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Groupoids and an index theorem for conical pseudo-manifolds

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Abstract. We define an analytical index map and a topological index map for conical pseudomanifolds. These constructions generalize the analogous constructions used by Atiyah and Singer in the proof of their topological index theorem for a smooth, compact manifold M. A main new ingredient in our proof is a non-commutative algebra that plays in our setting the role of $\mathscr{C}_0(T^*M)$. We prove a Thom isomorphism between non-commutative algebras which gives a new example of wrong way functoriality in K-theory. We then give a new proof of the Atiyah-Singer Index Theorem using deformation groupoids and show how it generalizes to conical pseudomanifolds. We thus prove a topological index theorem for conical pseudomanifolds.

Introduction

Let V be a closed, smooth manifold and let P be an elliptic pseudo-differential operator acting between Sobolev spaces of sections of two vector bundles over V. The ellipticity of P ensures that P has finite dimensional kernel and cokernel. The difference

$$\operatorname{Ind} P := \dim(\operatorname{Ker} P) - \dim(\operatorname{Coker} P)$$

is called the *Fredholm index* of P and turns out to depend only on the K-theory class $[\sigma(P)] \in K^0(T^*V)$ of the principal symbol of P (we always use K-theory with compact supports). Since every element in $K^0(T^*V)$ can be represented by the principal symbol of an elliptic pseudo-differential operator, one obtains in this way a group morphism

$$(0.1)\qquad \operatorname{Ind}_a^V : K^0(T^*V) \to \mathbb{Z}, \quad \operatorname{Ind}_a^V\big(\sigma(P)\big) = \operatorname{Ind} P,$$

called the *analytical index*, first introduced by M. Atiyah and I. Singer [4].

At first sight, the map Ind_a^V seems to depend essentially on the analysis of elliptic equations. In [4], Atiyah and Singer used embedding of V in Euclidean space to give a topological construction of the index map, and this leads to the so called *topological index* map Ind_t^V. The main result of [4] is that the topological index map Ind_t^V and the Fredholm index map Ind_a^V coincide. See [17] for review of these results, including an extension to non-compact manifolds.

The equality of the topological and Fredholm indices then allowed M. Atiyah and I. Singer to obtain a formula for the index of an elliptic operator P in terms of the Chern classes of $[\sigma(P)]$. Their formula, the celebrated Atiyah-Singer Index Formula, involves, in addition to the Chern character of the principal symbol of P, also a universal characteristic class associated with the manifold, the so called Todd class of the given manifold.

It is a natural and important question then to search for extensions of the Atiyah-Singer results. It is not the place here to mention all existing generalizations of the Atiyah-Singer index theory, but let us mention here the fundamental work of A. Connes on foliations [20], [21], [22], [23], [24] as well as [7], [32], [54], [55]. The index theorem for families and Bismut's superconnection formalism play an important role in the study of the so called "anomalies" in physics [9], [10], [12], [30]. A different but related direction is to extend this theory to singular spaces [3]. An important step in the index problem on singular manifolds was made by Melrose [44], [45] and Schulze [60], [61] who have introduced the "right class of pseudodifferential operators" for index theory on singular spaces. See also [1], [28], [29], [43], [53], [62]. Generalizations of this theory to singular spaces may turn out to be useful in the development of efficient numerical methods [6].

In this paper, we shall focus on the case of a pseudomanifold X with isolated conical singularities. In earlier work [27], the first two authors defined a C^*-algebra A_X that is dual to the algebra of continuous functions on X from the point of view of K-theory (i.e. A_X is a "K-dual of X" in the sense of [22], [24], [37]), which implies that there exists a natural isomorphism

$$K_0(X) \xrightarrow{\Sigma_X} K_0(A_X) \tag{0.2}$$

between the K-homology of X and the K-theory of $T^{\mathsf{S}}X$. The C^*-algebra A_X is the C^*-algebra of a groupoid denoted $T^{\mathsf{S}}X$.

One of the main results in [40], see also [47], [52], [51], [59] for similar results using different methods, is that the inverse of the map Σ_X of Equation (0.2) can be realized, as in the smooth case, by a map that assigns to each element in $K_0(A_X)$ an elliptic operator. Thus elements of $K_0(A_X)$ can be viewed as the symbols of some natural elliptic pseudo-differential operators realizing the K-homology of X. Of course, in the singular setting, one has to explain what is meant by "elliptic operator" and by "symbol" on X. An example of a convenient choice of elliptic operator in our situation is an elliptic pseudodifferential operator in the b-calculus [44], [60] or Melrose's c-calculus. As for the symbols, the notion is more or less the same as in the smooth case. On a manifold V, a symbol is a function on T^*V. For us, it will be convenient to view a symbol as a pointwise multiplication operator on $C_c^\infty(T^*V)$. A Fourier transform will allow us then to see a symbol as a family of convolution operators on $C_c^\infty(T_xV)$, $x \in V$. Thus symbols on V appear to be pseudo-differential operators on the *groupoid* TV. This picture generalizes then right away to our singular setting. In particular, it leads to a good notion of symbol for conical pseudomanifolds and enables us to interpret (0.2) as the principal symbol map.

In order to better explain our results, we need to introduce some notation. If G is an amenable groupoid, we let $K^0(G)$ denote $K_0\big(C^*(G)\big)$. The analytical index is then defined exactly as in the regular case by

$$\begin{aligned} \mathrm{Ind}_a^X : K^0(T^{\mathsf{S}}X) &\to \mathbb{Z}, \\ [a] &\mapsto \mathrm{Ind}\big(\Sigma_X^{-1}(a)\big), \end{aligned}$$

where $\mathrm{Ind} : K_0(X) \to \mathbb{Z}$ is the usual Fredholm index on compact spaces. Moreover one can generalise the *tangent groupoid* of A. Connes to our situation and get a nice description of the analytical index.

Following the spirit of [4], we define in this article a topological index Ind_t^X that generalizes the classical one and which satisfies the equality:

$$\mathrm{Ind}_a^X = \mathrm{Ind}_t^X.$$

In fact, we shall see that all ingredients of the classical topological index have a natural generalisation to the singular setting.

- Firstly the embedding of a smooth manifold into $\mathbb{R}^N$ gives rise to a normal bundle N and a Thom isomorphism $K^0(T^*V) \to K^0(T^*N)$. In the singular setting we embed X into $\mathbb{R}^N$, viewed as the cone over S^{N-1}. This gives rise to a *conical vector bundle* which is a conical pseudomanifold called the *normal space* and we get an isomorphism: $K^0(T^{\mathsf{S}}X) \to K^0(T^{\mathsf{S}}N)$. This map restrict to the usual Thom isomorphism on the regular part and is called again the *Thom isomorphism*.

- Secondly, in the smooth case, the normal bundle N identifies with an open subset of $\mathbb{R}^N$, and thus provides an excision map $K(TN) \to K(T\mathbb{R}^N)$. The same is true in the singular setting: $T^{\mathsf{S}}N$ appears to be an open subgroupoid of $T^{\mathsf{S}}\mathbb{R}^N$ so we have an excision map $K^0(T^{\mathsf{S}}N) \to K^0(T^{\mathsf{S}}\mathbb{R}^N)$.

- Finally, using the Bott periodicity $K^0(T^*\mathbb{R}^N) \simeq K^0(\mathbb{R}^{2N}) \to \mathbb{Z}$ and a natural KK-equivalence between $T^{\mathsf{S}}\mathbb{R}^N$ with $T\mathbb{R}^N$ we obtain an isomorphism $K^0(T^{\mathsf{S}}\mathbb{R}^N) \to \mathbb{Z}$.

As for the usual definition of the topological index, this allows us to define our generalisation of the topological Ind_t for conical manifolds.

This construction of the topological index is inspired from the techniques of *deformation groupoids* introduced by M. Hilsum and G. Skandalis in [33]. Moreover, the demonstration of the equality between Ind_a and Ind_t will be the same in the smooth and in the singular setting with the help of deformation groupoids.

We claim that our index maps are straight generalisations of the classical ones. To make this claim more concrete, consider a closed smooth manifold V and choose a point $c \in V$. Take a neighborhood of c diffeomorphic to the unit ball in $\mathbb{R}^n$ and consider it as the cone over S^{n-1}. This provides V with the structure of a conical manifold. Then the index maps $\mathrm{Ind}_*^{\mathsf{S}} : K^0(T^{\mathsf{S}}V) \to \mathbb{Z}$ and $\mathrm{Ind}_* : K^0(TV) \to \mathbb{Z}$ both correspond to the canonical map $K_0(V) \to \mathbb{Z}$ through the Poincaré duality $K_0(V) \simeq K^0(T^*V)$ and $K_0(V) \simeq K^0(T^{\mathsf{S}}V)$. In other words both notions of indices coincide trough the KK-equivalence $TV \simeq T^{\mathsf{S}}V$.

We will investigate the case of general stratifications and the proof of an index formula in forthcoming papers.

The paper is organized as follows. In Section 2 we describe the notion of conical pseudomanifolds and conical bundles. Section 2 reviews general facts about Lie groupoids. Section 3 is devoted to the construction of tangent spaces and tangent groupoids associated to conical pseudomanifolds as well as other deformation groupoids needed in the subsequent sections. Sections 4 and 5 contain the construction of analytical and topological indices, and the last section is devoted to the proof of our topological index theorem for conical pseudomanifolds, that is, the proof of the equality of analytical and topological indices for conical pseudomanifolds.

1. Cones and stratified bundles

We are interested in studying conical pseudomanifolds, which are special examples of stratified pseudomanifolds of depth one [31]. We will use the notations and equivalent descriptions given by A. Verona in [64] or used by J. P. Brasselet, G. Hector and M. Saralegi in [14]. See [35] for a review of the subject.

1.1. Conical pseudomanifolds. If L is a smooth manifold, the *cone* over L is, by definition, the topological space

$$cL := L \times [0, +\infty[/L \times \{0\}. \tag{1.1}$$

Thus $L \times \{0\}$ maps into a single point c of cL. We shall refer to c as the *singular point of* L. If $z \in L$ and $t \in [0, +\infty[$ then $[z, t]$ will denote the image of (z, t) in cL. We shall denote by

$$\rho_{cL} : cL \to [0, +\infty[, \quad \rho_{cL}([z, t]) := t$$

the map induced by the second projection and we call it the *defining function* of the cone.

Definition 1.1. A *conical stratification* is a triplet $(X, \mathsf{S}, \mathscr{C})$ where:

(i) X is a Hausdorff, locally compact, and secound countable space.

(ii) $\mathsf{S} \subset X$ is a finite set of points, called the *singular set* of X, such that $X^\circ := X \backslash \mathsf{S}$ is a smooth manifold.

(iii) $\mathscr{C} = \{(\mathscr{N}_s, \rho_s, L_s)\}_{s \in \mathsf{S}}$ is the set of *control data*, where $\mathscr{N}_s$ is an open neighborhood of s in X and $\rho_s : \mathscr{N}_s \to [0, +\infty[$ is a surjective continuous map such that $\rho_s^{-1}(0) = s$.

(iv) For each $s \in \mathsf{S}$, there exists a homeomorphism $\varphi_s : \mathscr{N}_s \to cL_s$, called *trivialisation map*, such that $\rho_{cL_s} \circ \varphi_s = \rho_s$ and such that the induced map $\mathscr{N}_s \backslash \{s\} \to L_s \times]0, +\infty[$ is a diffeomorphism. Moreover, if $s_0, s_1 \in \mathsf{S}$ then either $\mathscr{N}_{s_0} \cap \mathscr{N}_{s_1} = \emptyset$ or $s_0 = s_1$.

Let us notice that it follows from the definition that the connected components of X° are smooth manifolds. These connected components are called the *regular strata* of X.

Definition 1.2. Two conical stratifications $(X, \mathsf{S}_X, \mathscr{C}_X)$ and $(Y, \mathsf{S}_Y, \mathscr{C}_Y)$ are called *isomorphic* if there is an homeomorphism $f : X \to Y$ such that:

(i) f maps S_X onto S_Y.

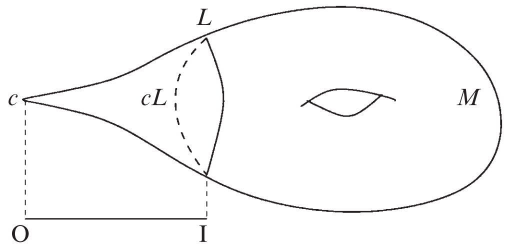
(ii) f restricts to a smooth diffeomorphism $f^\circ : X^\circ \to Y^\circ$.

(iii) The defining function ρ_s of any $s \in \mathsf{S}_X$ is equal to $\rho_{f(s)} \circ f$, where $\rho_{f(s)}$ is the defining function of $f(s) \in \mathsf{S}_Y$ (in particular $f(\mathscr{N}_s) = \mathscr{N}_{f(s)}$).

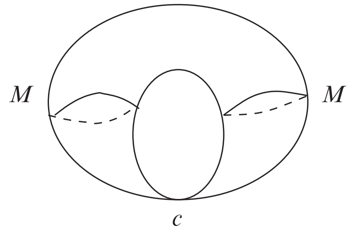
An isomorphism class of conical stratifications will be called a *conical pseudomanifold.*

In other words, a conical pseudomanifold is a locally compact, metrizable, second countable space X together with a finite set of points $\mathsf{S} \subset X$ such that $X^\circ = X \backslash \mathsf{S}$ is a smooth manifold and one can find a set of control data $\mathscr{C}$ such that $(X, \mathsf{S}, \mathscr{C})$ is a conical stratification.

Let M be a smooth manifold with boundary $L := \partial M$. An easy way to construct a conical pseudomanifold is to glue to M the *closed cone* $\overline{cL} := L \times [0,1]/L \times \{0\}$ along the boundary.



Notice that we do not ask the link L to be connected. For example, if M is a smooth manifold, the space $M \times S^1/M \times \{p\}$, $p \in S^1$, is a conical pseudomanifold with L consisting of two disjoint copies of M:



1.2. Conical bundles. We next introduce "conical bundles," a class of spaces not to be confused with vector bundles over conical manifolds. Assume that L is a smooth manifold, cL is the cone over L, $\pi_\xi : \xi \to L$ is a smooth vector bundle over L, and $c\xi$ is the cone over ξ. We define $\pi : c\xi \to cL$ by $\pi([z,t]) = [\pi_\xi(z), t]$ for $(z,t) \in \xi \times [0, +\infty[$. The set $(c\xi, \pi)$ is the *cone* over the vector bundle (ξ, π_ξ). Let us notice that the fiber above the singular point of cL is the singular point of $c\xi$. In particular, $c\xi$ is *not* a vector bundle over cL.

Definition 1.3. Let $(X, \mathsf{S}_X, \mathscr{C}_X)$ be a conical stratification. A *conical vector bundle* (E, π) over X is a conical stratification $(E, \mathsf{S}_E, \mathscr{C}_E)$ together with a continuous surjective map $\pi : E \to X$ such that:

(1) π induces a bijection between the singular sets S_E and S_X.

(2) If $E^\circ := E \backslash \mathsf{S}_E$, the restriction $\pi^\circ : E^\circ \to X^\circ$ is a smooth vector bundle.

