Topology of Algebraic Morphisms

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This paper is dedicated to LÊ Dũng Trang.

Abstract. We report on topological properties of a projective morphism $f : X \to V$ of algebraic varieties, namely the degeneration of Leray's spectral sequence as a consequence of Hard Lefschetz theorem and Hodge theory. A formula of the decomposition theorem, well adapted to a Hodge theoretic proof is given and is illustrated in the isolated singularity case with coefficients.

1. Introduction

1.1. From Lefschetz theorems to the decomposition theorem. The classical Hard Lefschetz theorem on a non singular complex projective variety $X \hookrightarrow \mathbb{P}^m$ of dimension n with a class $\eta \in H^2(X, \mathbb{Q})$ of an hyperplane section $X_t = H_t \cap X$, states that the iterated cup-product

$$H^{n-i}(X, \mathbb{Q}) \xrightarrow{\eta^i} H^{n+i}(X, \mathbb{Q}) \tag{1.1}$$

is an isomorphism for $i \in [0, n]$.

With the development of sheaf theoretical constructions of cohomology around 1950, a general tendency in the school of algebraic geometry was to extend results on varieties to families (the relative case with opposition to the absolute case). For a smooth projective morphism $f : X \to V$ where $\dim X = n$ and $\dim V = s$, the class of an hyperplane section defines a section $\eta \in R^2 f_* \mathbb{Q}_X$ inducing by cup-product a map

$$R^{n-s-i} f_* \mathbb{Q}_X \xrightarrow{\eta^i} R^{n-s+i} f_* \mathbb{Q}_X \tag{1.2}$$

which is an isomorphism.

If we imagine to study the cohomology of X via the fibration on V, we see that we need to consider the cohomology of V with coefficients in $R^i f_* \mathbb{Q}_X$. In this case the coefficients are local systems said to be of geometric origin, a terminology which refers to specific properties of these local systems due to the fact that the stalk at a point v is isomorphic to the cohomology of a fibre $H^i(X_v, \mathbb{Q})$ at v so that

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such local systems underly various geometric invariants. One of their important features is that they underlie polarized variations of Hodge structures, and many of the important theorems on *geometric local systems* follow uniquely from this underlying Hodge theory.
For example, in the abelian category of finite local systems which is noetherian and artinian, *the sheaves* $R^i f_* \mathbb{Q}_X$ *are semisimple*, which means that they split into a finite direct sum of irreducible local subsystems (with no nontrivial local subsystems).
The study of such geometric local systems has become a central object in the study of cohomology of algebraic varieties.

The isomorphisms η^i in the relative case are compatible with the Hodge structure on the cohomology of the fibres X_t of f. Deligne, using Hodge theory, deduced the degeneration of Leray's spectral sequence, from the relative version of Lefschetz result. By definition, such spectral sequence is associated to the canonical filtration τ on $Rf_*\mathbb{Q}_X$ defined by truncation, that is to say that if $\mathcal{I}_X$ is an injective resolution of $\mathbb{Q}_X$, then the filtered complex $(f_*\mathcal{I}_X, \tau)$ is defined up to a filtered quasi-isomorphism in the derived category of filtered complexes on the base V; then the associated spectral sequence corresponding to the filtration τ is defined up to isomorphism. Such filtration τ defines a filtration L on the cohomology of X

$$H^j(V, R^i f_* \mathbb{Q}_X) \Longrightarrow Gr_L^i H^{i+j}(X, \mathbb{Q})$$

and the degeneration statement asserts that there exists natural isomorphisms:

$$Gr_L^i H^{i+j}(X, \mathbb{Q}) \simeq H^j(V, R^i f_* \mathbb{Q}_X)$$

In particular there exists non canonical isomorphisms of rational vector spaces

$$H^n(X, \mathbb{Q}) \simeq \oplus_{i+j=n} H^j(V, R^i f_* \mathbb{Q}_X)$$

The degeneration translates in the derived category of sheaves on V, into a non-canonical decomposition of the derived direct image complex as a direct sum of its cohomology

$$Rf_*\mathbb{Q}_X \simeq \oplus R^i f_* \mathbb{Q}_X[-i]. \tag{1.3}$$

Local Invariant cycle. Let γ be a path in V then $f^{-1}(\gamma)$ is a smooth locally trivial fibre bundle (Ehresmann theorem) hence trivial when lifted on $[0,1]$, that is we can define a diffeomorphism $[0,1] \times F \xrightarrow{\varphi} (f^{-1}(\gamma))$ and deduce a diffeomorphism $X_{\gamma(0)} \simeq X_{\gamma(1)}$ as a composition of the following isomorphisms $X_{\gamma(0)} \simeq F \times \{0\} \simeq F \times \{1\} \simeq X_{\gamma(1)}$. When γ is a loop at v, such diffeomorphism is called the "Monodromy " acting on X_v and denoted by T_γ; it defines a linear action

$$T_\gamma^* : H^i(X_v, \mathbb{Q}) \to H^i(X_v, \mathbb{Q})$$

which is defined up to homotopy, since the action on cohomology of two homotopical loops is the same, so that we can consider the action of the fundamental group $\Pi_1(V, v)$ with base point $v \in V$ on $H^i(X_v, \mathbb{Q})$, giving rise to a representation.
The relation with Hodge theory is an important tool to study the variation of the complex structure of the fibre, since Ehresmann theorem does not apply to the complex structure but only to the underlying differential structure. Fixing a point v_0 and transporting the Hodge structure from neighbouring points v will result in a " variation of Hodge structures" on the cohomology at v_0. The general study of this variation is illustrated in a series of articles by Griffiths that lead to the definition of polarized VHS.
Another feature applied by Deligne to prove the invariant cycle theorem, is that

the invariants of the monodromy action form a constant sub-Hodge structure of the various Hoge structures of the fibres, compatible with the isomorphism:

$$H^0(V, R^i f_* \mathbb{Q}_X) \simeq H^i(X_v, \mathbb{Q})^{\Pi_1(V,v)} \tag{1.4}$$

These techniques apply also in the following local situation.
Let $f : X \to D$ be a projective morphism onto the complex disc, smooth outside $0 \in D$ and consider the action of the fundamental group $\Pi_1(D^*, t)$ on the cohomology of a fibre $H^i(X_t, \mathbb{Q})$ at a point t of D^*, then the following restriction morphism is surjective

$$H^i(X, \mathbb{Q}) \to H^i(X_t, \mathbb{Q})^{\Pi_1(D^*,t)} \tag{1.5}$$

1.2. The decomposition theorem for projective morphisms. A natural question is to find how far we can relax the hypothesis and keep in the same time the result. In fact the theorems as stated are false for a non necessarily smooth projective morphism. In order to formulate similar results in the presence of singularities of spaces as well of the morphism, various objects and tools introduced in the last two decades proved to be fundamental objects in the study of the topology as well of the geometry of varieties and lead to spectacular extensions of the results [1]. In particular the following are basic subjects in the theory
- Thom-Whitney stratification
- Perverse sheaves and Intermediate extension of a local system.
- General Intersection theory on cohomology.

The theorem of Ehresman which lead to the structure of local system on the higher direct image $R^i f_* \mathbb{Q}_X$ fails for a general morphism. For algebraic morphisms, Thom - Whitney stratification theory give the following structure of a proper morphism $f : X \to V$.

There exist finite Whitney stratifications $\mathfrak{X}$ of X and $\mathcal{S} = \{S_l\}_{l \leq d}$ of V (dimension $S_l = l$) such that for each connected component S of an $\mathcal{S}$ stratum S_l of V:

(1) *$f^{-1}S$ is a topological fibre bundle over S, union of connected components of strata of $\mathfrak{X}$, each mapped submersively to S.*
(2) *Local topological triviality: for all $v \in S$, there exist an open neighbourhood $U(v)$ in S and a stratum preserving homeomorphism $h : U \times f^{-1}(v) \simeq f^{-1}(U) : f \circ h = p_U$ where p_U is the projection on U.*

This statement can be found in an article by Lê D. T. and Teissier B. [9].

Since the restriction to a strata f/S is a locally trivial topological bundle, the higher direct cohomology sheaf $(R^i f_* \mathbb{Q}_X)/S$ is locally constant on S. Then we say that $R^i f_* \mathbb{Q}_X$ is constructible on V and $Rf_* \mathbb{Q}_X$ is cohomologically constructible on V.

The category of perverse sheaves. At this point we need to introduce the derived category $D^+(V, \mathbb{Q})$ of $\mathbb{Q}-$sheaves on a variety V. When we have a sequence of two morphisms $X \xrightarrow{f} V \xrightarrow{g} Z$, Leray's spectral sequence for the composition morphism $h = g \circ f$ starts with the terms $R^p g_* R^q f_* \mathbb{Q}$ and converges to a graded cohomology $Gr^L R^{p+q} h_* \mathbb{Q}$ with respect to a filtration L, which means in particular that we cannot recover from $R^q f_* \mathbb{Q}$ on V the original cohomology of the fibres of X but only the graded part. To remedy such problem, Verdier introduced the derived category $D^+(V, \mathbb{Q})$ of complexes of $\mathbb{Q}-$sheaves on V, so to make a complex equivalent to its various resolutions needed in the definition of derived functors. He kept the complexes as object of the category but modified the morphisms in two steps.

First by considering an equivalence of morphisms of complexes up to homotopy, second by inverting the quasi-isomorphisms (that is morphisms of complexes inducing isomorphisms on cohomology). In this way, in the previous example, we keep the complex $Rf_*\mathbb{Q}$ to which we apply Rg_* so that $Rg_*(Rf_*\mathbb{Q}) \simeq Rh_*\mathbb{Q}$ in $D^+(Z,\mathbb{Q})$, thus we recover completely the cohomology of the fibres of h.
Inside the category $D^+(V,\mathbb{Q})$, we did distinguish the subcategory of cohomologically constructible sheaves.
Once more a subcategory which is abelian, called the category of perverse sheaves, has been introduced in [1] following earlier work in [7]. It appeared to be a fundamental object in the study of topological and geometrical properties of the morphism f.

A complex of sheaves K in $D^+(V,\mathbb{Q})$ is defined to be perverse if the following property is satisfied: there exists a stratification $\mathcal{S}$ of V such that for each strata S, $i_S : S \to V$, the restriction $H^n(i_S^* K) = 0$ for $n > -dimS$ and $H^n(Ri_S^! K) = 0$ for $n < -dimS$.

When K is constructible, these conditions show that the restriction of K to the open strata, is reduced to a local system $\mathcal{L}$ in degree $-dimX$.

The perverse truncation. The main interest in the subcategory of perverse sheaves follows from the construction of a cohomological functor defined on the derived category $D^+(V,\mathbb{Q})$ with value in the category of perverse sheaves, constructed inductively with respect to a stratification of V. Namely, the notion of perverse truncation ${}^p\tau^i$ of a complex K is constructed in [1] and then the notion of $i-$th perverse cohomology ${}^p\mathcal{H}^i(K)$ is defined as the cone of the morphism ${}^p\tau^{i-1}(K) \to {}^p\tau^i(K)$ so to fit in a triangle ${}^p\tau^{i-1}(K) \to {}^p\tau^i(K) \to {}^p\mathcal{H}^i(K)$. Perverse cohomology sheaves in various degrees fit together in a long exact sequence in the abelian category and in fact such exact sequence is the best way to compute these objects, as in any cohomology theory.

The intermediate extension. Research in the above field has been motivated first by the discovery by Goresky and MacPherson [7] of special objects called Intersection complexes. They are uniquely defined by local systems on locally closed subsets of V. Their construction use the above Whitney stratification on a singular variety V and in an essential way the local topological triviality of the various strata S_l.
Normal section and Link. Near each point $v \in S_l \subset V$, we can suppose locally V embedded in an affine space $\mathbb{C}^n$, then an affine subspace intersecting transversally the tangent space T_{S_l} at v intersects an adapted neighbourhood $U(v)$ of V in a subspace $N_{S_{l,v}}$ called the normal section to S_l at v such that $N_{S_{l,v}} - \{v\}$ retracts by deformation onto the link $L_{S_{l,v}}$ which is a fundamental topological invariant of the singularity at v.
In the abelian category of perverse sheaves which is noetherian and artinian, the irreducible sheaves are Intersection complexes defined by irreducible local systems.The following is Deligne's construction of Intersection complexes.

Let $\mathcal{S} = \{S_l\}_{l \leq d}$ of V (dimension $S_l = l$) and let $j_0 : V - \overline{S}_0 \to V$, $j_l : V - \overline{S}_l \to V - \overline{S}_{l-1}$ for $0 < l < d$ denotes the embedding. The Intermediate extension compatible with $\mathcal{S}$ of a local system $\mathcal{L}$ on the big open strata S_d is defined as:

$$j_{!*}\mathcal{L}[d] = \tau_{\leq -1} Rj_{0*} \cdots \tau_{\leq -l-1} Rj_{l*} \cdots \tau_{\leq -d} Rj_{d-1*}\mathcal{L}[d]$$

where for all sheaves $\mathcal{F}$ constructible with respect to $\mathcal{S}$, we have

$$(Rj_{l*}\mathcal{F})_v \simeq R\Gamma(L_{S_{l,v}}, \mathcal{F})$$

where $L_{S_{l,v}}$ is the link of S_l at v, then $\tau_{\leq -l-1}$ truncates the cohomology up to degree $\leq -l-1$.

The Decomposition theorem. Now we are in a position where we can state the version of the decomposition for a projective morphism and a geometric local system ((1.1), [1], 6.2.4) so that we don't need to invoke explicitly the Hodge theoretical properties which underly such local system and can refer to ([1], 6.2.10)

Let $f : X \to V$ be a projective morphism and $\mathcal{L}$ be a geometric local system on a smooth open subset U of X, $j : U \to X$, dim. $X = n$. The map defined by iterated cup-product with the class of an hyperplane section $\eta \in R^2 f_* \mathbb{Q}_X$

$$ {}^p\mathcal{H}^{-i}(Rf_*(j_{!*}\mathcal{L}[n])) \xrightarrow{\eta^i} {}^p\mathcal{H}^i(Rf_*(j_{!*}\mathcal{L}[n])) \tag{1.6}$$

is an isomorphism.

Then, the degeneration of the perverse Leray's spectral sequence defined by the perverse filtration on $Rf_*(j_{!*}\mathcal{L}[n])$ follows and leads to the decomposition:

There exists a non canonical isomorphism in the derived category

$$Rf_*(j_{!*}\mathcal{L}[n])) \xrightarrow{\simeq} \oplus_i {}^p\mathcal{H}^i(Rf_*(j_{!*}\mathcal{L}[n]))[-i] \tag{1.7}$$

Differential modules. The category of local systems is equivalent to the category of flat connections on bundles. Each flat connection $(\mathcal{E}, \nabla)$ defines a DeRham complex denoted by $\Omega^*(\mathcal{E})$ which is a resolution of the corresponding local system.
More generally the category of holonomic coherent modules $\mathcal{M}$ with ring operators defined by the sheaf of differential operators $\mathcal{D}_X$ on a smooth manifold V correspond to the category of perverse sheaves on V via the DeRham realization of a module. In this context an adequate Hodge theory constructed in [10] leads to the above decomposition theorem for polarized variation of Hodge structures.
A summary of the main ingredients of the theory in the normal crossing divisor (NCD) case figures in the last section of the appendix.
In the text , for any functor Γ, we denote its derived functor by $R\Gamma$, in particular for a closed embedding $i : Y \to X$, $i^!$ is the functor of cohomology with support denoted sometimes $\underline{\Gamma}_Y$ its derived functor is $Ri^! = R\underline{\Gamma}_Y$ hence $R^j i^! = \underline{H}^j_Y$ while $R\Gamma(Y, i^!) = R\Gamma_Y$. Now we can discuss the results in this article.

2. Statement of the results

First we introduce the notations and invariants needed to state a decomposition formula well adapted to the geometry of the morphism and use of Hodge theory.

Adapted stratification. Let $f : X \to V$ be an algebraic morphism where dim.$V = d$ and consider a local system $\mathcal{L}$ on a smooth open subset U of X of dimension n. Let $j : U \to X$ denotes the embedding and $j_{!*}\mathcal{L}[n]$ the intermediate extension, then we can choose the above stratifications $\mathfrak{X}$ of X and $\mathcal{S} = \{S_l\}_{l \leq d}$ of V (dimension $S_l = l$) such that the complexes $Ri^!_{Y_l}(j_{!*}\mathcal{L}[n])$ of cohomology with support in $Y_l = f^{-1}(S_l)$ and $i^*_{Y_l}(j_{!*}\mathcal{L}[n])$ are cohomologically locally constant on each connected component of the various strata in Y_l.

Intersection morphism. Let $f : X \to V$ be a morphism of algebraic varieties and suppose X of dim. n over $\mathbb{C}$. We consider a point $v \in V$ with fibre $X_v =$

$f^{-1}(v)$ and the embedding $\alpha : X_v \hookrightarrow X$. We recall here a cohomological version of the Intersection product which appears in the construction of the mixed Hodge structure (MHS) of an isolated singularity [5] as well in the proof of the purity theorem in [2].

Definition 2.1. The Intersection morphism I on the fibre of $f : X \to V$ is defined as the composition morphism

$$H^i_{X_v}(X,\mathbb{Q}) \xrightarrow{\alpha_*} H^i(X,\mathbb{Q}) \xrightarrow{\alpha^*} H^i(X_v,\mathbb{Q}), \quad I = \alpha^* \circ \alpha_* \tag{2.1}$$

For a smooth X, I is related to the Intersection product.

More generally, the Intersection morphism plays an important role in the study of the decomposition theory with coefficients for general morphism as it does in [2] for constant coefficients.

Let $S_l \subset V$ be a strata of dimension l, $i_{S_l} : S_l \to V$, $i_{Y_l} : Y_l \to X$ the locally closed embeddings and $j_{S_l} : S_l \to \overline{S_l}$ (resp. $i_{\overline{S_l}} : \overline{S_l} \to V$) the embeddings into the closure of S_l (resp. V). Let $Y_l := f^{-1}(S_l) \subset X$, $i_{Y_l} : Y_l \hookrightarrow X$, $j : X - Y_{d-1} \to X$, $f_l : Y_l \to S_l$ the restriction of f, then the Intersection morphism is defined for a local system $\mathcal{L}$ on $X - Y_{d-1}$ as follows:

$$I : R^i f_*(i_{Y_l!} R i^!_{Y_l} j_{!*}\mathcal{L}[n]) \to R^i f_* j_{!*}\mathcal{L}[n] \to R^i f_*(i_{Y_l *} i^*_{Y_l} j_{!*}\mathcal{L}[n]) \tag{2.2}$$

Equivalently I is defined on the fibre of a point $v \in S_l$ as follows: let $N_{S_l}(v)$ be a normal section to S_l at $v \in V$, $N_{Y_l}(v) := f^{-1}(N_{S_l}(v))$, $X_v = f^{-1}(v)$, then the Intersection morphism is given by the composition of the morphisms

$$H^i_{X_v}(N_{Y_l}(v), j_{!*}\mathcal{L}[n]) \to H^i(N_{Y_l}(v), j_{!*}\mathcal{L}[n]) \to H^i(X_v, j_{!*}\mathcal{L}[n])$$

2.1. Decomposition formula. The starting point of our approach of the decomposition theorem is the following formula in terms of the above invariants

With the above notations, given a polarized variation of Hodge structures $\mathcal{L}$ on an open subset U of the variety X of dimension n, $j : U \to X$ the open embedding and a stratification of $f : X \to V$ adapted to $\mathcal{L}$. Let

$$\mathcal{L}^i_l = Im(R^{-l+i} f_{l*}(R i^!_{Y_l} j_{!*}\mathcal{L}[n]) \xrightarrow{I} R^{-l+i} f_{l*}(i^*_{Y_l} j_{!*}\mathcal{L}[n])) \tag{2.3}$$

denotes the image local system on S_l.

