Strange Attractors*

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Introduction: Deterministic Systems with a Touch of Fantasy

Systems with an irregular, non periodic, "chaotic" time evolution are frequently encountered in physics, chemistry, and biology. Think for example of the smoke rising in still air from a cigarette. Oscillations appear at a certain height in the smoke column, and they are so complicated as to apparently defy understanding. Although the time evolution obeys strict deterministic laws, the system seems to behave according to its own free will. Physicists, chemists, biologists, and also mathematicians have tried to understand this situation. We shall see how they have been helped by the concept of *strange attractor,* and by the use of modern computers.

A strange attractor consists of a infinity of points, in the plane as shown on Figure 1A, or in m-dimensional space. These points correspond to the states of a chaotic system. Strange attractors are relatively abstract mathematical objects, but computers give them some life, and draw pictures of them. (See the illustrations, and note that the computer may mark only a finite number of points.) It may well be that the reader has access to a computer, and can reproduce some of the "experiments" described below.

The Description of Time Evolution: Dynamical Systems

We specify the state of a physical, chemical, or biological system by parameters $x_1, x_2, \ldots, x_m$. A chemical system for example would be described by the concentrations of various reactants. The parameters vary with time, and we denote by

$$x_1(t), x_2(t), \ldots, x_m(t)$$

their values at time t. For simplicity we shall consider first only integer values of t (time expressed in seconds, or in years). We shall come back later to the case of continuously varying time.

How do we determine the time evolution of the system, in other words its *dynamics*? We shall admit that the parameters specifying the system at time $t + 1$ are given functions of the parameters at time t. We may thus write

$$\left.\begin{aligned} x_1(t+1) &= F_1(x_1(t), x_2(t), \ldots, x_m(t)) \\ x_2(t+1) &= F_2(x_1(t), x_2(t), \ldots, x_m(t)) \\ &\ldots \\ x_m(t+1) &= F_m(x_1(t), x_2(t), \ldots, x_m(t)) \end{aligned}\right\} \qquad (1)$$

We assume that the functions $F_1, F_2, \ldots, F_m$ are continuous and have continuous derivatives. This "technical" differentiability condition will be satisfied in our examples. We shall see later why it is important.

Given *initial values* $x_1(0), x_2(0), \ldots, x_m(0)$ for the parameters we can, using (1), compute $x_1(t), x_2(t), \ldots, x_m(t)$ successively for all positive integer times t. Thus, knowing the state of the system at time zero one may compute its state at time t. We say that the functions $F_1, F_2, \ldots, F_m$ determine a discrete time *dynamical system*. It is a *differentiable* dynamical system because we have assumed that the functions $F_1, F_2, \ldots, F_m$ have continuous derivatives.

An Example: The Hénon Attractor

Let us now examine a concrete case. Let $m = 2$, and write x, y instead of x_1, x_2. We are given

$$F_1(x, y) = y + 1 - ax^2$$
$$F_2(x, y) = bx$$

with $a = 1.4$ and $b = 0.3$. The relations (1) thus take the form

$$\left.\begin{aligned} x(t+1) &= y(t) + 1 - ax(t)^2 \\ y(t+1) &= bx(t) \end{aligned}\right\} \qquad (2)$$

Given $x(0), y(0)$ we may compute $x(t)$ and $y(t)$ for $t = 1, 2, \ldots, 10{,}000$ for instance, keeping everywhere sixteen significant figures. Done by hand this calculation would take many months and, since its interest is not obvious, nobody undertook it. For a digital computer on

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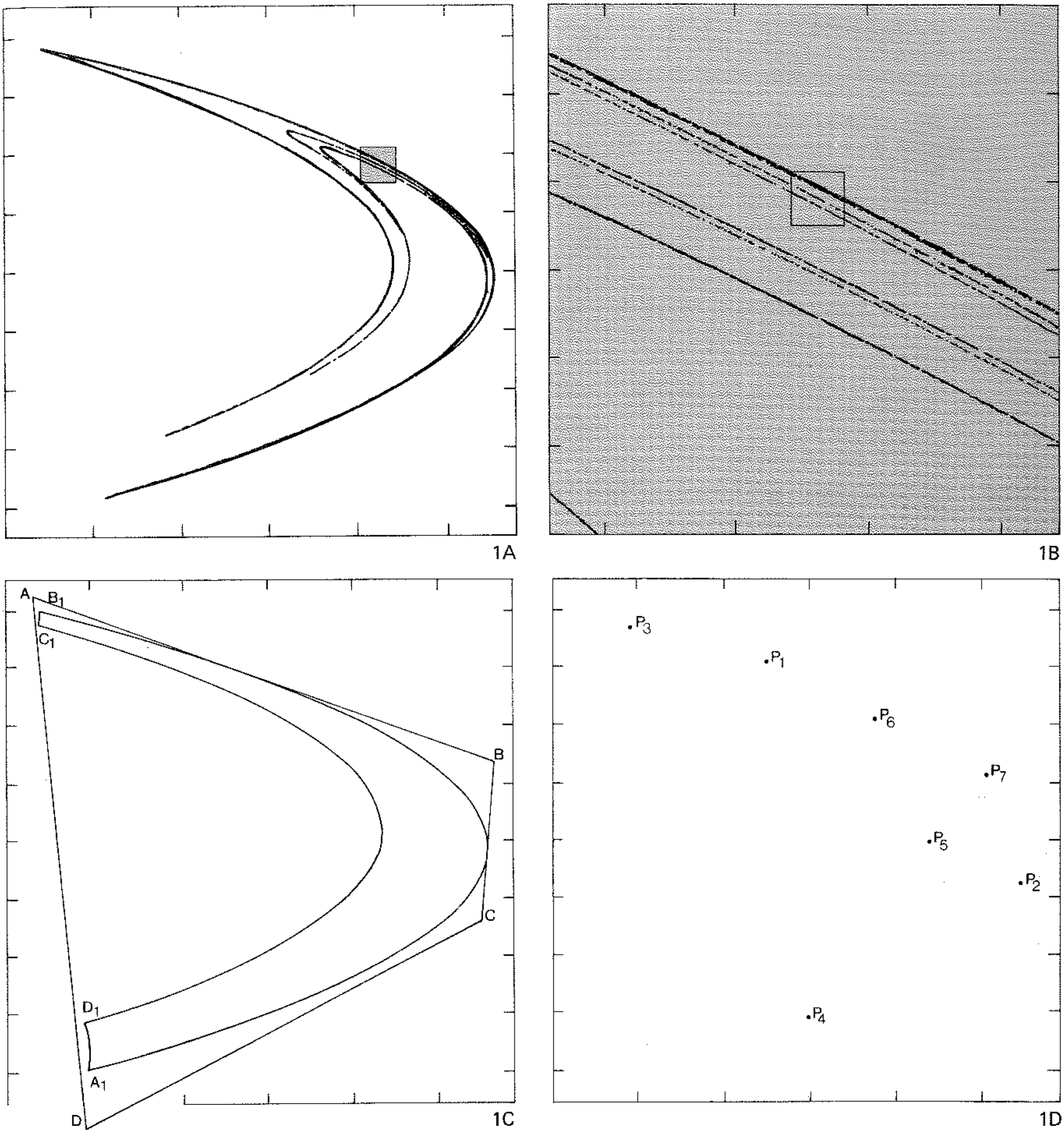