(3) The control data $\{\mathscr{M}_z, \rho_z, \xi_z\}_{z \in \mathsf{S}_E}$ of E and $\{\mathscr{N}_s, \rho_s, L_s\}_{s \in \mathsf{S}_X}$ of X satisfy: $\mathscr{M}_z = \pi^{-1}(\mathscr{N}_{\pi(z)})$ and $\rho_z = \rho_{\pi(z)} \circ \pi$. Moreover for $z \in \mathsf{S}_E$ and $s = \pi(z) \in \mathsf{S}_X$, the restriction $\pi_z : \mathscr{M}_z \to \mathscr{N}_s$ is a cone over the vector bundle ξ_z. More precisely, we have the following commutative diagram:

$$\begin{array}{ccc} \mathscr{M}_z & \xrightarrow{\pi_z} & \mathscr{N}_s \\ {\scriptstyle \Psi_z}\downarrow & & \downarrow{\scriptstyle \varphi_s} \\ c\xi_z & \longrightarrow & cL_s \end{array}$$

where $\xi_z \to L_s$ is a smooth vector bundle over L_s, the bottom horizontal arrow is the cone over $\xi_z \to L_s$ and Ψ_z, φ_s are trivialisation maps.

If X is a conical pseudomanifold, the isomorphism class of a conical vector bundle over a conical stratification $(X, \mathsf{S}_X, \mathscr{C}_X)$ will be called again a conical vector bundle over X.

We are interested in conical vector bundles because they allow us to introduce the right notion of tubular neighborhood in the class of conical manifolds.

Let L be a compact manifold and cL the cone over L. For $N \in \mathbb{N}$ large enough, we can find an embedding $j_L : L \to S^{N-1}$ where $S^{N-1} \subset \mathbb{R}^N$ denotes the unit sphere. Let $\mathscr{V}_L \to L$ be the normal bundle of this embedding. We let $c\mathscr{V}_L = \mathscr{V}_L \times [0, +\infty[/\mathscr{V}_L \times \{0\}$ be the cone over $\mathscr{V}_L$; it is a conical vector bundle over cL. Notice that the cone cS^{N-1} over S^{N-1} is isomorphic to $\mathbb{R}^N_\bullet$ which is $\mathbb{R}^N$ with 0 as a singular point. We will say that cL is *embedded* in $\mathbb{R}^N_\bullet$ and that $c\mathscr{V}_L$ is the *tubular neighborhood of this embedding*.

Now, let $X = (X, \mathsf{S}, \mathscr{C})$ be a compact conical stratification. Let

$$\mathscr{C} = \{(\mathscr{N}_s, \rho_s, L_s), s \in \mathsf{S}\}$$

be the set of control data, where $\mathscr{N}_s$ is a cone over L_s and choose a trivialisation map $\varphi_s : \mathscr{N}_s \to cL_s$ for each singular point s. For $N \in \mathbb{N}$ large enough, one can find an embedding $j : X^\circ = X \backslash \mathsf{S} \to \mathbb{R}^N$ such that:

- For any $s \in \mathsf{S}$, $j \circ \varphi_c^{-1}(L_s \times \{\lambda\})$ lies on a sphere $S(O_s, \lambda)$ centered on O_s and of radius λ for $\lambda \in]0, 1[$.

- The open balls $B(O_s, 1)$ centered on O_s and of radius 1 are disjoint.

- For each singular point s there is an embedding $j_{L_s} : L_s \to S(O_c, 1) \subset \mathbb{R}^N$ such that

$$\psi_s \circ j \circ \varphi_c^{-1}|_{L\times]0,1[} = j_{L_s} \times \mathrm{Id},$$

where $\psi_s : B(O_s, 1)\backslash\{O_s\} \to S(O_s, 1) \times]0, 1[$ is the canonical diffeomorphism.

Let $\mathscr{V}_{L_s} \to L_s$ be the normal bundle of the embedding j_{L_s} and $\mathscr{V} \to X^\circ$ be the normal bundle of the embedding j. Then we can identify the restriction of $\mathscr{V}$ to $\mathscr{N}_s|_{]0,1[} := \{z \in \mathscr{N}_s \,|\, 0 < \rho_s(z) < 1\}$ with $\mathscr{V}_{L_s} \times]0, 1[$. Let $c\mathscr{V}_{L_s} = \mathscr{V}_{L_s} \times [0, 1[/_{\mathscr{V}_{L_s}\times\{0\}}$ be the cone over L_s. We define the conical manifold

$$\mathscr{W} = \mathscr{V} \bigcup_{s\in \mathsf{S}} c\mathscr{V}_{L_s}$$

by glueing with $T\varphi_s$ the restriction of $\mathscr{V}$ over $\mathscr{N}_s|_{]0,1[}$ with $c\mathscr{V}_{L_s}\backslash\{s\}$. The conical manifold $\mathscr{W}$ is a conical vector bundle over X. It follows that $\mathscr{W}$ is a sub-stratified pseudomanifold of $(\mathbb{R}^N)^{\mathsf{S}}$ which is $\mathbb{R}^N$ with $\{O_s\}_{s\in\mathsf{S}}$ as singular points. We will say that $\mathscr{W}$ is the *tubular neighborhood of the embedding of X in $(\mathbb{R}^N)^{\mathsf{S}}$*.

2. Lie groupoids and their Lie algebroids

We refer to [58], [16], [42] for the classical definitions and construction related to groupoids and their Lie algebroids.

2.1. Lie groupoids. Groupoids, and especially differentiable groupoids will play an important role in what follows, so we recall the basic definitions and results needed for this paper. Recall first that a *groupoid* is a small category in which every morphism is an isomorphism.

Let us make the notion of a groupoid more explicit. Thus, a groupoid $\mathscr{G}$ is a pair $(\mathscr{G}^{(0)}, \mathscr{G}^{(1)})$ of sets together with structural morphisms $u : \mathscr{G}^{(0)} \to \mathscr{G}^{(1)}$, $s, r : \mathscr{G}^{(1)} \to \mathscr{G}^{(0)}$, $\iota : \mathscr{G}^{(1)} \to \mathscr{G}^{(1)}$, and, especially, the multiplication μ which is defined for pairs $(g, h) \in \mathscr{G}^{(1)} \times \mathscr{G}^{(1)}$ such that $s(g) = r(h)$. Here, the set $\mathscr{G}^{(0)}$ denotes the set of objects (or units) of the groupoid, whereas the set $\mathscr{G}^{(1)}$ denotes the set of morphisms of $\mathscr{G}$. Each object of $\mathscr{G}$ can be identified with a morphism of $\mathscr{G}$, the identity morphism of that object, which leads to an injective map $u : \mathscr{G}^{(0)} \to \mathscr{G}$. Each morphism $g \in \mathscr{G}$ has a "source" and a "range." We shall denote by $s(g)$ the *source* of g and by $r(g)$ the *range* of g. The inverse of a morphism g is denoted by $g^{-1} = \iota(g)$. The structural maps satisfy the following properties:

(i) $r(gh) = r(g)$ and $s(gh) = s(h)$, for any pair g, h satisfying $s(g) = r(h)$.

(ii) $s\big(u(x)\big) = r\big(u(x)\big) = x$, $u\big(r(g)\big)g = g$, $gu\big(s(g)\big) = g$.

(iii) $r(g^{-1}) = s(g)$, $s(g^{-1}) = r(g)$, $gg^{-1} = u\big(r(g)\big)$, and $g^{-1}g = u\big(s(g)\big)$.

(iv) The partially defined multiplication μ is associative.

We shall need groupoids with smooth structures.

Definition 2.1. A *Lie groupoid* is a groupoid

$$\mathcal{G} = (\mathcal{G}^{(0)}, \mathcal{G}^{(1)}, s, r, \mu, u, \iota)$$

such that $\mathcal{G}^{(0)}$ and $\mathcal{G}^{(1)}$ are manifolds with corners, the structural maps s, r, μ, u, and ι are differentiable, the domain map s is a submersion and $\mathcal{G}_x := s^{-1}(x)$, $x \in M$, are all Hausdorff manifolds without corners.

The term "differentiable groupoid" was used in the past instead of "Lie groupoid", whereas "Lie groupoid" had a more restricted meaning [42]. The usage has changed however more recently, and our definition reflects this change.

An example of a Lie groupoid that will be used repeatedly below is that of *pair groupoid*, which we now define. Let M be a smooth manifold. We let $\mathcal{G}^{(0)} = M$, $\mathcal{G}^{(1)} = M \times M$, $s(x, y) = y$, $r(x, y) = x$, $(x, y)(y, z) = (x, z)$, and embedding $u(x) = (x, x)$. The inverse is $\iota(x, y) = (y, x)$.

The infinitesimal object associated to a Lie groupoid is its "Lie algebroid," which we define next.

Definition 2.2. A *Lie algebroid* A over a manifold M is a vector bundle $A \to M$, together with a Lie algebra structure on the space $\Gamma(A)$ of smooth sections of A and a bundle map $\varrho : A \to TM$ whose extension to sections of these bundles satisfies

(i) $\varrho([X, Y]) = [\varrho(X), \varrho(Y)]$, and

(ii) $[X, fY] = f[X, Y] + \big(\varrho(X)f\big)Y$,

for any smooth sections X and Y of A and any smooth function f on M.

The map ϱ is called the *anchor map* of A. Note that we allow the base M in the definition above to be a manifold with corners.

The Lie algebroid associated to a differentiable groupoid $\mathcal{G}$ is defined as follows [42]. The vertical tangent bundle (along the fibers of s) of a differentiable groupoid $\mathcal{G}$ is, as usual,

$$T_{\mathrm{vert}}\mathcal{G} = \ker s_* = \bigcup_{x \in M} T\mathcal{G}_x \subset T\mathcal{G}. \tag{2.1}$$

Then $A(\mathcal{G}) := T_{\mathrm{vert}}\mathcal{G}|_M$, the restriction of the s-vertical tangent bundle to the set of units, defines the vector bundle structure on $A(\mathcal{G})$.

We now construct the bracket defining the Lie algebra structure on $\Gamma\big(A(\mathcal{G})\big)$. The right translation by an arrow $g \in \mathcal{G}$ defines a diffeomorphism

$$R_g : \mathcal{G}_{r(g)} \ni g' \mapsto g'g \in \mathcal{G}_{d(g)}.$$

A vector field X on $\mathcal{G}$ is called *s-vertical* if $s_*\big(X(g)\big) = 0$ for all g. The s-vertical vector fields are precisely the vector fields on $\mathcal{G}$ that can be restricted to vector fields on the submanifolds $\mathcal{G}_x$. It makes sense then to consider right-invariant vector fields on $\mathcal{G}$. It is not difficult to see that the sections of $A(\mathcal{G})$ are in one-to-one correspondence with s-vertical, right-invariant vector fields on $\mathcal{G}$.

The Lie bracket $[X, Y]$ of two s-vertical, right-invariant vector fields X and Y is also s-vertical and right-invariant, and hence the Lie bracket induces a Lie algebra structure on the sections of $A(\mathscr{G})$. To define the action of the sections of $A(\mathscr{G})$ on functions on M, let us observe that the right invariance property makes sense also for functions on $\mathscr{G}$, and that $\mathscr{C}^\infty(M)$ may be identified with the subspace of smooth, right-invariant functions on $\mathscr{G}$. If X is a right-invariant vector field on $\mathscr{G}$ and f is a right-invariant function on $\mathscr{G}$, then $X(f)$ will still be a right invariant function. This identifies the action of $\Gamma\big(A(\mathscr{G})\big)$ on $\mathscr{C}^\infty(M)$.

2.2. Pull back groupoids. Let $G \rightrightarrows M$ be a groupoid with source s and range r. If $f : N \to M$ is a surjective map, the *pull back* groupoid ${}^*f^*(G) \rightrightarrows N$ of G by f is by definition the set

$$ {}^*f^*(G) := \{(x, \gamma, y) \in N \times G \times N \mid r(\gamma) = f(x), s(\gamma) = f(y)\} $$

with the structural morphisms given by

(1) the unit map $x \mapsto \big(x, f(x), x\big)$,

(2) the source map $(x, \gamma, y) \mapsto y$ and range map $(x, \gamma, y) \mapsto x$,

(3) the product $(x, \gamma, y)(y, \eta, z) = (x, \gamma\eta, z)$ and inverse $(x, \gamma, y)^{-1} = (y, \gamma^{-1}, x)$.

The results of [50] apply to show that the groupoids G and ${}^*f^*(G)$ are Morita equivalent.

Let us assume for the rest of this subsection that G is a smooth groupoid and that f is a surjective submersion, then ${}^*f^*(G)$ is also a Lie groupoid. Let $\big(\mathscr{A}(G), q, [\,,]\big)$ be the Lie algebroid of G (which is defined since G is smooth). Recall that $q : \mathscr{A}(G) \to TM$ is the anchor map. Let $\big(\mathscr{A}\big({}^*f^*(G)\big), p, [\,,]\big)$ be the Lie algebroid of ${}^*f^*(G)$ and $Tf : TN \to TM$ be the differential of f. Then we claim that there exists an isomorphism

$$ \mathscr{A}\big({}^*f^*(G)\big) \simeq \{(V, U) \in TN \times \mathscr{A}(G) \mid Tf(V) = q(U) \in TM\} $$

under which the anchor map $p : \mathscr{A}\big({}^*f^*(G)\big) \to TN$ identifies with the projection $TN \times \mathscr{A}(G) \to TN$. In particular, if $(U, V) \in \mathscr{A}\big({}^*f^*(G)\big)$ with $U \in T_xN$ and $V \in \mathscr{A}_y(G)$, then $y = f(x)$.

2.3. Quasi-graphoid and almost injective Lie algebroid. Our Lie groupoids arise mostly as Lie groupoids with a given Lie algebroid. This is because often in analysis, one is given the set of derivations (differential operators), which forms a Lie algebra under the commutator. The groupoids are then used to "quantize" the given Lie algebra of vector fields to algebra of pseudodifferential operators [1], [45], [48], [57]. This has motivated several works on the integration of Lie algebroids [25], [26], [56]. We recall here some useful results of the first named author [26] on the integration of some Lie algebroids. See also [25], [42], [56].

Proposition 2.3. *Let $G \overset{s}{\underset{r}{\rightrightarrows}} M$ be a Lie groupoid over the manifold M. Let us denote by s its domain map, by r its range map, and by $u : M \to G$ its unit map. The two following assertions are equivalent*:

(1) *If* $v : V \to G$ *is a local section of* s *then* $r \circ v = 1_V$ *if, and only if,* $v = u|_V$.

(2) *If* N *is a manifold,* f *and* g *are two smooth maps from* N *to* G *such that*

(i) $s \circ f = s \circ g$ *and* $r \circ f = r \circ g$,

(ii) *one of the maps* $s \circ f$ *and* $r \circ f$ *is a submersion,*

then $f = g$.

Definition 2.4. A Lie groupoid that satisfies one of the two equivalent properties of Proposition 2.3 will be called a *quasi-graphoid.*

Suppose that $G \rightrightarrows M$ is a quasi-graphoid and denote by $\mathscr{A}G = (p : \mathscr{A}G \to TM, [\,,]_{\mathscr{A}})$ its Lie algebroid. A direct consequence of the previous definition is that the anchor p of $\mathscr{A}G$ is injective when restricted to a dense open subset of the base space M. In other words the anchor p induces an injective morphism $\tilde{p}$ from the set of smooth local sections of $\mathscr{A}G$ onto the set of smooth local tangent vector fields over M. In this situation we say that the Lie algebroid $\mathscr{A}G$ is *almost injective*.

