CONTACT STRUCTURES

AND

SYMPLECTIC FIBRATIONS OVER THE CIRCLE

I

CONSTRUCTION OF CONTACT MANIFOLDS

Contact forms and structures

A *contact form* on a $2n+1$-manifold V is a 1-form α whose product with $(d\alpha)^n$ nowhere vanishes.

A tangent hyperplane field ξ on V is a *contact structure* if it is the kernel field of a contact form α. Thus $d\alpha$ induces a symplectic form on ξ at each point whose conformal class is independent of α.

A contact manifold is a manifold equipped with a contact structure.

The geometry of a contact form has two components:

- the contact structure $\xi = \ker \alpha$;
- the kernel of $d\alpha$, which is a line field transversal to ξ.

The unique vector field ∂_α in $\ker d\alpha$ such that $\alpha(\partial_\alpha) = 1$ is called the *Reeb vector field* of α.

The universal local model

The staandard contact structure ξ_0 on $\mathbf{R}^{2n+1}$ is the kernel field of the 1-form

$$\alpha_0 = dz + \sum_{i=1}^{n} x_i \, dy_i,$$

where $(x, y, z) \in \mathbf{R}^n \times \mathbf{R}^n \times \mathbf{R}$.

The Reeb vector field of α_0 is

$$\partial_{\alpha_0} = \partial_z \, .$$

Theorem (Darboux). *In dimension $2n+1$, every contact structure is locally modeled on ξ_0 and every contact form is locally modeled on α_0.*

Stability and contact isotopies

Theorem (Libermann). *For any contact manifold* (V, ξ)*, projection* $TV \to TV/\xi$ *induces a bijection between vector fields preserving* ξ *and sections of* TV/ξ*.*

Theorem (Gray). *On a closed manifold, every component of the space of contact structures is an isotopy class.*

Two contact structures are *isotopic* is there exists a diffeomorphism isotopic to the identity that takes one to the other.

Corollary. *On a closed manifold, contact structures form at most countably many isotopy classes.*

Proof of Gray's Theorem

Given a path $\xi_t = \ker \alpha_t$ of contact structures, the required isotopy ϕ_t must satisfy

$$\phi_t^* \alpha_t \wedge \alpha_0 = 0 \quad \text{for all } t \in [0,1].$$

This condition is equivalent to

$$\frac{d}{dt}(\phi_t^* \alpha_t) \wedge \phi_t^* \alpha_t = 0,$$

and so to

$$\big((\nu_t \cdot \alpha_t) + \dot{\alpha}_t\big) \wedge \alpha_t = 0$$

where ν_t is the infinitesimal generator of ϕ_t.

Define ν_t as the vector field in ξ_t that is $d\alpha_t$-dual to $-\dot{\alpha}_t \,|\, _{\xi_t}$, and integrate ν_t to get ϕ_t.

Contact structures and strict pseudoconvexity

W: a complex manifold;
$V \subset W$: a cooriented real hypersurface.

The complex hyperplanes of TW tangent to V form a (co) oriented hyperplane field ξ on V.

If the Levi form

$$Q = d\alpha(.,\sqrt{-1}.)\,|_{\xi} \quad (\text{where } \xi = \ker \alpha)$$

is non-degenerate, ξ is a contact structure.

A typical case is when V is *strictly pseudoconvex*, *i.e.*, when Q is positive definite.

Example. The complex hyperplanes tangent to the boundary of the unit ball in $\mathbf{C}^{n+1}$ form the standard contact structure on $\mathbf{S}^{2n+1}$.

Strictly plurisubharmonic functions

F: a complex manifold;
$J: TF \to TF$: the associated almost complex structure (multiplication by $\sqrt{-1}$).

A function $f: F \to \mathbf{R}$ is *strictly plurisubharmonic* if the 2-form

$$\omega_f = -dd^{\mathbf{c}} f, \quad \text{where} \quad d^{\mathbf{c}} f = (df \circ J),$$

satisfies $\omega_f(v, Jv) > 0$ for all $v \neq 0$.