Figure 1. *The Hénon attractor.* A computer has been asked to mark points of coordinates $x(t)$, $y(t)$ for t going from 1 to 10,000. The point $(x(0), y(0))$ is given, and the following points are determined by

$$x(t+1) = y(t) + 1 - ax(t)^2, \quad y(t) = bx(t)$$

with $a = 1.4$ and $b = 0.3$. Figure 1A shows the result. The 10,000 points distribute themselves on a complex system of lines: the *Hénon attractor.* It is an example of a *strange attractor.* Magnification of the little square in Figure 1A yields 1B, and magnification of the little square in 1B would again yield a similar picture. Each new magnification resolves lines into more lines. The Hénon attractor is associated with a map of the plane which sends the point (x, y) to $(F_1(x, y), F_2(x, y))$, with $F_1(x, y) = y + 1 - ax^2$, $F_2(x, y) = bx$. In particular, the quadrilateral $ABCD$ of Figure 1C is mapped inside itself into $A_1B_1C_1D_1$. Notice that F_1, F_2 are polynomials, and therefore have continuous derivatives

$$\partial F_1/\partial x = -2ax \qquad \partial F_1/\partial y = 1$$

$$\partial F_2/\partial x = b \qquad \partial F_2/\partial y = 0$$

One can see that the surface of $A_1B_1C_1D_1$ is equal to three tenths of the surface of $ABCD$ (the factor $b = 0.3$ is given, up to sign, by the determinant of the above derivatives). In Figure 1D one has kept $b = 0.3$ but taken $a = 1.3$. The strange attractor disappears, and is replaced by the seven points of a periodic attractor.

the other hand, this boring and repetitive task is not a problem. Michel Hénon, of the observatory in Nice, did the first calculations with an HP-65 programmable pocket computer. He then went on to a more powerful machine (IBM 7040). That computer had a plotter, which marked on a sheet of paper the points with coordinates $x(t), y(t)$, for t ranging from 1 to 10,000. Figure 1A shows the picture obtained. Unexpectedly, the ten thousand points lie on a system of lines with complex structure. If the little square of Figure 1A is magnified, Figure 1B is obtained. If the square of Figure 1B were magnified, one would obtain again a similar picture, and so on, each magnification revealing lines which were not previously visible [1].

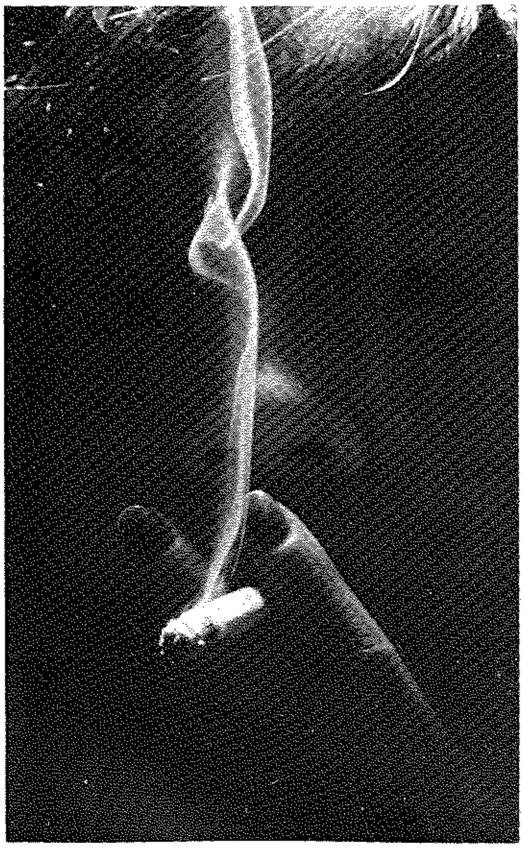
What happens if the initial point $(x(0), y(0))$ is changed? Well, for a "bad" choice $(x(t), y(t))$ will go to infinity (and in particular, leave the sheet of paper). For a "good" choice, $(x(1), y(1))$, $(x(2), y(2))$, . . ., will rapidly get close to the "noodle" of Figure 1A, and the general aspect of this picture will be reproduced after a few thousand points have been marked.

Our "noodle" is the *Hénon attractor*. It is an example of a *strange attractor*. Let me mention, among other curiosities, that the attractor may suddenly disappear when the parameters a, b in (2) are changed. Taking for instance $a = 1.3$ and $b = 0.3$ one sees the points $(x(t), y(t))$ approaching, when t increases, a set of seven points $P_1, \ldots, P_7$ (Figure 1D). Instead of a strange attractor we now have a *periodic attractor* (of period 7).

In trying to understand the Hénon attractor, it is helpful to consider the map F of the plane to itself defined by (2). If X has coordinates x and y, $F(X)$ has coordinates

$$F_1(x, y) = y + 1 - ax^2, \quad F_2(x, y) = bx$$

Call X_t the point with coordinates $x(t), y(t)$. Then $X_1 = F(X_0)$, $X_2 = F(F(X_0))$, etc . . ., X_t is obtained from X_0 by applying t times the map F. Figure 1C shows a quadrilateral $ABCD$, and its *image* $A_1B_1C_1D_1$ by F. This image is by definition the set of points $F(X)$ with X in the quadrilateral $ABCD$. Hénon has chosen the quadrilateral $ABCD$ in such a manner that it contains the image $A_1B_1C_1D_1$. Figure 1C shows that the quadrilateral is "folded in two" by the map F. If the initial point X_0 is in $ABCD$, then X_1 is in the image $A_1B_1C_1D_1$, and thus again in $ABCD$. All the points $X_1, X_2, \ldots, X_t, \ldots$ are therefore in the quadrilateral $ABCD$, and the Hénon attractor is also contained in that quadrilateral.



Smoke rising from a cigarette. – The atmosphere of Jupiter. Two of the many examples of systems whose evolution through time involves oscillations which can be described by strange attractors. (Clichés E. Rousseau & IPS)

Another Example: The Solenoid

We shall now examine an attractor in three dimensions, i.e., we shall take $n = 3$ in the formulae (1). Instead of writing explicit expressions for the functions F_1, F_2, F_3, we describe geometrically the map F of three-dimensional space to itself which they define. (This map F sends the point with coordinates x_1, x_2, x_3 to the point with coordinates $F_1(x_1, x_2, x_3), F_2(x_1, x_2, x_3), F_3(x_1, x_2, x_3)$). We suppose that F takes a ring A (the solid torus of Figure 2A), stretches it, makes it thinner, folds it, and places it in the manner drawn in Figure 2B. This figure shows both A, and its image $F(A)$ by the map F. The image $F(A)$ winds twice around the central hole of the ring A.

Starting from a point X_0 in the ring A, we write $X_1 = F(X_0)$, $X_2 = F(X_1)$, . . . Figure 2C shows the five thousand points $X_{51}, X_{52}, \ldots, X_{5050}$ (together with the set $F(A)$). A new strange attractor appears. Since the point X_0 is arbitrary in A, it is not in general on the attractor, but $X_1, X_2, X_3, \ldots$ get progressively closer to it. This is why we have marked the points starting at X_{51}. It is fascinating to observe the plotter (of the HP 9830A) draw the picture. About once per second a click is produced and a point is marked, in an apparently random manner. It takes a fairly long time before one can guess the final form of the attractor.