A less obvious remarkable property of a quasi-graphoid is that its s-connected component is determined by its infinitesimal structure. Precisely:

Proposition 2.5 ([26]). *Two s-connected quasi-graphoids having the same space of units are isomorphic if, and only if, their Lie algebroids are isomorphic.*

Note that we are not requiring the groupoids in the above proposition to be s-simply connected. The main result of [26] is the following:

Theorem 2.6. *Every almost injective Lie algebroid is integrable by an s-connected quasi-graphoid (uniquely by the above proposition).*

Finally, let $\mathscr{A}$ be a smooth vector bundle over a manifold M and $p : \mathscr{A} \to TM$ a morphism. We denote by $\tilde{p}$ the map induced by p from the set of smooth local sections of $\mathscr{A}$ to the set of smooth local vector fields on M. Notice that if $\tilde{p}$ is injective then $\mathscr{A}$ can be equipped with a Lie algebroid structure over M with anchor p if, and only if, the image of $\tilde{p}$ is stable under the Lie bracket.

Examples 2.7. *Regular foliation.* A smooth regular foliation $\mathscr{F}$ on a manifold M determines an integrable subbundle F of TM. Such a subbundle is an (almost) injective Lie algebroid over M. The holonomy groupoid of $\mathscr{F}$ is the s-connected quasi-graphoid which integrates F ([66]).

Tangent groupoid. One typical example of a quasi-graphoid is the tangent groupoid of A. Connes [22]. Let us denote by $A \sqcup B$ the *disjoint union* of the sets A and B. If M is a smooth manifold, the tangent groupoid of M is the disjoint union

$$\mathscr{G}_M^t = TM \times \{0\} \sqcup M \times M \times]0,1] \rightrightarrows M \times [0,1].$$

In order to equip $\mathscr{G}_M^t$ with a smooth structure, we choose a riemannian metric on M and we require that the map

$$V \subset TM \times [0,1] \to \mathscr{G}_M^t,$$

$$(x, V, t) \mapsto \begin{cases} (x, V, 0) & \text{if } t = 0, \\ \big(x, \exp_x(-tV), t\big) & \text{if } t \neq 0, \end{cases}$$

be a smooth diffeomorphism onto its image, where V is open in $TM \times [0,1]$ and contains $TM \times \{0\}$. The tangent groupoid of M is the s-connected quasi-graphoid which integrates the almost injective Lie algebroid:

$$p_M^t : \mathscr{A}\mathscr{G}_M^t = TM \times [0,1] \to T(M \times [0,1]) \simeq TM \times T[0,1],$$

$$(x, V, t) \mapsto (x, tV; t, 0).$$

2.4. Deformation of quasi-graphoids. In this paper, we will encounter deformation groupoids. The previous results give easy arguments to get sure that these deformation groupoids can be equipped with a smooth structure. For example, let $G_i \rightrightarrows M$, $i = 1, 2$, be two s-connected quasi-graphoids over the manifold M and let $\mathscr{A}G_i = (p_i : \mathscr{A}G_i \to TM, [\,,]_{\mathscr{A}_i})$ be the corresponding Lie algebroid. Suppose that:

- The bundles $\mathscr{A}G_1$ and $\mathscr{A}G_2$ are isomorphic.

- There is a morphism $p : \mathscr{A} := \mathscr{A}G_1 \times [0,1] \to TM \times T([0,1])$ of the form:

$$p(V, 0) = \big(p_1(V); 0, 0\big) \quad \text{and} \quad p(V, t) = \big(p_2 \circ \Phi(V, t); t, 0\big) \quad \text{if } t \neq 0,$$

where $\Phi : \mathscr{A}G_1 \times]0,1] \to \mathscr{A}G_2 \times]0,1]$ is an isomorphism of bundles over $M \times]0,1]$. Moreover the image of $\tilde{p}$ is stable under the Lie bracket.

In this situation, $\mathscr{A}$ is an almost injective Lie algebroid that can be integrated by the groupoid $H = G_1 \times \{0\} \cup G_2 \times]0,1] \rightrightarrows M \times [0,1]$. In particular, there is a smooth structure on H compatible with the smooth structure on G_1 and G_2.

3. A non-commutative tangent space for conical pseudomanifolds

In order to obtain an Atiyah-Singer type topological index theorem for our conical pseudomanifold X, we introduce in this chapter a suitable notion of tangent space to X and a suitable normal space to an embedding of X in $\mathbb{R}^{N+1}$ that sends the singular point to 0 and X° to $\{x_1 > 0\}$.

3.1. The S-tangent space and the tangent groupoid of a conical space. We recall here a construction from [27] that associates to a conical pseudomanifold X a groupoid $T^{\mathsf{S}}X$ that is a replacement of the notion of tangent space of X (for the purpose of studying K-theory) in the sense the C^*-algebras $C^*(T^{\mathsf{S}}X)$ and $C(X)$ are K-dual [27].

Let $(X, \mathsf{S}, \mathscr{C})$ be a conical pseudomanifold. Without loss of generality, we can assume that X has only one singular point. Thus $\mathsf{S} = \{c\}$ is a single point and $\mathscr{C} = \{(\mathscr{N}, \rho, L)\}$, where $\mathcal{N} \simeq cL$ is a cone over L and ρ is the defining function of the cone. We set $\rho = +\infty$ outside $\mathcal{N}$. We let $X^\circ = X \backslash \{c\}$. Recall that X° is a smooth manifold. We denote by O_X the open set $O_X = \{z \in X^\circ \mid \rho(z) < 1\}$.

At the level of sets, the S-*tangent space* of X is the groupoid:

$$T^{\mathsf{S}}X := TX^\circ|_{X^\circ \backslash O_X} \sqcup O_X \times O_X \rightrightarrows X^\circ.$$

Here, the groupoid $TX^\circ|_{X^\circ \backslash O_X} \rightrightarrows X^\circ \backslash O_X$ is the usual tangent vector bundle TX° of X° restricted to the closed subset $X^\circ \backslash O_X = \{z \in X^\circ \mid \rho(z) \geqq 1\}$. The groupoid $O_X \times O_X \rightrightarrows O_X$ is the pair groupoid over O_X.

The *tangent groupoid* of X is, as in the regular case [22], a deformation of its "tangent space" to the pair groupoid over its units:

$$\mathcal{G}_X^t := T^{\mathsf{S}}X \times \{0\} \sqcup X^\circ \times X^\circ \times]0,1] \rightrightarrows X^\circ \times [0,1].$$

Here, the groupoid $X^\circ \times X^\circ \times]0,1] \rightrightarrows X^\circ \times]0,1]$ is the product of the pair groupoid on X° with the set $]0,1]$.

In order to equip $\mathcal{G}_X^t$, and so $T^{\mathsf{S}}X$, with a smooth structure we have to choose a *gluing function*. First choose a positive smooth map $\tau : \mathbb{R} \to \mathbb{R}$ such that $\tau([0,+\infty[) = [0,1]$, $\tau^{-1}(0) = [1,+\infty[$ and $\tau'(t) \neq 0$ for $t < 1$. We denote by $\tau_X : X \to \mathbb{R}$ the map which assigns $\tau(\rho(x))$ to $x \in X^\circ \cap \mathcal{N}$ and 0 elsewhere. Thus $\tau_X(X^\circ) = [0,1[$, τ_X restricted to $O_X = \{z \in \mathcal{N} \mid 0 < \rho(z) < 1\}$ is a submersion and $\tau_X^{-1}(0) = X^\circ \backslash O_X$.

Proposition 3.1 ([27]). *There is a unique structure of Lie groupoid on $\mathcal{G}_X^t$ such that its Lie algebroid is the bundle $TX^\circ \times [0,1]$ with anchor*

$$p : (x, V, t) \in TX^\circ \times [0,1] \mapsto \big(x, \big(t + \tau_X^2(x)\big) V; t, 0\big) \in TX^\circ \times T[0,1].$$

Let us notice that the map p is injective when restricted to $X^\circ \times]0,1]$, which is a dense open subset of $X^\circ \times [0,1]$. Thus there exists one, and only one, structure of (almost injective) Lie algebroid on $TX^\circ \times [0,1]$ with p as anchor since the family of local vector fields on X° induced by the image by p of local sections of $TX^\circ \times [0,1]$ is stable under the Lie bracket. We know from [26], [56] that such a Lie algebroid is integrable. Moreover, according to Theorem 2.6, there is a unique Lie groupoid which integrates this algebroid and restricts over $X^\circ \times]0,1]$ to $X^\circ \times X^\circ \times]0,1] \rightrightarrows X^\circ \times]0,1]$.

Let us give an alternative proof of the previous proposition.

Proof. Recall that the (classical) tangent groupoid of X° is

$$\mathcal{G}_{X^\circ}^t = TX^\circ \times \{0\} \sqcup X^\circ \times X^\circ \times]0,1] \rightrightarrows X^\circ \times [0,1]$$

and that its Lie algebroid is the bundle $TX^\circ \times [0,1]$ over $X^\circ \times [0,1]$ with anchor $(x, V, t) \in TX^\circ \times [0,1] \mapsto (x, tV, t, 0) \in TX^\circ \times T[0,1]$. Similary, one can equip the groupoid $H = TX^\circ \times \{(0,0)\} \sqcup X^\circ \times X^\circ \times [0,1]^2 \backslash \{(0,0)\}$ with a unique smooth structure such that its Lie algebroid is the bundle $TX^\circ \times [0,1]^2$ with anchor the map

$$p : \mathscr{A} = T\mathscr{N}_1 \times [0,1] \times [0,1] \to T\mathscr{N}_1 \times T([0,1]) \times T([0,1]),$$
$$(x, V, t, l) \mapsto \big(x, (t+l)V; t, 0; l, 0\big).$$

Let $\delta : H \to \mathbb{R}$ be the map which sends any $\gamma \in H$ with source $s(\gamma) = (y, t, l)$ and range $r(\gamma) = (x, t, l)$ to $\delta(\gamma) = l - \tau_X(x)\tau_X(y)$. One can check that δ is a smooth submersion, so $H_\delta := \delta^{-1}(0)$ is a submanifold of H. Moreover $H_\delta := \delta^{-1}(0)$ inherits from H a structure of Lie groupoid over $X^\circ \times [0,1]$ whose Lie algebroid is given by

$$TX^\circ \times [0,1] \to TX^\circ \times T([0,1]),$$
$$(x, V, t) \mapsto \big(x, \big(t + \tau_X^2(x)\big)V; t, 0\big).$$

The groupoid H_δ is (obviously isomorphic) to $\mathscr{G}_X^t$. $\square$

We now introduce the tangent groupoid of a stratified pseudomanifold.

Definition 3.2. The groupoid $\mathscr{G}_X^t$ equipped with the smooth structure associated with a gluing function τ as above is called a *tangent groupoid* of the stratified pseudomanifold $(X, \mathsf{S}, \mathscr{C})$. The corresponding S-tangent space is the groupoid $T^{\mathsf{S}}X \simeq \mathscr{G}_X^t|_{X^\circ \times \{0\}}$ equipped with the induced smooth structure.

Remark 3.3. We will need the following remarks. See [27] for a proof.

(i) If X has more than one singular point, we let, for any $s \in \mathsf{S}$,

$$O_s := \{z \in X^\circ \cap \mathscr{N}_s \,|\, \rho_s(z) < 1\},$$

and we define $O = \bigsqcup_{s \in \mathsf{S}} O_s$. The S-*tangent space to* X is then

$$T^{\mathsf{S}}X := TX^\circ|_{X^\circ \setminus O} \bigsqcup_{s \in \mathsf{S}} O_s \times O_s \rightrightarrows X^\circ,$$

with the analogous smooth structure. In this situation the Lie algebroid of $\mathscr{G}_X^t$ is defined as previously with $\tau_X : X \to \mathbb{R}$ being the map which assigns $\tau\big(\rho_s(z)\big)$ to $z \in X^\circ \cap \mathscr{N}_s$ and 0 elsewhere.

(ii) The orbit space of $T^{\mathsf{S}}X$ is topologically equivalent to X: there is a canonical isomorphism between the algebras $C(X)$ and $C(X/T^{\mathsf{S}}X)$.

(iii) The tangent groupoid and the S-tangent space depend on the gluing. Nevertheless the K-theory of the C^*-algebras $C^*(\mathscr{G}_X^t)$ and $C^*(T^{\mathsf{S}}X)$ do not.

(iv) The groupoid $T^{\mathsf{S}}X$ is a continuous field of amenable groupoids parametrized by X, thus $T^{\mathsf{S}}X$ is amenable as well. It follows that $\mathscr{G}_X^t$ is also amenable as a continuous field of amenable groupoids parametrised by $[0,1]$. Hence the reduced and maximal C^*-algebras of $T^{\mathsf{S}}X$ and of $\mathscr{G}_X^t$ are equal and they are nuclear.

Examples 3.4. Here are two basic examples.

(i) When X is a smooth manifold, that is $X_0 = \emptyset$ and $X^\circ = X$, the previous construction gives rise to the usual tangent groupoid

$$\mathscr{G}_X^t = TX \times \{0\} \sqcup X \times X \times]0,1] \rightrightarrows X \times [0,1].$$

Moreover, $T^{\mathsf{S}}X = TX \rightrightarrows X$ is the usual tangent space.

(ii) Let L be a manifold and consider the (trivial) cone $cL = L \times [0,+\infty[/L \times \{0\}$ over L. In this situation $X^\circ = L \times]0,+\infty[$, $O_X = L \times]0,1[$ and

$$T^{\mathsf{S}}X = T(L \times [1,+\infty[) \sqcup \underbrace{L \times]0,1[\times L \times]0,1[}_{\text{the pair groupoid}} \rightrightarrows L \times]0,+\infty[,$$

where $T(L \times [1,+\infty[)$ denotes the restriction to $L \times [1,+\infty[$ of the tangent space $T(L \times \mathbb{R})$. The general case is always locally of this form.

3.2. The deformation groupoid of a conical vector bundle. Let $(E, \mathsf{S}_E, \mathscr{C}_E)$ be a conical vector bundle over $(X, \mathsf{S}_X, \mathscr{C}_X)$ and denote by $\pi : E \to X$ the corresponding projection. From the definition, π restricts to a smooth vector bundle map $\pi^\circ : E^\circ \to X^\circ$. We let $\pi_{[0,1]} = \pi^\circ \times \mathrm{id} : E^\circ \times [0,1] \to X^\circ \times [0,1]$.

We consider the tangent groupoids $\mathscr{G}_X^t \rightrightarrows X^\circ$ for X and $\mathscr{G}_E^t \rightrightarrows E^\circ$ for E equipped with a smooth structure constructed using the same gluing function τ (in particular $\tau_X \circ \pi = \tau_E$). We denote by ${}^*\pi_{[0,1]}^*(\mathscr{G}_X^t) \rightrightarrows E^\circ \times [0,1]$ the pull back of $\mathscr{G}_X^t$ by $\pi_{[0,1]}$.

Our next goal is to associate to the conical vector bundle E a deformation groupoid $\mathscr{T}_E^t$ using ${}^*\pi_{[0,1]}^*(\mathscr{G}_X^t)$ to $\mathscr{G}_E^t$. More precisely, we define:

$$\mathscr{T}_E^t := \mathscr{G}_E^t \times \{0\} \sqcup {}^*\pi_{[0,1]}^*(\mathscr{G}_X^t) \times]0,1] \rightrightarrows E^\circ \times [0,1] \times [0,1].$$

In order to equip $\mathscr{T}_E^t$ with a smooth structure, we first choose a smooth projection $P : TE^\circ \to \mathrm{Ker}(T\pi)$.