Suppose there exists a decomposition of the derived direct image into a finite direct sum of intermediate extensions, then necessarily the decomposition satisfy the following formula:

$$R f_* j_{!*}\mathcal{L}[n] \simeq \oplus_{l \le d, i \in \mathbb{Z}} \; i_{\overline{S_l}*} j_{S_l !*} \mathcal{L}^i_l[l-i] \tag{2.4}$$

where i is the degree of the perverse cohomology: ${}^p\mathcal{H}^i(R f_* j_{!*}\mathcal{L}[n]) \simeq \oplus_{l \le d} j_{S_l !*}\mathcal{L}^i_l[l]$.

This result is proved in *proposition* 3.1.

The main difficulty in the transcendental proof of the decomposition consists in the fact that the topological definition of perverse sheaves is not fit to the use of Hodge theory, while this formula shows that what we need is a Hodge theory on the cohomology with support on the fibres of f as well on their cohomology and then the image of the Intersection morphism. This can be done by reducing to the normal crossing divisor case (NCD) which corresponds to Deligne's work in his paper Hodge II [3 (5)] but with coefficients. The main points of such theory are recalled in the appendix.

Our second result is *theorem* 2.3 which is an extension to the complex case of Gabber's proof of purity for an isolated singular point in [3 (2)].
This result was first understood with constant coefficients and topologically interpreted in [7 (2)] and by others.

However, better than translating the proof from positive characteristic, the advantage of combining it with a proof of the decomposition theorem that came later, was clear since they are in fact related.
Thus we come to our third result in *theorem* 4.1 which proves the decomposition formula in the case of an isolated critical value for f.

From this point of view theorem 4.1 is a crucial step and applied to a normal section to the various strata S_l, should fit into a general new proof of the decomposition formula by induction. The second part of the proof of proposition 3.1 by induction on the strata is the example to follow.
However the main ingredient of the proof should be an adequate Hodge theory corresponding to Deligne's Hodge III [3 (6)]. An option is to clarify the work of Saito in [10] as in [8]. Another option is to discover different points of view as in [2],[7] and this paper.

Remark 2.2. The local systems $\mathcal{L}_l^i$ are fundamental objects in the geometry of the morphism f. For example they vanish for $l < d$ if f and V are smooth, such result becomes clear only after the interpretation of I as intersection of cycles on the fibre in the ambient space [2].

2.2. Local purity for an isolated singularity. Let $\mathcal{L}$ be a polarized variation of Hodge structures (VHS) on the complement V^* of an isolated singularity v in an algebraic variety V of dimension n. We give a transcendental geometrical proof of the local purity result below, following a similar case in positive characteristic in [3(2)]. The statement is based on the construction of a Hodge structure on the cohomology with coefficients of the link of an isolated singularity and on its properties described just below.

Theorem 2.3 (local purity). *Let $K = i_v^* Rj_{v*}\mathcal{L}[n]$. The weight ω of the cohomology $H^i(B_v - \{v\}, \mathcal{L}[n]) \simeq H^i(K)$ defined below and isomorphic to the cohomology of the link at v, satisfy: $\omega \leq m+n+i$ if $i \leq -1$ and $\omega > m+n+i$ if $i \geq 0$, that is:*

$$\begin{aligned} &Gr_q^W H^i(B_v - \{v\}, \mathcal{L}[n]) \simeq Gr_q^W H^i(K) = 0 \text{ for } q > m+n+i \text{ if } i \leq -1 \\ &Gr_q^W H^i(B_v - \{v\}, \mathcal{L}[n]) \simeq Gr_q^W H^i(K) = 0 \text{ for } q < m+n+i+1 \text{ if } i \geq 0. \end{aligned} \tag{2.5}$$

Remark 2.4. The two statements are equivalent by the duality isomorphism $DK \simeq K[-1]$ (duality on the link of v) as follows

$$\begin{aligned} D(Gr_{m+n-q}^W H^{-j}(K)) \simeq Gr_{m+n+q}^W H^j(DK) &\simeq Gr_{m+n+q}^{W[-1]} H^j(K[-1]) \\ &\simeq Gr_{m+n+q+1}^W H^{j-1}(K) \end{aligned}$$

where $W[-1]$ is W with a shift of the indices.

The proof is based on two concepts: the properties of the weights of the MHS on open smooth or proper varieties used in exact sequences to deduce semi or local purity and Lefschetz fibrations.
The plan of the proof consists of two steps guided by earlier work of Deligne in Weil

II [3 (4)]:
i) A generalized Artin-Lefshetz hyperplane section theorem with coefficients in perverse sheaves reduces the result by induction to an ultimate case.
ii) The crucial step follows from a Deligne-Lefschetz Hard theorem type.

Properties of Hodge theory used in the proof.
We refer to [6] and [10] for the construction of a theory of Hodge structures with coefficients on the complement of a NCD, recalled with proofs in the appendix.
We list here three specific properties needed in the proof. Let Y be a NCD, $i_Y : Y \to X$ the embedding and $\mathcal{L}$ a polarized VHS of weight m on $X - Y$ of dimension n. We view $\mathcal{L}$ as a complex of sheaves concentrated in degree 0 and we denote by $\mathcal{L}[n]$ the complex shifted by n to left so to get a perverse sheaf. We exhibit in the appendix a filtration on a combinatorial logarithmic complex quasi-isomorphic to $Rj_*\mathcal{L}[n]$

$$(\Omega^*\mathcal{L}[n], \mathcal{W}, F)$$

from which we deduce the structure of mixed Hodge structure on the cohomology $H^*(X, \mathcal{L}[n])$ as well $H^r_Y(X, j_{!*}\mathcal{L}[n])$ and $H^r(Y, j_{!*}\mathcal{L}[n])$. The following first two conditions follow easily from the definition of the weight.

1) The weight ω of $H^r_Y(X, j_{!*}\mathcal{L}[n])$ satisfy: $\omega \geq m + n + r$
2) The weight of $H^r(Y, j_{!*}\mathcal{L}[n])$ satisfy: $\omega \leq m + n + r$.

The third condition is more subtle. Let Z be a smooth variety of dimension d, E a NCD in Z, $i_E : E \to Z$, $k : Z - E \to Z$, $g : X \to Z$ a projective morphism s.t. $Y = g^{-1}(E)$ is a NCD and the restriction $g' := g_{|X-Y}$ smooth, then we can check the following isomorphisms

$${}^p\mathcal{H}^i(Rg_*Rj_*\mathcal{L}[n])) \simeq {}^p\mathcal{H}^i(Rk_*Rg'_*\mathcal{L}[n])) \simeq Rk_*(R^{i-d}g'_*\mathcal{L}[n])[d]$$

since the last complex is perverse, so that we can deduce the degeneration of the perverse Leray's spectral sequence on $Rg_*Rj_*\mathcal{L}[n]$ from the corresponding degeneration of Rg'_* on $Z - E$ and by restriction to E, using $Rg_*Rj_* \simeq Rk_*Rg'_*$, we get a decomposition

$$i_E^*Rg_*Rj_*\mathcal{L}[n] \simeq \oplus_{l\leq d, i\in\mathbb{Z}}\ i_E^*{}^p\mathcal{H}^i(Rg_*Rj_*\mathcal{L}[n]))[-i] \simeq i_E^*Rk_*R^{i-d}g'_*\mathcal{L}[n][d-i]$$

We will need the following strong property stating that the degeneration occurs in the category of MHS, that is

3) The perverse ${}^p\tau$ filtration defined by g on the ambient space and induced on $H^r(Y, i_Y^*Rj_*\mathcal{L}[n])$ is a filtration by sub-MHS and we have isomorphisms of MHS

$$\begin{aligned} Gr_s^{{}^p\tau}H^r(Y, i_Y^*Rj_*\mathcal{L}[n]) \simeq & H^{r-s}(E, i_E^*{}^p\mathcal{H}^s(Rk_*Rg'_*\mathcal{L}[n])) \\ \simeq & H^{r-s}(E, i_E^*Rk_*R^{s-d}g'_*\mathcal{L}[n+d]) \end{aligned}$$

In the case where Z is a curve, this result follows from the work of Steenbrink on nearby cycles and it is equivalent to the local invariant cycle theorem [5(3)]. In general it follows from a result on bigraded modules with Hodge Lefschetz properties [8], [10].

2.2.1. *MHS on the cohomology of the Link.* The main problem here is to describe a natural MHS on the link L_v of the singular point. Given a resolution of the singularity $\pi : V' \to V$ such that $Y = \pi^{-1}(v)$ is a NCD, let $i : Y = \pi^{-1}(v) \to V'$ (resp. $j : (V' - Y) \to V'$) be the embedding of the fibre at v in V (resp. of its open complement).

It will be helpful to distinguish $\mathcal{L}' := \pi^{-1}_{|V'-Y}\mathcal{L}$ *from* $\mathcal{L}$ *on* $V - \{v\}$*, although they are isomorphic.*
Let B_v be a ball with center v induced from an ambient smooth space and let $B_Y = \pi^{-1}(B_v)$, then the cohomology of $B_Y - Y$ is isomorphic to the cohomology of $B_v - \{v\}$ which is isomorphic to the cohomology of the link L_v by retract deformation. In the first step we define the MHS on $B_Y - Y$.

2.2.2. *The Intersection form* I *and the functor* i^*Rj_*. We consider on Y

$$Ri^!j_{!*}\mathcal{L}[n] \xrightarrow{I} i^*j_{!*}\mathcal{L}[n] \to i^*Rj_*\mathcal{L}[n]$$

the triangle obtained by restriction from the classical triangle on X so that the pushforward of I on X is defined as the composition of the restriction $j_{!*}\mathcal{L}[n] \to i_*i^*j_{!*}\mathcal{L}[n]$ with the canonical map $i_*Ri^!j_{!*}\mathcal{L}[n] \to j_{!*}\mathcal{L}[n]$, hence I can be interpreted as an Intersection form defined by the intersection product in the ambient space X of two cycles on Y [2].
The morphism $I : Ri^!j_{!*}\mathcal{L} \to i^*j_{!*}\mathcal{L}$ can be realized as a morphism of MHC so that a structure of MHC on $i^*Rj_*\mathcal{L}[n]$ can be deduced as a mixed cone over I.
It happens that the filtration $\mathcal{W}$ mentioned above on $Rj_*\mathcal{L}[n]$ (and computed in the appendix (definitions A.15 and A.39)) is defined for all indices r positive or negative by a unique formula on the combinatorial complex and describes the weight on $i^*Rj_*\mathcal{L}[n]$.
The complex $i^*Rj_*\mathcal{L}[n]$ *underlies a MHC* $(i^*\Omega^*\mathcal{L}[n], W, F)$ *with natural induced filtrations* (W, F) *where* W *is defined by:*

$$W_{r+m+n} = i^*\mathcal{W}_r \text{ for } r \geq 0 \text{ and } W_{r+m+n} = i^*\mathcal{W}_{r-1} \text{ for } r < 0$$

In the following definition we use the isomorphism with the cohomology of $B_Y - Y$ to deduce the MHS on L_v from the NCD case. This is an important step since the whole article will consist on the proof of a fundamental property of this MHS stated below as local purity.

Definition 2.5. Let $\mathcal{L}'$ denotes $\pi^{-1}_{|V'-Y}\mathcal{L}$, $i_v : v \to V$ and $j_v : (V - \{v\}) \to V$ denote the embeddings.
i) The MHS on the cohomology of $V - \{v\}$ is defined via the isomorphism
$R\Gamma(V - \{v\}, \mathcal{L}[n]) \simeq R\Gamma(V', Rj_*\mathcal{L}'[n])$.
ii) The MHS on the link L_v is deduced from the structure of MHC on $i^*Rj_*\mathcal{L}'[n]$ by the following isomorphisms

$$\begin{aligned} R\Gamma(L_v, \mathcal{L}[n]) \simeq R\Gamma(B_v - \{v\}, \mathcal{L}[n]) \simeq i_v^*Rj_{v*}\mathcal{L}[n] &\simeq R\Gamma(Y, i^*Rj_*\mathcal{L}'[n]) \\ &\simeq R\Gamma(B_Y - Y, \mathcal{L}'[n]) \end{aligned}$$

Such MHS is independent of the choice of the desingularisation:
let V'' be another desingularisation of V, we can cover V' and V'' by a third desingularisation, so we can suppose V'' above V', then the third property on degeneration of perverse Leray's filtration apply, although this case is far easier to check directly since we have a birational morphism.

3. Explicit decomposition formula and the proof of local purity

Given a stratification and a decomposition of a complex K as a direct sum of intermediate extensions of local systems $\mathcal{L}_l^i$ with respect to such stratification, these local systems can be recovered from K by an explicit formula in terms of the intersection morphism I. This simple remark is of fundamental importance in a

proof based on Hodge theory since such theory apply easily to I, while it is not clear how to use Hodge theory with the topological definition of perverse cohomology.

PROPOSITION 3.1. *Let K be a cohomologically constructible complex of sheaves on an algebraic variety V, which decomposes as a direct sum of its perverse cohomology via a quasi-isomorphism $K \simeq \oplus_{i\in\mathbb{Z}} {}^p\mathcal{H}^i(K)[-i]$. Moreover suppose there exists a finite algebraic Whitney stratification $\mathcal{S} = \{S_l\}_{l\leq d}$ of V (dimension $S_l = l$) and local systems $\mathcal{L}_l^i$ on the various strata S_l, considered as complexes in degree zero, such that each perverse cohomology group decomposes into a direct sum of intermediate extensions of $\mathcal{L}_l^i$ shifted in degree $-l$, that is*

$$ {}^p\mathcal{H}^i(K) \simeq \oplus_{l\leq d} k_{S_l\,!*}\mathcal{L}_l^i[l]$$

where $k_{S_l} : S_l \to \overline{S}_l$ (resp.$i_{S_l} : S_l \to V$ below) denote the locally closed embeddings, then

$$\mathcal{L}_l^i \simeq Im(\underline{H}_{S_l}^{i-l}(K) \xrightarrow{I} \underline{H}^{i-l}(i_{S_l}^* K))$$

where $\underline{H}_{S_l}^{i-l}$ is the sheaf of cohomology with support $H^{i-1}(Ri_{S_l}^!)$, $\underline{H}^{i-l}(i_{S_l}^* K)$ is the sheaf of cohomology and I is the composition morphism $\underline{H}_{S_l}^{i-l}(K) \to \underline{H}^{i-l}(K) \to \underline{H}^{i-l}(i_{S_l}^* K)$.

PROOF. 1) *Zero dimensional stratum case.* We can suppose the strata S_0 reduced to one point $v = S_0$, since the proof is local at v. Let $j_v : V - \{v\} \to V$ and $i_v : \{v\} \to V$. With the above hypothesis, the cohomology with support in v (resp. restricted to v) decomposes as:

$H_v^i(K) = \oplus_j H_v^i({}^p\mathcal{H}^j(K)[-j]) = \oplus_j H_v^{i-j}({}^p\mathcal{H}^j(K)),\ H^i(i_v^* K) = \oplus_j H^{i-j}(i_v^{*p}\mathcal{H}^j(K))$

where the various terms satisfy the perverse properties

$$H_v^{i-j}({}^p\mathcal{H}^j(K)) = 0 \text{ for } i < j, \quad H^{i-j}(i_v^{*p}\mathcal{H}^j(K)) = 0 \text{ for } i > j.$$

The morphism induced by I:

$$H_v^i(K) \to H^i(i_v^* K)$$

reduces by the decomposition to a direct sum of morphisms indexed by j:

$$H_v^{i-j}({}^p\mathcal{H}^j(K)) \to H^{i-j}(i_v^{*p}\mathcal{H}^j(K))$$

Taking into account the above conditions, we get

$$\oplus_{j\leq i} H_v^{i-j}({}^p\mathcal{H}^j(K)) \to \oplus_{j\geq i} H^{i-j}(i_v^{*p}\mathcal{H}^j(K))$$

Now we use the decomposition into intermediate extensions to write:

$${}^p\mathcal{H}^j(K) = {}^p\mathcal{H}^j(K)(v) \oplus {}^p\mathcal{H}^j(K)(*v)$$

where ${}^p\mathcal{H}^j(K)(v)$ is a vector space with support on v and

$${}^p\mathcal{H}^j(K)(*v) = (j_v)_{!*}({}^p\mathcal{H}^j(K)_{|V-\{v\}}).$$

Since the support is $\{v\}$, $H_v^{i-j}({}^p\mathcal{H}^j(K)(v)) = H^{i-j}(v, {}^p\mathcal{H}^j(K)(v))$ are equal to zero for $j \neq i$, while $H_v^{i-j}({}^p\mathcal{H}^j(K)(*v)) = 0$ for $i \leq j$ and dually: $H^{i-j}(v, {}^p\mathcal{H}^j(K)(*v)) = 0$ for $i \geq j$. Hence we deduce for $i = j$,

$\mathcal{L}_0^i \simeq H_v^0({}^p\mathcal{H}^i(K)) \simeq H_v^0({}^p\mathcal{H}^i(K)(v)) \simeq H^0(i_v^*({}^p\mathcal{H}^i(K)(v))) \simeq H^0(i_v^*({}^p\mathcal{H}^i(K)))$

$\simeq Im(H_v^i(K) \xrightarrow{I} H^i(i_v^* K))$.

2) *The proof extends by induction* to each strata and reduces to the previous result applied to a normal section. We start by induction on the open strata S_d where the restriction of K decomposes into the direct sum of its classical cohomologies shifted by $-d$.