The attractor of Figure 2C has been called a *solenoid*. Indeed the picture is suggestive of electric wires around an axis. To understand this structure, note that the solenoid is contained not only in the ring A of Figure 2A, but also in its image $F(A)$ drawn in Figure 2C, and also in $F(F(A))$, $F(F(F(A)))$, . . . The image $F(A)$ is the inside of a tube which winds twice around the central hole of A, $F(F(A))$ is in a thinner tube which winds four times around the hole, $F(F(F(A)))$ is still thinner and winds around eight times, etc. . . . The solenoid is thus contained in very thin tubes winding around many times, and this explains how it looks.

Sensitive Dependence with Respect to Initial Conditions: How Errors Grow with Time

Remember that the parameters $x_1(t), x_2(t), \ldots, x_m(t)$ are supposed to describe a physical, chemical, or biological

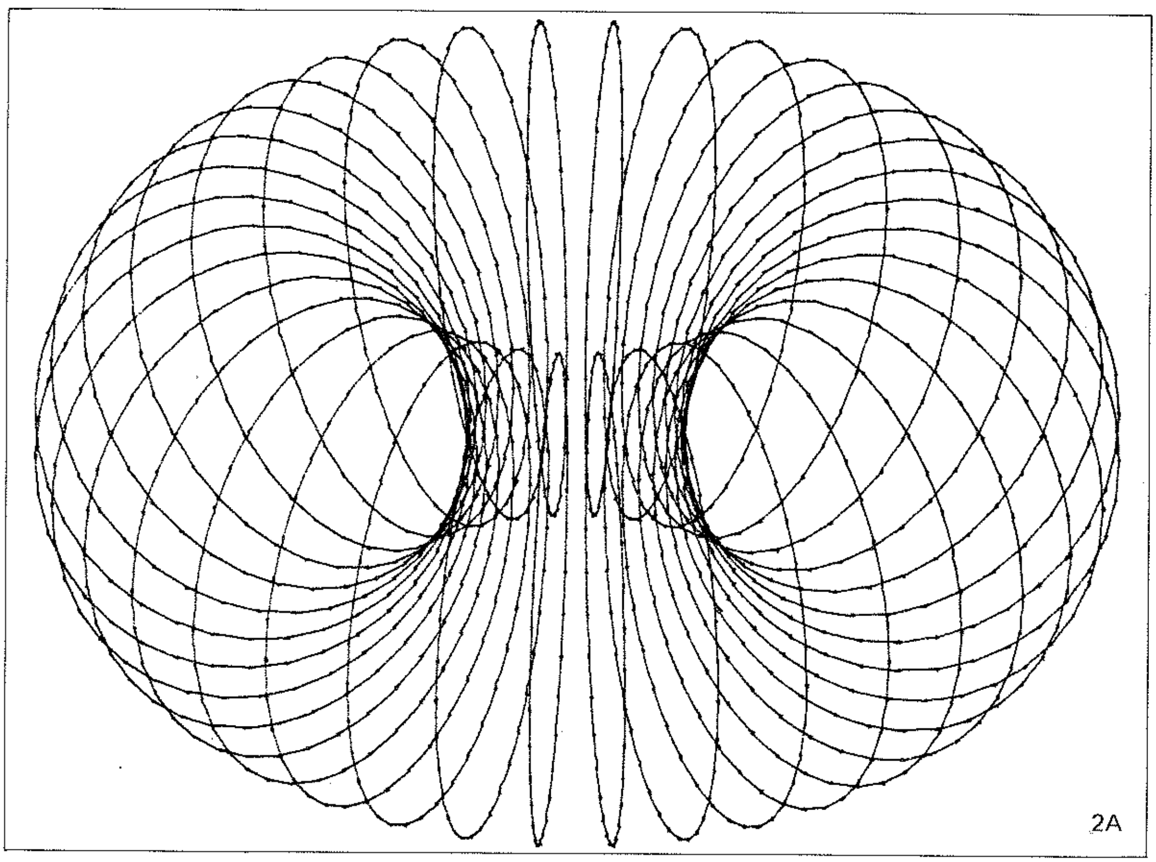
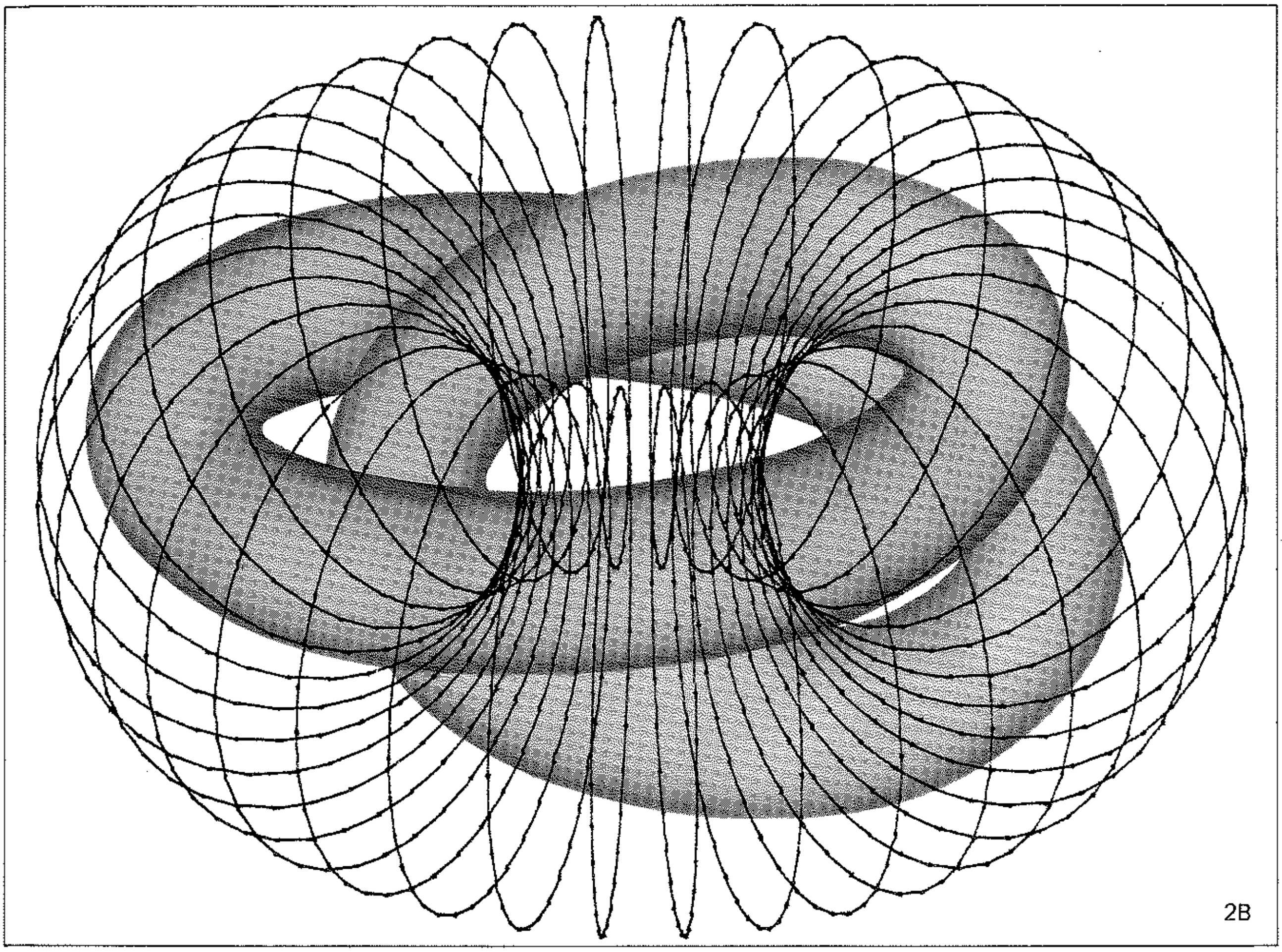
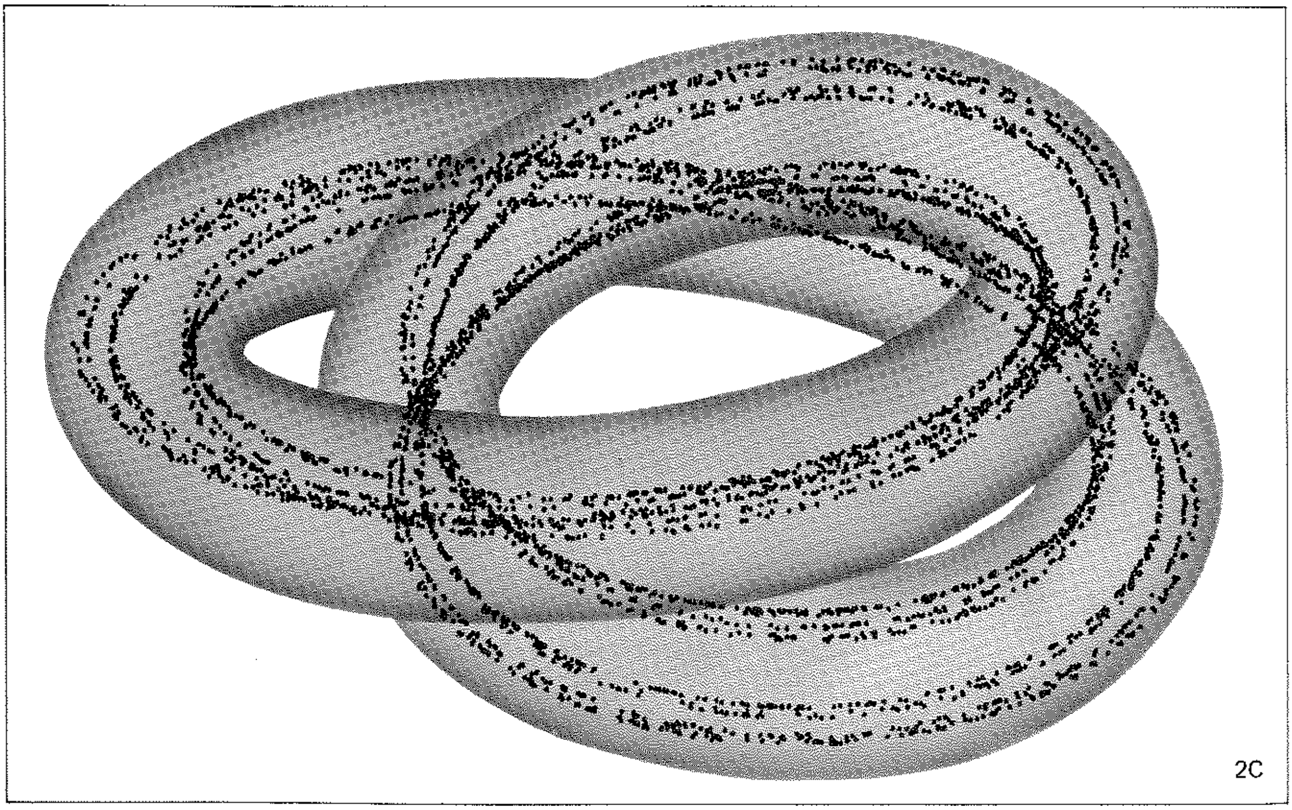