A simple calculation shows that the Lie algebroid of ${}^*\pi_{[0,1]}^*(\mathscr{G}_X^t)$ is isomorphic to the bundle $TE^\circ \times [0,1]$ endowed with the almost injective anchor map

$$(x, V, t) \mapsto \big(x, P(x,V) + \big(t + \tau_E(x)^2\big)\big(V - P(x,V)\big); t, 0\big).$$

We consider the bundle $\mathscr{A} = TE^\circ \times [0,1] \times [0,1]$ over $E^\circ \times [0,1] \times [0,1]$ and the almost injective morphism:

$$p : \mathscr{A} = TE^\circ \times [0,1] \times [0,1] \to TX^\circ \times T[0,1] \times T[0,1],$$
$$(x, V, t, l) \mapsto \big(x, \big(t + \tau_E^2(x)\big)V + lP(x,V)\big).$$

The image of $\tilde{p}$ is stable under the Lie bracket, thus $\mathscr{A}$ is an almost injective Lie algebroid. Moreover, the restriction of $\mathscr{A}$ to $E^\circ \times [0,1] \times \{0\}$ is the Lie algebroid of $\mathscr{G}_E^t$ and its restriction to $E^\circ \times [0,1] \times]0,1]$ is isomorphic to the Lie algebroid of ${}^*\pi^*_{[0,1]}(\mathscr{G}^t_X) \times]0,1]$. Thus $\mathscr{A}$ can be integrated by $\mathscr{T}^t_E$. In particular, $\mathscr{T}^t_E$ is a smooth groupoid. In conclusion, the restriction of $\mathscr{T}^t_E$ to $E^\circ \times \{0\} \times [0,1]$ leads to a Lie groupoid:

$$\mathscr{H}_E = T^{\mathsf{S}}E \times \{0\} \sqcup {}^*\pi^*(T^{\mathsf{S}}X) \times]0,1] \rightrightarrows E^\circ \times [0,1],$$

called a *Thom groupoid* associated to the conical vector bundle E over X.

The following example explains what these constructions become if there are no singularities.

Example 3.5. Suppose that $p : E \to M$ is a smooth vector bundle over the smooth manifold M. Then $\mathsf{S}_E = \mathsf{S}_M = \emptyset$, $\mathscr{G}^t_E = TE \times \{0\} \sqcup E \times E \times]0,1] \rightrightarrows E \times [0,1]$ and $\mathscr{G}^t_M = TM \times \{0\} \sqcup M \times M \times]0,1] \rightrightarrows M \times [0,1]$ are the usual tangent groupoids. In these examples associated to a smooth vector bundle, τ_E is the zero map. The groupoid $\mathscr{T}^t_E$ will then be given by

$$\mathscr{T}^t_E = TE \times \{0\} \times 0\} \sqcup {}^*p^*(TM) \times \{0\} \times]0,1] \sqcup E \times E \times]0,1] \times [0,1] \rightrightarrows E \times [0,1] \times [0,1]$$

and is smooth. Similarly, the Thom groupoid will be given by:

$$\mathscr{H}_E := TE \times \{0\} \sqcup {}^*p^*(TM) \times]0,1] \rightrightarrows E \times [0,1].$$

We now return to the general case of a conical vector bundle.

Remark 3.6. The groupoids $\mathscr{T}_E$ and $\mathscr{H}_E$ are continuous fields of amenable groupoids parametrized by $[0,1]$. Thus they are amenable, their reduced and maximal C^*-algebras are equal, and are nuclear.

4. The analytical index

Let X be a conical pseudomanifold, and let

$$\mathscr{G}^t_X = X^\circ \times X^\circ \times]0,1] \sqcup T^{\mathsf{S}}X \times \{0\} \rightrightarrows X^\circ \times [0,1]$$

be the tangent groupoid (unique up to isomorphism) for X for a given gluing function. Also, let $T^{\mathsf{S}}X \rightrightarrows X^\circ$ be the corresponding S-tangent space.

Since the groupoid $\mathscr{G}^t_X$ is a deformation groupoid of amenable groupoids, it defines a KK-element [27], [33]. More precisely, let

$$e_1 : C^*(\mathscr{G}^t_X) \to C^*(\mathscr{G}^t_X|_{X^\circ \times \{1\}}) = \mathscr{K}\big(L^2(X^\circ)\big)$$

be the evaluation at 1 and let $[e_1] \in KK\big(C^*(\mathscr{G}^t_X), \mathscr{K}\big(L^2(X^\circ)\big)\big)$ the element defined by e_1 in Kasparov's bivariant K-theory. Similarly, the evaluation at 0 defines a morphism $e_0 : C^*(\mathscr{G}^t_X) \to C^*(\mathscr{G}^t_X|_{X^\circ \times \{0\}}) = C^*(T^{\mathsf{S}}X)$ and then an element

$$[e_0] \in KK\big(C^*(\mathscr{G}^t_X), C^*(T^{\mathsf{S}}X)\big).$$

The kernel of e_0 is contractible and so e_0 is KK-invertible. We let:

$$\tilde{\partial} = [e_0]^{-1} \otimes [e_1] \in KK\big(C^*(T^{\mathsf{S}}X), \mathscr{K}\big),$$

be the Kasparov product over $C^*(\mathscr{G}_X^t)$ of $[e_1]$ and the K-inverse of $[e_0]$. Take b to be a generator of $KK(\mathscr{K}, \mathbb{C}) \simeq \mathbb{Z}$. We set $\partial = \tilde{\partial} \otimes b$. The element ∂ belongs to $KK\big(C^*(T^{\mathsf{S}}X), \mathbb{C}\big)$.

Definition 4.1. The map $(e_0)_* : K_0\big(C^*(\mathscr{G}_X^t)\big) \to K_0\big(C^*(T^{\mathsf{S}}X)\big)$ is an isomorphism and we define the *analytical index map* by

$$\operatorname{Ind}_a^X := (e_1)_* \circ (e_0)_*^{-1} : K_0\big(C^*(T^{\mathsf{S}}X)\big) \to K_0(\mathscr{K}) \simeq \mathbb{Z}, \tag{4.1}$$

or in other words, as the map defined by the Kasparov product with ∂.

Remarks 4.2. (1) Notice that in the case of a smooth manifold with the usual definition of tangent space and tangent groupoid, this definition leads to the classical definition of the analytical index map ([22], II.5).

(2) One can associate to a Lie groupoid a different analytical map. More precisely, when $G \rightrightarrows M$ is smooth, one can consider the adiabatic groupoid which is a deformation groupoid of G on its Lie algebroid $\mathscr{A}G$ [57]:

$$\mathscr{G}^t := \mathscr{A}G \times \{0\} \sqcup G \times]0,1] \rightrightarrows M \times [0,1].$$

Under some asumption $\mathscr{G}^t$ defines a KK-element in $KK\big(C^*(\mathscr{A}G), C^*(G)\big)$ and thus a map from $K_0\big(C^*(\mathscr{A}G)\big)$ to $K_0\big(C^*(G)\big)$.

Now, let X be a conical pseudomanifold and S its set of singular points. Choose a singular point $s \in \mathsf{S}$. Let us denote $X_{s,+} := X^\circ \backslash O_s$. The S-tangent space of X is then

$$T^{\mathsf{S}}X = O_s \times O_s \sqcup T^{\mathsf{S}}X_{s,+} \rightrightarrows X^\circ$$

where $O_s \times O_s \rightrightarrows O_s$ is the pair groupoid and $T^{\mathsf{S}}X_{s,+} := T^{\mathsf{S}}X|_{X_{s,+}}$. Then we have the following exact sequence of C^*-algebras:

$$0 \to \underbrace{C^*(O_s \times O_s)}_{=\mathscr{K}(L^2(O_s))} \xrightarrow{i} C^*(T^{\mathsf{S}}X) \xrightarrow{r_+} C^*(T^{\mathsf{S}}X_{s,+}) \to 0, \tag{4.2}$$

where i is the inclusion morphism and r_+ comes from the restriction of functions.

Proposition 4.3. *The exact sequence* (4.2) *induces the short exact sequence*

$$0 \longrightarrow K_0(\mathscr{K}) \simeq \mathbb{Z} \xrightarrow{i_*} K_0\big(C^*(T^{\mathsf{S}}X)\big) \xrightarrow{(r_+)_*} K_0\big(C^*(T^{\mathsf{S}}X_{s,+})\big) \longrightarrow 0.$$

Moreover $\operatorname{Ind}_a^X \circ i_* = \operatorname{Id}_{\mathbb{Z}}$, *thus*

$$\big(\operatorname{Ind}_a^X, (r_+)_*\big) : K_0\big(C^*(T^{\mathsf{S}}X)\big) \to \mathbb{Z} \oplus K_0\big(C^*(T^{\mathsf{S}}X_{s,+})\big)$$

is an isomorphism.

Proof. In order to prove the first statement, let us first consider the six terms exact sequence associated to the exact sequence of C^*-algebras of (4.2). Then recall that $K_1(\mathscr{K}) = 0$. It remains to show that i_* is injective. This point is a consequence of the second statement which is proved here'after. Let $\mathscr{G}_X^t \rightrightarrows X^\circ \times [0,1]$ be a tangent groupoid for X. Its restriction $\mathscr{G}_X^t|_{O_s \times [0,1]}$ to $O_s \times [0,1]$ is isomorphic to the groupoid $(O_s \times O_s) \times [0,1] \rightrightarrows O_s \times [0,1]$, the pair groupoid of O_s parametrized by $[0,1]$. The inclusion of $C_0(\mathscr{G}_X^t|_{O_s \times [0,1]})$ in $C_0(\mathscr{G}_X^t)$ induces a morphism of C^*-algebras

$$i^t : C^*(\mathscr{G}_X^t|_{O_s \times [0,1]}) \simeq \mathscr{K}\big(L^2(O_s)\big) \otimes C([0,1]) \to C^*(\mathscr{G}_X^t).$$

Moreover, we have the following commutative diagram of C^*-algebra morphisms:

$$\begin{tikzcd}
C^*(T^{\mathsf{S}}X) & C^*(\mathscr{G}_X^t) \arrow[l, "e_0"'] \arrow[r, "e_1"] & \mathscr{K}\big(L^2(X^\circ)\big) \\
\mathscr{K}\big(L^2(O_s)\big) \arrow[u, "i"] & \mathscr{K}\big(L^2(O_s)\big) \otimes C([0,1]) \arrow[l, "ev_0"] \arrow[r, "ev_1"'] \arrow[u, "i^t"'] & \mathscr{K}\big(L^2(O_s)\big) \arrow[u, "i_{\mathscr{K}}"']
\end{tikzcd}$$

where $i_{\mathscr{K}}$ is the isomorphism induced by the inclusion of the pair groupoid of O_s in the pair groupoid of X°, and ev_0, ev_1 are the evaluation maps at 0 and 1. The KK-element $[ev_0]$ is invertible and

$$[ev_0]^{-1} \otimes [ev_1] = 1 \in KK\big(\mathscr{K}\big(L^2(O_s)\big), \mathscr{K}\big(L^2(O_s)\big)\big).$$

Moreover $\cdot \otimes [i_{\mathscr{K}}]$ induces an isomorphism from $KK\big(\mathbb{C}, \mathscr{K}\big(L^2(O_s)\big)\big) \simeq \mathbb{Z}$ onto $KK\big(\mathbb{C}, \mathscr{K}\big(L^2(X^\circ)\big)\big) \simeq \mathbb{Z}$. Thus $[i] \otimes \tilde{\partial} = [i_{\mathscr{K}}]$, which proves that $\mathrm{Ind}_a^X \circ i_* = \mathrm{Id}_{\mathbb{Z}}$ and ensures that i_* is injective. $\square$

5. The inverse Thom map

Let $(E, \mathsf{S}_E, \mathscr{C}_E)$ be a conical vector bundle over $(X, \mathsf{S}_X, \mathscr{C}_X)$ and $\pi : E \to X$ the corresponding projection. We let

$$\mathscr{H}_E := T^{\mathsf{S}}E \times \{0\} \sqcup {}^*\pi^*(T^{\mathsf{S}}X) \times]0,1] \rightrightarrows E^\circ \times [0,1] \tag{5.1}$$

be the Thom groupoid of E, as before. The C^*-algebra of $\mathscr{H}_E$ is nuclear as well as the C^*-algebra of $T^{\mathsf{S}}E$. Thus $\mathscr{H}_E$ defines a KK-element:

$$\partial_{\mathscr{H}_E} := [\varepsilon_0]^{-1} \otimes [\varepsilon_1] \in KK\big(C^*(T^{\mathsf{S}}E), C^*(T^{\mathsf{S}}X)\big), \tag{5.2}$$

where $\varepsilon_1 : C^*(\mathscr{H}_E) \to C^*(\mathscr{H}_E|_{E^\circ \times \{1\}}) = C^*\big({}^*\pi^*(T^{\mathsf{S}}X)\big)$ is the evaluation map at 1 and $\varepsilon_0 : C^*(\mathscr{H}_E) \to C^*(\mathscr{H}_E|_{E^\circ \times \{0\}}) = C^*(T^{\mathsf{S}}E)$, the evaluation map at 0 is K-invertible.

Definition 5.1. The element $\partial_{\mathscr{H}_E} \in KK\big(C^*(T^{\mathsf{S}}E), C^*(T^{\mathsf{S}}X)\big)$ defined by equation (5.2) will be called the *inverse Thom element*.

Definition-Proposition 5.2. *Let $\mathscr{M}$ be the isomorphism induced by the Morita equivalence between $T^{\mathsf{S}}X$ and ${}^*\pi^*(T^{\mathsf{S}}X)$ and let $\cdot \otimes \partial_{\mathscr{H}_E}$ be the right Kasparov product by $\partial_{\mathscr{H}_E}$ over $C^*(T^{\mathsf{S}}E)$. Then the following diagram is commutative*:

$$\begin{array}{ccc}
K\big(C^*(T^{\mathsf{S}}E)\big) & \xrightarrow{\mathrm{Ind}_a^E} & \mathbb{Z} \\
{\scriptstyle \cdot\otimes\partial_{\mathscr{H}_E}}\downarrow & & \uparrow{\scriptstyle \mathrm{Ind}_a^X} \\
K\big(C^*\big({}^*\pi^*(T^{\mathsf{S}}X)\big)\big) & \underset{\mathscr{M}}{\xrightarrow{\simeq}} & K\big(C^*(T^{\mathsf{S}}X)\big).
\end{array}$$

The map $T_{\mathrm{inv}} := \mathscr{M} \circ (\cdot \otimes \partial_{\mathscr{H}_E})$ *is called the* inverse Thom map.

Proof. First consider the deformation groupoid $\mathscr{T}_E^t$:

$$\mathscr{T}_E^t := \mathscr{G}_E^t \times \{0\} \sqcup {}^*\pi_{[0,1]}^*(\mathscr{G}_X^t) \times]0,1] \rightrightarrows E^\circ \times [0,1] \times [0,1].$$

One can easily see that

$$\begin{aligned}
\mathscr{T}_E^t &= T^{\mathsf{S}}E \times \{0\} \times \{0\} \sqcup {}^*\pi^*(T^{\mathsf{S}}X) \times \{0\} \times]0,1] \sqcup E^\circ \times E^\circ \times]0,1] \times [0,1] \\
&\simeq \mathscr{H}_E \times \{0\} \sqcup (E^\circ \times E^\circ \times [0,1]) \times]0,1].
\end{aligned}$$

The groupoid $\mathscr{T}_E^t$ is equipped with a smooth structure compatible with the smooth structures of $\mathscr{G}_E^t \times \{0\}$, ${}^*\pi_{[0,1]}^*(\mathscr{G}_X^t) \times]0,1]$ as well as with the smooth structures of $\mathscr{H}_E$ and $(E^\circ \times E^\circ \times [0,1]) \times]0,1]$.

