B y \in KK(A,C)$ when $x \in KK(A,B)$ and $y \in KK(B,C)$.

The operation $\tau_C : KK(A,B) \to KK(A \otimes C, B \otimes C)$ allows the construction of the general form of the Kasparov product:

$$KK(A_1, B_1 \otimes C) \times KK(A_2 \otimes C, B_2) \to KK(A_1 \otimes A_2, B_1 \otimes B_2), \tag{2}$$

$$(x,y) \mapsto x \bigotimes_C y := \tau_{A_2}(x) \otimes \tau_{B_1}(y). \tag{3}$$

For $x \in KK(A, B \otimes C)$ and $y \in KK(B \otimes C, E)$, there is an ambiguity in the definition of $\tau_B(x) \otimes \tau_B(y)$: it can be defined by (3) with $B = A_2 = B_1$ or by $\tau_B(x \otimes y)$. These two products are different in general. Indeed, in the first case, the two copies of B involved in x and y play different roles, contrary to the second case. To remove this ambiguity, we adopt the following convention:

$$\tau_B(x \otimes y) = \tau_B(x) \otimes \tau_B(y)$$

and

$$x \bigotimes_C y = \tau_{\underline{B}}(x) \otimes \tau_B(y) \text{ or } \tau_B(x) \otimes \tau_{\underline{B}}(y).$$

Moreover, let $f_B : B\otimes B \to B\otimes B, a\otimes b \mapsto b\otimes a$ be the *flip* automorphism and let $[f_B]$ be the corresponding element of $KK(B\otimes B, B\otimes B)$. The morphism f_B exchanges the two copies of B, so

$$\tau_B(x)\otimes \tau_C[f_B]\otimes \tau_B(y) = x \bigotimes_C y.$$

1.5. KK-elements associated to deformation groupoids

We explain here a classical construction [5,10].

A smooth groupoid G is called a *deformation groupoid* if:

$$G = G_1 \times \{0\} \cup G_2 \times]0,1] \rightrightarrows G^{(0)} = M \times [0,1],$$

where G_1 and G_2 are smooth groupoids with unit space M. That is, G is obtained by gluing $G_2\times]0,1] \rightrightarrows M\times]0,1]$ which is the groupoid G_2 over M parameterized by $]0,1]$ with the groupoid $G_1 \times \{0\} \rightrightarrows M \times \{0\}$.

In this situation one can consider the saturated open subset $M\times]0,1]$ of $G^{(0)}$. Using the isomorphisms $C^*(G|_{M\times]0,1]}) \simeq C^*(G_2)\otimes C_0(]0,1])$ and $C^*(G|_{M\times\{0\}}) \simeq C^*(G_1)$, we obtain the following exact sequence of C^*-algebras:

$$0 \to C^*(G_2)\otimes C_0(]0,1]) \xrightarrow{i_{M\times]0,1]}} C^*(G) \xrightarrow{\mathrm{ev}_0} C^*(G_1) \to 0,$$

where $i_{M\times]0,1]}$ is the inclusion map and ev_0 is the *evaluation map* at 0, that is ev_0 is the map coming from the restriction of functions to $G|_{M\times\{0\}}$.

We assume now that $C^*(G_1)$ is nuclear. Since the C^*-algebra $C^*(G_2)\otimes C_0(]0,1])$ is contractible, the long exact sequence in KK-theory shows that the group homomorphism $(\mathrm{ev}_0)_* = \cdot\otimes[\mathrm{ev}_0] : KK(A, C^*(G)) \to KK(A, C^*(G_1))$ is an isomorphism for each C^*-algebra A.

In particular with $A = C^*(G)$ we get that $[\mathrm{ev}_0]$ is invertible in KK-theory: there is an element $[\mathrm{ev}_0]^{-1}$ in $KK(C^*(G_1), C^*(G))$ such that $[\mathrm{ev}_0]\otimes[\mathrm{ev}_0]^{-1} = 1_{C^*(G)}$ and $[\mathrm{ev}_0]^{-1}\otimes[\mathrm{ev}_0] = 1_{C^*(G_1)}$.

Let $\mathrm{ev}_1 : C^*(G) \to C^*(G_2)$ be the evaluation map at 1 and $[\mathrm{ev}_1]$ the corresponding element of $KK(C^*(G), C^*(G_2))$.

The *KK-element associated to the deformation groupoid* G is defined by

$$\delta = [\mathrm{ev}_0]^{-1}\otimes[\mathrm{ev}_1] \in KK(C^*(G_1), C^*(G_2)).$$

Example 1. (1) Let G be a smooth groupoid and let $\mathscr{A}G$ be its Lie algebroid.

The adiabatic groupoid of G [5,16,17]:

$$G_{\mathrm{ad}} = \mathscr{A}G \times \{0\} \cup G\times]0,1] \rightrightarrows G^{(0)} \times [0,1],$$

is a deformation groupoid. Here, the vector bundle $\pi : \mathscr{A}G \to G^{(0)}$ is considered as a groupoid in the obvious way.

Since $C_0(\mathscr{A}^*G)$ is nuclear, the previous construction applies and the associated KK-element $\delta \in KK(C_0(\mathscr{A}^*G), C^*(G))$ gives rises to a map:

$$\cdot \otimes \delta : K_0(C_0(\mathscr{A}^*G)) \to K_0(C^*(G))$$

This map is defined in [16] as the analytic index of the groupoid G.

(2) A particular case of (1) is given by the tangent groupoid of $\mathbb{R}$: $\mathrm{Tan}(\mathbb{R}) = \mathbb{R} \times \mathbb{R} \times]0,1] \cup T\mathbb{R} \times \{0\} \rightrightarrows \mathbb{R} \times [0,1]$ [5]. The corresponding KK-element δ_B, which belongs to $KK(C_0(\mathbb{R}^2), \mathscr{K})$ is the dual Bott element. Precisely, the map $(\cdot \otimes \delta_B)$ induces an isomorphism from $K(C_0(\mathbb{R}^2))$ into $K(\mathscr{K}) \simeq \mathbb{Z}$.

1.6. Pseudodifferential calculus on groupoids

We recall here some definitions and results of [4,16,17,20] (see also [6]).

Let G be a smooth groupoid, possibly with a boundary [16].

Let $U_\gamma : C^\infty(G_{s(\gamma)}) \to C^\infty(G_{r(\gamma)})$ be the isomorphism induced by right multiplication: $U_\gamma f(\gamma') = f(\gamma'\gamma)$. An operator $P : C_c^\infty(G) \to C^\infty(G)$ is a *G-operator* if there exists a family $P_x : C_c^\infty(G_x) \to C^\infty(G_x)$ such that $P(f)(\gamma) = P_{s(\gamma)}(f|_{G_{s(\gamma)}})(\gamma)$ and $U_\gamma P_{s(\gamma)} = P_{r(\gamma)} U_\gamma$.

A G-operator P is a *pseudodifferential operator on G* (resp. of order m) if for any open local chart $\Phi : \Omega \to s(\Omega) \times W$ of G such that $s = pr_1 \circ \Phi$ and any cut-off function $\chi \in C_c^\infty(\Omega)$, we have $(\Phi^*)^{-1}(\chi P \chi)_x \Phi^* = a(x, w, D_w)$ where $a \in S^*(s(\Omega) \times T^*W)$ is a classical symbol (resp. of order m).