Figure 2. *The solenoid.* Figure 2A is a perspective view of a ring A in three dimensional space. A map F stretches A, makes it thinner, folds it, and places the image $F(A)$ inside A so that $F(A)$ turns twice around the central hole of A, as shown in Figure 2B. Figure 2C shows $F(A)$ again, and also 5,000 points successively defined by $X_{t+1} = F(X_t)$ starting from some initial point X_0. The 5,000 points produce a wiry structure. It is a new strange attractor, called *solenoid*.



2B



2C

system at time t. We assume that the system has a deterministic time-evolution defined by the equations (1). With what precision can we predict the evolution if the choice of the initial values $x_1(0), x_2(0), \ldots, x_m(0)$ is slightly in error, as is always the case for experimental data? How will the error increase (or decrease) with increasing t? The answer will of course depend on the given functions $F_1, F_2, \ldots, F_m$, and on the initial values $x_1(0), x_2(0), \ldots, x_m(0)$. For the two strange attractors which we have examined (the Hénon attractor and the solenoid) a small error (or uncertainty) on the initial values gives an error (or uncertainty) at time t, which increases rapidly with t.

Let us verify this assertion for the Hénon attractor. We know that there is, around the attractor, a quadrilateral *ABCD* such that the map F folds the quadrilateral in two. As Figure 1C shows, the folding in two is accompanied by stretching. Thus if X_t and X'_t correspond to initial data X_0 and X'_0 close to each other, the distance $d(X_t, X'_t)$ generally increases with t. At least this is the case as long as this distance remains small; when the distance from X_t to X'_t becomes of the order of the total size of the attractor it cannot increase any more. Numerically one finds

$$d(X_t, X'_t) \sim d(X_0, X'_0).a^t \tag{3}$$

with $a \approx 1.52$. Since $a > 1$, the factor a^t increases rapidly (exponentially) with t. Therefore *the error $d(X_t, X'_t)$ increases exponentially with time.* The rate of exponential increase is determined by a (or by its logarithm $\lambda = \ln a$ called characteristic exponent, here $\lambda \approx 0.42$).

We may argue similarly for the solenoid. The map F stretches a tube containing the solenoid and, because of this stretching, formula (3) remains valid, with a different choice of $a > 1$.

The exponential increase of errors described by formula (3) is expressed by saying that the dynamical system under consideration has *sensitive dependence on initial condition.*

Notice that to give a precise meaning to (3) we have to take $d(X_0, X'_0)$ "infinitesimal". The assumption that $F_1, F_2, \ldots, F_m$ have continuous derivatives is used here. Notice also that, for given X_0, there may be exceptional X'_0 for which the error does not grow as indicated by (3) (it may for instance decrease).

A Little Bit of Mathematics: A Definition of Strange Attractors

Let us come back to the general dynamical system described by the equations (1). We call F the map of m dimensional space to itself which sends X with coordinates $x_1, \ldots, x_m$ to $F(X)$ with coordinates $F_1(x_1, \ldots, x_m), \ldots, F_m(x_1, \ldots, x_m)$. We shall say that a bounded set A in m-dimensional space is a *strange attractor* for the map F if there is a set U with the following properties:

(a) U is an m-dimensional *neighborhood* of A, i.e., for each point X of A, there is a little ball centered at X and entirely contained in U. In particular A is contained in U.