We therefore have the following commutative diagram of evaluation morphisms of C^*-algebras of groupoids:

$$\begin{array}{ccccc}
C^*(E^\circ \times E^\circ) & \xleftarrow{ev_0} & C^*(E^\circ \times E^\circ \times [0,1]) & \xrightarrow{ev_1} & C^*(E^\circ \times E^\circ) \\
{\scriptstyle e_1^E}\uparrow & & \uparrow{\scriptstyle q_{1,\cdot}} & & \uparrow{\scriptstyle {}^*\pi^* e_1^X} \\
C^*(\mathscr{G}_E^t) & \xleftarrow{q_{\cdot,0}} & C^*(\mathscr{T}_E^t) & \xrightarrow{q_{\cdot,1}} & C^*\big({}^*\pi_{[0,1]}^*(\mathscr{G}_X^t)\big) \\
{\scriptstyle e_0^E}\downarrow & & \downarrow{\scriptstyle q_{0,\cdot}} & & \downarrow{\scriptstyle {}^*\pi^* e_0^X} \\
C^*(T^{\mathsf{S}}E) & \underset{\varepsilon_0}{\longleftarrow} & C^*(\mathscr{H}_E) & \underset{\varepsilon_1}{\longrightarrow} & C^*\big({}^*\pi^*(T^{\mathsf{S}}X)\big).
\end{array}$$

In this diagram, the KK-elements $[e_0^E]$, $[{}^*\pi^* e_0^X]$, $[q_{\cdot,0}]$, $[\varepsilon_0]$, $[ev_1]$ and $[ev_0]$ are invertible. Let $\mathscr{M} : K\big(C^*\big({}^*\pi^*(T^{\mathsf{S}}X)\big) \to K\big(C^*(T^{\mathsf{S}}X)\big)\big)$ be the isomorphism induced by the Morita equivalence between ${}^*\pi^*(T^{\mathsf{S}}X)$ and $T^{\mathsf{S}}X$. Also, let x belong to

$$K\big(C^*\big({}^*\pi^*(T^{\mathsf{S}}X)\big)\big) = KK\big(\mathbb{C}, {}^*\pi^*(T^{\mathsf{S}}X)\big).$$

Then one can easily check the equality

$$\mathscr{M}(x) \otimes \tilde{\partial} = x \otimes [{}^*\pi^* e_0^X]^{-1} \otimes [{}^*\pi^* e_1^X].$$

Of course $[ev_0]^{-1} \otimes [ev_1] = 1 \in KK\big(C^*(E^\circ \times E^\circ), C^*(E^\circ \times E^\circ)\big)$. Thus the previous diagram implies that for any $x \in K\big(C^*(T^{\mathsf{S}}E)\big) = KK\big(\mathbb{C}, C^*(T^{\mathsf{S}}E)\big)$ we have:

$$\begin{aligned}\mathrm{Ind}_a^X \circ T_{\mathrm{inv}}(x) &= x \otimes [\varepsilon_0]^{-1} \otimes [\varepsilon_1] \otimes [{}^*\pi^* e_0^X]^{-1} \otimes [{}^*\pi^* e_1^X] \otimes b \\ &= x \otimes [e_0^E]^{-1} \otimes [e^{E_1}] \otimes [ev_0]^{-1} \otimes [ev_1] \otimes b \\ &= \mathrm{Ind}_a^E(x). \quad \square\end{aligned}$$

6. Index theorem

In this section, we state and prove our main theorem, namely, a topological index theorem for conical pseudomanifolds in the setting of groupoids. We begin with an account of the classical Atiyah-Singer topological index theorem in our groupoid setting.

6.1. A variant of the proof of Atiyah-Singer index theorem for compact manifolds using groupoids. Let $\mathscr{V}$ be the normal bundle of an embedding of a smooth manifold M in some euclidean space. In this subsection, we shall first justify the terminology of "inverse Thom map" we introduced for the map T_{inv} of Proposition 5.2 by showing that it coincides with the inverse of the classical Thom isomorphism when $E = T\mathscr{V}$ and $X = TM$.

In fact, we will define the Thom isomorphism when X is a locally compact space and $E = N \otimes \mathbb{C}$ is the complexification of a real vector bundle $N \to X$. As a consequence, we will derive a simple proof of the Atiyah-Singer Index Theorem for closed smooth manifolds. Our approach has the advantage that it extends to the singular setting.

Let us recall some classical facts [2], [4]. If $p : E \to X$ is a complex vector bundle over a locally compact space X, one can define a Thom map

$$i_! : K^0(X) \to K^0(E), \tag{6.1}$$

which turns to be an isomorphism. This Thom map is defined as follows. Let $x \in K^0(X)$ be represented by $[\xi_0; \xi_1; \alpha]$ where ξ_0, ξ_1 are complex vector bundles over X and $\alpha : \xi_0 \to \xi_1$ is an isomorphism outside a compact subset of X. With no loss of generality, one can assume that ξ_0, ξ_1 are hermitian and that α is unitary outside a compact subset of X.

Let us consider next the endomorphism of the vector bundle $p^*(\Lambda E) \to E$ given by

$$(C\omega)(v) = C(v)\omega(v) = \frac{1}{\sqrt{1 + \|v\|^2}}\big(v \wedge \omega(v) - v^* \llcorner \omega(v)\big).$$

The endomorphism $C\omega$ is selfadjoint, of degree 1 with respect to the $\mathbb{Z}_2$-grading $\Lambda_0 = \Lambda^{\mathrm{even}}E$, $\Lambda_1 = \Lambda^{\mathrm{odd}}E$ of the space of exterior forms. Moreover, we have that $(C\omega)^2 \to 1$ as ω approaches infinity in the fibers of E. Then, as we shall see in the next proposition, the Thom morphism $i_!$ of equation (6.1) can be expressed, in terms of the Kasparov products, as

$$i_!(x) := \left[\xi_0 \otimes \Lambda_0 \oplus \xi_1 \otimes \Lambda_1; \xi_0 \otimes \Lambda_1 \oplus \xi_1 \otimes \Lambda_0; \theta = \begin{pmatrix} N(1 \otimes C) & M(\alpha^* \otimes 1) \\ M(\alpha \otimes 1) & -N(1 \otimes C) \end{pmatrix}\right]$$

where M and N are the multiplication operators by the functions $M(v) = \dfrac{1}{\|v\|^2 + 1}$ and $N = 1 - M$, respectively.

Proposition 6.1. *Let $p : E \to X$ be a complex vector bundle over a locally compact base space X and $i_! : K^0(X) \to K^0(E)$ the corresponding Thom map. Denote by T the Kasparov element*

$$T := \big(C_0\big(E, p^*(\Lambda E)\big), \rho, C\big) \in KK\big(C_0(X), C_0(E)\big)$$

where ρ is multiplication by functions. Then $i_!(x) = x \otimes T$ for any $x \in K^0(X)$.

Proof. The isomorphism $K^0(X) \simeq KK\big(\mathbb{C}, C_0(X)\big)$ is such that to the triple $[\xi_0; \xi_1; \alpha]$ there corresponds to the Kasparov module:

$$x = \big(C_0(X, \xi), 1, \tilde{\alpha}\big), \quad \xi = \xi_0 \oplus \xi_1 \quad \text{and} \quad \tilde{\alpha} = \begin{pmatrix} 0 & \alpha^* \\ \alpha & 0 \end{pmatrix}.$$

Similarly, $i_!(x)$ corresponds to $(\mathscr{E}, \tilde{\theta})$ where

$$\mathscr{E} = C_0(X, \xi) \underset{\rho}{\otimes} C_0\big(E, p^*(\Lambda E)\big) \simeq C_0\big(E, p^*(\xi \otimes \Lambda E)\big) \quad \text{and} \quad \tilde{\theta} = \begin{pmatrix} 0 & \theta^* \\ \theta & 0 \end{pmatrix} \in \mathscr{L}(\mathscr{E}).$$

We next use the language of [13], [65], where the notion of "connection" in the framework of Kasparov's theory was defined. It is easy to check that $M(\tilde{\alpha} \hat{\otimes} 1)$ is a 0-connection on $\mathscr{E}$ and $N(1 \hat{\otimes} C)$ is a C-connection on $\mathscr{E}$ (the symbol $\hat{\otimes}$ denotes the graded tensor product), which yields that

$$\tilde{\theta} = M(\tilde{\alpha} \hat{\otimes} 1) + N(1 \hat{\otimes} C)$$

is a C-connection on $\mathscr{E}$. Moreover, for any $f \in C_0(X)$, we have

$$f[\tilde{\alpha} \hat{\otimes} 1, \tilde{\theta}] f^* = 2M|f|^2 \tilde{\alpha}^2 \hat{\otimes} 1 \geqq 0,$$

which proves that $(\mathscr{E}, \tilde{\theta})$ represents the Kasparov product of x and T. □

It is known that T is invertible in KK-theory ([36], paragraph 5, theorem 8). We now give a description of its inverse via a deformation groupoid when the bundle E is the complexification of a real euclidean bundle N. Hence let us assume that $E = N \otimes \mathbb{C}$ or, up to a $\mathbb{C}$-linear vector bundle isomorphism, let us assume that the bundle E is the Withney sum $N \oplus N$ of two copies of some real euclidean vector bundle $p_N : N \to X$ with the complex structure given by $J(v, w) = (-w, v)$, $(v, w) \in N \oplus N$. We endow the complex bundle E with the induced hermitian structure. We then define the *Thom groupoid* as follows:

$$\mathscr{T}_N := E \times [0,1] \rightrightarrows N \times [0,1]$$

with structural morphism given by

$$r(v,w,0) = s(v,w,0) = (v,0),$$
$$r(v,w,t) = (v,t),$$
$$s(v,w,t) = (w,t), \quad t > 0,$$
$$(v,w,0)\cdot(v,w',0) = (v,w+w',0) \quad \text{and}$$
$$(v,w,t)\cdot(w,u,t) = (v,u,t), \quad t > 0.$$

Thus, for $t = 0$, the groupoid structure of E corresponds to the vector bundle structure given by the first projection $E = N \oplus N \to N$ while for $t > 0$ the groupoid structure of E corresponds to the pair groupoid structure in each fiber $E_x = N_x \times N_x$.

The topology of $\mathscr{I}_N$ is inherited from the usual tangent groupoid topology, in particular $\mathscr{I}_N$ is a Hausdorff topological groupoid that can be viewed as a continuous field of groupoids over X with typical fiber the tangent groupoid of the typical fiber of the vector bundle $N \to X$. More precisely, the topology of $\mathscr{I}_N$ is such that the map $E \times [0,1] \to \mathscr{I}_N$ sending (u,v,t) to $(u,u+tv,t)$ if $t > 0$ and equal to identity if $t = 0$ is a homeomorphism.

The family of Lebesgue measures on euclidean fibers N_x, $x \in X$, gives rise to a continuous Haar system on $\mathscr{T}_N$ that allows us to define the C^*-algebra of $\mathscr{I}_N$ as a continuous field of amenable groupoids. Therefore, $\mathscr{I}_N$ is amenable. We also get an element of $KK\big(C^*(E), C_0(X)\big)$, denoted by T_{inv} and defined as usual by

$$T_{\mathrm{inv}} := [e_0]^{-1} \otimes [e_1] \otimes \mathscr{M}.$$

Here, as before, the morphism $e_0 : C^*(\mathscr{I}_N) \to C^*(\mathscr{I}_N|_{t=0}) = C^*(E)$ is the evaluation at 0, the morphism $e_1 : C^*(\mathscr{I}_N) \to C^*(\mathscr{I}_N|_{t=1})$ is the evaluation at 1, and $\mathscr{M}$ is the natural Morita equivalence between $C^*(\mathscr{I}_N|_{t=1})$ and $C_0(X)$. For instance, $\mathscr{M}$ is represented by the Kasparov module $(\mathscr{H}, m, 0)$ where $\mathscr{H}$ is the continuous field over X of Hilbert spaces $\mathscr{H}_x = L^2(N_x)$, $x \in X$, and m is the action of $C^*(\mathscr{I}_N|_{t=1}) = C^*(N \underset{X}{\times} N)$ by compact operators on $\mathscr{H}$.

We denote $T_0 = (\mathscr{E}_0, \rho_0, F_0) \in KK\big(C_0(X), C^*(E)\big)$ the element corresponding to the Thom element T of Proposition 6.1 through the isomorphism $C_0(E) \simeq C^*(E)$. This isomorphism is given by the Fourier transform applied to the second factor in $E = N \oplus N$ provided with the groupoid structure of $\mathscr{I}_N|_{t=0}$. The $C^*(E)$-Hilbert module $\mathscr{E}_0 = C^*(E, \Lambda E)$ is the natural completion of $C_c\big(E, p^*(\Lambda E)\big)$ (p is the bundle map $E \to X$). The representation ρ_0 of $C_0(X)$ and the endomorphism F_0 of $\mathscr{E}_0$ are given by

$$\rho_0(f)\omega(v,w) = f(x)\omega(v,w),$$

$$F_0\omega(v,w) = \int_{(w',\xi) \in N_x \times N_x^*} e^{i(w-w').\xi} C(v + i\xi)\omega(v,w')\, dw'\, d\xi.$$

In the above formulas, $f \in C_0(X)$, $\omega \in C_c\big(E, p^*(\Lambda E)\big)$ and $(v,w) \in E_x$. We can therefore state the following result.

Theorem 6.2. *The elements T_{inv} and T_0 are inverses to each other in KK-theory.*

Proof. We know ([36], paragraph 5, theorem 8) that T, hence T_0, is invertible so it is enough to check that $T_0 \otimes T_{\mathrm{inv}} = 1 \in KK\big(C_0(X), C_0(X)\big)$.

Since $T_{\mathrm{inv}} := [e_0]^{-1} \otimes [e_1] \otimes \mathscr{M}$ where e_t are restriction morphisms at $t = 0, 1$ in the groupoid $\mathscr{I}_N$ we first compute $\tilde{T} = T_0 \otimes [e_0]^{-1}$, that is, we look for

$$\tilde{T} = (\mathscr{E}, \rho, F) \in KK\big(C_0(X), C^*(\mathscr{I}_N)\big)$$

such that

$$(e_0)_*(\tilde{T}) = \big(\mathscr{E} \underset{e_0}{\otimes} C^*(E), \rho, F \otimes 1\big) = T_0.$$

Let $\mathscr{E} = C^*(\mathscr{I}_N, \Lambda E)$ be the $C^*(\mathscr{I}_N)$-Hilbert module completion of $C_c\big(\mathscr{I}_N, (r')^*\Lambda E\big)$, where $r' = p \circ \mathrm{pr}_1 \circ r : \mathscr{I}_N \to X$. Let us define a representation ρ of $C_0(X)$ on $\mathscr{E}$ by

$$\rho(f)\omega(v, w, t) = f\big(p(v)\big)\omega(v, w, t) \quad \text{for all } f \in C_0(X),\ \omega \in \mathscr{E},\ (v, w, t) \in \mathscr{I}_N.$$

Let F be the endomorphism of $\mathscr{E}$ densely defined on $C_c\big(\mathscr{I}_N, (r')^*\Lambda E\big)$ by

$$F\omega(v, w, t) = \int_{(v', \xi) \in N_x \times N_x^*} e^{i\left(\frac{v - v'}{t}\right) \cdot \xi} C(v + i\xi)\omega(v', w, t) \frac{dv'}{t^n} d\xi,$$

if $t > 0$ and by $F\omega(v, w, 0) = F_0\omega(v, w, 0)$ if $t = 0$. The integer n above is the rank of the bundle $N \to X$. One can check that the triple $(\mathscr{E}, \rho, F)$ is a Kasparov $\big(C_0(X), C^*(\mathscr{I}_N)\big)$-module and that under the obvious isomorphism

$$q\mathscr{E} \underset{e_0}{\otimes} C^*(E) \simeq \mathscr{E}_0,$$

ρ coincides with ρ_0 while $F \otimes 1$ coincides with F_0.