Then the induction on the strata is based on the following morphisms defined by I on $S_l \subset (V - \overline{S_{l+1}})$:

$$Ri^!_{S_l}K \xrightarrow{I} i^*_{S_l}K, \quad Ri^!_{S_l}{}^p\mathcal{H}^i(K) \xrightarrow{I} i^*_{S_l}{}^p\mathcal{H}^i(K)$$

Taking a local normal section N_l of a stratum S_l at a point y, we reduce the proof of the formula to the case of the stratum in dimension zero as follows. By the basic local topological triviality of the stratification, asserted by Thom and proved by Mather, we deduce:

$${}^p\mathcal{H}^j(K)_{|N_l}[-l] = {}^p\mathcal{H}^j(K_{|N_l})$$

hence the restriction to y of $\underline{H}^{i-l}(Ri^!_{S_l}K) \xrightarrow{I} \underline{H}^{i-l}(i^*_{S_l}K)$ is a direct sum of restrictions to y of

$$Im([\underline{H}^{i-l-j}_{S_l}({}^p\mathcal{H}^jK)]_{|N_l} \xrightarrow{I} [\underline{H}^{i-l-j}(i^*_{S_l}{}^p\mathcal{H}^jK)]_{|N_l})$$

is equal to

$$Im[H^{i-j}_y({}^p\mathcal{H}^j(K_{|N_l})) \xrightarrow{I} H^{i-j}(i^{*p}_y\mathcal{H}^j(K_{|N_l}))]$$

which reduces to the statement for the complex $K_{|N_l}$ on N_l at the zero dimensional stratum y. □

Local purity. The proof of the theorem, by induction on dimension of V, is in two steps. In the first we reduce by Lefschetz hyperplane theorem type to the case $i = 0$, then in the second step we apply the decomposition theorem to a Lefschetz fibration over a curve.

Proposition 3.2. The MHS on $H^i_c(V - \{v\}, \mathcal{L}[n])$ is pure of weight $m + n + i$ for $i > 0$. Dually, $H^i(V - \{v\}, \mathcal{L}[n])$ is pure of weight $m + n + i$ for $i < 0$.

Proof. Let H be a smooth hyperplane section of V not containing the singular point. Since $Rj_{v!}\mathcal{L}[n]$ is in the category ${}^pD^{\leq 0}(V)$, a generalization of Lefschetz hyperplane section theorem due to M.Artin, shows that:
$H^i_c(V - H, \mathcal{L}[n]) = H^i(V - H, Rj_{v!}\mathcal{L}[n]) \simeq 0$ for $i > 0$ since $V - H$ is affine.
The Gysin morphism of MHS in the exact sequence
$H^{i-2}(H, Rj_{v!}\mathcal{L}[n]) \simeq H^i(V, Rj_{v!}\mathcal{L}[n]) \to H^i(V, Rj_{v!}\mathcal{L}[n]) \to H^i(V - H, Rj_{v!}\mathcal{L}[n])$
shows that for $i > 0$, the pure Hodge structure $H^{i-2}(H, \mathcal{L}[n])(-1)$ is surjective on $H^i(V, Rj_{v!}\mathcal{L}[n]) = H^i_c(V - \{v\}, \mathcal{L}[n])$.
Gysin morphisms are compatible with MHS. To check the compatibility of Gysin morphisms with MHS we are reduced by construction to the following case.
Let H' be a smooth divisor on the desingularisation V' of V intersecting transversally Y (NCD in V') and let $i_{H'}$ denotes the embedding morphism. The Gysin morphism $G(i_{H'})$ and the restriction $\rho(i_{H'})$ can be defined on the level of mixed Hodge complexes (which can be easily checked on the model in the appendix since the transverse intersection $H' \cap Y$ is a NCD in H') and induce morphisms of MHS

$$G(i_{H'}) : H^i(H' - (H' \cap Y), \mathcal{L}') \to H^{i+2}_{H'}(V' - Y, \mathcal{L}')$$
$$\rho(i_{H'}) : H^i(V' - Y, \mathcal{L}') \to H^i(H' - (H' \cap Y), \mathcal{L}').$$

□

Corollary 3.3. i) $H^i(B_v - \{v\}, \mathcal{L}[n])$ is of weight $\geq m + n + i$ for $i \geq 0$.
ii) Dually: $H^i(B_v - \{v\}, \mathcal{L}[n])$ is of weight $\leq m + n + i + 1$ for $i \leq -1$.

Proof. The duality is Poincaré's duality on the link of v of dimension $2n-1$ as mentioned in the remark (2.4). It can be checked to be compatible with MHS on the model of MHC in the appendix.
We prove (i), since (ii) follows by duality. Let

$$H^i(V-\{v\},\mathcal{L}[n]) \to H^i(B_v-\{v\},\mathcal{L}[n]) \to H^{i+1}_c(V-\{v\},\mathcal{L}[n])$$

be the long exact sequence defined by the triangle:

$$Rj_{v!}\mathcal{L}[n] \to Rj_{v*}\mathcal{L}[n] \to i_{v*}i_v^*Rj_{v*}\mathcal{L}[n].$$

Here we will use the first properties of the weight recalled above and proved in the appendix. For $i \geq 0$, the weight of $H^i(B_v-\{v\},\mathcal{L}[n])$ is squeezed between the left side of weight $\geq m+n+i$ and the right side of weight equal to $m+n+i+1$ since $i+1>0$, hence the assertion (i) follows. □

The corollary doesn't cover the full result in the theorem. There remains to prove the following cases

(1) $Gr^W_{i+m+n+1}H^i(B_v-\{v\},\mathcal{L}[n]) = 0$ if $\quad i \leq -1$,

(2) $Gr^W_{i+m+n}H^i(B_v-\{v\},\mathcal{L}[n]) = 0$ if $i \geq 0$.

The proof uses a general hyperplane section through the singular point v for $i<-1$ and dually for $i>0$. The crucial case being for $i=0$ or dually $i=-1$.

Proof for $i>0$. The proof is similar to the note [3(2)]. Let H be an hyperplane section through v. Since $B_v-(B_v\cap H_v)$ is Stein, we have $H^i(B_v-(B_v\cap H_v),\mathcal{L}[n]) \simeq 0$ for $i>0$ so that :

$$H^i_{B_v\cap H_v-\{v\}}(B_v-\{v\},\mathcal{L}[n]) \simeq H^i(B_v-\{v\},\mathcal{L}[n]) \text{ for } i>1,$$
$$H^1_{B_v\cap H_v-\{v\}}(B_v-\{v\},\mathcal{L}[n]) \to H^1(B_v-\{v\},\mathcal{L}[n]) \to 0$$

hence it is enough to prove $Gr^W_{i+m+n}H^i_{B_v\cap H_v-\{v\}}(B_v-\{v\},\mathcal{L}[n]) = 0$.
Since $B_v-\{v\}$ and $H_v-\{v\}$ are smooth, we have a Gysin isomorphism

$$H^{i-2}(B_v\cap H_v-\{v\},\mathcal{L}[n])(-1) \simeq H^i_{B_v\cap H_v-\{v\}}(B_v-\{v\},\mathcal{L}[n])$$

then we note the isomorphism

$$H^{i-2}(B_v\cap H_v-\{v\},\mathcal{L}[n]) \simeq H^{i-1}(B_v\cap H_v-\{v\},\mathcal{L}[n-1])$$

and apply the inductive hypothesis on the hyperplane section H of dim $n-1$ to deduce $Gr^W_{i-1+m+n-1}H^{i-1}(B_v\cap H_v-\{v\},\mathcal{L}[n-1]) = 0$ for $i-1 \geq 0$.

3.1. the case $i=0$. $Gr^W_{m+n}H^{-1}(B_v-\{v\},\mathcal{L}[n]) \simeq 0$.
The proof is by induction on dim V. By duality the proof of $Gr^W_{m+n}H^0(B_v-\{v\},\mathcal{L}[n]) \simeq 0$ for $i=0$ is reduced to the proof of $Gr^W_{m+n}H^{-1}(B_v-\{v\},\mathcal{L}[n]) \simeq 0$ in the case $i=-1$.
Consider the triangle on V

$$Rj_{v!}\mathcal{L}[n] \to Rj_{v*}\mathcal{L}[n] \to i_{v*}i_v^*Rj_{v*}\mathcal{L}[n]$$

and its associated long exact sequence:

$$H^{-1}(V-\{v\},\mathcal{L}[n]) \to H^{-1}(B_v-v,\mathcal{L}[n]) \xrightarrow{\partial'_V} H^0(V,Rj_{v!}\mathcal{L}[n])$$

Since $H^{-1}(V-\{v\},\mathcal{L}[n])$ is pure of weight $m+n-1$ by the proposition, the proof is reduced to the following statement:
For any element $a \in Gr^W_{m+n}H^{-1}(B_v-\{v\},\mathcal{L}[n])$ the image $\partial'_V(a) \in Gr^W_{m+n}H^0(V,Rj_{v!}\mathcal{L}[n])$ vanishes.
We use the corresponding triangle on V'

$$Rj_!\mathcal{L}'[n] \to Rj_*\mathcal{L}'[n] \to i_{Y*}i_Y^*Rj_*\mathcal{L}'[n]$$

and the diagram

$$\begin{array}{ccc} H^{-1}(B_Y - Y, \mathcal{L}'[n]) & \stackrel{\partial'}{\to} & H^0(V', Rj_!\mathcal{L}'[n]) \\ \simeq\uparrow \pi^* & & \simeq\uparrow \pi^* \\ H^{-1}(B_v - \{v\}, \mathcal{L}[n]) & \stackrel{\partial'_V}{\to} & H^0(V, Rj_{v!}\mathcal{L}[n]) \end{array}$$

Since π^* are isomorphisms, the statement reduces to:
the image $\pi^*(\partial'_V(a)) = \partial'(\pi^*(a)) \in Gr^W_{m+n}H^0(V', Rj_!\mathcal{L}'[n])$ *vanishes.*
Then we deduce from the triangle

$$Rj_!\mathcal{L}'[n] \to j_{!*}\mathcal{L}'[n] \to i_{Y*}i_Y^*j_{!*}\mathcal{L}'[n]$$

a long exact sequence

$$H^{-1}(Y, j_{!*}\mathcal{L}'[n]) \to H^0(V', Rj_!\mathcal{L}'[n]) \stackrel{\alpha'}{\to} H^0(V', j_{!*}\mathcal{L}'[n])$$

where the weight of $H^{-1}(Y, j_{!*}\mathcal{L}'[n])$ is $< m+n$ since Y is closed, hence the morphism $Gr^W_{m+n}\alpha'$ is injective on $Gr^W_{m+n}H^0(V', j_{!*}\mathcal{L}'[n])$; moreover since $H^0(V', j_{!*}\mathcal{L}'[n])$ is pure of weight $m+n$, we don't need to work on Gr^W_{m+n} anymore, so the statement reduces to

Lemma 3.4 (Main lemma). *Let* $\partial\colon = \alpha' \circ \partial'$, *it is equivalent to prove:*
For any element $a \in W_{m+n}H^{-1}(B_Y - Y, \mathcal{L}'[n])$, *the obstruction* $\partial(a)$ *vanishes in* $\in H^0(V', j_{!*}\mathcal{L}'[n])$.

Proof. *The hard Lefschetz type proof* in [3(2)] is by induction on the dimension of V to allow the use of the purity theorem on the restriction to a general hyperplane section of V, in a long beautiful geometric argument. A similar proof would also apply here. Precisely instead of that long argument the decomposition theorem can answer the problem and on its turn it is proved by induction on dimension at the same time. From now on we consider an element $a \in W_{m+n}H^{-1}(B_Y - Y, \mathcal{L}'[n])$. We divide the proof in two steps.

(1) we prove that the obstruction $\partial(a)$ is a primitive element.
(2) we use the polarizations to prove that the primitive element $\partial(a)$ vanishes.

We start with the second assertion, easier to prove. Let Q denotes a polarization of $IH^0(V', j_{!*}\mathcal{L})[n])$, and consider the diagram

$$\begin{array}{ccccc} & & Gr^W_{m+n}H^0(V', j_{!*}\mathcal{L}'[n]) & & \\ & \stackrel{A}{\nearrow} & & \stackrel{A^*}{\searrow} & \\ Gr^W_{m+n}H^0_Y(V', \mathcal{L}[n]) & & \stackrel{d=I}{\longrightarrow} & & Gr^W_{m+n}H^0(Y, j_{!*}\mathcal{L}'[n]) \end{array}$$

We consider the non-degenerate pairing defined by duality

$$P : Gr^W_{m+n}H^0_Y(V', j_{!*}\mathcal{L}'[n]) \otimes Gr^W_{m+n}H^0(Y, j_{!*}\mathcal{L}'[n]) \to \mathbb{C}$$

then the duality between A and A^* is defined for all $b \in Gr^W_{m+n}(H^0_Y(V', j_{!*}\mathcal{L}'[n])$ and $c \in H^0(V', j_{!*}\mathcal{L}'[n])$ by the formula :

$$Q(A(b), c) = P(b, A^*(c)).$$

To the element $a \in Gr^W_{m+n}H^{-1}(B_v - \{v\}), \mathcal{L}[n])$ corresponds $a \in Gr^W_{m+n}H^{-1}(B_Y - Y), \mathcal{L}'[n])$ whose image by the connecting morphism is $\overline{a} \in Gr^W_{m+n}H^0_Y(V', j_{!*}\mathcal{L}'[n])$ such that $\partial(a) = A(\overline{a})$. Let C be the Weil operator defined by the HS on $H^0_Y(V', j_{!*}\mathcal{L}'[n])$, then: $Q(C.\partial(a), \partial(a)) = P(C\overline{a}, A^* \circ A(\overline{a})) = P(C.\overline{a}, I(\overline{a}))$

and since $I(\overline{a}) = 0$, we can deduce $Q(C.\partial(a), \partial(a)) = 0$, hence $\partial(a) = 0$ by polarization, which ends the proof (in fact this proof apply also for $i < -1$ if we prove by induction on H the hard Lefshetz theorem).

The proof of the first assertion will be subdivided into two lemmas.
We choose a general hyperplane section H of V' transversal to all strata Y_K. The image $\pi(H)$ is a subvariety of V with an isolated singularity. Let $k : (H - Y \cap H) \to H$ and $k_v : (\pi(H) - \{v\}) \to \pi(H)$ denote the embeddings, $\tilde{\mathcal{L}}$ (resp. $\tilde{\mathcal{L}}'$) the restriction of $\mathcal{L}$ to $\pi(H) - \{v\}$ (resp. to $H - Y$) and let R_H (resp. $R_{\pi(H)}$) denotes the restriction between corresponding elements on V' and H (resp. V and $\pi(H)$).

LEMMA 3.5. *The element $\partial(a) \in H^0(V', j_{!*}\mathcal{L}'[n])$ is primitive. Precisely its restriction, to a general hyperplane section H, $R_H(\partial(a)) \in H^0(H, k_{!*}\tilde{\mathcal{L}}'[n])$ vanishes.*

To prove the lemma we consider the triangles:

$$Rk_{v!}\tilde{\mathcal{L}}[n] \to Rk_{v*}\tilde{\mathcal{L}}[n] \to i_{v*}i_v^* Rk_{v*}\tilde{\mathcal{L}}[n], \quad Rk_!\tilde{\mathcal{L}}'[n] \to Rk_*\tilde{\mathcal{L}}'[n] \to i_{H*}i_H^* Rk_*\tilde{\mathcal{L}}'[n]$$

and the connection morphisms they define as shown in the diagram

$$\begin{array}{ccccc}
H^{-1}((B_Y - Y), \mathcal{L}'[n]) & \stackrel{\partial'}{\to} & H^0(V', Rj_!\mathcal{L}'[n]) & \stackrel{\alpha'}{\to} & H^0(V', j_{!*}\mathcal{L}'[n]) \\
\downarrow R_H & & \downarrow R_H & & \downarrow R_H \\
H^{-1}((B_Y - Y) \cap H, \tilde{\mathcal{L}}'[n]) & \stackrel{\partial'_H}{\to} & H^0(H, Rk_!\tilde{\mathcal{L}}'[n]) & \stackrel{\alpha'_H}{\to} & H^0(H, k_{!*}\tilde{\mathcal{L}}'[n]) \\
\uparrow & & \uparrow\simeq & & \\
H^{-1}(B_v \cap \pi(H) - \{v\}, \tilde{\mathcal{L}}[n]) & \stackrel{\partial'_{VH}}{\to} & H^0(\pi(H), Rk_{v!}\tilde{\mathcal{L}}[n]) & \stackrel{\alpha_{VH}}{\stackrel{\simeq}{\to}} & H^0(\pi(H), k_{v!*}\tilde{\mathcal{L}}[n]) \\
\uparrow R_{\pi(H)} & & \uparrow R_{\pi(H)} & & \uparrow R_{\pi(H)} \\
H^{-1}(B_v - \{v\}, \mathcal{L}[n]) & \stackrel{\partial'_V}{\to} & H^0(V, Rj_{v!}\mathcal{L}[n]) & \stackrel{\alpha_V}{\to} & H^0(V, j_{v!*}\mathcal{L}[n])
\end{array}$$

The morphism α_{VH} is an isomorphism as it can be deduced from the short exact sequence defined by the triangle

$$Rk_{v!}\tilde{\mathcal{L}}[n] \to k_{v!*}\tilde{\mathcal{L}}[n] \to i_{v*}i_v^*\tau_{\leq -2}Rk_{v*}\tilde{\mathcal{L}}[n]$$

By transversality, the connecting morphisms commute with the restrictions on H as well on $\pi(H)$. The compatibility between ∂'_V on V and ∂' on V' is considered with value in $H^0_c(V - \{v\}, \mathcal{L}[n]) \simeq H^0_c(V' - Y, \mathcal{L}'[n])$.

3.1.1. *Lefschetz fibration on $\pi(H)$ of dimension $n-1$ over a curve.* Let L denotes the parameter space of varying hyperplane sections H' of $\pi(H)$ in a Lefschetz pencil s.t. the general hyperplane section of $\pi(H)$ does not contain v. By blowing up the axis of the pencil (not containing v) in H as well in $\pi(H)$, we obtain $\overline{H}$ (resp. $\overline{\pi(H)}$) in the diagram

$$\overline{\mathcal{L}'} = q^*_{|H-Y}\tilde{\mathcal{L}}', \qquad \begin{array}{ccccc}
 & & \overline{H} & \stackrel{q}{\to} & H \\
 & \stackrel{f'_L}{\swarrow} & \downarrow \tilde{\pi} & & \downarrow \pi \\
L & \stackrel{f_L}{\leftarrow} & \overline{\pi(H)} & \to & \pi(H)
\end{array}$$

where f_L is defined by the Lefschetz pencil and $f'_L = f_L \circ \tilde{\pi}$. Recall that $\tilde{\mathcal{L}}'$ (resp. $\tilde{\mathcal{L}}$) is the restriction of $\mathcal{L}$ to $H - (Y \cap H)$ (resp. $\pi(H) - v \simeq H - (Y \cap H)$) and let $\overline{\mathcal{L}'} = q^*_{|H-Y}\tilde{\mathcal{L}}'$ its lift to $\overline{H}$, the blow-up of the axis away from Y.