(b) For every initial point X_0 in U, the point X_t with coordinates $x_1(t), \ldots, x_m(t)$ remains in U for positive t; it becomes and stays as close as one wants to A for t large enough. This means that A is *attracting.*

(c) There is sensitive dependence on initial condition when X_0 is in U. This makes A a *strange* attractor.

In the case of the Hénon attractor one can take for U the quadrilateral *ABCD* (Figure 1C), in the case of the solenoid one can take for U the solid torus A (Figure 2).

The above definition allows the practical determination of strange attractors in computer studies, but it is not quite complete mathematically. It is desirable also to impose the following condition.

(d) One can choose a point X_0 in A such that, arbitrarily close to each other point Y in A, there is a point X_t for some positive t. This *indecomposability condition* implies that A cannot be split into two different attractors.

It would also be necessary to make the notion of sensitive dependence on initial condition more precise. This however, leads to questions which are not too well understood. It must be said that the mathematical theory of strange attractors is difficult and, in part, still in its infancy. The solenoid is well understood, thanks to the work of Steve Smale [2] of Berkeley. By contrast, it has not been *proved* that Figures 1A and 1B do not just show a periodic orbit of very long period. The fact that the Hénon attractor exists as a strange attractor is for the time being a *belief* based on computer calculations! Perhaps our definition of strange attractors will have to be changed to adapt to more general situations. Do not take it too seriously.

It seems that the phrase "strange attractor" first appeared in print in a paper by Floris Takens (of Groningen) and myself [3]. I asked Floris Takens if he had created this remarkably successful expression. Here is his answer: "Did you ever ask God whether he created this damned universe? . . . I don't remember anything . . . I often create without remembering it . . ." The creation of strange attractors thus seems to be surrounded by clouds and thunder. Anyway, the name is beautiful, and well suited to these astonishing objects, of which we understand so little.

Besides strange attractors, we should remember that there are also non strange attractors. For instance *attracting fixed points.* The point A is an attracting fixed point if X_t gets arbitrarity close to A when t increases, provided X_0 is in a neighborhood U of A. In that case of course errors decrease when t increases, and there is no sensitive dependence on initial conditions. Attracting fixed points belong to the *periodic attractors*, which we have already met (Figure 1D). A periodic attractor has a finite number of points.

Attracting fixed points have been known for a long time. They describe an asymptotically stationary situation, i.e., for large t, X_t practically no longer depends on t. In the same manner the periodic attractors describe an asymptotically periodic situation. Scientists had got used to the notion that the asymptotic behavior of natural phenomena should be stationary, or perhaps periodic. Only recently did interest arise in the "chaotic" behavior, with sensitive dependence on initial condition, which occurs in many natural phenomena.

Strange Attractors in Nature

To describe the systems which they encounter, physicists, chemists, and biologists use equations of the type (1), or differential equations in the case of continuous time. One should not underestimate the amount of idealization implied by such a description. Certain parameters are selected as variables $x_1, \ldots, x_m$, others are ignored, and various simplifications are made. Idealization is a basic ingredient of all natural sciences, and a serious scientist must show that the natural system which he considers obeys deterministic laws of the type (1) with a good approximation. He may then look for strange attractors, either by the direct study of experimental results, or by computer simulation. In this manner, the "chaos" which occurs in certain phenomena becomes understandable, and it may be hoped that this understanding will lead to practical applications.

The study of "chaotic" or "turbulent" time evolutions in natural phenomena is now only at its beginnings. Progress is slow, due in part to experimental difficulties, in part to the insufficient development of the theory. In the absence of a satisfactory mathematical theory, computers play an important role in the interpretation of data.

We shall now discuss some examples of chaotic phenomena, and in particular the problem of fluid turbulence. In order to do this we shall have to use a continuous time t rather than a discrete time.

The Lorenz Attractor, and Meteorological Predictions

In order to define differentiable dynamical systems with continuous time we replace the equations (1) by differential equations

$$\left.\begin{aligned} &\frac{d}{dt}x_1(t) = G_1(x_1(t), \ldots, x_m(t)) \\ &\ldots \\ &\frac{d}{dt}x_m(t) = G_m(x_1(t), \ldots, x_m(t)) \end{aligned}\right\} \quad (4)$$

If $G_1, \ldots, G_m$ satisfy certain conditions (existence of continuous derivatives, etc.) the equations (4) uniquely determine the functions $x_1(t), \ldots, x_m(t)$ of time t when the initial data $x_1(0), \ldots, x_m(0)$ are known. The equations (4) thus define a deterministic evolution with continuous time, just as the equations (1) defined a deterministic evolution with discrete time.

Let us take for example $m = 3$, and write $x_1(t) = x$, $x_2(t) = y$, $x_3(t) = z$. We consider the differential equations

$$\left.\begin{aligned} \frac{dx}{dt} &= -\sigma x + \sigma y \\ \frac{dy}{dt} &= -xy + rx - y \\ \frac{dz}{dt} &= xy - bz \end{aligned}\right\} \quad (5)$$

with $\sigma = 10$, $b = 8/3$, and $r = 28$. Figure 3 shows the trajectory of the point (x, y, z) corresponding to the solution of these equations with initial condition (0, 0, 0). It appears that we have here again a strange attractor, and one can show that there is indeed sensitive dependence on initial condition.

The attractor of Figure 3 is the *Lorenz attractor*, named after Edward Lorenz, professor in the Meteorology department of the Massachusetts Institute of Technology. The equations (5) were indeed first written and studied by Lorenz [4]. These equations give an approximate description of a horizontal fluid layer heated from below. The warmer fluid formed at the bottom is lighter. It tends to rise, creating convection currents. If the heating is sufficiently intense, the convection takes place in an irregular, turbulent manner. This phenomenon takes place for instance in the earth atmosphere, and since it has sensitive dependence on initial condition, it is understandable that meteorologists cannot predict the state of the atmosphere with precision a long time in advance. The work of Ed Lorenz thus gives some theoretical excuse to the well-known unreliability of weather forecasts.

Fluid Turbulence: One of the Great Unsolved Problems of Theoretical Physics

Turbulence is a phenomenon easily produced by opening the tap over the bath tub or the kitchen sink. The nature

