Groupoids and pseudodifferential calculus

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We recall how pseudodifferential operators on a groupoid G can be expressed as integrals of kernels on the adiabatic groupoid G_{ad} of G and investigate several generalisations of pseudodifferential operators of the Boutet de Monvel calculus [4, 5].

1. Pseudodifferential operators as integral kernels [4]

A key ingredient here will be the adiabatic groupoid of a groupoid G which is a generalisation of the famous *tangent groupoid* of A. Connes (see [3]).

1.1. **The adiabatic groupoid.** Let $G \rightrightarrows G^{(0)}$ be a smooth groupoid, denote by $\mathfrak{A}G$ its Lie algebroid and $\sharp$ the corresponding anchor map. The *adiabatic groupoid* is the deformation to the normal cone of the inclusion $G(0) \subset G$ (see [7, 8, 9]) :

$$G_{ad} = G \times \mathbb{R}^* \sqcup \mathfrak{A}G \times \{0\} \rightrightarrows G^{(0)} \times \mathbb{R} \ .$$

It can be equipped with a (unique) smooth structure such that its Lie algebroid is the bundle $\mathfrak{A}G_{ad} = \mathfrak{A}G \times \mathbb{R}$ with anchor map :

$$\sharp_{ad} : \mathfrak{A}G \times \mathbb{R} \to TG^{(0)} \times T\mathbb{R} \ ; \ (x, X, t) \mapsto (\sharp(x, tX), (t, 0)) \ .$$

The scaling action of $\mathbb{R}^*$ on $G \times \mathbb{R}^*$ extends to a smooth action of $\mathbb{R}^*$ on G_{ad} which is free and proper outside the units $G^{(0)} \times \mathbb{R}$. For $\lambda \in \mathbb{R}^*$ it is given by :

$$\lambda \cdot (\gamma, t) = (\gamma, \lambda t) \ \text{ for } \ t \neq 0 \ \text{ and } \lambda \cdot (x, X, 0, \lambda) = (x, \frac{1}{\lambda} X, 0) \ .$$

The previous constructions applied to the product groupoid of G with the group $\mathbb{R}$ leads to a *local compactification* of G_{ad} :

$$\overline{G_{ad}} := \big((G \times \mathbb{R})_{ad} \setminus G^{(0)} \times \{0\} \times \mathbb{R}\big)/_{\mathbb{R}^*} = G_{ad} \sqcup G \setminus G^{(0)} \sqcup \mathcal{P}(\mathfrak{A}G) \ .$$

Notice that the map equal to identity on $G \times \mathbb{R}^*$ and which sends $\mathfrak{A}G \times \{0\}$ on $G^{(0)} \times \{0\}$ extends to a proper map $\tau : \overline{G_{ad}} \longrightarrow G \times \mathbb{R}$.

1.2. **Spaces of functions on G_{ad}.** A smooth function f on G_{ad} will be denoted $f = (f_t)_{t \in \mathbb{R}}$ where $f_t \in \mathcal{C}^\infty(G)$ for $t \neq 0$ and $f_0 \in \mathcal{C}^\infty(\mathfrak{A}G)$. We may introduce several spaces of functions on G_{ad} and on its crossed product by the $\mathbb{R}^*$ action:

- *The Schwartz algebra:* $S_c(G_{ad})$ is the restriction to G_{ad} of smooth functions on $\overline{G_{ad}}$ which are ∞-flat (i.e. vanish as well as all the derivatives) outside G_{ad} and whose support is sent by τ on a compact subset of $G \times \mathbb{R}$.
- *The ideal:* $\mathcal{J}_0(G) \subset C_c^\infty(G_{ad})$ of *rapidly decreasing functions* at 0 is made of smooth functions which are ∞-flat outside $G \times \mathbb{R}^*$.
- *The ideal:* $\mathcal{J}(G) \subset S_c(G_{ad})$ is the set of functions $f = (f_t)_{t \in \mathbb{R}}$ which satisfy that for any $g \in C_c^\infty(G)$, $(f_t * g)_{t \in \mathbb{R}^*}$ and $(g * f_t)_{t \in \mathbb{R}^*}$ belong to $\mathcal{J}_0(G)$.
- *The $\rtimes\mathbb{R}^*$ version :* we define similarly $\mathcal{S}_c(G_{ad} \rtimes \mathbb{R}^*)$ and the ideal $\mathcal{J}(G)_\rtimes \subset \mathcal{S}_c(G_{ad} \rtimes \mathbb{R}^*)$.