One says that $K \subset G$ is a *support* of P if $\mathrm{supp}(Pf) \subset K.\mathrm{supp}(f)$ for all $f \in C_c^\infty(G)$. When P has a compact support we say that P is *uniformly supported.*

These definitions extend immediately to the case of operators acting on sections of bundles on $G^{(0)}$ (pulled back to G with r), and we denote by $\Psi^*(G, E)$ the algebra of uniformly supported pseudodifferential operators on G acting on sections of E. Thanks to the invariance property, each operator $P \in \Psi^*(G; E)$ has a principal symbol $\sigma(P) \in C_c^\infty(S^*G, \hom \pi^*E)$ where S^*G is the sphere bundle associated to $\mathscr{A}^*G$ and π its natural projection onto $G^{(0)}$. The following inclusions hold: $\Psi^0(G, E) \subset \mathscr{L}(C^*(G, E))$ and $\Psi^{-1}(G, E) \subset \mathscr{K}(C^*(G, E))$. Moreover, the symbol map extends by continuity and gives rise to the following exact sequence of C^*-algebras:

$$0 \to \mathscr{K}(C^*(G, E)) \to \Psi_0(G, E) \xrightarrow{\sigma} C_0(S^*G, \hom \pi^*E) \to 0. \tag{4}$$

where $\Psi_0(G, E) = \overline{\Psi^0(G, E)}^{\mathscr{L}(C^*(G,E))}$. Finally a linear section Op_G of the symbol map can be defined by the following formula:

$$\mathrm{Op}_G(a)(u)(\gamma) = \int_{\substack{\xi \in \mathscr{A}^*_{r(\gamma)}G; \\ \gamma' \in G_{s(\gamma)}}} e^{i\langle E_G^{-1}(\gamma'\gamma^{-1}), \xi\rangle} a(r(\gamma), \xi) \phi(\gamma'\gamma^{-1}) p_{r(\gamma'), r(\gamma)} u(\gamma')\, d\gamma'\, d\xi.$$

Here $\phi \in C^\infty(G)$ is supported in the range of an exponential map $E_G : \mathscr{V}(\mathscr{A}G) \to G$ where $\mathscr{V}(\mathscr{A}G)$ denotes a small neighborhood of the zero section in $\mathscr{A}G$; moreover ϕ is assumed to be equal to one on a neighborhood of V in G. We have used a parallel transport p to get local trivializations of the bundle E. It is implicit in the formula that the symbol $a \in C_c^\infty(S^*G, \hom \pi^* E)$ has been extended in the usual way to $\mathscr{A}^*G$.

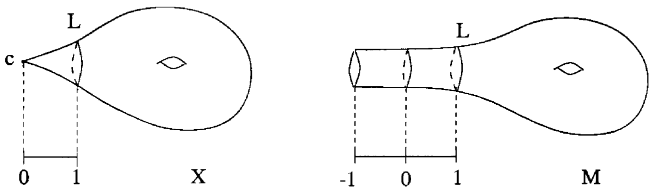
2. The geometry

Let X_1 be an m-dimensional compact manifold with boundary L. We attach to each connected component L_i of the boundary the cone $cL_i = L_i \times [0,1]/L_i \times \{0\}$, using the obvious map $L_i \times \{1\} \to L_i \subset \partial X_1$. The new space $X = \sqcup_{i=1}^p cL_i \cup X_1$ is a compact pseudomanifold with isolated singularities [9]. In general, there is no manifold structure around the vertices of the cones. From now on, we assume that L is connected, i.e. X has only one singularity denoted by c. The general case follows by exactly the same methods.

For any $\varepsilon \in]0,1]$, we will refer to $c_\varepsilon L = L \times [0,\varepsilon]/L \times \{0\}$ as a compact cone over L, to $\overset{o}{c_\varepsilon} L = L \times [0,\varepsilon[/L \times \{0\}$ as an open cone over L and we let $X_\varepsilon = L \times [\varepsilon, 1] \cup X_1$.

We define the manifold M by attaching to X_1 a cylinder $L \times]-1,1]$. We fix on M a Riemannian metric which is of product type on $L \times]-1,1]$ and we assume that its injectivity radius is bigger than 1.

We will use the following notations: M^+ denotes $L \times]0,1] \cup X_1$, $\overline{M_+}$ its closure in M and $M^- = L \times]-1,0[$. If y is a point of the cylindrical part of M or $X \setminus \{c\}$, we will write $y = (y_L, k_y)$ where $y_L \in L$ and $k_y \in]-1,1]$ are the tangential and radial coordinates. We extend the map k on X_1 to a smooth defining function for its boundary; in particular, $k^{-1}(1) = \partial X_1$ and $k(X_1) \subset [1, +\infty[$.



2.1. The tangent space of the conical pseudomanifold X

Let us consider $T\overline{M^+}$, the restriction to $\overline{M^+}$ of the tangent bundle of M. As a $\mathscr{C}^\infty$ vector bundle, it is a smooth groupoid with unit space $\overline{M^+}$. We define the groupoid G as the disjoint union:

$$G = M^- \times M^- \cup T\overline{M^+} \underset{r}{\overset{s}{\rightrightarrows}} M,$$

where $M^- \times M^- \rightrightarrows M^-$ is the pair groupoid.

In order to endow G with a smooth structure, compatible with the usual smooth structure on $M^- \times M^-$ and on TM^+, we have to take care of what happens around points of $T\overline{M^+}|_{\partial\overline{M^+}}$.

Let τ be a smooth positive function on $]-1,+\infty[$ such that $\tau^{-1}(\{0\}) = \mathbb{R}^+$. We let $\tilde{\tau}$ be the smooth map from M to $\mathbb{R}^+$ given by $\tilde{\tau}(y) = \tau(k_y)$.

Let (U,ϕ) be a local chart for M around $z \in \partial\overline{M^+}$. Setting $U^- = U \cap M^-$ and $\overline{U^+} = U \cap \overline{M^+}$, we define a local chart of G by

$$\tilde{\phi} : U^- \times U^- \cup T\overline{U^+} \to \mathbb{R}^m \times \mathbb{R}^m,$$

$$\tilde{\phi}(x,y) = \left(\phi(x), \frac{\phi(y)-\phi(x)}{\tilde{\tau}(x)}\right) \quad \text{if } (x,y) \in U^- \times U^- \tag{5}$$

and

$$\tilde{\phi}(x,V) = (\phi(x), (\phi)_*(x,V)) \quad \text{elsewhere.}$$

Let us explain why the range of $\widetilde{\phi}$ is open. We can assume that $\phi(U) = \mathbb{R}^m$ and $\phi(U^-) = \mathbb{R}^m_- = \mathbb{R}^{m-1} \times]-\infty, 0[$. Let B be a open ball in $\mathbb{R}^m$. Since $\widetilde{\tau}$ vanishes on $\overline{U^+}$ there exists an open neighborhood W of $\partial\overline{U^+}$ such that $\{\widetilde{\tau}(x)p + \phi(x) \mid x \in W \cap U^-,\ p \in B\} \subset \mathbb{R}^m_-$. Then $\widetilde{\phi}(T\partial\overline{U^+}) \cap \mathbb{R}^m \times B \subset \phi(W) \times B \subset \mathrm{Im}\widetilde{\phi}$.

We define in this way a structure of smooth groupoid on G.

Remark 2. (1) If τ is C^l then the atlas defined above provides G with a structure of C^{l-1} groupoid (it is easy to see that the source, target and inversion maps have the same regularity as the atlas).

(2) At the topological level, the space of orbits M/G of G is equivalent to X: there is a canonical isomorphism between the algebras $C(X)$ and $C(M/G)$.

Definition 3. The smooth groupoid $G \rightrightarrows M$ is called a tangent space of X.

It is important to remark that the Lie algebroid of $G \rightrightarrows M$ is the bundle $\mathscr{A}G = TM$ over M with anchor $p_G : \mathscr{A}G = TM \to TM, (x,V) \mapsto (x, \tilde{\tau}(x)V)$; in particular p_G is the zero map in restriction to $T\overline{M^+}$. The exponential map exp of the Riemannian manifold M provides an exponential map E_G for the groupoid G (for a description of exponential maps for groupoids, see e.g. [17,7]). More precisely:

$$E_G : \mathscr{V}(TM) \to G,$$

$$E_G(y,V) = (y,V) \text{ when } y \in \overline{M^+}$$

and

$$E_G(y,V) = (y, exp_y(-\tilde{\tau}(y)V)) \text{ when } y \in M^-,$$

where $\mathscr{V}(TM) = \{(y,V) \in TM \mid ||\tilde{\tau}(y)V|| < 1 \text{ and } exp_y(-\tilde{\tau}(y)V)) \in M^- \text{ if } y \in M^-\}$.

The map E_G is a diffeomorphism onto a neighborhood of the unit space M in G. In fact, we could have defined the smooth structure of G using the map E_G.

Remark 4. (1) There exists a slightly different groupoid which could naturally play the role of the tangent space of X. We will call it the tangent space with tail. It is defined by

$$G_q = L \times L \times T(]-1,0[) \cup T\overline{M^+} \rightrightarrows M.$$

As a groupoid, G_q is the union of two groupoids: the bundle $T\overline{M^+} \rightrightarrows \overline{M^+}$ and the groupoid $L \times L \times T(]-1,0[) \rightrightarrows L \times]-1,0[= M^-$ which is the product of the pair groupoid over L with the vector bundle $T(]-1,0[) \rightrightarrows]-1,0[$. One can equip G_q with a smooth structure similarly as we did for G. We will see that the C^*-algebras of G and G_q are KK-equivalent.