Next, we evaluate $\tilde{T}$ at $t = 1$ and $T_1 := (e_1)_*(\tilde{T}) \in KK\big(C_0(X), C^*(N \underset{X}{\times} N)\big)$ is represented by $(\mathscr{E}_1, \rho_1, F_1)$ where $\mathscr{E}_1 = C^*(N \underset{X}{\times} N, \Lambda E)$ is the $C^*(N \underset{X}{\times} N)$-Hilbert module completion of $C_c\big(\mathscr{I}_N|_{t=1}, (p \circ r)^*\Lambda E\big)$ and ρ_1, F_1 are given by the formulas above where t is replaced by 1.

Now, applying the Morita equivalence $\mathscr{M}$ to T_1 gives:

$$(\mathscr{E}_1, \rho_1, F_1) \otimes (\mathscr{H}, m, 0) = (\mathscr{H}_{\Lambda E}, \phi, F_1),$$

where $\mathscr{H}_{\Lambda E} = \big(L^2(N_x, \Lambda E_x)\big)_{x \in X}$, ϕ is the obvious action of $C_0(X)$ on $\mathscr{H}_{\Lambda E}$ and F_1 is the same operator as above identified with a continuous family of Fredholm operators acting on $L^2(N_x, \Lambda E_x)$:

$$F_1\omega(x, v) = \int_{(v', \xi) \in N_x \times N_x^*} e^{i(v - v') \cdot \xi} C(v + i\xi)\omega(x, v')\, dv'\, d\xi.$$

By [24], lemma 2.4, we know that $(\mathscr{H}_{\Lambda E}, \phi, F_1)$ represents 1 in $KK\big(C_0(X), C_0(X)\big)$ (the key point is again that the equivariant O_n-index of F_1 restricted to even forms is 1, see also [34]) and the theorem is proved. □

Now let us consider the vector bundle $p : T\mathscr{V} \to TM$, where M is a compact manifold embedded in some $\mathbb{R}^N$ and $\mathscr{V}$ is the normal bundle of the embedding. We let $q : TM \to M$ be the canonical projection and to simplify notations, we denote again by p the bundle map $\mathscr{V} \to M$ and by $\mathscr{V}$ the pull-back of $\mathscr{V}$ to TM via q.

Using the identifications $T_xM \oplus \mathscr{V}_x \simeq T_{(x,v)}\mathscr{V}$ for all $x \in M$ and $v \in \mathscr{V}_x$, we get the isomorphism of vector bundles over TM:

$$q^*(\mathscr{V} \oplus \mathscr{V}) \ni (x, X, v, w) \mapsto (x, v; X + w) \in T\mathscr{V}.$$

It follows that $T\mathscr{V}$ inherits a complex structure from $\mathscr{V} \oplus \mathscr{V} \simeq \mathscr{V} \otimes \mathbb{C}$ and we take the Atiyah-Singer convention: via the above isomorphism, the first parameter is real and the second is imaginary.

The previous construction leads to the groupoid $\mathscr{I}_{\mathscr{V}}$ giving the inverse of the Thom isomorphism. Actually, we slighty modify to retain the natural groupoid structure carried by the base space TM of the vector bundle $T\mathscr{V}$ (it is important in the purpose of extending the Thom isomorphism to the singular setting). Thus, we set

$$\mathscr{H}_{\mathscr{V}} = T\mathscr{V} \times \{0\} \sqcup {}^*p^*(TM) \times]0,1] \rightrightarrows \mathscr{V} \times [0,1].$$

This is the Thom groupoid defined in the Section 3.2. The groupoids $\mathscr{I}_{\mathscr{V}}$ and $\mathscr{H}_{\mathscr{V}}$ are not isomorphic, but a Fourier transform in the fibers of TM provides an isomorphism of their C^*-algebras: $C^*(\mathscr{I}_{\mathscr{V}}) \simeq C^*(\mathscr{H}_{\mathscr{V}})$. Moreover, this isomorphism is compatible with the restriction morphisms and we can rewrite Theorem 6.2:

Corollary 6.3. *Let $\partial_{\mathscr{H}_{\mathscr{V}}} = [\varepsilon_0]^{-1} \otimes [\varepsilon_1]$ be the KK-element associated with the deformation groupoid $\mathscr{H}_{\mathscr{V}}$ and let $\mathscr{M}$ be the natural Morita equivalence between $C^*\big({}^*p^*(TM)\big)$ and $C^*(TM)$. Then $T_{\mathrm{inv}} = \partial_{\mathscr{H}_{\mathscr{V}}} \otimes \mathscr{M} \in KK\big(C^*(T\mathscr{V}), C^*(TM)\big)$ gives the inverse of the Thom isomorphism $T \in KK\big(C_0(T^*M), C_0(T^*\mathscr{V})\big)$ through the isomorphisms $C_0(T^*M) \simeq C^*(TM)$ and $C_0(T^*\mathscr{V}) \simeq C^*(T\mathscr{V})$.*

Remarks 6.4. (1) Let us assume that M is a point and $\mathscr{V} = \mathbb{R}^N$. The groupoid $\mathscr{H}_{\mathscr{V}}$ is equal in that case to the tangent groupoid of the manifold $\mathbb{R}^N$ and the associated KK-element $\partial_{\mathscr{H}_{\mathscr{V}}} \otimes \mathscr{M}$ gives the Bott periodicity between the point and $\mathbb{R}^{2N}$.

(2) Let M_+ be a compact manifold with boundary and M the manifold without boundary obtained by doubling M_+. Keeping the notations above, let $\mathscr{V}_+$ be the restriction of $\mathscr{V}$ to M_+. All the previous constructions applied to M restrict to M_+ and give the inverse T^+_{inv} of the Thom element $T^+ \in KK\big(C_0(T^*M_+), C_0(T^*\mathscr{V}_+)\big)$.

With this description of the (inverse) Thom isomorphism in hand, the equality between the analytical and topological indices of Atiyah and Singer [2] follows from a commutative diagram. Let us denote by $p_{[0,1]}$ the map $p \times \mathrm{Id} : \mathscr{V} \times [0,1] \to M \times [0,1]$. We consider the deformation groupoid (cf. Example 3.5)

$$\mathscr{T}^t_{\mathscr{V}} = \mathscr{G}^t_{\mathscr{V}} \times \{0\} \sqcup {}^*p^*_{[0,1]}(\mathscr{G}^t_M) \simeq \mathscr{H}_{\mathscr{V}} \times \{0\} \sqcup (\mathscr{V} \times \mathscr{V} \times [0,1]) \times]0,1] \rightrightarrows \mathscr{V} \times [0,1] \times [0,1].$$

We use the obvious notation for restriction morphisms (cf. proof of Definition-Proposition 5.2) and $\mathscr{M}$ for the various (but always obvious) Morita equivalence maps. To shorten the diagram, we set $K(G) := K_0\big(C^*(G)\big)$ for all the (amenable) groupoids met below. We have:

(6.2)

$$\begin{tikzcd}
K(\cdot) & K(\cdot) \arrow[l, "="'] & K([0,1]) \arrow[l, "{[ev_0]}"'] \arrow[r, "{[ev_1]}"] & K(\cdot) \arrow[r, "="] & K(\cdot) \\
K(\mathbb{R}^N \times \mathbb{R}^N) \arrow[u, "\mathscr{M}"] & K(\mathscr{V} \times \mathscr{V}) \arrow[l, "j_1"'] \arrow[u, "\mathscr{M}"] & K(\mathscr{V} \times \mathscr{V} \times [0,1]) \arrow[l, "{[ev_0]}"'] \arrow[r, "{[ev_1]}"] \arrow[u, "\mathscr{M}"] & K(\mathscr{V} \times \mathscr{V}) \arrow[r, "\mathscr{M}"] \arrow[u, "\mathscr{M}"] & K(M \times M) \arrow[u, "\mathscr{M}"] \\
K(\mathscr{G}^t_{\mathbb{R}^N}) \arrow[u, "{[e_1^{\mathbb{R}^N}]}"] \arrow[d, "{[e_0^{\mathbb{R}^N}]}"'] & K(\mathscr{G}^t_{\mathscr{V}}) \arrow[l, "j"'] \arrow[u, "{[e_1^{\mathscr{V}}]}"] \arrow[d, "{[e_0^{\mathscr{V}}]}"'] & K(\mathscr{T}^t_{\mathscr{V}}) \arrow[l, "{[q_{\cdot,0}]}"'] \arrow[r, "{[q_{\cdot,1}]}"] \arrow[u, "{[q_{1,\cdot}]}"'] \arrow[d, "{[q_{0,\cdot}]}"] & K\big({}^*p^*_{[0,1]}(\mathscr{G}^t_M)\big) \arrow[r, "\mathscr{M}"] \arrow[u, "{[{}^*p^*e_1^M]}"'] \arrow[d, "{[{}^*p^*e_0^M]}"] & K(\mathscr{G}^t_M) \arrow[u, "{[e_1^M]}"'] \arrow[d, "{[e_0^M]}"] \\
K(T\mathbb{R}^N) & K(T\mathscr{V}) \arrow[l, "j_0"] & C^*(\mathscr{H}_{\mathscr{V}}) \arrow[l, "{[\varepsilon_0]}"] \arrow[r, "{[\varepsilon_1]}"'] & K\big({}^*p^*(TM)\big) \arrow[r, "\mathscr{M}"'] & K(TM).
\end{tikzcd}$$

The commutativity of this diagram is obvious. From the previous remark we deduce that the map $K(T\mathbb{R}^N) \to \mathbb{Z}$ associated with $[e_0^{\mathbb{R}^N}]^{-1} \otimes [e_1^{\mathbb{R}^N}] \otimes \mathscr{M}$ on the left column is equal to the Bott periodicity isomorphism β. Thanks to Corollary 6.3, the map $K(TM) \to K(T\mathscr{V})$ associated with $\mathscr{M}^{-1} \otimes \partial^{-1}_{H_{\mathscr{V}}} = T_0$ on the bottom line is equal to the Thom isomorphism, while j_0 is the usual excision map resulting from the identification of $\mathscr{V}$ with an open subset of $\mathbb{R}^N$. It follows that composing the bottom line with the left column produces the map

$$\beta \circ j_0 \circ T_0 : K(TM) \to \mathbb{Z}$$

which is exaclty the Atiyah-Singer's definition of the topological index map. We already know that the map $K(TM) \to \mathbb{Z}$ associated with $[e_0^M]^{-1} \otimes [e_1^M] \otimes \mathscr{M}$ on the right column is the analytical index map. Finally, the commutativity of the diagram and the fact that the map associated with $[ev_0]^{-1} \otimes [ev_1]$ on the top line is identity, completes our proof of the Atiyah-Singer Index Theorem.

Another proof of the usual Atiyah-Singer Index Theorem in the framework of deformation groupoids can be found in [49].

6.2. An index theorem for conical pseudomanifolds. We define for a conical manifold a topological index and prove the equality between the topological and analytical indices. Both indices are straight generalisations of the ones used in the Atiyah-Singer Index Theorem: indeed, if we apply our constructions to a smooth manifold and its tangent space, we find exaclty the classical topological and analytical indices. Thus, the egality of indices we proove can be presented as the index theorem for conical manifolds. Moreover, the K-theory of $T^{\mathsf{S}}X$ is exhausted by *elliptic symbols* associated with pseudo-differential operators in the b-calculus [40] and the analytical index can be interpreted via the Poincaré duality [27], as their Fredholm index.

Let X be a compact conical pseudomanifold embedded in $(\mathbb{R}^N)^{\mathsf{S}}$ for some N and let $\mathscr{W}$ be a tubular neighborhood of this embedding as in 1. We first assume that X has only one singularity. We denote by

$$\mathscr{H}_{\mathscr{W}} = T^{\mathsf{S}}\mathscr{W} \times \{0\} \sqcup {}^*\pi^*(T^{\mathsf{S}}X) \times]0,1] \rightrightarrows \mathscr{W}^\circ \times [0,1]$$

the Thom groupoid associated with $\pi : \mathscr{W} \to X$ and by

$$\mathscr{H}_+ = T\mathscr{W}_+ \times \{0\} \sqcup {}^*\pi_+^*(TX_+) \times]0,1] \rightrightarrows \mathscr{W}_+ \times [0,1]$$

the Thom groupoid associated with $\pi_+ : \mathscr{W}_+ \to X_+$. Here

$$\mathscr{W}_+ = \mathscr{W} \backslash O_{\mathscr{W}} = \{(z,V) \in \mathscr{W} \mid \rho(z) \geqq 1\} \quad \text{and} \quad X_+ = X \backslash O_X = \{z \in X \mid \rho(z) \geqq 1\},$$

where ρ is in both cases the defining function of the singularity. We denote by T_{inv} and T_{inv}^+ the respective inverse-Thom elements. Recall (cf. Proposition 4.3) that we have the two following short exact sequences coming from inclusion and restriction morphisms:

$$0 \longrightarrow K\big(\mathscr{K}\big(L^2(O_{\mathscr{W}})\big)\big) \xrightarrow{i_{\mathscr{V}*}} K\big(C^*(T^{\mathsf{S}}\mathscr{W})\big) \xrightarrow{r_{\mathscr{W}*}} K\big(C^*(T\mathscr{W}_+)\big) \longrightarrow 0,$$
$$0 \longrightarrow K\big(\mathscr{K}\big(L^2(O_X)\big)\big) \longrightarrow K\big(C^*(T^{\mathsf{S}}X)\big) \longrightarrow K\big(C^*(TX_+)\big) \longrightarrow 0.$$

Definition-Proposition 6.5. *The following diagram commutes*:

(6.3)
$$\begin{array}{ccccccccc}
0 & \longrightarrow & K\big(\mathscr{K}\big(L^2(O_{\mathscr{W}})\big)\big) & \xrightarrow{i_{\mathscr{W}*}} & K\big(C^*(T^{\mathsf{S}}\mathscr{W})\big) & \xrightarrow{r_{\mathscr{W}*}} & K\big(C^*(T\mathscr{W}_+)\big) & \longrightarrow & 0 \\
 & & \Big\downarrow{\scriptstyle \mathscr{M}} & & \Big\downarrow{\scriptstyle \cdot\otimes T_{\mathrm{inv}}} & & \Big\downarrow{\scriptstyle \cdot\otimes T_{\mathrm{inv}}^+} & & \\
0 & \longrightarrow & K\big(\mathscr{K}\big(L^2(O_X)\big)\big) & \longrightarrow & K\big(C^*(T^{\mathsf{S}}X)\big) & \longrightarrow & K\big(C^*(TX_+)\big) & \longrightarrow & 0
\end{array}$$

where $\mathscr{M}$ is the natural Morita equivalence map. In particular, the map

$$\cdot \otimes T_{\mathrm{inv}} : K\big(C^*(T^{\mathsf{S}}\mathscr{W})\big) \to K\big(C^*(T^{\mathsf{S}}X)\big) \tag{6.4}$$

is an isomorphism. Its inverse is denoted by T and called the Thom isomorphism.