The ideal $\mathcal{J}(G)$ enables us to recover the pseudodifferential operators on G, precisely we have :

Theorem 1. *For $f = (f_t)_{t\in\mathbb{R}} \in \mathcal{J}(G)$ and $m \in \mathbb{Z}$ (and even $m \in \mathbb{C}$) let*

$$P = \int_0^{+\infty} t^m f_t \frac{dt}{t} \quad \text{and} \quad \sigma : (x,\xi) \in \mathfrak{A}^*G \setminus G^{(0)} \mapsto \int_0^{+\infty} t^m \widehat{f}(x, t\xi, 0)\frac{dt}{t}$$

Then P is a pseudodifferential operator of order $-m$ on G and its principal symbol is σ. Moreover any pseudodifferential operator on G is of this form.

Let us denote by $\mathcal{J}_+(G)$ the image of $\mathcal{J}(G)$ under the restriction of functions to $\overline{G_{ad+}} := \tau^{-1}(G \times \mathbb{R}_+)$ and $J_+(G)$ its closure in $C^*(G_{ad})$.

Theorem 2. *A completion of $\mathcal{J}_+(G)$ into a bimodule $\mathcal{E}$ leads to a Morita equivalence between $\Psi_0^*(G)$ and $J_+(G) \rtimes \mathbb{R}_+^*$.*

In the special case of the pair groupoid $G = V \times V$ over a smooth manifold V, this last theorem was proved abstractly by Aastrup-Melo-Monthubert-Schrohe in [1]. In this situation G_{ad} is the tangent groupoid and the previous construction leads to an ideal $\mathcal{J}$ of $\mathcal{S}_c(G_{ad})$ which can be (restricted and) completed into a full $\Psi_0^*(G) = \Psi_0(V)$ module $\mathcal{E}$ which satisfies $\mathcal{K}(\mathcal{E}) \simeq J_+ \rtimes \mathbb{R}_+^*$.

2. The Boutet de Monvel calculus

Let $M = V \times \mathbb{R}_+$ be a manifold with boundary embeded in the smooth manifold $\widetilde{M} = V \times \mathbb{R}$. The aim here is to define a pseudodifferential calculus adapted to M.

Let P be a pseudodifferential operator on $\widetilde{M}$, $f \in \mathcal{C}_c^\infty(M)$ and $\tilde{f}$ the extension of f by 0 on $\widetilde{M}$. The computation $P(\tilde{f})$ gives a function on $M \setminus \partial M$ which may not admit a limit on $\partial M = V \times \{0\}$. This leads to the notion of *transmitting property* : the operator P has the transmitting property when for any smooth function $f \in \mathcal{C}_c^\infty(M)$, $P(\tilde{f})$ coincides on $M \setminus \partial M$ with a smooth function on $\widetilde{M}$. In such a situation we let P_+ be the corresponding operator on $\mathcal{C}_c^\infty(M)$ and we denote by $\mathcal{P}_+^M$ the set of such operators.

When P, Q are pseudodifferential operators on $\widetilde{M}$ with the transmitting property it may happen that $P_+Q_+ \neq (PQ)_+$, thus $\mathcal{P}_+^M$ is not an algebra. To solve this problem, Boutet de Monvel defined the algebra $\mathbb{G}_M$ of *singular Green* operators whose typical elements are $P_+Q_+ - (PQ)_+$. The space $\mathcal{P}_+^M + \mathbb{G}_M$ is now an algebra. Moreover he also defined the spaces $\mathcal{K}$ of *singular Poisson* operators $\mathcal{C}_c^\infty(\partial M) \to \mathcal{C}_c^\infty(M)$ and $\mathcal{T}$ of *singular Trace* operators $\mathcal{C}_c^\infty(M) \to \mathcal{C}_c^\infty(\partial M)$, adjoint of each other and formed an algebra of 2×2 matrices :

$$\begin{pmatrix} P_+ + G & K \\ T & Q \end{pmatrix} : \mathcal{C}_c^\infty(M) \oplus \mathcal{C}_c^\infty(\partial M) \to \mathcal{C}_c^\infty(M) \oplus \mathcal{C}_c^\infty(\partial M)$$

where $P_+ \in \mathcal{P}_+^M$, $G \in \mathbb{G}$, $K \in \mathcal{K}$, $T \in \mathcal{T}$ and Q is an ordinary pseudodifferential operator on $V = \partial M$. See [2] for the original construction and [6] for a detailed description.