We apply in the next steps the decomposition theorem for the Lefschetz fibration .
The lemma follows if we prove

LEMMA 3.6. *The inverse image of $R_H(\partial a)$ in $H^0(\overline{H}, k_{!*}\tilde{\mathcal{L}}'[n])$ has two different perverse weights with respect to the above Lefschetz fibration, hence it vanishes.*

PROOF. We consider three steps. Let $k' : (\overline{H} - q^{-1}(Y)) \to \overline{H}$, since the blowup is away from Y, we have $Y \simeq q^{-1}(Y)$ and $q^*k_{!*}\tilde{\mathcal{L}}' \simeq k'_{!*}\overline{\mathcal{L}'}$. We want to apply the decomposition to the direct image $R(f'_L)_*k'_{!*}(\overline{\mathcal{L}'}[n-1])$ of the perverse sheaf $k'_{!*}(\overline{\mathcal{L}'}[n-1])$ on $\overline{H}$ of dimension $n-1$.
i) Since $dim\,\pi(H) = n-1$ and $k_{v!}\tilde{\mathcal{L}}[n-1])$ is in the category ${}^pD^{\leq 0}(\pi(H))$, the cohomology $H^0(\pi(H) - H', k_{v!}\tilde{\mathcal{L}}[n])$ of the affine space $\pi(H) - H'$ vanishes by weak Artin - Lefschetz theorem, hence the restriction of $\partial'_{VH}R_{\pi(H)}(a)$ vanishes on $\pi(H) - H'$, then $\partial'_{VH}R_{\pi(H)}(a)$ is image of an element $\xi \in H^0_{H'}(\pi(H), k_{v!}\tilde{\mathcal{L}}[n]) \simeq H^{-2}(\pi(H) \cap H', \tilde{\mathcal{L}}[n])$, equivalently $\alpha_{VH} \circ \partial'_{VH}R_{\pi(H)}(a)$ is image of an element $\xi \in H^0_{H'}(\pi(H), k_{v!*}\tilde{\mathcal{L}}[n]) \simeq H^{-2}(H', \tilde{\mathcal{L}}[n])$.
By induction on the dimension $k_{v!*}\tilde{\mathcal{L}}[n-1]$ has pure cohomology on $\pi(H)$, hence we can apply the following result on Lefschetz fibration.
The parameters of good hyperplane sections form an open subset L^* whose fundamental group Π_1 acts on the cohomology of a fixed general H', then the restriction morphism

$$H^{-2}(\pi(H), k_{v!*}\tilde{\mathcal{L}}[n]) \to H^{-2}(H', \tilde{\mathcal{L}}[n])^{\Pi_1}$$

is an isomorphism onto the fixed part by Π_1. The dual statement asserts that the Gysin morphism induces an isomorphism

$$H^{-2}(H', \tilde{\mathcal{L}}[n])^{\Pi_1} \xrightarrow{\simeq} H^0(\pi(H), k_{v!*}\tilde{\mathcal{L}}[n]).$$

Then we can choose as in [3] a normalized $\xi \in H^{-2}(H', \tilde{\mathcal{L}}[n])^{\Pi_1}$ for a fixed general section H' with image $\alpha_{VH} \circ \partial'_{VH}R_{\pi(H)}(a) \in H^0(\pi(H), k_{v!*}\tilde{\mathcal{L}}[n])$.
ii) *Class section in $H^0(L, s_{!*}(R^1((f'_L)_{|L^*})_*\overline{\mathcal{L}'}[n-1])) \subset H^0(L, {}^p\mathcal{H}^2(R(f'_L)_*(k'_{!*}\overline{\mathcal{L}'}[n-1])))$.*
Let $s : L^* \to L$ denotes the embedding of the regular strata with respect to f'_L. In particular the point $f'_L(Y) = f_L(v)$ is outside L^*.
Since the element ξ is invariant, it defines outside $Y \cap H$ a section
$\xi_{f_L} \in H^0(L^*, R^{-2}(f'_L)_*(\overline{\mathcal{L}'}[n]))$. Moreover

$$H^0(L^*, R^{-2}((f'_L)_{|L^*})_*\overline{\mathcal{L}'}[n]) \simeq H^0(L, s_{!*}(R^{-1}((f'_L)_{|L^*})_*\overline{\mathcal{L}'}[n-1]))$$

The image by Gysin on H of ξ is $\partial_H R_H(a)$. It lifts by q to

$$q^*(\partial_H R_H(a)) \in H^0(\overline{H}, q^*k_{!*}\tilde{\mathcal{L}}'[n]) \simeq H^1(\overline{H}, k'_{!*}\overline{\mathcal{L}'}[n-1])$$

and defines a section in $H^0(L, s_{!*}(R^1((f'_L)_{|L^*})_*\overline{\mathcal{L}'}[n-1]))$ obtained from the section defined by ξ above by cup-product with an hyperplane section class.
iii) *Class section in $H^0(L, \overline{\mathcal{L}'}^1_0) \subset H^0(L, {}^p\mathcal{H}^1(R(f'_L)_*(k'_{!*}\overline{\mathcal{L}'}[n-1])))$.*
On the other side the obstruction $\partial_H R_H(a) \in H^0(H, k_{!*}\tilde{\mathcal{L}}'[n])$ is obtained by composition of the morphisms
$H^{-1}((B_Y - Y) \cap H, \tilde{\mathcal{L}}'[n]) \to H^0_{Y\cap H}(H, k_{!*}\tilde{\mathcal{L}}'[n]) \to H^0(H, k_{!*}\tilde{\mathcal{L}}'[n])$
so that $\partial_H R_H(a)$ is in the image of
$H^0_{\overline{H}\cap Y}(\overline{H}, q^*k_{!*}\tilde{\mathcal{L}}'[n]) \simeq H^0_{\overline{H}\cap Y}(\overline{H}, k'_{!*}\overline{\mathcal{L}'}[n]) \simeq H^1_{\overline{H}\cap Y}(\overline{H}, k'_{!*}\overline{\mathcal{L}'}[n-1])$
that is in the component of *the decomposition formula* denoted by $\overline{\mathcal{L}'}^1_0$.
We deduce from ii) and iii) that $\partial_H R_H(a)$ has perverse weights 1 and 2, hence must vanish; thus the proposition would follow from the decomposition on $\pi(H)$ which

is of dimension $n-1$. In turn, once the purity theorem is proved in dimension n, it can be applied to prove the decomposition in dimension n. $\square$

This ends the proof of the proposition if we prove the decomposition theorem on perverse cohomology for the Lefschetz projection f'_L over a line. We address this question in a more general setting in order to clarify the general use of Hodge theory. $\square$

4. Hodge theory and the decomposition theorem

We give a proof of the decomposition theorem, by induction on $dim\ V$, in a setting general enough to cover our need and clarify the role of Hodge structure.

THEOREM 4.1. *Let $f : X \to V$ denotes a projective morphism from a smooth variety X of dimension n to an algebraic variety V of dimension d with an isolated singular point v such that $Y = f^{-1}(v)$ is a NCD and f smooth on $X - Y$, $\mathcal{L}$ a polarized variation of Hodge structures of weight m on the open subset $X - Y$, $j : X - Y \to X$, $i_v : v \to V$, $j_v : V - v \to V$ the embeddings. Define*

$$K = Rf_* j_{!*}\mathcal{L}[n], \quad \mathcal{L}_d^j \simeq (R^{-d+j} f_* K)_{|V-\{v\}}, \quad \mathcal{L}_0^i \simeq Im(H_v^i(K) \xrightarrow{I^i} H^i(i_v^* K))$$

and consider the triangle on V

$$i_{v,*} R i_v^!(K) \xrightarrow{\alpha} K \xrightarrow{\rho} R j_{v*} K_{|V-\{v\}} \xrightarrow{[1]} \tag{4.1}$$

and its associated long exact sequence of perverse cohomology

$${}^p\mathcal{H}^i(i_{v,*} R i_v^! K) \xrightarrow{{}^p\alpha^i} {}^p\mathcal{H}^i(K) \xrightarrow{{}^p\rho^i} {}^p\mathcal{H}^i(R j_{v*} K_{|V-\{v\}}) \xrightarrow{{}^p\delta^i} \tag{4.2}$$

then the perverse cohomology decomposes as

$${}^p\mathcal{H}^i(K) \simeq Im\, {}^p\alpha^i \oplus Im\, {}^p\rho^i \simeq i_{v,*}\mathcal{L}_0^i \oplus j_{v!*}\mathcal{L}_d^i[d]$$

Moreover

$$ker\ {}^p\alpha^i \simeq Im\ {}^p\delta^{i-1} \simeq \oplus_{i-1-j\geq 0} R^{i-1-j} j_{v,*}(\mathcal{L}_d^j[d]) \tag{4.3}$$

The statement of the result relies on the introduction of the abelian category of perverse sheaves and perverse cohomology in [1].
The proof is given in the last of the following three parts. The first part is a computation in perverse cohomology and the second states relevant result in usual cohomology.

4.1. Perverse cohomology. We give first as an example, a result on the structure of the perverse cohomology ${}^p\mathcal{H}^i(R j_{v*} K_{|V-\{v\}})$ useful in the proof.

LEMMA 4.2. *Let $\mathcal{L}'$ be a local system on $V - v$.*
i) We have an exact sequence
$0 \to j_{v!*}\mathcal{L}'[d] \to {}^p\mathcal{H}^0(R j_{v*}\mathcal{L}'[d]) \to R^0 j_{v,*}(\mathcal{L}'[d]) \to 0.$
ii) ${}^p\mathcal{H}^i(R j_{v*}\mathcal{L}'[d]) = 0$ for $i < 0$ and ${}^p\mathcal{H}^i(R j_{v*}\mathcal{L}'[d]) = R^i j_{v,*}(\mathcal{L}'[d])$ for $i > 0$
iii) $H^0({}^p\mathcal{H}^0(R j_{v*}\mathcal{L}'[d])) \simeq R^0 j_{v,*}(\mathcal{L}'[d])$
iv) $H_v^i(j_{v!*}\mathcal{L}'[d]) \simeq H_v^i({}^p\mathcal{H}^0(R j_{v*}\mathcal{L}'[d])$ for $i \geq 2$ and $H_v^i({}^p\mathcal{H}^0(R j_{v*}\mathcal{L}'[d]) = 0$ otherwise.

PROOF. i) The long perverse exact sequence defined by the cone C_v of the the morphism $\mathbf{j}_{v!*}\mathcal{L}'[d]) \to Rj_{v,*}\mathcal{L}'[d]$

$$^p\mathcal{H}^i(j_{v!*}\mathcal{L}'[d]) \to {}^p\mathcal{H}^i(Rj_{v*}\mathcal{L}'[d]) \to {}^p\mathcal{H}^i(C_v) \to$$

reduces to

$$^p\mathcal{H}^i(Rj_{v*}\mathcal{L}'[d]) = 0 \text{ for } i < 0,\ {}^p\mathcal{H}^i(Rj_{v*}\mathcal{L}'[d]) = R^i j_{v,*}(\mathcal{L}'[d]) \text{ for } i > 0$$

since C_v is supported on v in degree ≥ 0 and $^p\mathcal{H}^i(Rj_{v!*}\mathcal{L}'[d]) = 0$ for $i \neq 0$. In degree zero, we get the exact sequence

$$0 \to j_{v!*}\mathcal{L}'[d] \to {}^p\mathcal{H}^0(Rj_{v*}\mathcal{L}'[d]) \to R^0 j_{v,*}(\mathcal{L}'[d]) \to 0.$$

The assertion (ii) follows from this last exact sequence as well the isomorphism $R^i j_{v,*}(\mathcal{L}'[d]) \simeq H_v^{i+1}(j_{v!*}\mathcal{L}'[d])$ for $i \geq 0$. □

LEMMA 4.3. *Let $K' = \oplus_k \mathcal{L}'^k[d-k]$ be a direct sum of local systems on $V - v$*
i) The perverse cohomology of $Rj_{v,*}K'$ fits into the short exact sequence

$$0 \to j_{v!*}(\mathcal{L}'^i[d]) \to {}^p\mathcal{H}^i(Rj_{v,*}K') \to \oplus_{k\leq i} R^i j_{v,*}(\mathcal{L}'^k[d-k]) \to 0. \tag{4.4}$$

moreover
ii) $H^0({}^p\mathcal{H}^i(Rj_{v,*}K')) \simeq \oplus_{k\leq i} R^i j_{v,*}(\mathcal{L}'^k[d-k]) \simeq R^i j_{v,*}({}^p\tau_{\leq i}K')$

PROOF. We deduce from the previous lemma, for any integer b
$^p\mathcal{H}^i(Rj_{v*}\mathcal{L}'[b]) = 0$ for $i < d-b$, $^p\mathcal{H}^i(Rj_{v*}\mathcal{L}'[b]) = R^i j_{v,*}(\mathcal{L}'[b])$ for $i > d-b$
and the short exact sequence
$0 \to j_{v!*}\mathcal{L}'[d] \to {}^p\mathcal{H}^{d-b}(Rj_{v*}\mathcal{L}'[b]) \to R^{d-b} j_{v,*}(\mathcal{L}'[b]) \to 0.$
Then the lemma follows. □

4.2. Proof of the theorem. First we show how *Hodge theory* is used in the proof. Let $i_Y : Y \to X$ denotes the embedding and $f_v : Y \to v$ the restriction of f. We consider the triangle (on left side below) associated to $j_{!*}\mathcal{L}[n]$ on X and restricted to Y and its direct image by f on V (on the right side), a triangle defined by $K\colon = f_* j_{!*}\mathcal{L}[n]$ and restricted to the singular point v

$$Ri_Y^! j_{!*}\mathcal{L}[n] \xrightarrow{I^i} i_Y^* j_{!*}\mathcal{L}[n] \to i_Y^* Rj_*\mathcal{L}[n] \xrightarrow{[1]};\ Ri_v^!(K) \xrightarrow{I^i} i_v^* K \to i_v^* Rj_{v*}K_{|V-\{v\}} \xrightarrow{[1]} \tag{4.5}$$

The restriction of α to v is transformed into the Intersection morphism I. Taking cohomology we get the diagram (4.6)

$$\begin{array}{ccccccc}
H^{r-1}(Y, Rj_*\mathcal{L}[n]) & \xrightarrow{\delta'^{r-1}} & H_Y^r(X, j_{!*}\mathcal{L}[n]) & \xrightarrow{I'^r} & H^r(Y, j_{!*}\mathcal{L}[n]) & \xrightarrow{\rho'^r} & H^r(Y, Rj_*\mathcal{L}[n]) \\
\simeq\uparrow & & \simeq\uparrow & & \simeq\uparrow & & \simeq\uparrow \\
H^{r-1}(i_v^* Rj_{v*}K) & \xrightarrow{\delta^{r-1}} & H_v^r(K) & \xrightarrow{I^r} & H^r(i_v^* K) & \xrightarrow{\rho^r} & H^r(i_v^* Rj_{v*}K_{|V-\{v\}})
\end{array}$$

To suggest to the reader a rough idea of the proof, we apply in the first step, Deligne's decomposition for the smooth restriction of f to $V - v$ and deduce the existence of decompositions

$$K_{|V-\{v\}} \simeq \oplus_j \mathcal{L}_d^j[d-j] \text{ where } \mathcal{L}_d^j \simeq (R^{-d+j} f_* j_{!*}\mathcal{L}[n])_{|V-\{v\}}$$

inducing a decomposition on cohomology (4.7)

$$H^r(i_v^* Rj_{v*}K_{|V-\{v\}}) \simeq (\oplus_{r-j<0} H^{r-j}(i_v^* Rj_{v*}\mathcal{L}_d^j[d])) \oplus (\oplus_{r-j\geq 0} H^{r-j}(i_v^* Rj_{v*}\mathcal{L}_d^j[d]))$$

where we isolated for convenience, in the first summand on the right side, terms relevant to the Intermediate extension.
The fact is that Hodge theory can distinguish between the two summands as we will see later, namely via the following conditions on the weight ω of the MHS on the cohomology $H^{r-i}(i_v^* Rj_{v*}\mathcal{L}_d^j[d])$ (see local purity theorem above applied to $\mathcal{L}_d^j[d]$)

$$\omega \leq m+n+r \text{ if } r-j \leq -1 \text{ and } \omega > m+n+r \text{ if } r-j \geq 0. \tag{*}$$

since $\mathcal{L}^j_d$ (resp. $\mathcal{L}^j_d[d]$) is of weight $m+n+j-d$ (resp. $m+n+j$).
Moreover we need to use the following conditions on the weight:
i) *The weight* ω *of* $H^{r+1}_v(K) \simeq H^r_Y(X, j_{!*}\mathcal{L}[n])$ *satisfy:* $\omega \geq m+n+r$ *since* Y *is a NCD.*
ii) *The weight of* $H^r(i^*_v K) \simeq H^r(Y, j_{!*}\mathcal{L}[n])$ *satisfy:* $\omega \leq m+n+r$ *since* Y *is closed.*

It follows from these conditions that the image of ρ^r is the first summand, however for this conclusion we need to assume one more compatibility result: the Hodge structure on $H^r(Y, Rj_*\mathcal{L}[n])$ is computed directly on X while on each component $H^r(i^*_v Rj_{v*}\mathcal{L}^j_d[d])$ of the above decomposition (4.7), the MHS is computed on a desingularisation of V. Here we need to check compatibility between the various MHS, that is the degeneration of the perverse Leray's spectral sequence in the category of MHS in the case of NCD assumed earlier as property (3) (see 2.2).