Proof. Let us note again by π the (smooth) vector bundle map $\mathscr{W}^\circ \to X^\circ$ and consider the following diagram:

(6.5)
$$\begin{array}{ccccc}
0 \to C^*(O_{\mathscr{W}} \times O_{\mathscr{W}}) & \longrightarrow & C^*(T^{\mathsf{S}}\mathscr{W}) & \xrightarrow{r_+} & C^*(T\mathscr{W}_+) \to 0 \\
\Big\uparrow{\scriptstyle ev_0} & & \Big\uparrow{\scriptstyle ev_0} & & \Big\uparrow{\scriptstyle ev_0} \\
0 \to C^*(O_{\mathscr{W}} \times O_{\mathscr{W}} \times [0,1]) & \longrightarrow & C^*(\mathscr{H}_{\mathscr{W}}) & \xrightarrow{r_+} & C^*(\mathscr{H}_+) \to 0 \\
\Big\downarrow{\scriptstyle ev_1} & & \Big\downarrow{\scriptstyle ev_1} & & \Big\downarrow{\scriptstyle ev_1} \\
0 \to C^*\big({}^*\pi^*(O_X \times O_X)\big) & \longrightarrow & C^*\big({}^*\pi^*(T^{\mathsf{S}}X)\big) & \xrightarrow{r_+} & C^*\big({}^*\pi^*(TX_+)\big) \to 0
\end{array}$$

where the (Lie) groupoid isomorphism ${}^*\pi^*(O_X \times O_X) \simeq O_{\mathscr{W}} \times O_{\mathscr{W}}$ has been used. Applying the K functor and Morita equivalence maps to the bottom line to get rid of the pull back ${}^*\pi^*$ and using the fact that the long exact sequences in K-theory associated to the top and bottom lines split in short exact sequences, give the diagram (6.3). Since $\mathscr{M}$ and T_{inv}^+ are isomorphisms, the same is true for T_{inv}. □

Remarks 6.6. (1) When X has several singular points, the invertibility of $\cdot \otimes T_{\mathrm{inv}}$ remains true. This can be checked thanks to a recursive process on the number k of singular points. First choose a singular point $s \in \mathsf{S}$ and call again s its image in $\mathscr{W}$ by the embedding π. Denote by

$$\mathscr{H}_{s,+} = T\mathscr{W}_{s,+} \times \{0\} \sqcup {}^*\pi^*_{s,+}(T^{\mathsf{S}}X_{s,+}) \times]0,1] \rightrightarrows \mathscr{W}_{s,+} \times [0,1]$$

the Thom groupoid associated with $\pi_{s,+} : \mathscr{W}_{s,+} \to X_{s,+}$. Recall that

$$\mathscr{W}_{s,+} = \{(z,V) \in \mathscr{W} \mid \rho_s(z) \geqq 1\} \quad \text{and} \quad X_{s,+} = X \backslash O_s = \{z \in X \mid \rho_s(z) \geqq 1\},$$

where ρ_s is in both cases the defining function associated to s. We denote by $T^{s,+}_{\mathrm{inv}}$ the corresponding inverse-Thom element. The same proof as before gives that the map

$$\cdot \otimes T_{\mathrm{inv}} : K\big(C^*(T^{\mathsf{S}}\mathscr{W})\big) \to K\big(C^*(T^{\mathsf{S}}X)\big)$$

is an isomorphism as soon as

$$\cdot \otimes T^{s,+}_{\mathrm{inv}} : K\big(C^*(T^{\mathsf{S}}\mathscr{W}_{s,+})\big) \to K\big(C^*(T^{\mathsf{S}}X_{s,+})\big)$$

is. But now $X_{s,+}$ has $k-1$ singular points.

(2) The Thom map we define extends the usual one: this is exactly what is said by the commutativity of the diagram (6.3).

Let us recall that we started with an embedding of X into $(\mathbb{R}^N)^{\mathsf{S}}$ which is $\mathbb{R}^N$ with k singular points where k is the cardinal of S. The S-tangent space $T^{\mathsf{S}}\mathscr{W}$ of $\mathscr{W}$ is obviously isomorphic to an open subgroupoid of the S-tangent space $T^{\mathsf{S}}(\mathbb{R}^N)^{\mathsf{S}}$. Thus we get an excision homomorphism:

$$j : C^*(T^{\mathsf{S}}\mathscr{W}) \to C^*\big(T^{\mathsf{S}}(\mathbb{R}^N)^{\mathsf{S}}\big).$$

There is a natural identification of the K-theory group $K\big(T^{\mathsf{S}}(\mathbb{R}^N)^{\mathsf{S}}\big)$ with $\mathbb{Z}$, analog to the one given by Bott periodicity in the case of $T\mathbb{R}^N = \mathbb{R}^{2N}$ coming from its tangent groupoid (cf. Remark 6.4):

$$\partial_{(\mathbb{R}^N)^{\mathsf{S}}} = [e_0]^{-1} \otimes [e_1] \otimes \mathscr{M} : K\big(T^{\mathsf{S}}(\mathbb{R}^N)^{\mathsf{S}}\big) \to \mathbb{Z},$$
$$K\big(C^*\big(T^{\mathsf{S}}(\mathbb{R}^N)^{\mathsf{S}}\big)\big) \xleftarrow{[e_0]} K\big(C^*(\mathscr{G}^t_{(\mathbb{R}^N)^{\mathsf{S}}})\big) \xrightarrow{[e_1]} K\big(\mathscr{K}\big(L^2(\mathbb{R}^N)\big)\big) \xrightarrow{\mathscr{M}} K(\cdot) \simeq \mathbb{Z}.$$

We are now in position to extend the Atiyah-Singer topological index to conical pseudomanifolds:

Definition 6.7. The topological index of the conical pseudomanifold X is defined by

$$\mathrm{Ind}^X_t = \partial_{(\mathbb{R}^N)^{\mathsf{S}}} \circ [j] \circ T.$$

Moreover, we obtain the following extension of the Atiyah-Singer Index Theorem.

Theorem 6.8. *If X is a pseudomanifold with conical singularities then*

$$\mathrm{Ind}_a^X = \mathrm{Ind}_t^X .$$

Proof. The proof is similar to our proof of the Atiyah-Singer Index Theorem. Indeed, let us write down the analog of the diagram (6.2) for the singular manifold X:

(6.6)

$$\begin{tikzcd}
K(\cdot) & K(\cdot) \arrow[l, "="'] & K([0,1]) \arrow[l, "{[ev_0]}"'] \arrow[r, "{[ev_1]}"] & K(\cdot) \arrow[r, "="] & K(\cdot) \\
K(\mathbb{R}^{2N}) \arrow[u, "\mathscr{M}"] & K(\mathscr{W}^2) \arrow[l, "j_1"'] \arrow[u, "\mathscr{M}"] & K(\mathscr{W}^2 \times [0,1]) \arrow[l, "{[ev_0]}"'] \arrow[r, "{[ev_1]}"] \arrow[u, "\mathscr{M}"] & K(\mathscr{W}^2) \arrow[r, "\mathscr{M}"] \arrow[u, "\mathscr{M}"] & K\big((X^o)^2\big) \arrow[u, "\mathscr{M}"] \\
K(\mathscr{G}^t_{(\mathbb{R}^N)^{\mathsf{S}}}) \arrow[u, "{[e_1^{(\mathbb{R}^N)^c}]}"] \arrow[d, "{[e_0^{(\mathbb{R}^N)^c}]}"'] & K(\mathscr{G}^t_{\mathscr{W}}) \arrow[l, "j"'] \arrow[u, "{[e_1^{\mathscr{W}}]}"] \arrow[d, "{[e_0^{\mathscr{W}}]}"'] & K(\mathscr{T}^t_{\mathscr{W}}) \arrow[l, "{[q_{\cdot,0}]}"'] \arrow[r, "{[q_{\cdot,1}]}"] \arrow[u, "{[q_{1,\cdot}]}"'] \arrow[d, "{[q_{0,\cdot}]}"] & K\big({}^*\pi^*_{[0,1]}(\mathscr{G}^t_X)\big) \arrow[r, "\mathscr{M}"] \arrow[u, "{[{}^*\pi^* e_1^X]}"'] \arrow[d, "{[{}^*\pi^* e_0^X]}"] & K(\mathscr{G}^t_X) \arrow[u, "{[e_1^X]}"'] \arrow[d, "{[e_0^X]}"] \\
K\big(T^{\mathsf{S}}(\mathbb{R}^N)^{\mathsf{S}}\big) & K(T^{\mathsf{S}}\mathscr{W}) \arrow[l, "j_0"] & C^*(\mathscr{H}_{\mathscr{W}}) \arrow[l, "{[\varepsilon_0]}"] \arrow[r, "{[\varepsilon_1]}"'] & K\big({}^*\pi^*(T^{\mathsf{S}}X)\big) \arrow[r, "\mathscr{M}"'] & K(T^{\mathsf{S}}X).
\end{tikzcd}$$

This diagram involves various deformation groupoids associated to X and its embedding into $(\mathbb{R}^N)^{\mathsf{S}}$. The commutativity is obvious since everything comes from morphisms of algebras or from explicit Morita equivalences. As before, the convention $K(G) = K_0\big(C^*(G)\big)$ is used to shorten the diagram and intuitive notations are chosen to name the various restriction morphisms. Starting from the bottom right corner and following the right column gives the analytical index map. Starting from the bottom right corner and following the bottom line and next the left column gives the topological index map. $\square$

6.3. Signification of the index map. In the sequel we suppose that X has only one singularity.

In [27], a Poincaré duality in bivariant K-theory between $C(X)$ and $C^*(T^{\mathsf{S}}X)$ is proved. Taking the Kasparov product with the dual-Dirac element involved in this duality provides an isomorphism:

$$K_0(X) \overset{\Sigma_X}{\to} K^0(T^{\mathsf{S}}X). \tag{6.7}$$

When a K-homology class of X and a K-theory class of $T^{\mathsf{S}}X$ coincide trough this isomorphism, we say that they are Poincaré dual.

If $p : X \to \cdot$ is the trivial map, then (6.7) satisfies ([40])

$$\mathrm{Ind}_a^X \circ \Sigma_X = p_* : K_0(X) \to \mathbb{Z} \simeq K_0(\cdot). \tag{6.8}$$

Remember that cycles of $K_0(Y)$, for a compact Hausdorff space Y, are given by triples (H, π, F) where $H = H_+ \oplus H_-$ is a $\mathbb{Z}_2$-graded Hilbert space, π a degree 0 homomorphism of $C(Y)$ into the algebra of bounded operators on H and $F = \begin{pmatrix} 0 & F_- \\ F_+ & 0 \end{pmatrix}$ a bounded operator on H of degree 1 such that $F^2 - 1$ and $[\pi, F]$ are compact. Since

$$p_*(H, \pi, F) = \text{Fredholm-Index}(P),$$

the equality (6.8) implies that Ind_a^X produces indices of Fredholm operators. To make things more concrete and see what Fredholm operators come into the play, one needs to compute explicitely (6.7), or mimeting the case of smooth manifolds, interpret it as a *symbol map* associating *K-theory classes of the tangent space* to *elliptic pseudodifferential operators*. This is done in full details, and summarized below, for the 0-order case in [40]. We give also an account of the unbounded case, necessary to compare our index with the ones computed in [18], [15], [19], [39].

6.3.1. K-homology of the conical pseudomanifold and elliptic operators. Let Ψ_b^* be the algebra of the b-calculus [44] on $\overline{X^\circ}$ (the obvious compactification of X° into a manifold with boundary). A b-pseudodifferential operator P is said to be *fully elliptic* if its principal symbol $\sigma_{\mathrm{int}}(P)$, regarded as an ordinary pseudodifferential operator on X° is invertible and the indicial family $\big(\hat{P}(\tau)\big)_{\tau \in \mathbb{R}}$ is everywhere invertible [44] (that is, for all $\tau \in \mathbb{R}$, the pseudodifferential operator $\hat{P}(\tau)$ on $L = \partial \overline{X^\circ}$ is invertible). A *full parametrix* of P is then another b-operator Q such that PQ and QP are equal to 1 modulo a negative order b-operator with vanishing indicial family. When P is a zero order fully elliptic b-operator, it is Fredholm on the Hilbert space $L^{2,b} := L^2(X^\circ, d\mu_b)$ for the natural measure $d\mu_b = \dfrac{dh}{h} dy$ coming with an exact b-metric [44], and we get a canonically defined K-homology class of X:

$$[P] := [(H^b, \pi, \mathbf{P})] \in K_0(X) \tag{6.9}$$

where $\mathbf{P} = \begin{pmatrix} 0 & Q \\ P & 0 \end{pmatrix}$, $H^b = L^{2,b} \oplus L^{2,b}$ and $\pi : C(X) \to \mathscr{B}(H^b)$ is the homomorphism given by pointwise multiplication (for all $f \in C_c^\infty(X^\circ) \oplus \mathbb{C}$, $[\pi(f), \mathbf{P}]$ has negative order and vanishing indicial family, thus it is a compact operator on H^b [44]; since $C_c^\infty(X^\circ) \oplus \mathbb{C}$ is dense in $C(X)$, it proves that the commutators $[\pi, \mathbf{P}]$ are compact and $[P]$ is well defined).

6.3.2. K-theory of the noncommutative tangent space and symbols. To compute the Poincaré dual of K-classes given by (6.9), one uses a slightly different, but KK-equivalent, definition of $T^{\mathsf{S}}X$:

$$T^{\mathsf{S}}X := T]0,1[\times L \times L \sqcup TX_+ \rightrightarrows X^o. \tag{6.10}$$

The KK-equivalence between both definitions is explicit ([40]) and allows us to translate all the previous constructions to this variant of the tangent space.