(2) The groupoid G is obtained by gluing along their common boundary $TL \times \mathbb{R}$ the groupoids $T\overline{M^+}$ and a groupoid isomorphic to $\mathrm{Tan}(L) \rtimes \mathbb{R}_+^*$ obtained by the action of $\mathbb{R}_+^*$ (by multiplication on the real parameter) on the tangent groupoid $\mathrm{Tan}(L) = L \times L \times \mathbb{R}_+^* \cup TL \times \{0\}$ of L. The groupoid G_q is defined in the same way except that we consider the trivial action of $\mathbb{R}_+^*$.

2.2. The tangent groupoid of the pseudomanifold X

The following construction is a natural generalization of the tangent groupoid of a manifold defined by Connes [5]. We define the *tangent groupoid* $\mathscr{G}$ of the pseudomanifold X as a deformation of the pair groupoid over M into the groupoid G. This deformation process has a nice description at the level of Lie algebroids. Indeed, the Lie algebroid of $\mathscr{G}$ should be the (unique) Lie algebroid given by the fiber bundle $\mathscr{A}\mathscr{G} = [0,1] \times \mathscr{A}G = [0,1] \times TM$ over $[0,1] \times M$, with anchor map

$$\begin{array}{llll} p_{\mathscr{G}} : \mathscr{A}\mathscr{G} = [0,1] \times TM & \rightarrow & T([0,1] \times M) = T[0,1] \times TM \\ (\lambda, x, V) & \mapsto & (\lambda, 0, x, p_G(x,V) + \lambda V) = (\lambda, 0, x, (\tilde{\tau}(x) + \lambda)V). \end{array}$$

Such a Lie algebroid is almost injective, thus it is integrable [7,8].

We now define the *tangent groupoid*:

$$\mathscr{G} = M \times M \times]0,1] \cup G \times \{0\} \rightrightarrows M \times [0,1],$$

whose smooth structure is described hereafter.

Since $\mathscr{G} = (M \times M \times [0,1] \backslash (\overline{M^+} \times M \cup M \times \overline{M^+}) \times \{0\}) \cup T\overline{M^+} \times \{0\}$, we keep the smooth structure on $M \times M \times [0,1] \backslash (\overline{M^+} \times M \cup M \times \overline{M^+}) \times \{0\}$ as an open subset in the manifold with boundary $M \times M \times [0,1]$. We consider the following map:

$$\rho : \mathscr{V}(TM \times [0,1]) \to \mathscr{G} = M \times M \times]0,1] \cup G \times \{0\},$$

$$\rho(z, V, \lambda) = \begin{cases} (z, V, 0) & \text{if } z \in \overline{M^+} \text{ and } \lambda = 0, \\ (z, exp_z(-(\tilde{\tau}(z) + \lambda).V), \lambda) & \text{elsewhere,} \end{cases}$$

where $\mathscr{V}(TM \times [0,1])$ is an open subset in $TM \times [0,1]$ such that $\mathscr{V}(TM \times [0,1]) \cap TM \times \{0\} = \mathscr{V}(TM)$, and which is small enough so the exponential in the definition of ρ is well defined. Then $T\overline{M^+} \times \{0\}$ is in the image of ρ and we equip $\mathscr{G}$ around $T\overline{M^+} \times \{0\}$ with the smooth structure for which ρ is a diffeomorphism onto its image. One can easily check that it is compatible with the smooth structure of $M \times M \times [0,1] \backslash (\overline{M^+} \times M \cup M \times \overline{M^+}) \times \{0\}$.

The Lie algebroid of $\mathscr{G}$ is $\mathscr{AG}$ and ρ is an exponential map for $\mathscr{G}$.

2.3. The C^*-algebras

Let μ be the Riemannian measure on M and let v be the corresponding Lebesgue measure on the fibers of TM. The family $\{\lambda^x; x \in M\}$, where $d\lambda^x(y) = \frac{1}{\tilde{\tau}(y)^m} d\mu(y)$ if x belongs to M^-, and $d\lambda^x(V) = dv(V)$ if x is in $\overline{M^+}$, is a Haar system for G. We use this Haar system to define the convolution algebra of G.

Remark 5. (1) G is a continuous field of amenable groupoids parameterized by X. More precisely, $G = \sqcup_{x \in X} \pi^{-1}(x)$ where $\pi : G \to X$ is the obvious projection map. If $x \neq c$, $\pi^{-1}(x) = T_x M$ is amenable. If $x = c$, $\pi^{-1}(c) = M^- \times M^- \cup T\overline{M^+}|_{\partial \overline{M^+}}$ is isomorphic to the groupoid $H = \mathrm{Tan}(L) \rtimes \mathbb{R}$ of an action of $\mathbb{R}$ on the tangent groupoid $\mathrm{Tan}(L) = L \times L \times]0,1[\cup TL \times \{0\}$ of L. The groupoid H is an extension of the group $\mathbb{R}$ by $\mathrm{Tan}(L)$, both of them being amenable, according to [1, Theorem 5.3.14], H is amenable. Finally according to [1, Proposition 5.3.4], G is amenable. In the same way, $\mathscr{G}$ and G_q are amenable. Hence their reduced and maximal C^*-algebras are equal and they are nuclear.

(2) Using the KK-equivalence between $\mathscr{K}$ and $C_0(\mathbb{R}^2)$ (cf. Example 1(2)) one can establish a KK-equivalence between $C^*(G)$ and $C^*(G_q)$.

3. The Dirac element

The tangent groupoid $\mathscr{G} \rightrightarrows M \times [0,1]$ is a deformation groupoid and its C^*-algebra is nuclear, thus it defines a KK-element. We let $\widetilde{\partial}$ be the KK-element associated to $\mathscr{G}$.

More precisely:

$$\widetilde{\partial} = [e_0]^{-1} \otimes [e_1] \in KK(C^*(G), \mathscr{K}),$$

where $e_0 : C^*(\mathscr{G}) \to C^*(\mathscr{G}|_{M\times\{0\}}) = C^*(G)$, the evaluation map at 0 is K-invertible, and $e_1 : C^*(\mathscr{G}) \to C^*(\mathscr{G}|_{M\times\{1\}}) = \mathscr{K}(L^2(M))$ is the evaluation map at 1. Let b be the (positive) generator of $KK(\mathscr{K}, \mathbb{C}) \simeq \mathbb{Z}$. We set $\partial = \widetilde{\partial} \otimes b$.

The algebra $C(X)$ is isomorphic to the algebra of continuous functions on the orbits space M/G of G. Thus $C(X)$ maps to the multiplier algebra of $C^*(G)$ and we let Ψ be the morphism $\Psi : C^*(G) \otimes C(X) \to C^*(G)$ induced by the multiplication. In other words, if $a \in C_c^\infty(G)$ and $f \in C(X)$, $\Psi(a,f) \in C_c(G)$ is defined by

$$\Psi(a,f)(\gamma) = \begin{cases} a(\gamma)f(r(\gamma)) = a(\gamma)f(s(\gamma)) & \text{if } \gamma \in T\overline{M^+}, \\ a(\gamma)f(c) & \text{if } \gamma \in M^- \times M^-. \end{cases}$$

We denote by $[\Psi]$ the corresponding element in $KK(C^*(G) \otimes C(X), C^*(G))$.

Definition 6. The Dirac element is

$$D = [\Psi] \otimes \partial \in KK(C^*(G) \otimes C(X), \mathbb{C}).$$

4. The dual Dirac element

We first recall the construction of the dual Dirac element for a compact manifold V [6,15].

Let V be a smooth compact n dimensional Riemannian manifold, whose injectivity radius is at least 1. We denote by Λ the bundle of complex valued differential forms on V, and we keep this notation for its pull-back to T^*V and its restrictions to various subsets of V and T^*V. For $x \in V$, we denote by O_x the geodesic ball with radius $\frac{1}{4}$, H_x the Hilbert space $L^2(O_x, \Lambda)$ and we write H for the continuous field of Hilbert spaces $\bigcup_{x\in V} H_x$.