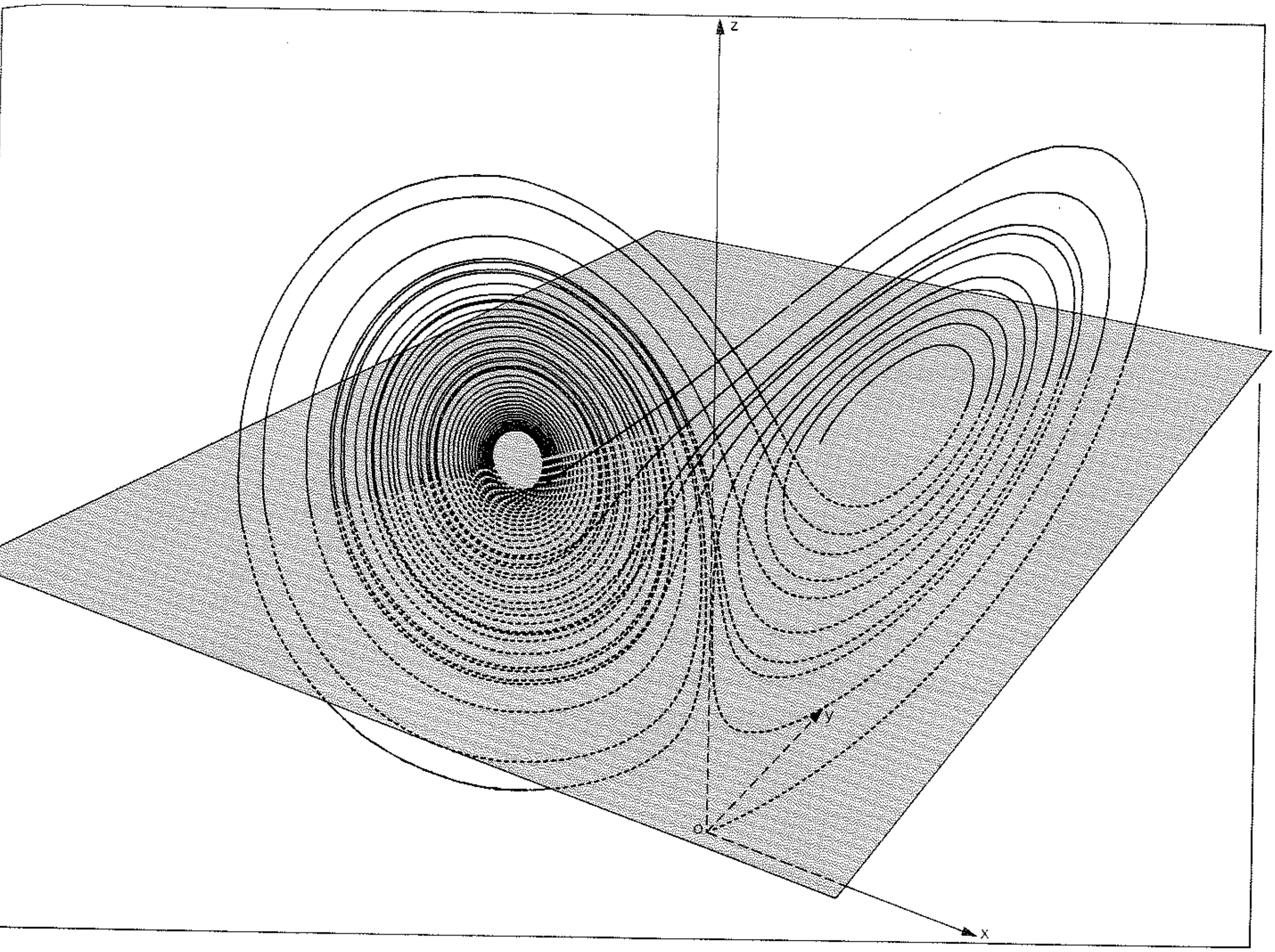


Figure 3. *The Lorenz attractor.* This beautiful figure has been obtained by Oscar Lanford, of Berkeley. It illustrates a new strange attractor, the *Lorenz attractor,* which is approached by the solutions of the Lorenz system of equations:

$$\frac{dx}{dt} = -10x + 10y, \quad \frac{dy}{dt} = -xz + 28x - y, \quad \frac{dz}{dt} = xy - \frac{8}{3}z.$$

Lanford has chosen the solution which starts from the origin (0, 0, 0) at time $t = 0$. It makes one loop to the right, then a few loops to the left, then to the right, and so on in irregular manner. One follows the solution here for fifty loops. The part below the plane $z = 27$ is drawn as a dotted line. If one would take, instead of (0, 0, 0), a nearby initial condition, the new solution would soon deviate from the old one, and the numbers of loops to the left and to the right would no longer be the same. There is *sensitive dependence with respect to initial conditions.* The Lorenz equations are suggested by a problem of atmospheric convection. Edward Lorenz has used the sensitive dependence on initial condition observed with the above equations to justify the imprecision of weather forecasting.

of turbulence remains however rather mysterious and controversial.

One may in principle describe the time evolution of a viscous fluid by equations of the form (4). The number m will have to be taken infinite, because the state of the fluid at a given instant of time requires an infinite number of variables for its description. We admit that there are no further problems, and write $X(t)$ and G instead of $x_1(t), x_2(t), \ldots$, and $G_1, G_2, \ldots$. The equations (4) can then be written in compact form as

$$\frac{d}{dt} X(t) = G_\mu(X(t)) \tag{6}$$

We have introduced a parameter μ in (6) to indicate the intensity of external action on the fluid. (If there is no external action, viscosity brings the fluid to rest, and there is no turbulence). In the example of the tap, μ might give the degree of opening of the tap. In the convection equations (5) of Lorenz, μ is replaced by r, which is proportional to the temperature difference between the top and the bottom of the fluid layer. In many hydrodynamical problems, the role of μ is taken by a parameter called *Reynolds number*.

If $\mu = 0$, i.e., if there is no external action, the fluid tends to a state of rest $X(t) = X_0$. This state corresponds to an attracting fixed point X_0 for our dynamical system.

For small μ one observes again a steady state $X(t) = X_\mu$. As μ is further increased, one often sees periodic oscillations in the fluid. This means that asymptotically

$$X(t) = f(\omega t)$$

where f is a function of period 2π and ω the frequency of the oscillations. This situation corresponds to a periodic attractor for continuous time, i.e., a circle or "attracting limit cycle". For sufficiently large μ, the fluid motion becomes irregular, chaotic: turbulence has set in.

When I became interested in turbulence, around 1970, Lorenz' paper of 1963 was not known to physicists and mathematicians. The most popular theory of turbulence was that of Lev D. Landau of Moscow [5]. According to this theory, the time evolution of a turbulent fluid is asymptotically given by