Roughly speaking, a noncommutative symbol on the pseudomanifold X is a pseudodifferential operator, in the groupoid sense ([48], [57]), on $T^{\mathsf{S}}X$. For technical reasons, one asks to these objects to be smooth up to $h = 0$, in other words we define the algebra of noncommutative symbols as

$$S^*(X) = \Psi^*(\overline{T^{\mathsf{S}}X}) \tag{6.11}$$

where $\overline{T^{\mathsf{S}}X} = \{0\} \times \mathbb{R} \times L \times L \cup T^{\mathsf{S}}X$ and the letter Ψ is reserved for the space of pseudodifferential operators on the indicated groupoid. See [40] for the precise assumptions on the Schwartz kernels of the operators in (6.11). Considering the closed saturated subspace $L = \partial\overline{X^\circ}$ of the space of units $\overline{X^\circ}$ of $\overline{T^{\mathsf{S}}X}$, we get a restriction homomorphism:

$$S^*(X) = \Psi^*(\overline{T^{\mathsf{S}}X}) \xrightarrow{\rho} \Psi^*(\mathbb{R} \times L \times L) \simeq \Psi^*_{\mathrm{susp}}(L) \tag{6.12}$$

where $\Psi^*_{\mathrm{susp}}(L)$ denotes the space of suspended pseudodifferential operators of R. Melrose [46]. A noncommutative symbol $a \in S^m(X)$ is *fully elliptic* if there exists $b \in S^{-m}(X)$ such that ab and ba are equal to 1 modulo $S^{-1}(X) \cap \ker\rho =: \mathscr{J}$. Fully elliptic symbols $a \in S^0(X)$ give canonically K-classes of the tangent space $T^{\mathsf{S}}X$:

$$[a] = [\mathscr{E}, \mathbf{a}] \in KK\big(\mathbb{C}, C^*(T^{\mathsf{S}}X)\big) = K^0(T^{\mathsf{S}}X) \tag{6.13}$$

where $\mathbf{a} = \begin{pmatrix} 0 & b \\ a & 0 \end{pmatrix}$, b any inverse of a modulo $\mathscr{J}$ and $\mathscr{E} = C^*(T^{\mathsf{S}}X) \oplus C^*(T^{\mathsf{S}}X)$.

6.3.3. $\mathrm{Ind}_a^{\mathsf{S}}$ as a Fredholm index. The main result of [40] is:

Theorem 6.9. *There exists a surjective linear map $\sigma_X : \Psi_b^* \to S^*$ such that:*

- *$P \in \Psi_b^*$ is fully elliptic if and only if $\sigma_X(P)$ is fully elliptic.*
- *For all zero order fully elliptic operators P,*

$$\Sigma_X([P]) = [\sigma_X(P)]. \tag{6.14}$$

See also [38] for a thorough study of the property of full ellipticity in b-calculus in the framework of groupoids.

Remarks 6.10. Allowing vector bundles E over $\overline{X^\circ}$ and defining the algebra of b-operators $\Psi_b^*(E)$ and the algebra of noncommutative symbols $S^*(X,E)$ accordingly, we get a full description of $K_0(X)$ in terms of b-operators and of $K^0(T^{\mathsf{S}}X)$ in terms of noncommutative symbols. This is also proved in [40]. Thus, for any $x \in K^0(T^{\mathsf{S}}X)$, we have

$$\mathrm{Ind}_a^X(x) = \text{Fredholm-index}(P_x)$$

where P_x is any b-operator such that $[\sigma_X(P_x)] = x$.

The reader should not be surprised by our definition of (noncommutative) symbols: if V is a smooth manifold, the algebra of ordinary symbols is isomorphic to the algebra of pseudodifferential operators, for a suitable choice of regularizing operators imposed by the use of the Fourier transform, on the *groupoid* TV.

6.3.4. The unbounded case and geometric operators. The symbol map σ_X constructed in [40] makes sense on differential b-operators. It turns out that natural geometric operators on X, when provided with a conical metric, can be written as b-differential operators with singular coefficients at $h = 0$, or, in the terminology of [39], Fuchs type operators. We explain in this paragraph how to relate the analysis of these operators ([18], [15], [19], [39]) to our K-theoretic constructions, for the case of a Dirac operator on X even dimensional with one conical point s.

Let g be a riemannian metric on X° which is conical on $O_X =]0,1[\times L$, that is: $g = dh^2 + h^2 g_L$. We assume to simplify the computations that the riemannian metric g_L on L is independant of h when $h \leqq 1$. We denote by $d\mathrm{vol}_X$ the corresponding volume form.

Let $\mathscr{E} = \mathscr{E}_+ \oplus \mathscr{E}_-$ be a Clifford module and c the corresponding Clifford multiplication ([8]).

If X° has a spin structure, then there exists a ($\mathbb{Z}_2$-graded) vector bundle $\mathscr{W}$ such that $\mathscr{E} \simeq \mathscr{W} \otimes \mathscr{S}$ where $\mathscr{S} = \mathscr{S}_+ \oplus \mathscr{S}_-$ is the spinor bundle.

In the case of a spin structure, using the canonical metric structure and Clifford connection $\nabla^{\mathscr{S}}$ of the spinor bundle, and using on $\mathscr{W}$ a metric structure of product type on O_x and a compatible connection $\nabla^{\mathscr{W}}$, we get on $\mathscr{E}$ a metric structure, such that $\mathscr{E}_+ \perp \mathscr{E}_-$ and $c(v)^* = -c(v)$ for unitary tangent vectors $v \in TX^\circ$, and a Clifford connection $\nabla^{\mathscr{E}}$ such that the corresponding Dirac operator D is symmetric, when considered as an unbounded operator on $L^2(\mathscr{E})$ with domain the space $C_c^\infty(\mathscr{E})$ of compactly supported sections. Recall that D is defined locally by the formula:

$$s \in C^\infty(\mathscr{E}), \quad Ds = \sum_{i=1}^n c(e_i) \nabla_{e_i}^{\mathscr{E}} s,$$

where $(e_1, \ldots, e_n)$ is a local basis of TX°.

If X° has no spin structure, then the isomorphism $\mathscr{E} \simeq \mathscr{W} \otimes \mathscr{S}$ remains true locally. Thus, one can still construct locally on $\mathscr{E}$ metrics and connections with the previous properties and then patch them with a partition of unity (U_i, ϕ_i) on X (that is (U_i) is a finite open covering of X° by open charts, $\phi_i \in C^\infty(U_i) \cap C_c(U_i \cup \{s\})$ and $\sum_i \phi(x) = 1$, $\forall x \in X^\circ$). The resulting Dirac operator is again symmetric, and all subsequent computations are exactly the same with or without spin structure.

Although the Hilbert space $L^2(\mathscr{E})$, whose scalar product is given by

$$(s,t) = \int_{X^\circ} \big(s(x), t(x)\big)_{\mathscr{E}_x} \, d\mathrm{vol}_X \tag{6.15}$$

is the most natural Hilbert space with respect to the given geometric data, computations are easier with $H_b(E)$ which is defined as, if $\pi : X^\circ \to X_+$ denotes the obvious retraction map and $\tilde{\mathscr{E}} = \mathscr{E}|_L$, the completion of $C_c^\infty\big(\pi^*(\mathscr{E}|_{X_+})\big)$ for the scalar product:

$$(s,t)_b = \int_{X_+} \big(s(x), t(x)\big)_{\mathscr{E}_x} \, d\mathrm{vol}_X + \int_{x=(h,y) \in O_X} \big(s(x), t(x)\big)_{\tilde{\mathscr{E}}} \frac{dh}{h} d\mathrm{vol}_Y(y). \tag{6.16}$$

One can choose an isometry $U : H_b(E) \to L^2(E)$ such that $U : C^\infty\big(\pi^*(\mathscr{E}|_{X_+})\big) \to C^\infty(\mathscr{E})$ is equal to identity on the complement of some open neighborhood of $\overline{O_X}$ and given on

O_X by

$$U(s)(h,y) = h^{-\frac{n}{2}} \theta^{\mathscr{E}}_{(1,y)\to(h,y)} s(h,y)$$

where $\theta^{\mathscr{E}}_{(1,y)\to(h,y)} : \mathscr{E}_{(1,y)} \to \mathscr{E}_{(h,y)}$ is the parallel transport associated with the connection and the canonical identification $C^\infty\big(\pi^*(\mathscr{E}|_{X_+})|_{O_X}\big) \simeq C^\infty\big(]0,1[, C^\infty(\tilde{\mathscr{E}})\big)$ has been used.

Then, a straight computation shows that ([11], [19], [41]) the following holds for sections s supported on O_X:

$$U^{-1}DUs = c(e_1)\cdot \frac{\partial s}{\partial h} + \frac{1}{h}\left(\tilde{D} - \frac{e_1}{2}\right)s \tag{6.17}$$

where $\left(e_1 = \dfrac{\partial}{\partial h}, e_2, \dots, e_n\right)$ is a local orthonormal basis in TO_X and $\tilde{D}$ is the differential operator on L, acting on the sections of $\tilde{\mathscr{E}}$, given by $\tilde{D}u = \sum_{i=2}^{n} c(\tilde{e}_i)\nabla^{\tilde{\mathscr{E}}}_{\tilde{e}_i} u$, where $\tilde{e}_i(y) = e_i(1,y)$ and $\nabla^{\tilde{\mathscr{E}}}$ is the connection on $\tilde{\mathscr{E}}$ induced by $\nabla^{\mathscr{E}}$. Moreover we have $\mathscr{E}_- = c(e_1)\cdot\mathscr{E}_+$, and the operator $U^{-1}DU$ is given by the matrix, in the decomposition $\mathscr{E} = \mathscr{E}_+ \oplus c(e_1)\cdot\mathscr{E}_+ \simeq \mathscr{E}_+^2$,

$$\begin{pmatrix} 0 & -\dfrac{\partial}{\partial h} + \dfrac{1}{h}\left(S + \dfrac{1}{2}\right) \\ \dfrac{\partial}{\partial h} + \dfrac{1}{h}\left(S - \dfrac{1}{2}\right) & 0 \end{pmatrix} \tag{6.18}$$

where $S = \sum_{i=2}^{n} c(\tilde{e}_i\tilde{e}_1)\nabla^{\tilde{\mathscr{E}}}_{\tilde{e}_i}$ is again a symmetric Dirac operator on the Clifford module $\widetilde{\mathscr{E}_+}$ over L.

It is of course equivalent to study D on $L^2(E)$ or $T = U^{-1}DU$ on $H_b(E)$. If we used dh instead of $\dfrac{dh}{h}$ in (6.16), which leads to the Hilbert space used in [15], [39] to study Fuchs type operators, S would appear without the extra terms $\pm\dfrac{1}{2}$ in (6.18).

The deformation process of [40] used to associate noncommutative symbols to b-pseudodifferential operators, can be applied to T and gives a family $(T_t)_{0\leqq t\leqq 1}$ where $t > 0$, $T_t \in \dfrac{1}{h}.\,\Psi^1_b(\mathscr{E})$ and $T_0 \in \dfrac{1}{h}.S^1(X,\mathscr{E})$ have the following expression on O_X:

$$t > 0, \quad T_t = \begin{pmatrix} 0 & -t\dfrac{\partial}{\partial h} + \dfrac{1}{h}\left(S + \dfrac{t}{2}\right) \\ t\dfrac{\partial}{\partial h} + \dfrac{1}{h}\left(S - \dfrac{t}{2}\right) & 0 \end{pmatrix} \tag{6.19}$$

and

$$\sigma_X(T) := T_0 = \frac{1}{h}\begin{pmatrix} 0 & -\dfrac{\partial}{\partial\lambda} + S \\ \dfrac{\partial}{\partial\lambda} + S & 0 \end{pmatrix}. \tag{6.20}$$

Observe that the family $D_t := UT_tU^{-1}$ coincides for $t > 0$ with the one given by the deformation of the conical metric $\frac{dh^2}{t^2} + h^2 g_L$ ([11]).

The natural questions are then: Does the noncommutative symbol T_0 give canonically a K-theory element of $T^{\mathsf{S}}X$? Does the operator T give canonically a K-homology class on X? Are the corresponding classes Poincaré dual?

The answer to the first two questions is negative in general, but becomes affirmative under some conditions on the spectrum of S and in that case the answer to the last question is affirmative too. Let us explain these phenomena.

Firstly, the noncommutative symbol $a := hT_0$ is fully elliptic if and only if

$$0 \notin \operatorname{spec} S. \tag{6.21}$$

We assume in the sequel that this condition is satisfied. Using the ellipticity of a as a pseudodifferential operator on $T^{\mathsf{S}}X$ and the invertibility of $a|_{h=0}$, we can prove thanks to [63] that $\left(1 + \sigma_X(T)^2\right)^{-1} \in h^2 \cdot S^{-2}(X, \mathscr{E}) \subset \mathscr{K}\left(C^*(T^{\mathsf{S}}X, \mathscr{E})\right)$. This implies that the closure of $\sigma_X(T)$ as an unbounded operator on the $C^*(T^{\mathsf{S}}X)$-Hilbert module $C^*(T^{\mathsf{S}}X, \mathscr{E})$ with domain $C^\infty(T^{\mathsf{S}}X, \mathscr{E})$ is selfadjoint, regular and provides an unbounded $\left(\mathbb{C}, C^*(T^{\mathsf{S}}X)\right)$-Kasparov bimodule ([5], [63]). We thus get here a well defined, canonical, element $[\sigma_X(T)] \in K^0(T^{\mathsf{S}}X)$.

We turn back now to the operators T_t, $1 \geqq t > 0$. It is well known that they always have a selfadjoint extension, not unique in general ([15], [19], [39]). Adpating for instance the computations of [15] to our particular choice of Hilbert space $H_b(\mathscr{E})$, we see that T_t, $t > 0$, with domain $C_c^\infty(E)$ is essentially selfadjoint if and only if $\operatorname{spec}(S) \cap \left]-\frac{t}{2}, \frac{t}{2}\right[= \emptyset$. Otherwise, any choice of an orthogonal decomposition of

$$W_t = \bigoplus_{-t/2<u<t/2} \mathbb{C}.e_u \tag{6.22}$$

where the e_u's describe an orthonormal system of eigenvectors of S, allows to define a selfadjoint extension of T ([15]).

Thus, for α small enough, thanks to the assumption (6.21), T_α is essentially selfadjoint. It is also Fredholm by [15], so T_α gives an unbounded $\left(C(X), \mathbb{C}\right)$-Kasparov bimodule, in other words a K-homology class $[T_\alpha] \in K_0(X)$, and we have

$$\Sigma_X([T_\alpha]) = [\sigma_X(T)].$$

To check this, one shows that the Woronowicz transform $q(T_\alpha) = T_\alpha.(1 + T_\alpha^2)^{-1/2}$ ([63], [5]) of T_α can be represented in K-homology by a zero order b-operator with noncommutative symbol equal to the Woronowicz transform $q\left(\sigma_X(T)\right) = \sigma_X(T).\left(1 + \sigma_X(T)^2\right)^{-1/2}$ of $\sigma_X(T)$, and then the Theorem 6.9 applies.

In particular:

$$\dim \ker(T_\alpha)_+ - \dim \ker(T_\alpha)_- = \mathrm{Ind}_a^X([\sigma_X(T)]).$$

In general, $T = T_1$ has several selfadjoint extensions, but using the splitness of

$$0 \to \mathbb{C} \to K_0(X) \to K_0(X^\circ) \to 0$$

one shows that two given selfadjoint extensions of T give the same K-homology class if and only if their Fredholm index is the same. Thus a selfadjoint extension T_Z, given by a choice of a decomposition $Z \oplus Z^\perp$ of (6.22), produces the same K-homology class as T_α (and then, is Poincaré dual to its noncommutative symbol) if and only if $2\dim Z = \dim W_1$.

Let us say a word about the case $0 \in \operatorname{spec} S$. For small t, the selfadjoint extensions of T_t are classified by the orthogonal decompositions of $\ker S$. There is a priori no canonical choice. On the other hand, the noncommutative symbol $\sigma_X(T)$ is not fully elliptic. We conjecture that the selfadjoint extensions of $\sigma_X(T)$, as an unbounded operator on the Hilbert module $C^*(T^{\mathsf{S}}X, \mathscr{E})$, are again classified by the orthogonal decomposition of $\ker S$ and give unbounded Kasparov modules which are in one-to-one correspondence, via Poincaré duality, with the selfadjoint extensions of T_t.

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