Properties. *Let $f: F \to \mathbf{R}$ be a strictly plurisubharmonic function. Then:*
- *ω_f is an exact Kähler – and hence symplectic – form;*
- *the regular level sets of f are strictly pseudoconvex hypersurfaces;*
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Stein manifolds

A complex manifold F (possibly with boundary) is a *Stein manifold* if it admits a positive, proper, and strictly pluriśubharmonic (Morse) function for which ∂F is a regular level set.

Theorem (Lefschetz). *Every Stein manifold of real dimension $2n$ has the homotopy type of a n-dimensional cell complex.*

Theorem (Grauert). *A complex manifold is a Stein manifold iff it embeds analytically and properly into a complex affine space.*

Exact symplectic manifolds

A symplectic manifold (W, ω) is *exact* if the symplectic form ω is exact.

any primitive λ of $\omega = d\lambda$ is a *Liouville form* and the ω-dual vector field defined by

$$\vec{\lambda} \vee \omega = \lambda$$

is a *Liouville vector field.*

Any Liouville vector field expands ω exponentially since

$$\vec{\lambda} \cdot \omega = d(\vec{\lambda} \vee \omega) = d\lambda = \omega.$$

Example. Let α be a contact form on V. The exact symplectic manifold $(V \times \mathbf{R}, \omega = d\lambda)$ where

$$\lambda = e^t \alpha, \quad t \in \mathbf{R},$$

is the *symplectization* of the contact manifold $(V, \xi = \ker \alpha)$.
The Liouville vector field of λ is $\vec{\lambda} = \partial_t$.

Contact structures and symplectic convexity

(W, ω): an exact symplectic manifold;
$V \subset W$: a cooriented hypersurface.

V is ω-*convex* if there exists a Liouville form λ that induces a positive contact form on V, or equivalently, whose Liouville vector field $\vec{\lambda}$ is positively transversal to V.

The induced contact structure on V depends on λ but its isotopy class does not provided V is closed.

Example. In $(\mathbf{R}^{2n}, \omega_0 = \sum dx_i\, dy_i)$, every hypersurface bounding a compact domain star-shaped around the origin is ω_0 convex since the radial vector field $\sum x_i \partial_{x_i} + y_i \partial_{y_i}$ is Liouville.

Convex symplectic manifolds

A symplectic manifold (F, ω) will be called *convex* if it is compact, exact and has ω-convex boundary.

The *completion* of (F, ω) is then the symplectic manifold $(\widetilde{F}, \widetilde{\omega})$ where

$$\widetilde{F} = F \cup_{\partial} \big(\partial F \times [0, \infty)\big)$$

and $\widetilde{\omega}(p,t) = e^t \omega(p)$, $(p,t) \in \partial F \times [0, \infty)$.

Theorem (Gromov-Eliashberg). *let F be a compact Stein manifold and $\omega_f = -dd^{\mathbf{c}} f$ for some positive strictly plurisubharmonic function f which is constant on ∂F. Then (F, ω_f) is a convex symplectic manifold and the i isotopy class of $\widetilde{\omega}_f$ on $\widetilde{F}$ is independent of f.*

Weinstein manifolds

(F, ω): a symplectic manifold.

A function $f\colon F \to \mathbf{R}$ is ω*-convex* if it admits a gradientlike vector field ν that expands ω exponentially, *i.e.*, satisfies $\nu \cdot \omega = \omega$.

(F, ω) is a *Weinstein manifold* if it admits a positive, proper, and ω-convex Morse function f for which ∂F is a regular level set.