$$X(t) = f_k(\omega_1 t, \omega_2 t, \ldots, \omega_k t) \tag{7}$$

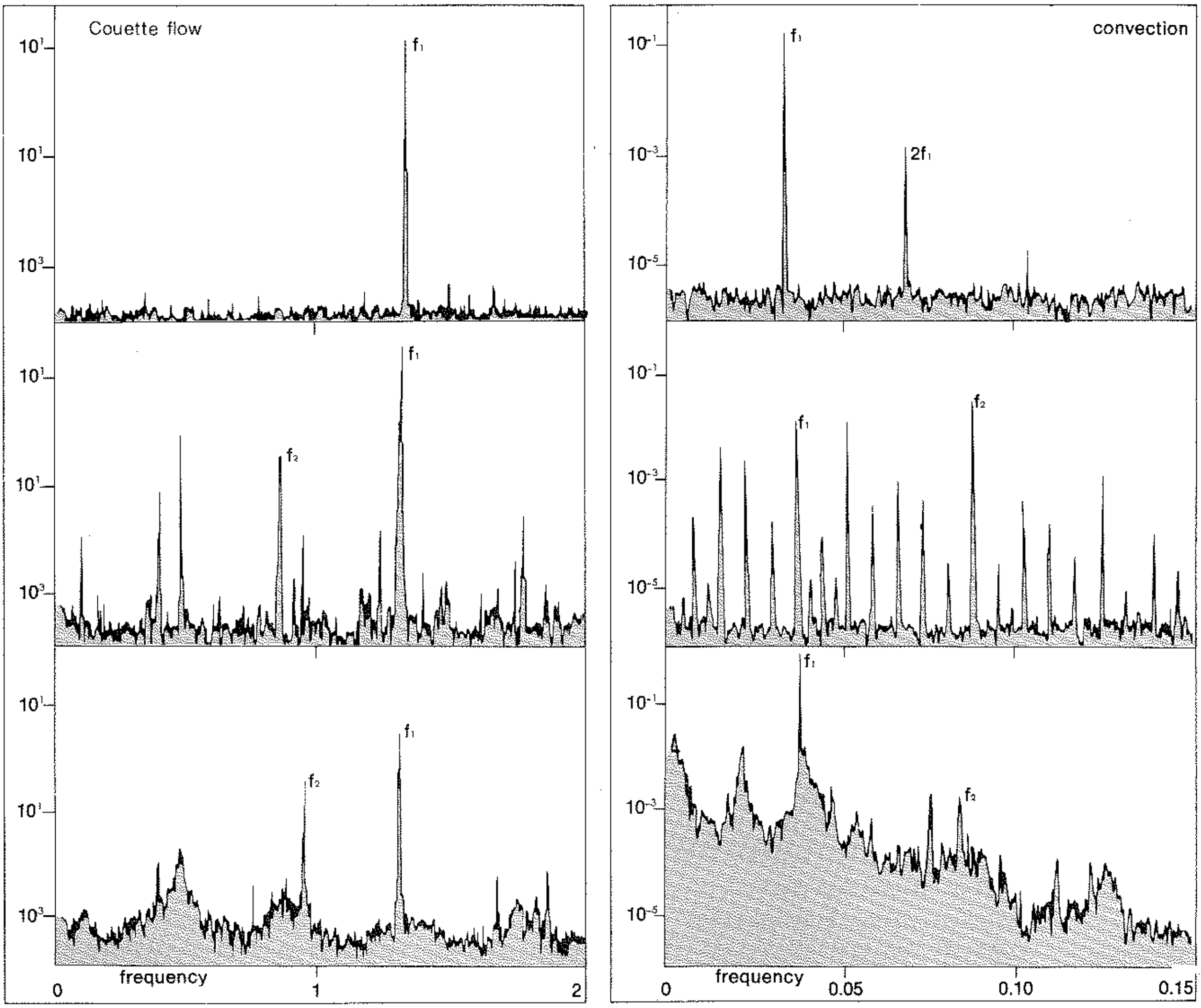


Figure 4. *Frequency spectra.* A frequency analysis of the time dependence of a phenomenon is possible, whether this dependence is periodic or not. One obtains thus a "frequency spectrum" giving the square of the amplitude associated with each frequency. The spectra on the left of the figure have been measured by R. Fenstermacher for the Couette flow (the interval between two coaxial circular cylinders is filled with fluid, and the inner cylinder is rotated at constant speed). The spectra on the right have been measured by S. Benson for a convective flow (a liquid layer is heated from below, the hot liquid formed below is lighter and rises, producing convection currents).

The different spectra shown correspond to different speeds of rotation (Couette) or different intensities of heating (convection). The spectra at the top contain isolated peaks corresponding to a certain frequency and its harmonics: the system is *periodic*. The spectra in the middle row exhibit several independent frequencies: the system is *quasi periodic*. The spectra at the bottom show some wide peaks on a background of *continuous spectrum*, this suggests that a strange attractor is present. Notice that the frequency spectra are shown with a logarithmic vertical scale.

where f_k is a periodic function of period 2π in each of its arguments, and $\omega_1, \omega_2, \ldots, \omega_k$ are independent frequencies. A function of t of the form (7) is called quasiperiodic. (One can see that the corresponding quasiperiodic attractor is a k-dimensional torus). A quasiperiodic function has a non periodic, irregular aspect, suggestive of turbulence. However a small change in initial conditions simply replaces $\omega_1 t, \ldots, \omega_k t$ by $\omega_1 t + \alpha_1, \ldots, \omega_k t + \alpha_k$ with small $\alpha_1, \ldots, \alpha_k$. There is thus no sensitive dependence on initial conditions.

It was tempting to appeal to strange attractors rather than quasiperiodic attractors to interpret turbulence. A mathematical argument against quasiperiodic attractors is their fragility. My attention had been drawn on this fragility, or absence of "structural stability" by the seminars of René Thom at the Institut des Hautes Etudes Scientifiques (Bures-sur-Yvette). By a small perturbation of (6) one can destroy a quasiperiodic attractor and, if $k \geqslant 3$, obtain a strange attractor. I had published this result with Floris Takens [3] in 1971, and we had on this occasion proposed the idea that turbulence is described by strange attractors. While structural stability may not be as important an aspect of things as we thought at the time, the connection between strange attractors and turbulence was a lucky idea.

It remained to be seen if strange attractors would give a better description of turbulence than quasiperiodic attractors. There is no direct experimental test of sensitive dependence on initial condition in hydrodynamics. One may however do a frequency analysis of the fluid velocity at a point, considered as a function of time. The function giving the square of the amplitude versus the frequency is called *frequency spectrum* (see Figure 4). For a quasiperiodic function the frequency spectrum is formed of discrete peaks at the frequencies $\omega_1, \ldots, \omega_k$ and their linear combinations with integer coefficients. By contrast if the time evolution is governed by a strange attractor one may obtain a continuous frequency spectrum.

It was known that the frequency spectrum of a turbulent fluid is continuous, but this fact was attributed to the accumulation of a large number of independent frequencies simulating, in the limit, a continuous spectrum. Recently (1974–75), delicate experiments performed by Guenter Ahlers at Bell Labs (Murray Hill, NJ), Jerry Gollub