Second we show how *Perverse cohomolgy* is used in the proof. The decomposition (4.7) is not canonical; the properties of perverse cohomology distinguish the terms of the first summand on the right side of the decomposition, according to the degree j, so to exhibit the various intermediate extensions $j_{v!*}\mathcal{L}^i_d$ in the statement of the theorem. For this we introduce the long exact sequence of perverse cohomology defined on V by the above triangle

$$^p\mathcal{H}^i(i_{v*}Ri^!_v K) \stackrel{^p\alpha^i}{\to} {}^p\mathcal{H}^i(K) \stackrel{^p\rho^i}{\to} {}^p\mathcal{H}^i(Rj_{v*}K_{|V-\{v\}}) \stackrel{^p\delta^i}{\to}$$

and the following morphism of triangles (4.8)

$$\begin{array}{ccccc} i^*_v{}^p\tau^{i-1}K & \to & i^*_v{}^p\tau^i K & \stackrel{\pi^i}{\to} & i^*_v{}^p\mathcal{H}^i(K)[-i] \\ \downarrow & & \downarrow {}^p\rho^i & & \downarrow {}^p\rho^i \\ i^*_v{}^p\tau^{i-1}(Rj_{v*}K_{|V-\{v\}}) & \to & i^*_v{}^p\tau^i(Rj_{v*}K_{|V-\{v\}}) & \stackrel{\pi^i}{\to} & i^*_v{}^p\mathcal{H}^i(Rj_{v*}K_{|V-\{v\}})[-i] \end{array}$$

To prove that $Im^p\rho^i = j_{v!*}\mathcal{L}^i_d[d])$ we need to prove, in view of lemma 2 (ii), that $^p\rho^i$ induced on $H^i(i^*_v{}^p\mathcal{H}^i(K)[-i])$ is equal to zero, or

$^p\rho^i_0 : H^0(^p\mathcal{H}^i(K)) \to H^0(^p\mathcal{H}^i(Rj_{v,*}K)) \simeq \oplus_{j\leq i} R^i j_{v,*}(\mathcal{L}^j[d-j])$

is equal to 0. The main idea is to factorize $^p\rho^i_0$ into the various $R^i j_{v,*}(\mathcal{L}^j[d-j]), j \leq i$, through $H^i(Y, j_{!*}\mathcal{L}[n]) \to H^i(B_Y - Y, j_{!*}\mathcal{L}[n]) \simeq H^i(B_v - v, K)$ where B_Y and B_v are small neighbourhood of Y and v. In the long exact sequence

$$H^r(i^*_v(^p\tau^{i-1}K)) \to H^r(i^*_v(^p\tau^i K)) \to H^r(i^*_v(^p\mathcal{H}^i(K)[-i])) \to H^{r+1}(i^*_v(^p\tau^{i-1}K))$$

the last term vanishes for $r \geq i-1$, then an element $a \in H^i(i^*_v(^p\mathcal{H}^i(K)[-i]))$ lifts to an element $a' \in H^i(i^*_v(^p\tau^i K))$. The induced morphism on complexes $^p\rho^i : {}^p\tau^i K \to {}^p\tau^i(Rj_{v*}K_{|V-\{v\}})$ and on cohomology

$$^p\rho^i : H^i(i^*_v(^p\tau^i K)) \to H^i(i^*_v(^p\tau^i(Rj_{v*}K_{|V-\{v\}})))$$

are well defined and the image $^p\rho^i(a')$ projects to $\pi^i(^p\rho^i(a')) = {}^p\rho^i(a)$. By proper base change the spaces $H^i(i^*_v(^p\tau^i K))$ define a filtration on $H^i(Y, j_{!*}\mathcal{L}[n])$ and the element a' considered in that space is of weight $\leq m+n+i$ hence $^p\rho^i(a')$ is also of weight $\leq m+n+i$ and projects to $^p\rho^i(a)$ in the space

$$H^i(i^*_v(^p\mathcal{H}^i(Rj_{v*}K_{|V-\{v\}})[-i])) \simeq H^0(i^*_v(^p\mathcal{H}^i(Rj_{v*}K_{|V-\{v\}}))) \simeq$$
$$\oplus_{j\leq i} i^*_v R^i j_{v,*}(\mathcal{L}^j_d[d-j]) \simeq \oplus_{i-j\geq 0} i^*_v R^{i-j} j_{v,*}(\mathcal{L}^j_d[d])$$

of weight $> m+n+i$, hence $^p\rho^i(a) = 0$ (here we use the compatibility of the weight mentioned above).

We conclude that the image of $^p\rho^i$ on the cohomology, restricted to v, is in the

cohomology of $i_v^* j_{v!*}\mathcal{L}_d^i[d]$, hence the image of ${}^p\rho^i$ is $j_{v!*}\mathcal{L}_d^i[d]$ which is the smallest perverse extension of the restriction to $V-v$ of ${}^p\mathcal{H}^i(K)$.

The image of ${}^p\alpha^i$ in (4.3) is easier to compute since the perverse cohomology of $i_v^!(K)$ coincides with the cohomology on $H_v^i(K)$, then the image of ${}^p\alpha^i$ in (4.2) is isomorphic to the cokernel $coker^p\delta^{i-1}$. Since I^i in the diagram (4.6) is the restriction to v of α^i in (4.1), we have an isomorphism: $coker({}^p\delta^{i-1}) \simeq coker({}^p\delta'^{i-1})$ (4.2) and (4.6), hence $Im\,{}^p\alpha^i$ coincides with the image of I'^i on $H_Y^i(X, j_{!*}\mathcal{L}[n])$.
In order to prove the decomposition ${}^p\mathcal{H}^iK \simeq Im^p\alpha^i \oplus j_{v!*}\mathcal{L}_d^i[d]$, and in view of the splitting criteria [2, (4.1.3)], we need to prove that the canonical morphism $H_v^0({}^p\mathcal{H}^iK) \simeq H^0({i_v}^*({}^p\mathcal{H}^iK))$ is an isomorphism, which follows from $H_v^0(j_{!*}\mathcal{L}_d^i[d]) \simeq H^0({i_v}^*(j_{!*}\mathcal{L}_d^i[d]))$. In fact both spaces are isomorphic to $Im\,I^i \simeq Im\,I'^i$.

4.3. Hard Lefschetz type theorems. To prove the degeneration of the perverse Leray's spectral sequence, we are reduced by the previous decomposition to prove Hard Lefschetz for

$$\mathcal{L}_0^i = Im(H_Y^i(X, j_{!*}\mathcal{L}[n]) \xrightarrow{I^i} H^i(Y, j_{!*}\mathcal{L}[n]))$$

since it is already true for $\mathcal{L}_d^i$ and its intermediate extension.

Proposition 4.4. i) The cup-product with an hyperplane section class induces isomorphisms $\eta^i : \mathcal{L}_0^{-i} \to \mathcal{L}_0^i$ for $i<0$.
ii) The HS $\mathcal{L}_0^{-i}$ for $i \le 0$, is Poincaré dual to $\mathcal{L}_0^i$. It is polarized via (i).

Proof. Let H be an hyperplane section intersecting the NCD Y normally and i_H denotes the closed embedding of H in X.

Lemma 4.5. *For an hyperplane section H normally intersecting Y, we have the following isomorphisms of perverse sheaves:*
1) $i_H^*(j_{!*}\mathcal{L}[n])[-1] \simeq (j_{H!*}i_{H\cap(X-Y)}^*\mathcal{L}[n-1]))$.
2) $i_{Y\cap H}^* Ri_Y^!(j_{!*}\mathcal{L}[n])[1] \simeq Ri_{Y\cap H}^! i_H^*(j_{!*}\mathcal{L}[n])[1]$.

The lemma is admitted.
The cup product with the class η of H is defined by composition of the morphisms

$$j_{!*}\mathcal{L}[n] \xrightarrow{\rho} i_{H*}i_H^* j_{!*}\mathcal{L}[n] \simeq i_{H!}Ri_H^!(j_{!*}\mathcal{L}[n])[2] \xrightarrow{G} j_{!*}(\mathcal{L}[n])[2]$$

by applying the functor $i_Y^!$ and i_Y^* to the above morphisms, η induces morphisms denoted with the same letter

$$\eta : Ri_Y^! j_{!*}\mathcal{L}[n] \to Ri_Y^!(j_{!*}\mathcal{L}[n])[2], \quad \eta : i_Y^*(j_{!*}\mathcal{L}[n]) \to i_Y^*(j_{!*}(\mathcal{L}[n])[2])$$

These morphisms commute with the the intersection morphisms

$$I : Ri_Y^! j_{!*}\mathcal{L}[n] \to i_Y^* j_{!*}\mathcal{L}[n] \text{ and } I_H : i_{Y\cap H}^* Ri_Y^!(j_{!*}\mathcal{L}[n]) \to i_{Y\cap H}^*(j_{!*}\mathcal{L}[n])$$

and fit in the diagram

$$\begin{array}{ccccc} H_Y^i(X, j_{!*}\mathcal{L}[n]) & \overset{\rho_{X,Y}}{\to} & H_{Y\cap H}^i(H, j_{!*}\mathcal{L}[n]) & \overset{G_{X,Y}}{\to} & H_Y^{i+2}(X, j_{!*}\mathcal{L}[n]) \\ I^i\downarrow & & I_H^i\downarrow & & I^{i+2}\downarrow \\ H^i(Y, j_{!*}\mathcal{L}[n]) & \overset{\rho_H}{\to} & H^i(Y\cap H, j_{!*}\mathcal{L}[n]) & \overset{G_H}{\to} & H^{i+2}(Y, j_{!*}\mathcal{L}[n]) \end{array}$$

By construction $\eta = G_{X,Y} \circ \rho_{X,Y}$ on the first line, $\eta = G_H \circ \rho_H$ on the second line, and $\mathcal{L}_0^i = ImI^i, (\mathcal{L}_{|H})_0^i = ImI_H^i, \mathcal{L}_0^{i+2} = ImI^{i+2}$ are the images of the vertical

maps defined by I from the top line to the bottom line s.t. we can factorize the Intersection forms as in the following diagram

$$\begin{array}{ccccc} H^i_Y(X, j_{!*}\mathcal{L}[n]) & \overset{\rho_{X,Y}}{\to} & H^i_{Y\cap H}(H, j_{!*}\mathcal{L}[n]) & \overset{G_{X,Y}}{\to} & H^{i+2}_Y(X, j_{!*}\mathcal{L}[n]) \\ I^i\downarrow & & I^i_H\downarrow & & I^{i+2}\downarrow \\ \mathcal{L}^i_0 & \overset{\rho'}{\to} & \mathcal{L}^i_{|H,0} & \overset{G'}{\to} & \mathcal{L}^{i+2}_0 \\ \downarrow & & \downarrow & & \downarrow \\ H^i(Y, j_{!*}\mathcal{L}[n]) & \overset{\rho_H}{\to} & H^i(Y\cap H, j_{!*}\mathcal{L}[n]) & \overset{G_H}{\to} & H^{i+2}(Y, j_{!*}\mathcal{L}[n]) \end{array}$$

ρ_H (resp. G_H) induces a morphism $\rho' : \mathcal{L}^i_0 \to \mathcal{L}^i_{|H,0}$ (resp. $G' : \mathcal{L}^i_{|H,0} \to \mathcal{L}^{i+2}_0$) fitting in commutative diagrams.

Lemma 4.6. *The morphisms $\rho_{X,Y} : H^i_Y(X, j_{!*}\mathcal{L}[n]) \to H^i_{Y\cap H}(H, j_{!*}\mathcal{L}[n])$ are isomorphisms for $i < 0$.*

Proof. It follows from Lefschetz hyperplane section applied to a perverse sheaf on X, in fact the complex $P_Y = Ri^!_Y j_{!*}\mathcal{L}[n+1]$ is a perverse sheaf on Y. The morphism
$H^i_Y(X, j_{!*}\mathcal{L}[n]) \simeq H^{i-1}(Y, P_Y) \overset{\rho_{X,Y}}{\to} H^{i-1}(Y\cap H, P_Y) \simeq H^i_{Y\cap H}(H, \mathbf{j}_{!*}\mathcal{L}[n])$
is an isomorphism for $i < 0$ and injective for $i = 0$. □

Proof of the proposition continued: ρ' is an isomorphism.
Let $K = Rf_* j_{!*}\mathcal{L}[n]$, $K(H) = R(f_{|H})_*(j_{|H})_{!*}\mathcal{L}_{|H}[n]$ and consider the diagram

$$\begin{array}{ccc} H^{i-1}(i^*_v Rj_{v*}K_{|V-\{v\}}) & \overset{\rho*}{\to} & H^{i-1}(i^*_v Rj_{v*}K(H)_{|V-\{v\}}) \\ \delta^i\downarrow & & \delta^i_H\downarrow \\ H^i_Y(X, j_{!*}\mathcal{L}[n]) \simeq H^i_v(V, K) & \overset{\rho_{X,Y}}{\to} & H^i_{Y\cap H}(H, j_{!*}\mathcal{L}[n]) \simeq H^i_v(V, K(H)) \\ I^i\downarrow & & I^i_H\downarrow \\ H^i(Y, j_{!*}\mathcal{L}[n]) \simeq H^i(i^*_v K) & \overset{\rho_H}{\to} & H^i(Y\cap H, j_{!*}\mathcal{L}[n]) \simeq H^i(i^*_v K(H)) \end{array}$$

By the previous result (theorem (4.1))

(1) $ker(I^i) = Im(\delta^i) = \delta^i(\oplus_{j<i} H^{i-1-j}(i^*_v Rj_{v*}\mathcal{L}^j_d[d])$
(2) $Im(\delta^i_H) = \delta^i_H(\oplus_{j<i} H^{i-1-j}((i^*_v Rj_{v*}(\mathcal{L}_{|H})^j_d[d])$

Since $i < 0$ and $j < i$ we have $j < -1$ s.t. we an deduce that restricion to H induces an isomrphism of the sheaf $\mathcal{L}^j_d \simeq R^{n-d+j} f_*\mathcal{L}_{|V-\{v\}}$ with the sheaf $(\mathcal{L}_{|H})^j_d$ by appliction of the hyperplane section theorem on the $n-d$ dimensional fibres of f on $V - \{v\}$. Let ρ_H, δ_H and $ker(I_H)$ correspond to the exact sequence defined by the Intersection form on H for the restriction $(\mathcal{L}_{|H})$ of $\mathcal{L}$. It follows that $\rho_{X,Y}$, which is an isomorphism by lemma 2, induces isomorphisms on $Im\delta^i = ker(I^i) \simeq Im\delta^i_H = ker(I^i_H)$, hence induces isomorphisms: $\rho' : \mathcal{L}^i_0 \simeq (\mathcal{L}_{|H})^i_0$ for $i < 0$. □

Corollary 4.7. The cup-product with an hyperplane section class induces isomorphisms $\eta^i : {}^p\mathcal{H}^{-i}(K) \to {}^p\mathcal{H}^i(K)$ for $i < 0$.

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Deligne-Hodge-DeRham theory with coefficients

Appendix A

A.1. Introduction. The subject of this section is to construct a mixed Hodge structure MHS on the cohomology of a local system $\mathcal{L}$ underlying a polarized variation of Hodge structures (VHS) on the complement $X^* = X - Y$ of a normal crossing divisor (NCD) Y in a smooth proper complex algbraic variety X of dimension n, and prove the various properties used in the article.

The MHS in the above case is fundamental in a general study. Our approach is based on the note [12] where we propose a MHC of the logarithmic type, constructed with the various weight filtrations of the local monodromies, from which we deduce the various results first in the open case then in the case of the nearby cycles defined by any local equation of Y. In this last case we can check that the weight filtration obtained on cohomology is the same as the one obtained by Saito (see last section A6), while it is known that the Hodge filtration defined by the poles in Saito's work gives the same filtration as the one deduced from the logarithmic complex. So we start by discussing various technical and theoretical aspects using only the concept of polarized VHS and mixed Hodge complexes (MHC).

Let $j : X^* \to X$ denotes the open embedding. Technically, we need to define a structure of MHC on the higher direct image $Rj_*\mathcal{L}$.

The work consists then in two parts, first to define the rational weight filtration W and second to construct the complex weight W and Hodge filtration F. In the rational case, we don't have a particular representative of $Rj_*\mathcal{L}$ by a distinguished complex, so the method is to use the theory of perverse sheaves to describe W.
In the complex case we need to construct a bi-filtered complex and we use the logarithmic complex with coefficients in *Deligne's analytic bundle* $\mathcal{L}_X$ *extension of* $\mathcal{L}\otimes\mathcal{O}_{X^*}$*, with logarithmic connection* ∇, since by Deligne's theorem [6]

$$Rj_*\mathcal{L}\cong\Omega^*_X(LogY)\otimes\mathcal{L}_X \tag{1.1}$$

By the subsequent work of Schmid, Cattani and Kaplan, Kashiwara and Kawai, the Hodge filtration F extends by sub-bundles. In this article we describe a bi-filtered complex

$$(\Omega^*\mathcal{L}[n],W,F) \tag{1.2}$$

where $n=\ dim\,X$, constructed as a sum of a combinatorial complex constantly equal to the logarithmic complex with coefficients and which underly the structure of MHC we are looking for. Although the existence of such bi-filtered complex is important in the general theory, we insist on basic results on $Gr^{\mathcal{W}}\mathcal{L}$ since they can be stated more easily and give a deep insight into topological and geometrical properties of the variety and the local system. Let us fix the hypothesis and the notations for the rest of the section.

A.1.1. *Hypothesis. Let* $\mathcal{L}$ *be a local system defined over* $\mathbb{Q}$*, on the complement of the* NCD Y *in a smooth analytic variety* X*,* $(\mathcal{L}_X,\nabla)$ *the canonical extension of* $\mathcal{L}_{X^*}=\mathcal{L}\otimes\mathcal{O}_{X^*}$ [6], [21 (2)] *with a meromorphic connection* ∇ *on* X *having logarithmic poles along* Y *and* $\Omega^*_X(LogY)\otimes\mathcal{L}_X$ *the associated DeRham logarithmic complex defined by* ∇*; moreover we suppose that* $\mathcal{L}^{\mathbb{C}}=\mathcal{L}\otimes\mathbb{C}$ *underlies a polarized* VHS *of weight* m*.*
In the text we write $\mathcal{L}$ for the rational $\mathcal{L}^r$ as well for the complex local system $\mathcal{L}^c$, $\mathcal{L}_X$ for the analytic extension and to simplify the exposition, we suppose $\mathcal{L}$ locally unipotent along Y.

A.1.2. *Notation.* We suppose the NCD, $Y=\cup_{i\in I}Y_i$ equal to the union of irreducible and smooth components Y_i for i in I. For all subset K of I, let $Y_K=\cap_{i\in K}Y_i$, $Y^*_K=Y_K-\cup_{i\notin K}Y_i$, and $j^K:Y^*_K\to Y_K$ the locally closed embedding, then $Y_K-Y^*_K$ is a NCD in Y_K and the open subsets Y^*_K of Y_K form with X^* a natural stratification of X. All extensions of $\mathcal{L}$ considered are constructible with respect to this stratification and even perverse.
We write $\mathcal{L}_{Y_K}$ for the restriction of $\mathcal{L}_X$ to Y_K, then the residue of the connection along a component Y_i of Y induces an endomorphism $Res_{Y_i}(\nabla)$ of $\mathcal{L}_{Y_i}$. The monodromy of the local system defines an automorphism of $\mathcal{L}_{X-Y}$ which extends to an automorphism T_X of $\mathcal{L}_X$ related to the residue along Y_i by the following basic formula of Deligne: $(T_X)_{|Y_i}=exp(-2i\pi)Res_{Y_i}(\nabla)$. Since we develop the theory in the case of a local system with locally unipotent monodromy along Y_i, the residues $Res_{Y_i}(\nabla)$ are nilpotent endomorphisms and will be denoted by $\mathcal{N}_i$. For each subset of indices K, we denote also by $\mathcal{N}_i$, $i\in K$ the restriction to $\mathcal{L}_{Y_K}$ so to introduce the filtration

$$\mathcal{W}^K_{Y_K}=W(\Sigma_{i\in K}\mathcal{N}_i) \tag{1.3}$$

by sub-bundles of $\mathcal{L}_{Y_K}$ defined by $\Sigma_{i\in K}\mathcal{N}_i$.
Let $i_K:Y^*_K\to X$, we introduce the local systems

$$\mathcal{L}^K = i_K^* R^{|K|} j_* \mathcal{L} \simeq i_K^* R\Gamma_Y^{|K|+1} j_{!*}\mathcal{L}, \quad \mathcal{L}'^K = i_K^* R^0 j_* \mathcal{L}$$

and the bundles

$$\mathcal{L}_{Y_K}^K = \mathcal{L}_{Y_K}/(\Sigma_{i\in K}\mathcal{N}_i\mathcal{L}_{Y_K}), \quad \mathcal{L}_{Y_K}'^K = \cap_{i\in K}(ker\mathcal{N}_i : \mathcal{L}_{Y_K} \to \mathcal{L}_{Y_K})$$

Although the analytic bundle extends, the local system does not; instead his associated local systems $\mathcal{L}^K$ and $\mathcal{L}'^K$, globally defined on Y_K^*, will be important in the theory. The following proposition is an application of the isomorphism (1.1). It will become clear through the text until we reach the proof of most of the results in Proposition (A.44).