With a model operator on $\mathbb{R}^n$, for instance those given in Theorem (19.2.12) of [11], we define a continuous family $P = (P_x)_{x\in V}$ of pseudodifferential operators $P_x \in \Psi^0(O_x, \Lambda)$ of order 0 satisfying the following conditions:

(1) P_x is *trivial at infinity of* O_x, which means that P_x is the sum of a compactly supported pseudodifferential operator and a smooth bounded section of the bundle $\mathrm{End}\Lambda \to O_x$, (in particular, this ensures boundedness on H_x).
(2) P_x is selfadjoint on H_x, and has degree one (i.e. $P_x = \begin{pmatrix} 0 & P_x^- \\ P_x^+ & 0 \end{pmatrix}$) with respect to the grading induced by $\Lambda = \Lambda^{\mathrm{ev}} \oplus \Lambda^{\mathrm{odd}}$.
(3) $P_x^2 - \mathrm{Id}$ is a compactly supported pseudodifferential operator of order -1; in particular it is compact on H_x.

(4) the family $P = (P_x)_{x \in V}$ has a trivial index bundle of rank one. In fact, for all x, P_x^+ is onto, it has a one-dimensional kernel and there exists a continuous section $V \ni x \mapsto e_x \in \ker P_x^+ \subset L^2(M, \Lambda)$.

Here the continuity of the family means that P is an endomorphism of the $C(V)$-Hilbert module H.

We let a_x be the principal symbol of P_x. Under the assumptions above, the Kasparov module

$$\lambda_x = [(C_0(T^* O_x, \Lambda), 1, a_x)]$$

is a generator of $K_0(C_0(T^* O_x)) \simeq \mathbb{Z}$. The following element:

$$\lambda_V = [(C_0(T^* O_x, \Lambda), 1, a_x)_{x \in V}] \in K_0(C_0(T^* V \times V))$$

is the dual Dirac element used in the proof the Poincaré duality between $C(V)$ and $C_0(T^* V)$ [15].

There is an alternative elegant description of λ_V [6]. Let us consider the map $f : V \to T^* V \times V, x \mapsto ((x, 0), x)$. This map is K-oriented so it gives rise to an element $f! \in KK(C(V), C_0(T^* V \times V))$. If p denotes the obvious map $\mathbb{C} \to C(V)$, then:

$$\lambda_V = [p] \otimes f!.$$

With the previous example in mind and keeping the same notations, we shall define an element $\Delta \in K_0(C^*(\mathcal{G} \times X))$ whose evaluation $\lambda = (e_0)_*(\Delta) \in K_0(C^*(G \times X))$ will be the appropriate dual Dirac element of the pseudomanifold X.

We define a map $h : X \backslash \{c\} \simeq M^+ \to M$ which pushes points in M^-. More precisely, $h(y) = y$ when $k_y \geqslant 1$, and $h(y) = (y_L, l(k_y))$ otherwise, where:

$$l(k) = \begin{cases} 3k - 2 & \text{if } 1/2 \leqslant k \leqslant 1, \\ -1/2 & \text{if } 0 < k \leqslant 1/2. \end{cases}$$

From now we fix $\varepsilon \in]0, 1/2[$. Recall that $X_\varepsilon = \{x \in X | k_x \geqslant \varepsilon\}$. We set:

$$\delta = (\lambda_{h(x)})_{t \in [0,1], x \in X_\varepsilon} \in K_0(C_0(\mathscr{A}^*(\mathcal{G}) \times X_\varepsilon)).$$

Here $\mathscr{A}^*(\mathcal{G}) \simeq T^* M \times [0, 1]$ and δ corresponds to the K-oriented map: $[0, 1] \times X_\varepsilon \to T^* M \times [0, 1] \times X_\varepsilon, (t, x) \mapsto ((h(x), 0), t, x)$.

Next, let us consider the adiabatic groupoid of $\mathcal{G}$ [5,16,17] (see Example 1(1)):

$$\mathscr{H} = \{0\} \times \mathscr{A}(\mathcal{G}) \cup]0, 1] \times \mathcal{G} \rightrightarrows [0, 1] \times \mathcal{G}^{(0)}.$$

We let $\Theta \in KK(C_0(\mathscr{A}^*(\mathcal{G})), C^*(\mathcal{G}))$ be the KK-element associated to $\mathscr{H}$, i.e.

$$\Theta = [\mathrm{ev}_0]^{-1} \otimes [\mathrm{ev}_1],$$

where $\mathrm{ev}_0 : C^*(\mathscr{H}) \to C_0(T^* M \times [0, 1])$ is the evaluation map at 0 composed with the Fourier transform $C^*(\mathscr{A}(\mathcal{G})) \xrightarrow{\simeq} C_0(\mathscr{A}^*(\mathcal{G}))$, and $\mathrm{ev}_1 : C^*(\mathscr{H}) \to C^*(\mathcal{G})$ is the evaluation at 1. We define $\varDelta_\varepsilon \in K_0(C^*(\mathscr{G} \times X_\varepsilon))$ by

$$\varDelta_\varepsilon = \delta \bigotimes_{C_0(\mathscr{A}^*(\mathscr{G}))} \Theta.$$

Proposition 7. (1) *The element $\varDelta_\varepsilon$ satisfies*

$$(e_1)_*(\varDelta_\varepsilon) = 1_{X_\varepsilon} \in K^0(X_\varepsilon) \simeq K_0(\mathscr{K} \otimes C(X_\varepsilon)),$$

where $e_1 : C^*(\mathscr{G}) \to C^*(\mathscr{G}|_{M\times\{1\}}) \simeq \mathscr{K}$ is the evaluation map at 1.

(2) *There exists $\varDelta_0 \in K_0(C^*(\mathscr{G} \times X|_{\mathscr{O}\times[0,1]}))$ extending $\varDelta_\varepsilon$, that is,*

$$r_* \circ (i_{\mathscr{O}\times[0,1]})_*(\varDelta_0) = \varDelta_\varepsilon,$$

where $\mathscr{O}$ is the open subset $\bigcup_{x\in M^+} O_{h(x)} \times \{x\} \cup M^- \times \overset{o}{c} L$ of $M \times X$, *$i_{\mathscr{O}\times[0,1]} : C^*(\mathscr{G} \times X|_{\mathscr{O}\times[0,1]}) \to C^*(\mathscr{G} \times X)$ is the inclusion morphism and $r : C^*(\mathscr{G} \times X) \to C^*(\mathscr{G} \times X_\varepsilon)$ is the restriction morphism.*

Proof. (1) Let us note $\mathscr{O}_\varepsilon = \bigcup_{x\in X_\varepsilon} O_{h(x)} \times \{x\}$. This is an open subset of $M \times X_\varepsilon$ the unit space of the groupoid $G \times X_\varepsilon$. Let $\widetilde{\mathscr{O}}_\varepsilon$ be its lift to $M \times [0,1]^2 \times X_\varepsilon$ which is the unit space of the groupoid $\mathscr{H} \times X_\varepsilon$. We let Ω be the groupoid:

$$\Omega = \mathscr{H} \times X_\varepsilon|_{\widetilde{\mathscr{O}}_\varepsilon}.$$

In fact, Ω identifies with the adiabatic groupoid of $\Omega_1 = \mathscr{G} \times X_\varepsilon|_{\mathscr{O}_\varepsilon\times[0,1]}$.

We shall use the pseudodifferential calculus on Ω to get an explicit representant of $\varDelta_\varepsilon$. The family $(a_{h(x)})_{x\in X_\varepsilon}$ depends smoothly on x and defines a symbol $a \in S^0(\mathscr{A}^*(\Omega), \mathrm{End}\Lambda)$. Note that this symbol is independent of the two real parameters coming from the lift of $\mathscr{O}_\varepsilon$ to $\widetilde{\mathscr{O}}_\varepsilon$. Let Op_Ω be a quantification map for Ω. Thanks to the properties of this calculus and the fact that each $a_{h(x)}(y,\xi)$ is of order 0, trivial at infinity (that is independent of ξ near the infinity of $O_{h(x)}$) and $a^2_{h(x)}(y,\xi) - 1$ is of order -1 and vanishes near the infinity of $O_{h(x)}$, we deduce from the exact sequence (4):

$$\mathrm{Op}_\Omega(a) \in \mathscr{L}(C^*(\Omega,\Lambda)) \text{ and } \mathrm{Op}^2_\Omega(a) - \mathrm{Id} \in \mathscr{K}(C^*(\Omega,\Lambda)).$$