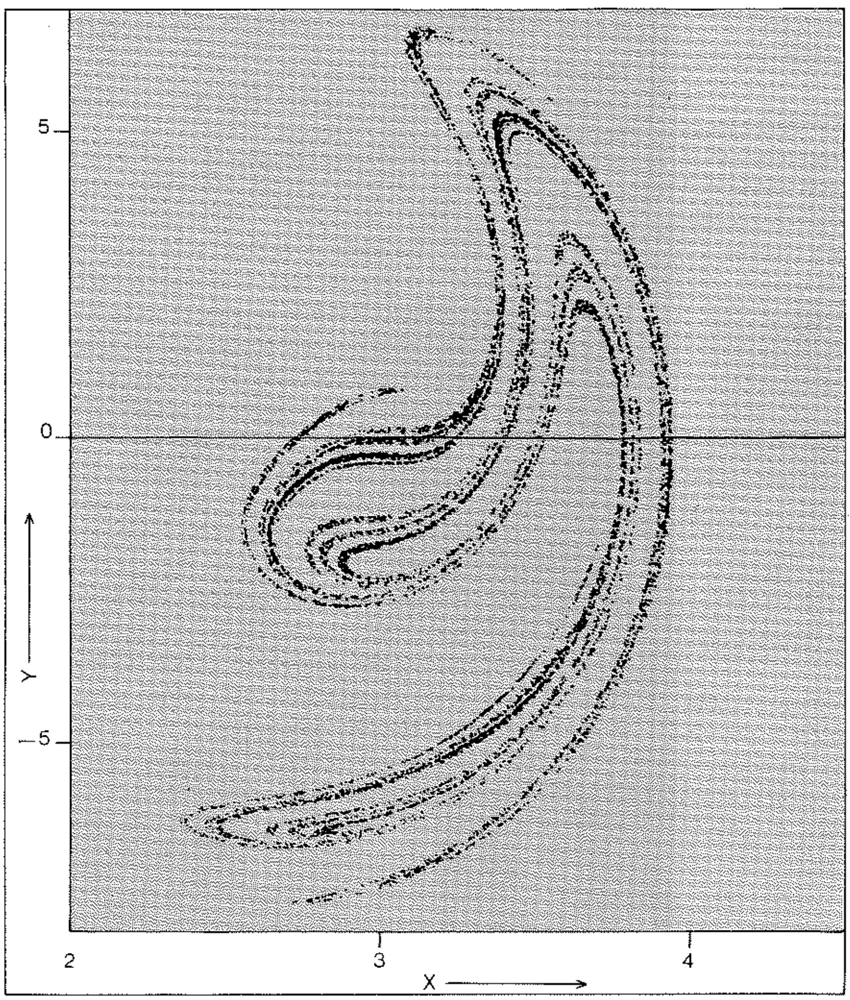


Figure 5. *A Japanese attractor.* This picture shows a strange attractor invented by Yoshisuke Ueda, of Kyoto University. It is obtained by solving the differential equation

$$\frac{d^2x}{dt^2} + k\frac{dx}{dt} + x^3 = B \cos t$$

for $k = 0.1$ and $B = 12$, and marking the points with coordinates $x(2n\pi)$, $\frac{d}{dt}x(2n\pi)$ for integer n (discrete time $t = 2n\pi$). Depending on the initial conditions one obtains either the above attractor, or a single point (attracting fixed point). Y. Ueda has studied strange attractors numerically for a number of years on analog and digital computers. Esthetically, his pictures are probably the finest obtained to this date.

and Harry Swinney at City College (New York) [6], and others, have shown that things happen differently. When one increases the parameter μ describing the system, the transition to the continuous spectrum characteristic of turbulence is rapid. There is no progressive accumulation of many independent discrete frequencies. So it seems that the onset of turbulence may well correspond to the appearance of strange attractors.

Other Chaotic Phenomena: Turbulence Everywhere

It should here be mentioned that frictionless mechanical systems (conservative systems) give rise neither to strange attractors, nor in fact to attractors at all. Actually, a theorem of mechanics, Liouville's theorem, asserts that time evolution preserves volumes in phase space. This prevents the volume contraction which occurs near an attractor. On the other hand, conservative systems often show sensitive dependence on initial condition.

The physico-chemical systems which give rise to strange attractors are the *dissipative systems,* i.e., those for which a "noble" form of energy (for instance mechanical, electrical, or chemical energy) is changed into heat [7]. These systems actually exhibit an interesting behavior only if they are constantly fed some noble energy, otherwise they go to rest.

One knows chemical reactions which are periodic in time (see inset). I asked in 1971 a chemist, specialist of these periodic reactions, if he thought that one would find chemical reactions with chaotic time dependence. He answered that if an experimentalist obtained a chaotic record in the study of a chemical reaction, he would throw away the record, saying that the experiment was unsuccessful. Things, fortunately, have changed, and we now have several examples of non periodic chemical reactions.

The magnetism of the earth perhaps gives an example of a strange attractor. It is known that the earth magnetic field reverses itself at irregular intervals. This phenomenon occurred at least sixteen times in the last four million years. Geophysicists have written "dynamo equations" with chaotic solutions which describe irregular changes of direction of a magnetic field. There is however as yet no quantitatively satisfactory theory.

Ecologists have studied non periodic models in population dynamics. If m species have, in the year $t + 1$, populations $x_1(t + 1), \ldots, x_m(t + 1)$ determined by the equations (1) in terms of the populations in the year t, one may expect strange attractors to occur. In fact, already for $m = 1$, the equation

$$x(t + 1) = Rx(t)\,(1 - x(t))$$

gives rise to nonperiodic behavior [8].

One imagines easily that strange attractors may play a role in economics, where periodic processes (economic cycles) are well-known. In fact, let us suppose that the macroeconomical evolution equations contain a parameter μ describing, say, the level of technological development. By analogy with hydrodynamics we would guess that for small μ the economy is in a steady state and that, as μ increases, periodic or quasiperiodic cycles may develop. For high μ chaotic behavior with sensitive dependence on initial condition would be present. This discussion is some

A Periodic Chemical Phenomenon: The Belousov-Zhabotinski Reaction

For about twenty years now, an oscillating reaction has been known to chemists. The oscillations have a period of the order of one minute, and continue for perhaps an hour, until the reagents are exhausted. If reagents are added continuously, while reaction products are removed, the oscillations proceed periodically forever. The reaction is, roughly speaking, the oxydization of malonate by bromate, catalyzed by Cerium. The experiment is fairly easy to realize: here is the recipe.

Malonic acid	0.3 M
Cerous nitrate	0.005–0.01 M
Sulfuric acid	3.0 M
Sodium bromate	0.05–0.01 M
Ferroin	a little

M means "molar", for instance sulfuric acid occurs at the concentration of 3 moles per liter. Ferroin is an oxidation reduction indicator (obtained by mixing in water a small amount of *o*-phenanthroline and ferrous sulfate). In practice one prepares one solution with part of the reagents (in water), and another solution with the rest of the reagents. The oscillating reaction starts when the two solutions are mixed. Perhaps the mathematical reader should be warned that diluting sulfuric acid produces heat and requires caution (see a chemistry text). The sulfuric solution should be allowed to cool before being mixed with the other solution, otherwise the oscillations will not be seen. During the reaction, the ferroin turns from blue to purple to red, making the oscillations visible. At the same time the Cerium ion changes from pale yellow to colorless, so that all kinds of hues are produced.

The Belousov-Zhabotinski reaction, which we just described, caused astonishment and some disbelief among chemists when it was discovered. Other periodic chemical reactions have now been discovered, in particular in systems of biological origin. One speculates on the physiological significance of these reactions, but little is really known with certainty.

what metaphorical, but its conclusions are suggestive, and a more detailed analysis may be useful.

To conclude this list of examples, let me mention a dynamical system of vital interest to everyone of us: the heart. The normal cardiac regime is periodic, but there are many nonperiodic pathologies (like ventricular fibrillation) which lead to the steady state of death. It seems that great medical benefit might be derived from computer studied of a realistic mathematical model which would reproduce the various cardiac dynamical regimes.

The application of the ideas which we have discussed often poses serious methodological problems. How does one maintain constant experimental conditions, and how does one make precise measurements? In any case, the recognition of the role of strange attractors in many problems is a great conceptual progress. The nonperiodic fluctuations of a dynamical system do not necessarity indicate an experiment spoilt by mysterious random forces; they often point to a strange attractor, which one may try to understand [9].

I have not spoken of the esthetic appeal of strange attractors. These systems of curves, these clouds of points suggest sometimes fireworks or galaxies, sometimes strange and disquieting vegetal proliferations. A realm lies there of forms to explore, and harmonies to discover.

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