Proposition A.1. i) $\mathcal{L}_{Y_K}^K$ (resp. $\mathcal{L}_{Y_K}'^K$) is a flat bundle with flat sections defined by the local system $\mathcal{L}^K$ (resp. $\mathcal{L}'^K$). Precisely, $\mathcal{L}_{Y_K}^K$ (resp. $\mathcal{L}_{Y_K}'^K$) is isomorphic to Deligne's extension of the corresponding complex local system.
ii) The filtration $\mathcal{W}_{Y_K}^K$ (1.3) induces a filtration by flat sub-bundles of $\mathcal{L}_{Y_K}^K$ (resp $\mathcal{L}_{Y_K}'^K$), hence induces a filtration by complex sub-local systems $\mathcal{W}^K$ of $\mathcal{L}^K$ (resp. $\mathcal{L}'^K$).
iii) The filtration $\mathcal{W}^K$ is defined on the rational local system $\mathcal{L}^K$ (resp. $\mathcal{L}'^K$).
iv) Let $\mathcal{L}_r^K := Gr_{r-|K|}^{\mathcal{W}^K}\mathcal{L}^K$ and $\mathcal{L}_{Y_K,r}^K := Gr_{r-|K|}^{\mathcal{W}^K}\mathcal{L}_{Y_K}^K$ for $r \geq |K|$, resp.
$\mathcal{L}_r^K := Gr_{r+|K|}^{W^K}\mathcal{L}'^K$ and $\mathcal{L}_{Y_K,r}^K := Gr_{r+|K|}^{W^K}\mathcal{L}_{Y_K}'^K$ for $r \leq -|K|$,
the system defined by $(\mathcal{L}_r^K, \mathcal{L}_{Y_K,r}^K, F)$ where F is the Hodge filtration induced from $\mathcal{L}_{Y_K}$, is a polarized VHS.

In the above proposition $|K| \geq 1$; for length 0 we may refer to $\mathcal{L}$ by $\mathcal{L}_0^K$.
Following the conventions, a VHS, $\mathcal{L}$ on X will be considered as a complex of sheaves in degree 0 and its shift $\mathcal{L}[n]$ in degree $-n$ is a Hodge complex of weight $m+n$ in case $\mathcal{L}$ is of weight m.
The cohomology sheaves $R^i j_*\mathcal{L}$ are constructible on X whose restrictions are local systems on Y_K^*.We need to introduce the intermediate extensions $j_{!*}^K\mathcal{L}_r^K[n-|K|]$ on Y_K in order to obtain Hodge complexes.
The duality between $\mathcal{L}'^K$ and $\mathcal{L}^K$ can be viewed fibrewise at a point y as a duality over a multiple tori in a normal section to Y_K at y, retract deformation of the open link $L_y - (L_y \cap Y)$.
Throughout this section we compute the fibre of $\mathcal{L}_r^K$ as the cohomology of $C_r^K L$ (see (A(3.3)) with a natural Hodge structure and prove the following results

Theorem A.2. *There exists a bifiltered complex $(\Omega^*\mathcal{L}[n], \mathcal{W}, F)$ attached to a polarized VHS satisfying the following decomposition property into intermediate extensions of polarized VHS*

$$(Gr_r^{\mathcal{W}}\Omega^*\mathcal{L}[n], F) \cong \oplus_{K\subset I} j_{!*}^K\mathcal{L}_r^K[n-|K|], F[-|K|]), \text{ for } r > 0$$

(the shift on the decreasing filtration F increases the index of F by $|K|$ and the weight by $2|K|$).

$$(Gr_0^{\mathcal{W}}\Omega^*\mathcal{L}[n], F) \cong 0$$

$$(Gr_r^{\mathcal{W}}\Omega^*\mathcal{L}[n], F) \cong \oplus_{K\subset I} j_{!*}^K\mathcal{L}_r^K[n+1-|K|], F), \text{ for } r < 0$$

Theorem A.3. *There exists a rational weight filtration $(Rj_*\mathcal{L}[n], \mathcal{W})$ and a quasi-isomorphism $(Rj_*\mathcal{L}[n], \mathcal{W}) \otimes_{\mathbb{Q}} \mathbb{C} \cong (\Omega^*\mathcal{L}[n], \mathcal{W})$, such that $\mathcal{W}_0 \cong j_{!*}\mathcal{L}[n]$.*

We obtain in the complement of a NCD case, a basic result on which the theory can be built in a more general setting. The use of the theory of differential modules by Saito in [22] is also built on the case of the complement of a NCD as it is recalled at

the end of this section. The MHS obtained here is the same as the one constructed by Saito as we can check below (see A.1.3) when we consider the nearby cycles case since both weight filtrations are compatible with the weight $W(\mathcal{N})$ suggested by the next formula A(1.4).
In particular the Intersection cohomology [13] is the fundamental ingredient in such theory and provides a new class of Hodge complexes not merely defined by complete non singular projective varieties.

THEOREM A.4. *For X proper of dim.n, there exists a natural MHS on the various cohomology groups with coefficients in $\mathcal{L}$ underlying a polarized VHS of weight m on $X-Y$, as follows*

i) The bi-filtered complex

$$(\Omega^*\mathcal{L}[n], W, F), \text{ where } W_i = \mathcal{W}_{i-m-n} \text{ for } i \geq m+n \text{ and } W_i = 0 \text{ for } i < m+n$$

underlies a MHC isomorphic to $Rj_*\mathcal{L}$ s.t. $H^i(X-Y, \mathcal{L}[n])$ is of weight $\geq i+m+n$. The filtration W is by perverse sheaves. Dually $H^i_c(X-Y, \mathcal{L}[n])$ is of weight $\leq i+m+n$.

ii) Let $i_Y : Y \to X$ denotes the embedding. The quotient complex $i_Y^*(\Omega^*\mathcal{L}[n]/W_{m+n})$ with the induced filtrations is a MHC quasi-isomorphic to $Ri^!_Y(j_{!*}\mathcal{L}[n])[1]$ hence $H^{i+1}_Y(X, j_{!*}\mathcal{L}[n])$ is of weight $\geq i+1+m+n$.

iii) The bi-filtered complex

$$(i_Y^*\mathcal{W}_1\Omega^*\mathcal{L}[n], W, F) \text{ where } W_i = \mathcal{W}_{i-m-n-1} \text{ for } i \leq m+n$$

underlies a MHC isomorphic to $i_Y^*j_{!*}\mathcal{L}[n]$, hence $H^i(Y, i_Y^*j_{!*}\mathcal{L}[n])$ is of weight $\leq i+m+n$.

We give now a more detailed discussion of the contents.

A.1.3. *Weight filtration.* The defining property of such filtration can be understood after a digression on the sheaves of nearby cycles. Under the additional hypothesis of the existence of a local equation f of Y, defining a morphism on an open subset $f : U \to D$ to a complex disc, such that $Y \cap U = f^{-1}(0)$, consider $\mathcal{N} = -\frac{1}{2i\pi} Log\mathcal{T}^u$ the logarithm of the unipotent part of the monodromy on on the complex of sheaves of nearby cycles Ψ_f. The filtration $W(\mathcal{N})$ on Ψ_f^u is defined by the nilpotent endomorphism $\mathcal{N}$ in the abelian category of rational (resp. complex) perverse sheaves. The isomorphism in the abelian category of perverse sheaves [2]

$$Coker(j_{!*}\mathcal{L}[n] \to Rj_*\mathcal{L}[n]) \simeq Coker\mathcal{N} \colon \Psi_f^u(\mathcal{L})[n-1] \to \Psi_f^u(\mathcal{L})[n-1] \tag{1.4}$$

suggest to start the weight filtration with $j_{!*}\mathcal{L}[n]$ and continue with $W(\mathcal{N})$ induced on $Coker\mathcal{N}$, then the main problem is to show, that the various weights defined locally on Y (for different local equations) glue together on $Coker(j_{!*}\mathcal{L}[n] \to Rj_*\mathcal{L}[n]$. For each local equation f, Ψ_f^u can be defined by a section of Verdier's monodromic specialization sheaf on the normal cone to Y in X [25]. For a different equation $g = uf$ with u invertible Ψ_g^u and Ψ_f^u are isomorphic but not canonically, so we cannot define a global complex Ψ_Y^u on Y and $W(\mathcal{N})$ on it, however we prove that the isomorphism becomes canonical on $Coker\mathcal{N}$ hence the induced $W(\mathcal{N})$ is globally defined on $Coker(j_{!*}\mathcal{L}[n] \to Rj_*\mathcal{L}[n])$.
Here we solve the problem in the case of NCD by a different approach. We define first the complex weight filtration and then prove it is rationally defined. The detailed study of the complex $\Omega^*\mathcal{L}[n]$, necessary for Hodge theory, leads to such approach of the rational weight.

THEOREM A.5. *For each local equation f of Y defined on an open set U of X, $\mathcal{N}$ acts on $\Psi_f^u\mathcal{L}[n-1]$; then the induced filtration by $\mathcal{W}(\mathcal{N})$ on Cokernel of $\mathcal{N}$ (the right term of* (1.4)*) coincides with the induced filtration by $\mathcal{W}$ on $\Omega^*\mathcal{L}[n]/\mathcal{W}_0 \simeq Coker(j_{!*}\mathcal{L}[n] \to Rj_*\mathcal{L}[n])$ (the left term).*

REMARK A.6. The existence of the global rational weight filtration has been proved for any divisor using the deformation to the normal cone and will appear later in a joint work with Lê D.T. and Migliorini L.

Let $p : C_YX - Y \to Y$ denotes the projection of the punctured normal cone onto Y. Then for any perverse sheaf $\mathcal{K}$, the filtration $\mathcal{W}$ on $Rj_*\mathcal{K}$ defined as $\mathcal{W}_0 = j_{!*}\mathcal{K}$ and for $i > 0$
$\mathcal{W}_i(Rj_*\mathcal{K}/j_{!*}\mathcal{K}) \simeq \mathcal{W}_{i-1}{}^p\mathcal{H}^0(Rp_*(\Psi_Y\mathcal{K})) := Ker[{}^p\mathcal{H}^0(Rp_*(\Psi_Y\mathcal{K}) \to {}^p\mathcal{H}^0(Rp_*(\Psi_Y\mathcal{K}/\mathcal{W}_i(\mathcal{N}|\Psi_Y\mathcal{K})]$
s.t. for each local equation $f = 0$ of Y defined on an open set U_f, we have on $Y_f = U_f \cap Y$
$\mathcal{W}_i(Rj_*\mathcal{K}/j_{!*}\mathcal{K})_{|Y_f} \simeq \mathcal{W}_{i-1}(Coker\mathcal{N}|(\Psi_f^u\mathcal{K}[-1])$.
Dually on $i^*j_{!*}\mathcal{K}[-1] \simeq {}^p\mathcal{H}^{-1}(Rp_*(\Psi_Y\mathcal{K})$ the weight is defined as
$(\mathcal{W}_i\, i^*j_{!*}\mathcal{K})[-1] \simeq {}^p\mathcal{H}^{-1}(Rp_*\mathcal{W}_{i+1}(\mathcal{N}|\Psi_Y\mathcal{K}))$

A.1.4. *Purity and decomposition (local results).* Working with the complex local system $\mathcal{L}$ we need to exhibit a bifiltered complex $(\Omega^*\mathcal{L}[n], \mathcal{W}, F)$ underlying a MHC. Its construction is suggested by an algebraic formula of the Intersection complex given by Kashiwara and Kawai in [20]. We explain now the main basic local results.

1- If we consider a point $y \in Y_M^*$, a VHS on $\mathcal{L}$ of weight m defines a nilpotent orbit L with a set of nilpotent endomorphisms $N_i, i \in M$. The nilpotent orbit theorem [4], [18] states that the VHS degenerates along Y_M^* into a variation of MHS with weight filtration $W^M = W(\Sigma_{i\in M}N_i)$ shifted by m.

However the difficulty in the construction of the weight is to understand what happens at the intersection of Y_M and Y_K for two subsets M and K of I. This difficulty couldn't be explained for the $\mathbb{Q}-$ structure until the discovery of perverse sheaves. In order to prove the decomposition of $Gr^{\mathcal{W}}\Omega^*\mathcal{L}[n]$ into intermediate direct image of various local systems on the components of $Y = \cup_{i\in I}Y_i$, we will introduce for $K \subset M$, the complex C_r^KL (see A(3.3)) constructed out of the nilpotent orbit $(L, N_i)_{i\in K}$ defined by $\mathcal{L}$ at a point $z \in Y_K^*$ and $W^K = W(\Sigma_{i\in K}N_i)$. We prove that C_r^KL has a unique non vanishing cohomology isomorphic to $Gr^{W^K}_{r-|K|}(L/\Sigma_{i\in K}N_iL)$ and that is the fibre at z of the rationally defined local system $\mathcal{L}_r^K$ previously introduced for $r > 0$.

Now Y_M is a subset of Y_K and we are interested in the fibre of $j_{!*}^K\mathcal{L}_r^K[n-|K|]$ at $y \in Y_M^*$, so we will introduce the complex $C_r^{KM}L$ (see A(3.2)), which is quasi-isomorphic to this fibre and will appear as a component of the decomposition (theorem A.30) of the graded part of the weight filtration.

These main results in the open case form the content of the second and third subsections.

2- The second major technical result (A(5.4), (lemma A.53)) is in the case of nearby co-cycles. If we consider a local equation f of Y, we show that $Gr_r^{W(\mathcal{N})}\psi_f^u\mathcal{L}[n]$ decomposes into a direct sum of complexes ν_r^KL and that C_r^KL is isomorphic to the primitive part , which proves that the complex weight filtration is compatible with our description of the rational weight.

A.1.5. *A Key result for a nilpotent orbit* $(L, N_i, i \in K)$ that enables us to give most of the proofs is the existence of a natural decomposition (lemma A.12)

$$Gr_r^{W^K} L = \oplus Gr_{m_{i_n}}^{W(N_{i_n})} \cdots Gr_{m_{i_1}}^{W(N_{i_1})} L \,:\, \Sigma_{i_j \in K} m_{i_j} = r$$

As a consequence the local systems on Y_K^* defined for $r > 0$ by $C_r^K L$ decompose into a direct sum of elementary components

$$\oplus Gr_{m_{i_k}}^{W(N_{i_k})} \cdots Gr_{m_{i_1}}^{W(N_{i_1})} (L/(N_{i_1} L + \cdots + N_{i_k} L)) \,:\, \Sigma_{i_j \in K} m_{i_j} = r + |K|, m_{i_j} \geq 2$$

the corresponding elementary complexes are introduced in (definition A.19) and are key ingredients in the proof.

The above results explain the subtle relation between the filtration $\mathcal{W}(\mathcal{N})$ on $\Psi_f^u(\mathcal{L})[n]$ which is hard to compute and the various local monodromy at points of Y.

Finally the local definition of the weight is in (A.3.2), the purity in (A.3.4), (prop. A.18), and the decomposition in (A.3.9) and (theorem A.30). The global definition of the weight is in (A.4.2), (definition A.39) and the decomposition in (theorem A.46). The weight of the nilpotent action on Ψ_f^u is in (A.5.1) and the comparison in (A.5.3). We suggest strongly to the reader to follow the proofs on an example, sometimes on the surface case as in the example (A.16); this example will be again useful for $\Psi_f \mathcal{L}[n]$ in (example A.54). For X a line and $Y = 0$ a point, the fibre at 0 of $Rj_* \mathcal{L}$ is a complex $L \xrightarrow{N} L$ where (L, N) is a nilpotent orbit of weight m and the weight $\mathcal{W}[m]$ on the complex is $(\mathcal{W}[m])_{r+m} = \mathcal{W}_r$ defined by the sub-complex $(W_{r+1} L \xrightarrow{N} W_{r-1} L)$; if we shift the complex by n to the left, the weight is shifted by $1 + m$ $(\mathcal{W}[m+1])$.

A.2. Local invariants of $\mathcal{L}$. We need a precise description of our objects in terms of the local invariants of the local system $\mathcal{L}$. We recall some preliminaries on $Rj_* \mathcal{L}$ and we give a basic local decomposition of weight filtrations defined by local monodromy.

A.2.1. *Local and global description of* $Rj_* \mathcal{L}$. In the neighbourhood of a point y in Y, we can suppose $X \simeq D^{n+k}$ and $X^* = X - Y \simeq (D^*)^n \times D^k$ where D is a complex disc, denoted with a star when the origin is deleted. The fundamental group $\Pi_1(X^*)$ is a free abelian group generated by n elements representing classes of closed paths around the origin, one for each D^* in the various axis with one dimensional coordinate z_i (the hypersurface Y_i is defined locally by the equation $z_i = 0$). Then the local system $\mathcal{L}$ corresponds to a representation of $\Pi_1(X^*)$ in a vector space L defined by the action of commuting automorphisms T_i for $i \in [1, n]$ indexed by the local components Y_i of Y and called monodromy action around Y_i. The automorphisms T_i decomposes as a product of commuting automorphisms, semi-simple and unipotent $T_i = T_i^s T_i^u$.

Classically L is viewed as the fibre of $\mathcal{L}$ at the reference point for the fundamental group $\Pi_1(X^*)$, however since we will need to extend the Hodge filtration on Deligne's extended bundle, it is important to view L as the vector space of multi-valued sections of $\mathcal{L}$ (that is the sections of the inverse of $\mathcal{L}$ on a universal covering of X^*).