Hence, we get an element $[(C^*(\Omega,\Lambda), \mathrm{Op}_\Omega(a))] \in K_0(C^*(\Omega))$ which gives using the inclusion $\Omega \subset \mathscr{H} \times X_\varepsilon$ an element $\widetilde{\varDelta} \in K_0(C^*(\mathscr{H} \times X_\varepsilon))$ satisfying

$$(\mathrm{ev}_0)_*(\widetilde{\varDelta}) = \delta \in K_0(C_0(T^*M \times [0,1] \times X_\varepsilon)).$$

Hence,

$$\varDelta_\varepsilon = (\mathrm{ev}_1)_*(\widetilde{\varDelta}) = [(C^*(\Omega_1,\Lambda), \mathrm{Op}_{\Omega_1}(a))],$$

where $\Omega_1 = \mathscr{G} \times X_\varepsilon|_{\mathcal{O}_\varepsilon \times [0,1]}$. Now consider the evaluation map $e_1 : C^*(\mathscr{G}) \to \mathscr{K}$. We get

$$(e_1)_*(\Delta_\varepsilon) = [(C^*(\Omega_{1,1}, \Lambda), \mathrm{Op}_{\Omega_{1,1}}(a))] \in K_0(\mathscr{K} \otimes C(X_\varepsilon)),$$

where we have set $\Omega_{1,1} = \mathscr{G} \times X_\varepsilon|_{\mathcal{O}_\varepsilon \times \{1\}}$. Note that

$$\Omega_{1,1} = \bigcup_{x \in X_\varepsilon} O_{h(x)} \times O_{h(x)} \times \{x\} \subset (M \times M) \times X_\varepsilon$$

and $\mathrm{Op}_{\Omega_{1,1}}$ is an ordinary quantification map which assigns to a symbol living on $T^*O_{h(x)} \simeq \mathscr{A}^*(O_{h(x)} \times O_{h(x)})$ a pseudodifferential operator on $O_{h(x)}$. Since $P|_{X_\varepsilon} = (P_{h(x)})_{x \in X_\varepsilon}$ has symbol equal to $(a_{h(x)})_{x \in X_\varepsilon}$ and has a trivial index bundle of rank one, the following holds:

$$(e_1)_*(\Delta_\varepsilon) = [(C^*(\Omega_{1,1}, \Lambda), \mathrm{Op}_{\Omega_{1,1}}(a))] = [(H|_{X_\varepsilon}, P|_{X_\varepsilon})] = 1_{X_\varepsilon} \in K^0(X_\varepsilon).$$

(2) The existence of Δ_0 follows immediately from:

Lemma 8. *If $r_L : C(X_\varepsilon) \to C(L)$ $(L = \partial X_\varepsilon)$ denotes the restriction homomorphism, and $i_{M^- \times [0,1]} : \mathscr{K} \otimes C([0,1]) \simeq C^*(\mathscr{G}|_{M^- \times [0,1]}) \to C^*(\mathscr{G})$ the inclusion morphism, then*

$$(r_L)_*(\Delta_\varepsilon) = (i_{M^- \times [0,1]})_*(1_L),$$

where 1_L is the unit of the ring $K_0(\mathscr{K} \otimes C([0,1] \times L)) \simeq K^0(L)$.

Proof. The element $(r_L)_*(\Delta_\varepsilon)$ is represented by

$$(C^*(\partial\Omega_1, \Lambda), \partial P) \in E(\mathbb{C}, C^*(\mathscr{G} \times L))$$

where $\partial\Omega_1 = \bigcup_{(t,x) \in [0,1] \times L} O_{h(x)} \times O_{h(x)} \times \{(t,x)\} \subset (M^- \times M^-) \times [0,1] \times L$ and $\partial P = (P_{h(x)})_{x \in L}$. Since $C^*(\partial\Omega_1, \Lambda)$ is also a $\mathscr{K} \otimes C([0,1] \times L)$-Hilbert module, we observe that

$$x = [(C^*(\partial\Omega_1, \Lambda), \partial P)] \in K_0(\mathscr{K} \otimes C([0,1] \times L))$$

is such that $(i_{M^- \times [0,1]})_*(x) = (r_L)_*(\Delta_\varepsilon)$. Moreover, under the isomorphism $K^0(L) \simeq K_0(\mathscr{K} \otimes C([0,1] \times L))$, the element x is represented by: $(H_{h(x)}, P_{h(x)})_{x \in L}$, which also represents the unit element $1_L \in K^0(L)$ thanks to the triviality of the index bundle of the family $(P_{h(x)})_{x \in L}$. $\square$

By a slight abuse of notation, $\mathscr{G} \times X_\varepsilon|_{\mathcal{O} \times [0,1]}$, $\mathscr{G} \times c_\varepsilon L|_{\mathcal{O} \times [0,1]}$ and $\mathscr{G} \times L|_{\mathcal{O} \times [0,1]}$ will denote, respectively, the restrictions of $\mathscr{G} \times X$ to $\mathcal{O} \times [0,1] \cap M \times X_\varepsilon \times [0,1]$, $\mathcal{O} \times [0,1] \cap M \times c_\varepsilon L \times [0,1] = M^- \times c_\varepsilon L \times [0,1]$ and $\mathcal{O} \times [0,1] \cap M \times \partial X_\varepsilon \times [0,1] = M^- \times L \times [0,1]$.

It is obvious from the concrete description of Δ_ε that it comes from an element $\Delta_{\varepsilon,\mathcal{O}} \in K_0(C^*(\mathscr{G} \times X_\varepsilon|_{\mathcal{O} \times [0,1]}))$ via the inclusion morphism. Now let $x_0 \in K_0(C^*(\mathscr{G} \times$

$c_\varepsilon L|_{\mathcal{O}\times[0,1]}))$ be the pushforward of $1\in K_0(\mathbb{C})$ via the obvious homomorphism:

$$\mathcal{K}\simeq C^*(M^-\times M^-)\to C^*(M^-\times M^-)\otimes C([0,1]\times c_\varepsilon L)\simeq C^*(\mathcal{G}\times c_\varepsilon L|_{\mathcal{O}\times[0,1]})).$$

The preceding lemma and the Mayer-Vietoris exact sequence in K-theory associated to the following commutative diagram:

$$\begin{array}{ccc} C^*(\mathcal{G}\times X|_{\mathcal{O}\times[0,1]}) & \xrightarrow{\quad r \quad} & C^*(\mathcal{G}\times X_\varepsilon|_{\mathcal{O}\times[0,1]}) \\ {\scriptstyle r_{cL}}\downarrow & & \downarrow{\scriptstyle r_L} \\ C^*(\mathcal{G}\times c_\varepsilon L|_{\mathcal{O}\times[0,1]}) & \xrightarrow[r_L]{} & C^*(\mathcal{G}\times L|_{\mathcal{O}\times[0,1]})\simeq\mathcal{K}\otimes C([0,1]\times L) \end{array}$$

show that there exists $\Delta_0\in K_0(C^*(\mathcal{G}\times X|_{\mathcal{O}\times[0,1]}))$ satisfying $r_*(\Delta_0)=\Delta_{\varepsilon,\mathcal{O}}$ and $(r_{cL})_*(\Delta_0)=x_0$. $\square$

Remark 9. In fact, $(r_L)_*(\Delta_{\varepsilon,\mathcal{O}})$ may be represented by the *trivial* vector bundle $\ker P|_L=\bigcup_{x\in L}\ker P_{h(x)}\to L$ while x_0 may be represented by the *product* vector bundle $cL\times\mathbb{C}\to cL$. The element Δ_0 is obtained by gluing these bundles along $L=L\times\{1\}\subset cL$. This involves a bundle isomorphism $\ker P|_L\simeq L\times\mathbb{C}$, which yields a continuous map $\psi:L\to GL_1(\mathbb{C})$ and a class $[\psi]\in K^1(L)$. One could be more careful with the construction of the family (P_x) to make sure that $[\psi]=0$, otherwise one may perturb Δ_0 by elements coming from $K^1(L)$. That will be done in the next proposition.

Proposition 10. *There exists* $\Delta_{\mathcal{O}}\in K_0(C^*(\mathcal{G}\times X|_{\mathcal{O}\times[0,1]}))$ *such that*:

(1) $r_*\circ(i_{\mathcal{O}\times[0,1]})_*(\Delta_{\mathcal{O}})=\Delta_\varepsilon$,
(2) $(e_1)_*\circ(i_{\mathcal{O}\times[0,1]})_*(\Delta_{\mathcal{O}})=1_X\in K^0(X)\simeq K_0(C(X)\otimes\mathcal{K})$.