Example. Every Stein manifold is a Weinstein manifold: the gradient ∇f of any spsh function f with respect to the metric $\omega_f(., J.)$ satisfies

$$\nabla f \cdot \omega_f = d(\nabla f \vee \omega_f) = -dd^{\mathbf{c}} f = \omega_f .$$

Theorem (Eliashberg). *Every Weinstein manifold is symplectically diffeomorphic to a Stein manifold.*

Construction of contact structures
(F,ω): a convex symplectic manifold;
$\phi\colon F \to F$: a symplectic diffeomorphism equal to the identity near $K = \partial F$;
$\Sigma(F,\phi)$: the mapping torus of ϕ, that is

$$\Sigma(F,\phi) = \big(F \times [0,1]\big)/\sim, \quad (p,1) \sim (\phi(p),0).$$

$\partial\,\Sigma(F,\phi) = K \times \mathbf{S}^1$ — because $\phi = \mathrm{id}$ near K – so

$$\overline{\Sigma}(F,\phi) = \Sigma(F,\phi) \cup_\partial (K \times \mathbf{D}^2)$$

is a closed manifold.

Theorem 1. *The manifold $\overline{\Sigma}(F,\phi)$ admits a contact structure.*

Contactization of diffeomorphisms

β: a Liouville form making the boundary ω-convex;
γ: the contact form induced by β on $K = \partial F$.

$\phi^*\omega = \omega$ so $\phi^*\beta - \beta$ is closed.
Assume that $\phi^*\beta - \beta = -dh$ is exact.

The 1-form $\widetilde{\alpha} = \beta + dt$ is a contact form on $\widetilde{V} = F \times \mathbf{R}$ and is invariant under the transformation

$$\widetilde{\phi}\colon (p,t) \longmapsto \big(\phi(p), t + h(p)\big).$$

Hence $\widetilde{\alpha}$ induces a contact from α on the quotient $\widetilde{V}/\widetilde{\phi}$ which is the mapping torus $\Sigma(F,\phi)$.

To conclude, extend $\xi = \ker\alpha$ to $\overline{\Sigma}(F,\phi)$ by letting

$$\xi = \ker(\gamma + r^2\,d\theta) \quad \text{on } K \times \mathbf{D}^2.$$

Isotopy Lemma

Lemma. *We can deform ϕ through symplectic diffeomorphisms equal to the identity near $K = \partial F$ to a diffeomorphism ϕ_1 for which $\phi_1^* \beta - \beta$ is an exact form.*

Proof. Let μ be the form $\beta - \phi^* \beta$ and $\vec{\mu}$ the ω-dual vector field. Since μ is closed, $\vec{\mu}$ preserves ω.

Let ψ_s denote the flow of $\vec{\mu}$. All maps $\phi \circ \psi_s$ are symplectic diffeomorphisms equal to the identity near the boundary and $\phi_1 = \phi \circ \psi_1$ works because

$$\phi_1^* \beta - \beta = -\psi_1^* \mu + \psi_1^* \beta - \beta$$

and $\psi_s^* \mu = \mu$ for all s while

$$\psi_s^* \beta - \beta = s\mu + dh_s$$

where $h_s = \int_0^s \psi_u^* (\vec{\mu} \vee \beta)\, du$.

Open books

An *open book* in a closed manifold V is a pair (K, θ) consisting of:

- a codimension 2 submanifold $K \subset V$ with trivial normal bundle;
- a fibration $\theta \colon V \setminus K \to \mathbf{S}^1$ which, in a neighborhood $K \times \mathbf{D}^2$ of K, is the normal angular coordinate.

The binding K and the fibers of θ are naturally cooriented.

Let (K, θ) be an open book and ν a vector field transversal to the fibers of θ and equal to ∂_θ near K.

If ϕ denotes the first return map of ν on an arbitrary fiber F of θ, then V can be identified with $\overline{\Sigma}(F, \phi)$.

Contact structures and open books

Definition. A contact structure ξ on a closed manifold V is *carried* by an open book (K, θ) if it is the kernel of a 1-form α satisfying the following:

- α induces a positive contact form on K;
- $d\alpha$ induces a positive symplectic form on each fiber F of θ.

We say that α is *adapted* to (K, θ).

Note that the contact orientation of (K, α) coincides with the boundary orientation of the symplectic manifold $(F, d\alpha)$.

Exercise. The contact structure obtained on $\overline{\Sigma}(F, \phi)$ by contactization (Theorem 1) is carried by the canonical open book.