Given a $\mathbb{Q}-$local system, locally unipotent along Y (to simplify the exposition) we consider $\mathcal{L}_{X^*} := \mathcal{L} \otimes_{\mathbb{Q}} \mathcal{O}_{X^*}$ and its Deligne's bundle extension $\mathcal{L}_X$ [6],[21(2)] which has a nice description as the subsheaf of $j_* \mathcal{L}_{X*}$ generated locally at a point y in Y by sections associated to multivalued sections of $\mathcal{L}$ as follows. The logarithm of the

unipotent monodromy, $N_i := -\frac{1}{2i\pi} Log T_i^u = \frac{1}{2i\pi}\Sigma_{k\geq 1}(1/k)(I-T_i^u)^k$ is defined as the sum of nilpotent endomorphisms $(I-T_i^u)$ so that this sum is finite.
A multivalued section v corresponds to a germ $\tilde{v} \in Rj_*\mathcal{L}_{X*}$ with an explicit description of the action of the connection by the formulas

$$\tilde{\mathrm{v}}(z) = (exp(\Sigma_{j\in J}(logz_j)N_j)).v, \quad \nabla\tilde{v} = \Sigma_{j\in J}\widetilde{N_j.v} \otimes \frac{dz_j}{z_j} \tag{2.1}$$

a basis of L is sent on a basis of $\mathcal{L}_{X,y}$.
The residue of the connection ∇ along each Y_j defines an endomorphism $\mathcal{N}_j$ on the restriction $\mathcal{L}_{Y_j}$ of $\mathcal{L}_X$.
The fibre at the origin of the complex $\Omega_X^*(LogY)\otimes\mathcal{L}_X$ is quasi-isomorphic to a Koszul complex as follows. We associate to $(L, N_i), i \in [1,n]$ a strict simplicial vector space such that for all sequences $(i.) = (i_1 < \cdots < i_p)$

$$L(i.) = L \qquad , \qquad N_{i_j} : L(i. - i_j) \to L(i.)$$

Definition A.7. The simple complex defined by the simplicial vector space above is the Koszul complex (or the exterior algebra) defined by (L, N_i) and denoted by $\Omega(L, N.)$. A general notation is $s(L(J), N.)_{J\subset[1,n]}$ where J is identified with the strictly increasing sequence of its elements and where $L(J) = L$.

It is quasi-isomorphic to the Koszul complex $\Omega(L, Id-T.)$ defined by $(L, Id-T_i), i \in [1,n]$. This local setting compares to the global case via Grothendieck and Deligne DeRham cohomology results.

Lemma A.8. *For $M \subset I$ and $y \in Y_M^*$, the above correspondence $v \mapsto \tilde{v}$, from L to $\mathcal{L}_{X,y}$, extended from $L(i_1,\ldots,i_j)$ to $(\Omega_X^*(LogY)\otimes\mathcal{L}_X)_y$ by $v \mapsto \tilde{v}\frac{dz_{i_1}}{z_{i_1}}\wedge\ldots\wedge\frac{dz_{i_j}}{z_{i_j}}$, induces quasi-isomorphisms*

$$(\Omega_X^*(LogY)\otimes\mathcal{L}_X)_y \cong \Omega(L, N_j, j\in M) \cong s(L(J), N.)_{J\subset[1,n]} \tag{2.2}$$

hence $(Rj_*\mathcal{L})_y \cong \Omega(L, N_j, j\in M)$.

This description of $(Rj_*\mathcal{L})_y$ is the model for the description of the next various perverse sheaves.

A.2.2. *The intermediate extension $j_{!*}\mathcal{L}[n]$.* Let $N_J = \Pi_{j\in J}N_j$ denotes a composition of endomorphisms of L, we consider the strict simplicial sub-complex of the DeRham logarithmic complex defined by ImN_J in $L(J) = L$.

Definition A.9. The simple complex defined by the above simplicial subvector space is denoted by

$$IC(L) := s(N_JL, N.)_{J\subset M},\ N_JL := N_{j_1}N_{j_2}\ldots N_{j_p}L, j_i \in J$$

Locally the germ of the intermediate extension $j_{!*}\mathcal{L}$ of $\mathcal{L}$ at a point $y \in Y_M^*$ is quasi-isomorphic to the above complex [18 (3)]

$$j_{!*}(\mathcal{L}[n])_y \simeq IC(L)[n] \simeq s(N_JL, N.)_{J\subset M}[n]$$

The corresponding global DeRham description is given as a sub-complex $IC(X,\mathcal{L})$ of $\Omega_X^*(LogY)\otimes\mathcal{L}_X$. In terms of a set of coordinates $z_i, i\in M$, defining Y_M in a neighbourhood of $y \in Y_M^*$, $IC^j(X,\mathcal{L})$ is the subanalytic complex of $\Omega_X^*(LogY)\otimes\mathcal{L}_X$ with fibre at y generated, as an $\Omega_{X,y}^*$ sub-module, by the sections $\tilde{v}\wedge_{j\in J}\frac{dz_j}{z_j}$ for $v \in N_JL$. This formula is independent of the choice of coordinates, since if we choose a different coordinate $z_i' = fz_i$ instead of z_i, with f invertible holomorphic

at y, the difference $\frac{dz'_i}{z'_i} - \frac{dz_i}{z_i} = \frac{df}{f}$ is holomorphic at y, hence the difference of the sections $\tilde{v} \wedge_{j\in J} \frac{dz'_j}{z'_j} - \tilde{v} \wedge_{j\in J} \frac{dz_j}{z_j}$ is still a section of the sub-complex $IC(X, \mathcal{L})$; moreover the restriction of the section is still defined in the sub-complex near y, since $N_J L \subset N_{J-i} L$ for all $i \in J$.

LEMMA A.10. *We have a quasi-isomorphism* $j_{!*}\mathcal{L}[n] \cong IC(X, \mathcal{L})[n]$.

A.2.3. *Hodge filtration and Nilpotent orbits. Variation of Hodge structures (VHS).* Consider the flat bundle $(\mathcal{L}_X, \nabla)$ in the previous hypothesis and suppose now that $\mathcal{L}_{X^*}$ underlies a VHS that is a polarized filtration by sub-bundles F of weight m satisfying Griffith's conditions [13].

The *nilpotent and the SL_2 orbit* theorems [15], [4], [18], [19] show that F extends to a filtration by sub-bundles F of $\mathcal{L}_X$ such that the restrictions to open intersections Y^*_M of components of Y underly *locally* a variation of mixed Hodge structures $VMHS$ where the weight filtration is defined by the nilpotent endomorphism $\mathcal{N}_M$, residue of the connection, (there is no flat bundle defined globally on Y^*_M, if $z_{i_1}, \ldots, z_{i_n}$ for $i_j \in M$ are local equations at $y \in Y^*_M$, then $\Psi_{z_{i_n}} \circ \ldots \circ \Psi_{z_{i_1}} \mathcal{L}$ is the underlying local system near y).

Local version. Near a point $y \in Y^*_M$ with $|M| = n$ a neighbourhood of y in the fibre of the normal bundle looks like a disc D^n and the above hypothesis reduces to

Local Hypothesis : Nilpotent orbits [4]. Let

$$(L, N_i, F, P, m, i \in M = [1, n]) \tag{2.3}$$

be defined by the VHS, that is a $\mathbb{Q}-$vector space L with endomorphisms N_i viewed as defined by the multivalued horizontal (zero) sections of the connection on $(D^*)^n$ (hence sections on the inverse image on the universal covering), a Hodge structure F on $L^{\mathbb{C}} = L \otimes_{\mathbb{Q}} \mathbb{C}$ viewed as the fibre of the vector bundle $\mathcal{L}_X$ at y (here $y = 0$), a natural integer m the weight and the polarization P.
The main theorem in [4] states that for all $N = \Sigma_{i\in M} \lambda_i N_i$ with $\lambda_i > 0$ positive reals, the filtration $W(N)$ (with center 0) is independent of N when λ_i vary and $W(N)[m]$ is the weight filtration of a graded polarized MHS called the limit MHS of weight m ($L, F, W(N)[m]$).

REMARK A.11. $W(N)[m]$ is $W(N)$ with indices shifted by m to the right: $(W(N)[m])_r := W_{r-m}(N)$, the convention being a shift to left for a decreasing filtration and to right for an increasing filtration.

It is important to notice that the orbits depend on the point z near y considered, in particular $F_z \neq F_y$. In this case when we restrict the orbit to $J \subset M$, we should write

$$(L, N_i, F(J), P, m, i \in J \subset M)$$

We write W^J for $W(N_J)$ where $N_J = \Sigma_{i\in J} N_i$. We will need the following result [4 p 505]:

Let $I, J \subset M$ then $W^{I\cup J}$ is the weight filtration of N_J relative to $W(N_I)$

$$\forall j, i \geq 0, N^i_J : Gr^{W^{I\cup J}}_{i+j} Gr^{W^I}_j \xrightarrow{\sim} Gr^{W^{I\cup J}}_{j-i} Gr^{W^I}_j .$$

A.2.4. *Properties of the relative weight filtrations.* Given a nilpotent orbit we may consider various filtrations $W^J = W(\Sigma_{i\in J} N_i)$ for various $J \subset M$. They are centered at 0 (that is we suppose here the weight of the nilpotent orbit equal to zero, otherwise the true weight of the MHS is defined up to a shift), preserved by N_i for $i \in M$ and shifted by -2 for $i \in J$: $N_i W^J_r \subset N_i W^J_{r-2}$. We need to know more about the action of N_i which is compatible with $W(N_j)$. The starting point of this study is the definition of the relative weight filtration by Deligne [9]

Let (L, W) be a finite dimensional vector space L endowed with an increasing filtration W and N a nilpotent endomorphism compatible with W. There may exists at most a unique filtration $M = M(N, W)$ satisfying
1) $N : M_j \simeq M_{j-2}$
2) $N^j : Gr^M_{k+j} Gr^W_k L \simeq Gr^M_{k-j} Gr^W_k L$,

A main result in [4] shows

The filtrations W^J defined by a polarized nilpotent orbit satisfy
1) For a subset $J \subset [1, n]$, $\forall j \in J, \forall \lambda_j > 0$, $N_J = \sum_{j\in J} \lambda_j N_j$, the filtration $W^J = W(N_J)$ is independent of $\lambda_j > 0$
2) For subsets J and J' in $[1, n]$, $A = J \cup J'$ we have for all $j \in \mathbb{N}, k \in \mathbb{Z}$:
$N^j_J : Gr^{W^A}_{k+j} Gr^{W^{J'}}_k L \simeq Gr^{W^A}_{k-j} Gr^{W^{J'}}_k L$,
that is W^A is the relative weight filtration of N_J acting on $(L, W^{J'})$.
We remark also that for all $J \subseteq B \subseteq A$, N_B and N_J induce the same morphism on $Gr^{W^{J'}}_k L$.
Finally we need the following result of Kashiwara ([19, thm 3.2.9, p 1002])

Let (L, N, W) consists of a vector space endowed with an increasing filtration W preserved by a nilpotent endomorphism N on L and suppose that the relative filtration $M = M(N, W)$ exists, then there exists a canonical decomposition:

$$Gr^M_l L = \oplus_k Gr^M_l Gr^W_k L$$

Precisely, Kashiwara exhibits a splitting of the exact sequence:
$0 \to W_{k-1} Gr^M_l L \to W_k Gr^M_l L \to Gr^W_k Gr^M_l L \to 0$.
by constructing a natural section of $Gr^W_k Gr^M_l L$ into $W_k Gr^M_l$. We will need later more precise relations between these filtrations that we discuss now.

Lemma A.12. *(Key lemma: Decomposition of the relative weight filtrations). Let $(L, N_i, i \in [1, n], F)$ be a polarized nilpotent orbit and for $A \subseteq [1, n]$ let $W^A := W(\sum_{i\in A} N_i)$ (all weights centered at 0), then :*
i) For all $i \in A$, the filtration W^A induces a trivial filtration on $Gr^{W^i}_k Gr^{W^{(A-i)}}_{k'} L$ of weight $k + k'$
ii) For $A = \{i_1, \dots, i_j\} \subset [1, n]$, of length $|A| = j$ we have a natural decomposition

$$Gr^{W^A}_r L \simeq \oplus_{m_{i.} \in X^A_r} Gr^{W^{i_j}}_{m_{i_j}} \cdots Gr^{W^{i_1}}_{m_{i_1}} L \text{ where } X^A_r = \{m_{i.} \in \mathbb{Z}^j : \Sigma_{i_l \in A} m_{i_l} = r\}$$

more precisely

$$Gr^{W^A}_r (\cap_{i_s \in A} W^{i_s}_{a_{i_s}} L) \simeq \oplus_{\{m_{i.} \in X^A_r, m_{i_s} \le a_{i_s}\}} Gr^{W^{i_j}}_{m_{i_j}} \cdots Gr^{W^{i_1}}_{m_{i_1}} L \tag{2.4}$$

iii) Let $A = B \cup C$, N'_i denotes the restriction of N_i to $Gr^{W^C}_c$ and $N'_B = \Sigma_{i\in B} N'_i$, then W^B_b induces $W_b(N'_B)$ on $Gr^{W^C}_c$, that is

$$Gr^{W^A}_{b+c} Gr^{W^C}_c L \simeq Gr^{W^B}_b Gr^{W^C}_c L \simeq Gr^{W^A}_{b+c} Gr^{W^B}_b L \simeq Gr^{W^C}_c Gr^{W^B}_b L$$

iv) The repeated graded objects in i) do not depend on the order of the elements in A.

Remark A.13. This result give relations between various weight filtrations in terms of the elementary ones $W^i := W(N_i)$ and will be extremely useful in the study later of the properties of the weight filtration on the mixed Hodge complex.

Proof. To stress the properties of commutativity of the graded operation for the filtrations, we prove first

Sublemma: For all subsets $[1,n] \supset A \supset \{B,C\}$, the isomorphism of Zassenhaus $Gr_b^{W^B} Gr_c^{W^C} L \simeq Gr_c^{W^C} Gr_b^{W^B} L$ is an isomorphism of MHS with weight filtration (up to a shift) $W = W^A$ and Hodge filtration $F = F_A$, hence compatible with the third filtration W^A or F_A.

Proof of the sub-lemma: Recall that both spaces $Gr_b^{W^B} Gr_c^{W^C}$ and $Gr_c^{W^C} Gr_b^{W^B}$ are isomorphic to $W_b^B \cap W_c^C$ modulo $W_c^C \cap W_{b-1}^B + W_b^B \cap W_{c-1}^C$. In this isomorphism a third filtration like F_A (resp. W^A) is induced on one side by $F'_k = (F_A^k \cap W_c^C) + W_{c-1}^C$ (resp. $W'_k = (W_k^A \cap W_c^C) + W_{c-1}^C$) and on the second side by $F''_k = (F_A^k \cap W_b^B) + W_{b-1}^B$ (resp. $W''_k = W_k^A \cap W_b^B) + W_{b-1}^B$). We introduce the third filtration $F'''_k = F_A^k \cap W_b^B \cap W_c^C$ (resp. $W'''_k = W_k^A \cap W_b^B \cap W_c^C$) and we notice that all these spaces are in the category of MHS, hence the isomorphism of Zassenhaus is strict and compatible with the third filtrations induced by F_A (resp W^A).

Proof of the key lemma i). Let $A \subset [1,n]$ and $i \in A$, then W^A exists on L and coincides with the relative weight filtration for N_i with respect to $W^{(A-i)}$ by a result of Cattani and Kaplan [4] (see A.2.4 and just before). Then we have by Kashiwara's result $Gr_l^{W^A} L \simeq \oplus_k Gr_l^{W^A} Gr_k^{W^{(A-i)}} L$. Let us attach to each point (k,l) in the plane the space $Gr_l^{W^A} Gr_k^{W^{(A-i)}} L$ and let $M_j = \oplus_l Gr_l^{W^A} Gr_{l-j}^{W^{(A-i)}} L$ be the direct sum along indices in the plane (k,l) on a parallel to the diagonal $(l = k+j)$. Then we have for $j > 0$

$$(N_i)^j : Gr_{k+j}^{W^A} Gr_k^{W^{(A-i)}} L \simeq Gr_{k-j}^{W^A} Gr_k^{W^{(A-i)}} L, \ (N_i)^j : M_j \simeq M_{-j}.$$

This property leads us to introduce the space $V = \oplus_l Gr_l^{W^A} L \simeq \oplus_{l,k} Gr_l^{W^A} Gr_k^{W^{(A-i)}} L$, then N_i on L extends to a nilpotent endomorphism on V, $N_i : V \to V$ inducing $N_i : Gr_l^{W^A} L \to Gr_{l-2}^{W^A} L$ on each $l-$component of V. We consider on V two increasing filtrations $W'_s := \oplus_{l-k \leq s} Gr_l^{W^A} Gr_k^{W^{(A-i)}} L$ and $W''_s := \oplus_l W_s^i Gr_l^{W^A} L$. Then N_i shift these filtrations by -2. In fact $N_i : W'_s \to W'_{s-2}$ sends $Gr_l^{W^A} Gr_k^{W^{(A-i)}} L$ to $Gr_{l-2}^{W^A} Gr_k^{W^{(A-i)}} L$ and $(N_i)^j$ induces an isomorphism $Gr_j^{W'} V \simeq Gr_{-j}^{W'} V$. As well we have an isomorphism $Gr_j^{W''} V \simeq Gr_{-j}^{W''} V$, since $(N_i)^j : (Gr_j^{W^i} L, W^A, F_A) \simeq (Gr_{-j}^{W_i} L, W^A, F_A)$ is an isomorphism of MHS up to a shift in indices, hence strict on W^A and F_A and induces an isomorphism $Gr_j^{W^i} Gr_l^{W^A} \simeq Gr_{-j}^{W_i} Gr_{l-2j}^{W^A}$. Then these two filtrations W'_s and W''_s are equal by uniqueness of the weight filtration of N_i on V, that is

$$W''_s = \oplus_{k,l} W_s^i Gr_l^{W^A} Gr_k^{W^{(A-i)}} L = W'_s = \oplus_{l-k \leq s} Gr_l^{W^A} Gr_k^{W^{(A-i)}} L$$

that is $W_s^i Gr_l^{W^A} Gr_k^{W^{(A-i)}} L = Gr_l^{W^A} Gr_k^{W^{(A-i)}} L$ if $l - k \leq s$ and $W_s^i Gr_l^{W^A} Gr_k^{W^{(A-i)}} L = 0$ otherwise, hence $Gr_l^{W^A} Gr_j^{W^i} L \simeq Gr_l^{W^A} Gr_j^{W^i} Gr_{l-j}^{W^{A-i}} L$,

and for all $l \neq k+k'$, $Gr_l^{W^A} Gr_k^{W^i} Gr_{k'}^{W^{(A-i)}} L \simeq 0$. which ends the proof of (i).
ii) Since W^A induces a trivial filtration on $Gr_k^{W^i} Gr_{k'}^{W^{(A-i)}} L$ of weight $k+k'$ we have

$$Gr_l^{W^A} L \simeq \oplus_k Gr_l^{W^A} Gr_k^{W^{A-i}} L \simeq \oplus_k Gr_l^{W^A} Gr_k^{W^{A-i}} Gr_{l-k}^{W^i} L \simeq \oplus_k Gr_k^{W^{(A-i)}} Gr_{l-k}^{W^i} L.$$

Now if we suppose by induction on length of A, the decomposition true for $A-i$, we deduce easily the decomposition for A from the above result.
iii) We restate here the property of the relative monodromy for W^A with respect to W^C and we apply ii).
iv)In the proof above we can start with any i in A, hence the decomposition is symmetric in elements in A. It follows that the graded objects of the filtrations $W^i, W^r, W^{\{i,r,j\}}$ commute and since W^j can be expressed using these filtrations, we deduce that W^i, W^r, W^j also commute, for example:
$Gr_{a+b+c}^{W^{\{i,r,j\}}} Gr_{a+b}^{W^{\{i,j\}}} Gr_a^{W^i} \simeq Gr_c^{W^r} Gr_b^{W^j} Gr_a^{W^i}$ is symmetric in i, j, r. □

A.3. The weight filtration and main theorems in the local case. To describe the weight filtration, we introduce a category $S(I) = S$ attached to a set I already used by Kashiwara and Kawai [20] for the intersection complex. We start with a local study, that is to say with the hypothesis of a polarized nilpotent orbit and we describe the weight filtration W on a combinatorial complex quasi-isomorphic to the DeRham complex $\Omega(L, N.)$. The features of the purity theory will appear relatively quickly. First we ask the reader to take some time to get acquainted with the new category $S(I)$ whose objects are indices for the combinatorial complex. The weights zero or -1 describe a complex $IC(L)$ quasi- isomorphic to the fibre of the intermediate extension of $\mathcal{L}$ and for the other weights we need to introduce the complexes $C_r^{KM} L$ for $K \subset M \subset I$ (A.3.3) which describe the purity theory (A.3.4), (prop.A.20) and the geometry of the decomposition theorem (A.3.9), (theorem A.30). A basic technique in the proof is the decomposition into elementary complexes (A.3.8), reflecting relations between the weight filtrations of the various N_i.