Proof. Firstly, we note that

$$r_*\circ(e_1)_*\circ(i_{\mathcal{O}\times[0,1]})_*(\Delta_0)=(e_1)_*\circ r_*\circ(i_{\mathcal{O}\times[0,1]})_*(\Delta_0)=(e_1)_*(\Delta_\varepsilon)=1_{X_\varepsilon}=r_*(1_X).$$

From the exact sequence:

$$0\to C_0(\overset{o}{c}_\varepsilon L)\xrightarrow{j}C(X)\xrightarrow{r}C(X_\varepsilon)\to 0$$

and the previous computation, we deduce

$$(e_1)_*(\Delta_0)-1_X\in\mathrm{Im}(j_*).$$

We choose $y_0\in K^1(L)\simeq K_0(C(\overset{o}{c}_\varepsilon L)\otimes\mathcal{K})$ such that

$$j_*(y_0)=(e_1)_*(\Delta_0)-1_X.$$

Moreover one deduces from the fact that the inclusion $\mathscr{K}(L^2(M^-)) \subset \mathscr{K}(L^2(M))$ induces an isomorphism in K-theory, that

$$(e_1)_* \circ i_* : K_0(C^*(\mathscr{G} \times \overset{o}{c}_\varepsilon L|_{\mathcal{O}\times[0,1]})) \to K_0(\mathscr{K} \otimes C_0(\overset{o}{c}_\varepsilon L))$$

is an isomorphism. Here $\mathscr{G} \times \overset{o}{c}_\varepsilon L|_{\mathcal{O}\times[0,1]}$ is the restriction of $\mathscr{G} \times X$ to $\mathcal{O} \times [0,1] \cap M \times [0,1] \times \overset{o}{c}_\varepsilon L$ and $i : C^*(\mathscr{G} \times \overset{o}{c}_\varepsilon L|_{\mathcal{O}\times[0,1]}) \to C^*(\mathscr{G} \times X)$ is the inclusion morphism.

Now let $\widetilde{y_0} \in K_0(C^*(\mathscr{G} \times \overset{o}{c}_\varepsilon L|_{\mathcal{O}\times[0,1]}))$ be the unique element such that $(e_1)_* \circ i_*(\widetilde{y_0}) = -y_0$. We set:

$$\Delta_{\mathcal{O}} = \Delta_0 + j_*(\widetilde{y_0}),$$

where we have still denoted by j the inclusion morphism from $C^*(\mathscr{G} \times \overset{o}{c}_\varepsilon L|_{\mathcal{O}\times[0,1]})$ to $C^*(\mathscr{G} \times X|_{\mathcal{O}\times[0,1]})$. Then $\Delta_{\mathcal{O}} \in K_0(C^*(\mathscr{G} \times X|_{\mathcal{O}\times[0,1]}))$ satisfies (1) and (2). $\square$

We define

$$\Delta = (i_{\mathcal{O}\times[0,1]})_*(\Delta_{\mathcal{O}}) \in K_0(C^*(\mathscr{G} \times X)).$$

Definition 11. The dual Dirac element $\lambda \in K_0(C^*(G) \otimes C(X))$ of the singular manifold X is defined by

$$\lambda = (e_0)_*(\Delta),$$

where $e_0 : C^*(\mathscr{G}) \to C^*(G)$ is the evaluation homomorphism at 0.

We have proved the following:

Proposition 12. (1) *The following equality holds*:

$$\lambda \underset{C^*(G)}{\bigotimes} \partial = 1_X \in K_0(C(X)) \simeq K^0(X),$$

where 1_X is the unit of the ring $K_0(C(X))$.

(2) *For each open subset $\mathcal{O}_\alpha$, $0 < \alpha < 1$, of $M \times X$ defined by*

$$\mathcal{O}_\alpha = \{(x,y) \in M \times X \mid d(x,y) < 1, k_y > 0\} \cup \{(x,y) \in M \times X \mid k_x < \alpha, k_y < \alpha\},$$

the dual Dirac element λ belongs to the range of:

$$(i_{\mathcal{O}_\alpha})_* : K_0(C^*(G \times X|_{\mathcal{O}_\alpha})) \to K_0(C^*(G \times X)).$$

Roughly speaking, every $\mathcal{O}_\alpha$ contains the "support" of λ. From now on, we choose $\mathcal{O} = \mathcal{O}_{1/2}$ and $\widetilde{\lambda}$ a preimage of λ for $(i_{\mathcal{O}})_*$.

5. The Poincaré duality

This section is devoted to the proof of our main result:

Theorem 13. *The Dirac element D and the dual Dirac element λ induce a Poincaré duality between $C^*(G)$ and $C(X)$, that is*

(1) $\lambda \bigotimes_{C^*(G)} D = 1_{C(X)} \in KK(C(X), C(X))$,

(2) $\lambda \bigotimes_{C(X)} D = 1_{C^*(G)} \in KK(C^*(G), C^*(G))$.

5.1. Computation of $\lambda \otimes_{C^*(G)} D$

Let $m : C(X) \otimes C(X) \to C(X)$ be the morphism of C^*-algebras induced by multiplication of functions.

Lemma 14. *The following equality holds*:

$$\lambda \bigotimes_{C^*(G)} [\Psi] = \lambda \bigotimes_{C(X)} [m].$$

Proof. According to Proposition 12 we have that $\lambda \otimes_{C^*(G)} [\Psi] = \tau_{C(X)}(\tilde{\lambda}) \otimes [H_0]$ and $\lambda \otimes_{C(X)} [m] = \tau_{C(X)}(\tilde{\lambda}) \otimes [\tilde{H}_1]$, where H_0 and $\tilde{H}_1$ are the morphisms from $C^*(G \times X|_{\mathcal{O}}) \otimes C(X)$ to $C^*(G \times X)$ defined by

$$H_0(B \otimes f)(\gamma, y) = f(r(\gamma))B(\gamma, y) \quad \text{and} \quad \tilde{H}_1(B \otimes f)(\gamma, y) = f(y)B(\gamma, y),$$

when $f \in C(X)$, $B \in C_c(G \times X|_{\mathcal{O}})$, $(\gamma, y) \in G \times X$, and $C(X)$ is identified with the algebra of continuous functions on M which are constant on M^-.

There is an obvious homotopy between $\tilde{H}_1$ and H_1 defined by

$$H_1(B \otimes f)(\gamma, y) = f(h(y))B(\gamma, y),$$

where h is the map constructed in Section 4 to push points in M^-.

Let $\mathcal{W} = \{(x, y) \in M \times X; d(x, y) < 1 \text{ and } k_y \geqslant \varepsilon\}$ viewed as a subset of $M^+ \times M^+$. Let c be the continuous function $\mathcal{W} \times [0, 1] \to M^+ \subset X$ such that $c(x, y, \cdot)$ is the geodesic path going from x to y when $(x, y) \in \mathcal{W}$.

We obtain an homotopy between H_0 and H_1 by setting for each $t \in [0,1]$:

$$H_t(B \otimes f)(\gamma, y) = \begin{cases} f(c(r(\gamma), h(x), t))B(\gamma, x) & \text{if } k_x \geqslant 1/2, \\ f(c)B(\gamma, x) & \text{elsewhere.} \end{cases}$$

□

Now we are able to compute the product $\lambda \otimes_{C^*(G)} D$:

$$\begin{aligned} \lambda \bigotimes_{C^*(G)} D &= \left(\lambda \bigotimes_{C^*(G)} [\Psi]\right) \bigotimes_{C^*(G)} \partial = \left(\lambda \bigotimes_{C(X)} [m]\right) \bigotimes_{C^*(G)} \partial \\ &= \tau_{C(X)}(\lambda) \otimes \left([m] \bigotimes_{\mathbb{C}} \partial\right) = \tau_{C(X)}(\lambda) \otimes \left(\partial \bigotimes_{\mathbb{C}} [m]\right) \\ &= \tau_{C(X)}\left(\lambda \bigotimes_{C^*(G)} \partial\right) \otimes [m] = 1_{C(X)}. \end{aligned}$$

The equality of the first line results from the previous lemma, the equality of the second line comes from the commutativity of the Kasparov product over $\mathbb{C}$ and the last equality follows from Proposition 12. This finishes the proof of the first part of Theorem 13.