The product of a singular Poisson operator with a singular Trace operator gives a singular Green operator. Thus, forgetting $\mathcal{P}_+^M$, one gets an algebra of 2×2 matrices which (almost) gives a Morita equivalence between the two corners : $\mathbb{G}_M$ and the algebra of pseudodifferential operators on V. The comparison of this phenomenon with the result of Theorem 2 leads to the following [5] :

Theorem 3. *For $f \in \mathcal{J}$ and $F \in \mathcal{J}_{\rtimes}$:*

- $K_f : \mathcal{C}_c^\infty(\partial M) \to \mathcal{C}_c^\infty(M),\ u_0 \mapsto (f_t * u_0)_{t\in\mathbb{R}_+^*}$ *is a singular Poisson operator.*
- $T_f : \mathcal{C}_c^\infty(M) \to \mathcal{C}_c^\infty(\partial M),\ u \mapsto \int_0^\infty f_t * u_t\ \frac{dt}{t}$ *is a singular Trace operator.*
- $G_F : \mathcal{C}_c^\infty(M) \to \mathcal{C}_c^\infty(M)\ ; u \mapsto F * u$ *is a singular Green operator.*

Moreover, we obtain in this way all the singular Green, Trace and Poisson operators of the Boutet de Monvel calculus up to smoothing operators.

This result enables to propose a natural extension of such a calculus.
If $M = V \times \mathbb{R}$ and $G \rightrightarrows V$ is a smooth groupoid on V, the above constructions still make sense : for $f \in \mathcal{J}(G)$ and $F \in \mathcal{J}(G)_{\rtimes}$ the same formulas give *Poisson type operators* $K_f : \mathcal{C}_c^\infty(G) \to \mathcal{C}_c^\infty(G \times \mathbb{R})$, *Trace type operators* $T_f : \mathcal{C}_c^\infty(G \times \mathbb{R}) \to \mathcal{C}_c^\infty(G)$ and *Green type operators* $G_F : \mathcal{C}_c^\infty(G \times \mathbb{R}) \to \mathcal{C}_c^\infty(G \times \mathbb{R})$.

One can go one step further by considering a groupoid $\mathbb{G} \rightrightarrows M$ which is transverse to a codimension one submanifold V of M. The transversality condition means that (locally) $\mathbb{G}$ is isomorphic to $\mathbb{G}_V^V \times \mathbb{R} \times \mathbb{R}$ around $V \subset \mathbb{G}^{(0)} \subset \mathbb{G}$. By replacing the groupoid $\mathbb{G}$ around V by $(\mathbb{G}_V^V)_{ad} \rtimes \mathbb{R}^*$ one gets a groupoid $\mathbb{G}_{cg} = \mathbb{G}_{M\setminus V}^{M\setminus V} \cup \mathfrak{A}\mathbb{G}_V^V \rtimes \mathbb{R}^*$. We can produce again *Poisson type operators* K : $\mathcal{C}_c^\infty(\mathbb{G}_V^V) \to \mathcal{C}^\infty(\mathbb{G}_V \setminus \mathbb{G}_V^V)$, *Trace type operators* $T : \mathcal{C}_c^\infty(\mathbb{G}_V) \to \mathcal{C}_c^\infty(\mathbb{G}_V^V)$ and *Green type operators* $G : \mathcal{C}_c^\infty(\mathbb{G}_V \setminus \mathbb{G}_V^V) \to \mathcal{C}^\infty(\mathbb{G}_V \setminus \mathbb{G}_V^V)$.

This is a first step in the way to establish generalized Boutet de Monvel index theorems. See [5] for details.

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