Let us notice the following consequence: for every C^*-algebras A and B, we have:

$$\left(\cdot \bigotimes_{C^*(G)} D\right) \circ \left(\lambda \bigotimes_{C(X)} \cdot\right) = \mathrm{Id}_{KK(C(X) \otimes A, B)} \text{ and } \left(\lambda \bigotimes_{C^*(G)} \cdot\right) \circ \left(\cdot \bigotimes_{C(X)} D\right) = \mathrm{Id}_{KK(A, B \otimes C(X))}.$$

In particular, this implies the following useful remark:

Remark 15. The equality $(\cdot \otimes_{C^*(G)} D)(\lambda \otimes_{C(X)} D) = (\cdot \otimes_{C^*(G)} D) \circ (\lambda \otimes_{C(X)} \cdot)(D) = D$ ensures that $\lambda \otimes_{C(X)} D - 1_{C^*(G)}$ belongs to the kernel of the map $(\cdot \otimes_{C^*(G)} D) \colon KK(C^*(G), C^*(G)) \to KK(C^*(G) \otimes C(X), \mathbb{C})$.

5.2. *Computation of* $\lambda \otimes_{C(X)} D$

The purpose here is to prove that

$$\lambda \bigotimes_{C(X)} D = \tau_{\underline{C^*(G)}}(\lambda) \otimes \tau_{C^*(G)}([\Psi] \otimes \partial) = 1_{C^*(G)}.$$

This problem leads us to study the invariance of $\lambda \otimes_{C(X)} [\Psi]$ under the *flip* automorphism $\tilde{f}$ of $C^*(G \times G) \simeq C^*(G) \otimes C^*(G)$.

Indeed, we have

$$\tau_{C^*(G)}([\Psi]) \otimes [\tilde{f}] \otimes \tau_{C^*(G)}(\partial) = [\Psi] \bigotimes_{\mathbb{C}} \partial = \partial \bigotimes_{\mathbb{C}} [\Psi] = \tau_{\underline{C^*(G)} \otimes C(X)}(\partial) \otimes [\Psi],$$

which implies (cf. Proposition 12):

$$\left(\left(\lambda \bigotimes_{C(X)} [\Psi]\right) \otimes [\tilde{f}]\right) \otimes \tau_{C^*(G)}(\partial) = \left(\lambda \bigotimes_{C^*(G)} \partial\right) \bigotimes_{C(X)} [\Psi] = 1_{C^*(G)}.$$

Hence,

$$\lambda \bigotimes_{C(X)} D - 1_{C^*(G)} = \left(\left(\lambda \bigotimes_{C(X)} [\Psi]\right) \otimes ([\mathrm{id}] - [\tilde{f}])\right) \otimes \tau_{C^*(G)}(\partial) \tag{6}$$

and the invariance of $\lambda \otimes_{C(X)} [\Psi]$ under $[\tilde{f}]$ would enable us to conclude the proof. Such an invariance would be analogous to Lemma 4.6 of [15]. Unfortunately, we are not able to prove that $\lambda \otimes_{C(X)} [\Psi]$ is invariant under the flip.

Put $C = L \times]-1, 1[$ and $F = M \times M \backslash C \times C$.

We let $\tilde{\mathcal{O}}$ be the inverse image of $\mathcal{O}$ by the canonical projection of $M \times M \to M \times X$. We denote $(i_{\tilde{\mathcal{O}}})_* : KK(C^*(G), C^*(G \times G|_{\tilde{\mathcal{O}}})) \to KK(C^*(G), C^*(G \times G))$ the morphism corresponding to the inclusion $i_{\tilde{\mathcal{O}}}$. A simple computation shows that:

Lemma 16. *The element $\lambda \otimes_{C(X)} [\Psi]$ belongs to the image of $(i_{\tilde{\mathcal{O}}})_*$.*

The set $F \cap \tilde{\mathcal{O}}$ is an open and symmetric subset of $M \times M$, hence the flip makes sense on $C^*(G \times G|_{F \cap \tilde{\mathcal{O}}})$.

Lemma 17. *The flip automorphism of $C^*(G \times G|_{F \cap \tilde{\mathcal{O}}})$ is homotopic to identity.*

Proof. The set $F \cap \tilde{\mathcal{O}} = \{(x, y) \in M \times M \mid d(x, y) < 1,\ k_x \geqslant 1 \text{ or } k_y \geqslant 1\}$ is a subset of $\overline{M^+} \times \overline{M^+}$, so the algebra $C^*(G \times G|_{F \cap \tilde{\mathcal{O}}})$ is isomorphic to $C_0(T^*(F \cap \tilde{\mathcal{O}}))$. To prove the lemma, it is sufficient to find a proper homotopy between the flip $f_{F \cap \tilde{\mathcal{O}}}$ of $T^*(F \cap \tilde{\mathcal{O}})$ and $\mathrm{id}_{T^*(F \cap \tilde{\mathcal{O}})}$.

The exponential map of M provides an isomorphism ϕ between $T^*(F \cap \tilde{\mathcal{O}})$ and $(T^* M_l)^{\oplus 3}$, where $M_l = \{x \in M \mid k_x \geqslant l\}$ for some $0 < l < 1$. Via this isomorphism, the flip becomes the automorphism of $C_0((T^* M_l)^{\oplus 3})$ induced by the map $g : (x, X, Y, Z) \in (T_x^* M_l)^3 \mapsto (x, -X, Z, Y)$. One can take for example: $\phi : (x, y, X, Y) \mapsto (m(x, y), \exp^{-1}_{m(x,y)}(x) - \exp^{-1}_{m(x,y)}(z), T(x, y, X), T(y, x, Y))$, where $m(x, y) = \exp_x(\frac{\exp_x^{-1} y}{2})$ is the middle point of the geodesic joining x to y and $T(x, y, \cdot)$: $T_x M \to T_{m(x,y)} M$ is the parallel transport along the geodesic joining x to $m(x, y)$.

Let $A : [0, 1] \to SO_3(\mathbb{R})$ be a continuous path from $\begin{pmatrix} -1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}$ to $\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$.

The map $[0, 1] \times (T^* M_l)^{\oplus 3} \to (T^* M_l)^{\oplus 3}$; $(t, x, V) \mapsto (x, A_t.V)$ is a proper homotopy between identity and g. $\square$

Note that $C \times C$ is a saturated open subset of $(G \times G)^{(0)}$. So we obtain the following commutative diagram of C^*-algebras:

$$\begin{tikzcd} 0 \arrow[r] & C^*(G \times G|_{C\times C\cap\tilde{\mathcal{O}}}) \arrow[r] \arrow[d] & C^*(G \times G|_{\tilde{\mathcal{O}}}) \arrow[r, "r_{F\cap\tilde{\mathcal{O}}}"] \arrow[d, "i_{\tilde{\mathcal{O}}}"] & C^*(G \times G|_{F\cap\tilde{\mathcal{O}}}) \arrow[r] \arrow[d, "i_{F\cap\tilde{\mathcal{O}}}"] & 0 \\ 0 \arrow[r] & C^*(G \times G|_{C\times C}) \arrow[r, "i_{C\times C}"'] & C^*(G \times G) \arrow[r, "r_F"'] & C^*(G \times G|_F) \arrow[r] & 0. \end{tikzcd} \tag{7}$$

Since flip automorphisms commute with restriction and inclusion morphisms, this commutative diagram and the previous lemma imply that the induced morphisms of KK groups satisfy:

$$(r_F)_* \circ \tilde{f}_* \circ (i_{\tilde{\mathcal{O}}})_* = (r_F)_* \circ (i_{\tilde{\mathcal{O}}})_*.$$

In other words, $(r_F)_* \circ (\mathrm{id} - \tilde{f})_* \circ (i_{\tilde{\mathcal{O}}})_*$ is the zero map.

Hence Lemma 16 implies that $(\lambda \otimes_{C(X)} [\Psi]) \otimes ([\mathrm{id}] - [\tilde{f}\,])$ belongs to the kernel of the map $(r_F)_* : KK(C^*(G), C^*(G \times G)) \to KK(C^*(G), C^*(G \times G|_F))$. It follows from the long exact sequence in KK-theory associated to the second short exact sequence of (7) that $(\lambda \otimes_{C(X)} [\Psi]) \otimes ([\mathrm{id}] - [\tilde{f}\,])$ belongs to the image of the map $(i_{C\times C})_* : KK(C^*(G), C^*(G \times G|_{C\times C})) \to KK(C^*(G), C^*(G \times G))$.

Remark 18. The C^*-algebra $C^*(G|_C)$ is KK-equivalent to $\mathcal{K}(L^2(M^-))$. Indeed, we have the following exact sequence:

$$0 \to \mathcal{K}(L^2(M^-)) \xrightarrow{i_{M^-,C}} C^*(G|_C) \to C^*(G|_{C\setminus M^-}) \simeq C_0(T^*(L \times [0,1[)) \to 0$$

and the C^*-algebra $C_0(T^*(L \times [0,1[))$ is contractible. So $[i_{M^-,C}]$ is an invertible element of $KK(\mathcal{K}(L^2(M^-)), C^*(G|_C))$.

In particular, $C^*(G \times G|_{C\times C})$ is KK-equivalent to $\mathcal{K} \otimes \mathcal{K}$. Furthermore, the flip automorphism of $\mathcal{K} \otimes \mathcal{K}$ is homotopic to identity. Together with our last result, this only shows that $(\lambda \otimes_{C(X)} [\Psi]) \otimes ([\mathrm{id}] - [\tilde{f}\,])$ is a torsion element (of order 2).

Lemma 19. *The element $\lambda \otimes_{C(X)} D - 1_{C^*(G)}$ belongs to the image of the map $(i_C)_*$: $KK(C^*(G), C^*(G|_C)) \to KK(C^*(G), C^*(G))$ induced by the inclusion morphism i_C.*

Proof. The proof follows from the equality $[i_{C\times C}] \otimes \tau_{C^*(G)}(\partial) = ([i_C] \otimes_{\mathbb{C}} [i_C]) \otimes \tau_{C^*(G)}(\partial) = \tau_{C^*(G|_C)}([i_C] \otimes \partial) \otimes [i_C]$ and the fact that $(\lambda \otimes_{C(X)} [\Psi]) \otimes ([\mathrm{Id}] - [\tilde{f}\,])$ is in the image of $(i_{C\times C})_*$. $\square$

Thus, with Remark 15 in mind, it remains to show that the map $(\cdot \otimes_{C^*(G)} D) \circ (i_C)_*$ is injective.

We consider the morphisms $i^- : \mathcal{K}(L^2(M^-)) \to C^*(G)$ and $i^{\mathcal{K}} : \mathcal{K}(L^2(M^-)) \to \mathcal{K}(L^2(M))$ induced by the inclusion of functions. Since $i^{\mathcal{K}}$ preserve the rank of operators, $(i^{\mathscr{K}})_*$ is an isomorphism. We let $e_c : C(X) \to \mathbb{C}$ be the evaluation map at c. The map e_c admits a right inverse, so e_c^* is injective.

Proposition 20. *For any C^*-algebra A, the following diagram is commutative*

$$\begin{tikzcd} KK(A, \mathscr{K}(L^2(M^-))) \arrow[r, "(i^-)_*"] \arrow[d, "(\cdot \otimes [i^{\mathscr{K}}] \otimes b)"'] & KK(A, C^*(G)) \arrow[d, "\left(\cdot \bigotimes_{C^*(G)} D\right)"] \\ KK(A, \mathbb{C}) \arrow[r, "e_c^*"'] & KK(A \otimes C(X), \mathbb{C}) \end{tikzcd}$$

Proof. For any $x \in KK(A, \mathscr{K}(L^2(M^-)))$ we write:

$$\left(\cdot \bigotimes_{C^*(G)} D\right) \circ (i^-)_*(x) = \tau_{C(X)}(x \otimes [i^-]) \otimes D = \tau_{C(X)}(x) \otimes \tau_{C(X)}([i^-]) \otimes [\Psi] \otimes \partial.$$

If $f \in C(X)$ and $k \in KK(L^2(M^-))$, we observe that $\Psi((i^-(k) \otimes f)) = f(c) i^-(k) = e_c(f) i^-(k)$. In particular, $\tau_{C(X)}([i^-]) \otimes [\Psi] = \tau_{\mathscr{K}}([e_c]) \otimes [i^-]$. It follows that

$$\left(\cdot \bigotimes_{C^*(G)} D\right) \circ (i^-)_*(x) = \tau_{C(X)}(x) \otimes \tau_{\mathscr{K}}([e_c]) \otimes [i^-] \otimes \partial.$$

Furthermore, the following commutative diagram of C^*-algebras:

$$\begin{tikzcd} \mathscr{K}(L^2(M^-)) \arrow[d, "i^{\mathscr{K}}"'] & C^*(\mathscr{G}|_{M^- \times [0,1]}) \simeq C([0,1]) \otimes \mathscr{K}(L^2(M^-)) \arrow[l, "\mathrm{ev}_1 \otimes \mathrm{id}"'] \arrow[r, "\mathrm{ev}_0 \otimes \mathrm{id}"] \arrow[d] & \mathscr{K}(L^2(M^-)) \arrow[d, "i^-"] \\ \mathscr{K}(L^2(M)) & C^*(\mathscr{G}) \arrow[l, "e_1"] \arrow[r, "e_0"'] & C^*(G) \end{tikzcd}$$

shows that $[i^-] \otimes \partial = [i^{\mathscr{K}}]$.

Finally, using that $\tau_{C(X)}(x) \otimes \tau_{\mathscr{K}}([e_c]) = x \bigotimes_{\mathbb{C}} [e_c]$, we get

$$\left(\cdot \bigotimes_{C^*(G)} D\right) \circ (i^-)_*(x) = e_c^* \circ (\cdot \otimes [i^{\mathscr{K}}] \otimes b)(x). \qquad \square$$

We have already noticed that $C^*(G|_C)$ is KK-equivalent to $\mathscr{K}(L^2(M^-))$ (cf. Remark 18), and $i_C \circ i_{M^-,C} = i^-$. So, using the previous proposition (applied to $A = C^*(G)$), we deduce

Corollary 21. *The morphism $(\cdot \otimes_{C^*(G)} D)$ is injective when restricted to the image of $(i_C)_*$ going from $KK(C^*(G), C^*(G|_C))$ to $KK(C^*(G), C^*(G))$.*

Combining Lemma 19, Remark 15 and Corollary 21, we conclude that

$$\lambda \bigotimes_{C(X)} D = 1_{C^*(G)}.$$

This finishes the proof of Theorem 13.

Remark 22. The K-duality for the pseudomanifold X is strongly related to a Poincaré duality for manifolds with boundary. Let us consider the following two exact sequences:

$$0 \to \mathscr{K}(L^2(M^-)) \xrightarrow{i^-} C^*(G) \xrightarrow{r} C_0(T^*\overline{M^+}) \to 0, \tag{8}$$

$$0 \leftarrow \mathbb{C} \underset{e_c}{\leftarrow} C(X) \underset{j}{\leftarrow} C_0(X \backslash \{c\}) \leftarrow 0. \tag{9}$$

Note that the exact sequence (9) is split, and that Proposition 20 ensures the injectivity of $(i^-)_*$. Hence both (8) and (9) give rise to short exact sequences in KK-theory, and invoking again Proposition 20, we get the following commutative diagram:

$$\begin{array}{ccccc}
0 \to KK(A, \mathscr{K}) & \xrightarrow{(i^-)_*} & KK(A, C^*(G)) & \xrightarrow{r_*} & KK(A, C_0(T^*\overline{M^+})) \to 0 \\
\Big\downarrow {\scriptstyle (\cdot \otimes [i^{\mathscr{K}}] \otimes b)} & & \Big\downarrow {\scriptstyle \left(\cdot \bigotimes_{C^*(G)} D\right)} & & \Big\downarrow \\
0 \to KK(A, \mathbb{C}) & \xrightarrow[e_c^*]{} & KK(A \otimes C(X), \mathbb{C}) & \xrightarrow[j^*]{} & KK(A \otimes C_0(M^+), \mathbb{C}) \to 0
\end{array}$$

The vertical arrows are isomorphisms: it is obvious for the left one and a consequence of Theorem 13 for the middle one. Hence there is an induced isomorphism $KK(A, C_0(T^*\overline{M^+})) \to KK(A \otimes C_0(M^+), \mathbb{C})$ making the diagram commutative.

Conversely, using [6], one can prove the K-duality between $C_0(T\overline{M^+})$ and $C_0(M^+)$, and obtain from this an alternative proof of Theorem 13. This will be used in a forthcoming paper to extend this work to general pseudomanifolds.

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