A.3.1. *Complexes with indices in the category $S(I)$.* The techniques are similar to the simplicial techniques in Deligne's paper. Here the singularities may come from the coefficients as well as the NCD in X. We introduce a category $S(I)$ attached to a set I, whose objects consist of sequences of decreasing subsets of I of the following form:

$$(s_.) = (I = s_1 \underset{\neq}{\supset} s_2 \ldots \underset{\neq}{\supset} s_p \neq \emptyset),\ (p > 0)$$

Subtracting a subset s_i from a sequence $s_.$ defines a morphism $\delta_i(s.) : (s. - s_i) \to s.$ and more generally $Hom(s.', s.)$ is equal to one element iff $(s.') \leq (s.)$ that is $(s.')$ is obtained from $(s.)$ by deleting some subsets. We write $s. \in S(I)$ and define its degree or length $|s.|$ as the number of subsets s_i in $(s_.)$.

Correspondence with an open simplex. If $I = \{1, \ldots, n\}$ is finite, $S(I)$ can be realized as a barycentric subdivision of the open simplex Δ_{n-1} of dimension $n-1$. A subset K corresponds to the barycenter of the vertices in K and a sequence of subsets to an oriented simplex defined by the vertices associated to the subsets.
For example, for $I = \{1, 2\}$, $S(I)$ consists of the barycenter $\{3/2\}$ of $]1, 2[$ defined by $\{1, 2\}$, and the open simplices $]1, 3/2[$, $]3/2, 2[$ defined resp. by the sequences $\{1, 2\} \supset \{1\}$ and $\{1, 2\} \supset \{2\}$.

Since all sequences contain I, all corresponding simplices must have the barycenter defined by I as vertex, that is a sub-simplex contained in the open simplex $\Delta^*_{n-1} = \Delta_{n-1} - \partial\Delta_{n-1}$. In this way we define an incidence relation $\varepsilon(s., s.')$ between two adjacent sequences equal to $+1$ or -1 according to orientation. Incidence relations $\varepsilon(\Delta_{n-1}, s.)$ are defined as well between Δ_{n-1} and the simplices corresponding to maximal sequences.

Combinatorial objects of an abelian category with indices in $S(I)$, that is functors, are thus defined, as well as complexes of such objects.
We need essentially the following construction. An algebraic or analytic variety over a fixed variety X with indices in $S(I)$ denoted by Π is a covariant functor defined by $\Pi(s.) : X_{s.} \to X$ and morphisms $\Pi(s.' \leq s.) : X_{s.'} \to X_{s.}$ over X for $s. \in S$. An abelian sheaf over Π (resp. complex of abelian sheaves) $\mathcal{F}$ is a contravariant functor of abelian sheaves (resp. complex of abelian sheaves) $\mathcal{F}_{s.}$ over $X_{s.}$ (with functorial morphisms $\varphi(s.' \leq s.) : \Pi^*\mathcal{F}_{s.} \to \mathcal{F}_{s.'}$ for $(s.') \leq (s.)$).
The direct image of an abelian sheaf over Π (resp. complex of sheaves) denoted $\Pi_*\mathcal{F}$ or preferably $s(\mathcal{F}_{s.})_{s.\in S}$ is the simple complex (resp.simple complex associated to a double complex) on X:

$$s(\mathcal{F}_{s.})_{s.\in S} := \Pi_*\mathcal{F} := \oplus_{s.\in S}(\Pi_*\mathcal{F}_{s.})[|s.|-|I|],\ d = \Sigma_{i\in[1,|s.|]}(-1)^{\varepsilon(s.,\delta_i(s.))}\varphi(\delta_i(s.) \leq s.)$$

Example A.14. The variety X defines the constant variety $X_{s.} = X$. The constant sheaf $\mathbb{Z}$ lifts to a sheaf on $X_{s.}$ such that the "diagonal morphism" : $\mathbb{Z}_X \to \oplus_{|(s.)|=|I|}\mathbb{Z}_{X_{s.}}$ (that is : $n \in \mathbb{Z} \to (\ldots, \varepsilon(\Delta_{n-1}, s.)n_{s.}, \ldots) \in \oplus_{|(s.)|=|I|}\mathbb{Z}$ defines a quasi-isomorphism $\mathbb{Z}_X \cong \Pi_*\Pi^*(\mathbb{Z}_X)$. This is true since $S(I)$ is isomorphic to the category defined by the barycentric subdivision of an open simplex of dimension $|I| - 1$.

A.3.2. *Local definition of the weight filtration.* Our hypothesis here consists again of the polarized nilpotent orbit $(L, (N_i)_{i\in M}, F, m)$ of weight m and the corresponding filtrations $(W^J)_{J\subset M}$ where $W^J = W(\sum_{i\in J} N_i)$.
We will use the category $S(M)$ attached to M whose objects consist of sequences of decreasing subsets of M of the form $(s_.) = (M = s_1 \underset{\neq}{\supset} s_2 \ldots \underset{\neq}{\supset} s_p \neq \emptyset)$, $p > 0$.
In this construction we will need double complexes, more precisely complexes of the previously defined exterior complexes, so we introduce *the category* $M^+_.$ *whose objects are the subsets* $J \subset M$ *including the empty set* so that the DeRham complex $\Omega(L, N.)$ *is written now as* $s(L_J)_{J\subset M}$ and we consider objects with indices in the category $M^+_. \times S(M)$.
Geometrically M corresponds to a normal section to Y^*_M in X and J to $\wedge_{i\in J}dz_i$ in the exterior DeRham complex on the normal section to Y^*_M. The decomposition $M^+_. \simeq (M - K)^+_. \times K^+_.$ corresponds to the isomorphism $\mathbb{C}^M \simeq \mathbb{C}^{(M-K)} \times \mathbb{C}^K$.
Notations. For each $s. \in S(M)$ let $W^{s_\lambda} = W(\sum_{i\in s_\lambda} N_i)$ centered at 0, for $J \subset M$ and an integer r, we define $a_{s_\lambda}(J, r) = |s_\lambda| - 2|s_\lambda \cap J| + r$, and for all $(J, s.) \in M^+_. \times S(M)$ the functorial vector spaces
$W_r(J, s.)L := \bigcap_{s_\lambda\in s.} W^{s_\lambda}_{a_{s_\lambda}(J,r)}L$, $\quad F^r(J, s.) := F^{r-|J|}L$, $\quad W^{s_\lambda} = W(\sum_{i\in s_\lambda} N_i)$,
then we consider for each $(s.)$ a DeRham complex $\Omega(L, N.)$

Definition A.15. The weight $\mathcal{W}$ (centered at zero) and Hodge F filtrations on the combinatorial DeRham complex $\Omega^*L = s(\Omega(L, N.))_{s.\in S(M)}$ are defined by "summing" over J and s.

$\mathcal{W}_r(\Omega^* L)\colon = s(\bigcap_{s_\lambda\in s.} W^{s_\lambda}_{a_{s_\lambda}(J,r)} L)_{(J,s.)\in M^+_.\times S(M)},\ \ a_{s_\lambda}(J,r) = |s_\lambda| - 2|s_\lambda\cap J| + r$
and $F^r(\Omega^* L)\colon = s(F^{r-|J|}L)_{(J,s.)\in M^+_.\times S(M)}$

$$(\Omega^* L, \mathcal{W}, F). \tag{3.1}$$

The filtrations can be constructed in two times, first by summing over J to get the sub-complexes
$W_r(s.) = s(W_r(J,s.))_{J\subset M}$ (weight) and $F^r(s.)\colon = s(F^r(J,s.))_{J\subset M}$ (Hodge).

Example A.16. in dimension 2.
Let $W^{1,2} = W(N_1+N_2), W^1 = W(N_1)$ and $W^2 = W(N_2)$, the weight $\mathcal{W}_r$ is a double complex:
$\mathcal{W}_r(\{1,2\} \underset{\neq}{\supset} 1) \oplus \mathcal{W}_r(\{1,2\} \underset{\neq}{\supset} 2) \to \mathcal{W}_r(\{1,2\})$
where the first line is the direct sum of:
$\mathcal{W}_r(\{1,2\} \underset{\neq}{\supset} 1) =$
$(W^{1,2}_{r+2}\cap W^1_{r+1} \overset{(N_1,\,N_2)}{\longrightarrow} W^{1,2}_r\cap W^1_{r-1} \oplus W^{1,2}_r\cap W^1_{r+1} \overset{(-N_2,\,N_1)}{\longrightarrow} W^{1,2}_{r-2}\cap W^1_{r-1})$
and
$\mathcal{W}_r(\{1,2\} \underset{\neq}{\supset} 2) =$
$(W^{1,2}_{r+2}\cap W^2_{r+1} \overset{(N_1,\,N_2)}{\longrightarrow} W^{1,2}_r\cap W^2_{r+1} \oplus W^{1,2}_r\cap W^2_{r-1} \overset{(-N_2,\,N_1)}{\longrightarrow} W^{1,2}_{r-2}\cap W^2_{r-1})$
The second line for $\{1,2\}$ is

$\mathcal{W}_r(\{1,2\}) = (W^{1,2}_{r+2} \overset{(N_1,\,N_2)}{\longrightarrow} W^{1,2}_r \oplus W^{1,2}_r \overset{(-N_2,\,N_1)}{\longrightarrow} W^{1,2}_{r-2})$.
which reduces to the formula in [20] for $r=-1$.

A.3.3. *The Complexes $C^{KM}_r L$ and $C^K_r L$.* To study the graded part of the weight, we need to introduce the following subcategories:
For each subset $K\subset M$, let $S_K(M) = \{s.\in S(M) : K\in s.\}$ (that is $\exists\lambda : K = s_\lambda$).
The isomorphism of categories:

$$S(K)\times S(M-K) \overset{\sim}{\to} S_K(M),\ (s., s.') \to (K\cup s.', s.)$$

will be of important use later. We consider the vector spaces with indices $(J,s.)\in M.^+\times S_K M$,
$C^{KM}_r L(J,s.)\colon = \bigcap_{K\neq s_\lambda\in s.} W^{s_\lambda}_{a_\lambda(J,r-1)} Gr^{W^K}_{a_{K(J,r)}} L$ and for each $(s.)$ the associated complex obtained by summing over J (resp. over $(s.)$:
$C^{KM}_r L(s.)\colon = s(C_r L(J,s.))_{J\in M^+_.},\ C^{KM}_r L\colon = s(C^{KM}_r L(s.))_{(s.)\in S_K M}$.
We write $C^K_r L(J,s.)$, $C^K_r L(s.)$ and $C^K_r L$ when $K=M$.

Definition A.17. For $K\subseteq M$ the complex $C^{KM}_r L$ is defined by summing over J and $(s.)$

$$C^{KM}_r L\colon = s\Big(\bigcap_{K\neq s_\lambda\in s.} W^{s_\lambda}_{a_\lambda(J,r-1)} Gr^{W^K}_{a_{K(J,r)}} L\Big)_{(J,s.)\in M.^+\times S_K(M)} \tag{3.2}$$

In the case $K=M$ we write $C^K_r L$

$$C^K_r L = s(C^K_r L(J,s.))_{(J,s.)\in K.^+\times S(K)} = s((\cap_{K\underset{\neq}{\supset} s_\lambda\in s.} W^{s_\lambda}_{a_{s_\lambda}(J,r-1)} Gr^{W^K}_{a_{K(J,r)}} L)_{(J,s.)\in K^+_.\times S(K)} \tag{3.3}$$

A.3.4. *Purity of the cohomology of the complex* $C_r^K L$. In this subsection we aim to prove that the filtration $\mathcal{W}$ will lead to the weight of what would be in the proper case a mixed Hodge complex in Deligne's terminology, that is the induced filtration by F on the graded parts $Gr^{\mathcal{W}}\Omega^* L$ is a Hodge filtration. For this we need to decompose the complex as a direct sum of intermediate extensions of variations of Hodge structures (which has a meaning locally) whose global cohomologies on $X - Y$ are pure Hodge structures [5] and [18]. The decomposition itself is in the next section. Here we prove the purity of the complex $C_r^K L$. Its unique non vanishing cohomology will the fibre of the variations of Hodge structures needed in the decomposition of $Gr^{\mathcal{W}}\Omega^* L$. The result here is a fundamental step in the general proof. The key lemma proved earlier provides what seems to be the elementary property at the level of a nilpotent orbit that leads to establish the purity and decomposition results. The proof of the theorem below will occupy the whole subsection. First we present a set of elementary complexes. Second we prove the purity result on the complexes $C_r^K L$ which behave as a direct sum of elementary complexes.

Let L be a polarized nilpotent orbit, then the complexes $C_r^K L$ satisfy the following properties

Proposition A.18. (Purity). The cohomology of the complex $C_r^K L$, concentrated in a unique degree, underlies a polarized HS.

The proof will occupy the whole section and is divided in two parts. Precise information can be found in the proposition below.

A.3.5. *Elementary complexes.* We suppose K of length $|K| = n$ and we identify K with the set of integers $[1, n]$, then the elementary complexes are defined by the following simplicial vector spaces. For $J \subset [1, n]$, let

$$K((m_1, \cdots, m_n), J)L = Gr^{W^n}_{m_n - 2|\{n\}\cap J|} \cdots Gr^{W^i}_{m_i - 2|\{i\}\cap J|} \cdots Gr^{W^1}_{m_1 - 2|\{1\}\cap J|} L \tag{3.4}$$

The endomorphism N_i induces a morphism denoted also
$N_i : K((m_1, \cdots, m_n), J)L \to K((m_1, \cdots, m_n), J \cup i)L$ trivial for $i \in J$.
Remark. Instead of L, we can consider such formulas for various natural spaces derived from L such as $L/N_J L$, $N_J L$ or $\cap_{s_\lambda \in s.} W^{s_\lambda}_{a_\lambda} L$ for a sequence $s.$ of subsets of $[1, m]$ containing $[1, n] \subset [1, m]$.

Definition A.19. The elementary complexes are the simple complexes associated to the simplicial vector spaces A(3.4) by summing over $J \subseteq [1, n]$

$$K(m_1, \cdots, m_n)L := s(K((m_1, \cdots, m_n), J)L, N_i)_{J \subseteq [1,n]} \tag{3.5}$$

Proposition A.20. For any $((m_1, \cdots, m_n) \in \mathbb{Z}^n$ let $J(m.) = \{i \in [1, n] : m_i > 1\}$. The cohomology of an elementary complex $K(m_1, \cdots, m_n)L$ is isomorphic to a sub-quotient of the vector space $K((m_1, \cdots, m_n), J(m.))L$ concentrated in degree $|J(m.)|$. Moreover it vanishes if there exists at least one $m_i = 1$.

More precisely, let

$$M(m.) := (\cap_{i \notin J(m.)}(ker N_i : L/(\Sigma_{j \in J(m.)} N_j L) \to L/(\Sigma_{j \in J(m.)} N_j L)$$

then, the cohomology is isomorphic to
$K((m_1, \cdots, m_n), J(m.))(M(m.)) \simeq Gr^{W^n}_{m_n - 2|\{n\}\cap J(m.)|} \cdots Gr^{W^1}_{m_1 - 2|\{1\}\cap J(m.)|}(M(m.))$.

Proof. The proof by induction on n is based on the fact that given an index i, we can view $K(m_1, \cdots, m_n)L$ as the cone over

$N_i : K(m_1, \cdots, \widehat{m_i}, \cdots, m_n)(Gr^{W^i}_{m_i} L) \xrightarrow{N_i} K(m_1, \cdots, \widehat{m_i}, \cdots, m_n)(Gr^{W^i}_{m_i-2} L)$.
It is enough to notice that $N_i : Gr^{W^i}_{m_i} L \to Gr^{W^i}_{m_i-2} L$ is injective if $m_i > 0$, surjective if $m_i < 2$ (bijective for $m_i = 1$). The associated morphism on the complex $K(m_1, \cdots, \widehat{m_i}, \cdots, m_n)$ will have the same property since the constituent vector spaces respect exact sequences by strictness of MHS.
Hence if $m_i > 0$ (resp. $m_i < 2$), N_i is injective on $Gr^{W^i}_{m_i} L$ (resp. surjective onto $Gr^{W^i}_{m_i-2} L$), then
$K(m_1, \cdots, m_i, \cdots, m_n)L \cong K(m_1, \cdots, \widehat{m_i}, \cdots, m_n)(Gr^{W^i}_{m_i-2}(L/N_i L))[-1]$
(resp. $K(m_1, \cdots, \widehat{m_i}, \cdots, m_n)(Gr^{W^i}_{m_i}(ker N_i : L \to L))$
where $K(m_1, \cdots, \widehat{m_i}, \cdots, m_n)$ is applied to the polarized nilpotent orbit $Gr^{W^i}_{m_i-2}(L/N_i L)$ (resp.$Gr^{W^i}_{m_i}(ker N_i : L \to L)$) with the nilpotent endomorphisms N'_j induced by N_j for $j \neq i$. □

Remark A.21. The cohomology space is symmetric in the operations kernel and cokernel and is isomorphic to
$K((m_1, \cdots, m_n), J(m.))[(\cap_{\{i:m_i<1\}} ker N_i)/(\Sigma_{\{j:m_j>1\}} N_j(\cap_{\{i:m_i<1\}} ker N_i))]$.
that is at each process of taking $Gr^{W^i}_{m_i}$ we apply the functor *ker* if $m_i \notin J(m.)$ and *coker* if $m_i \in J(m.)$

A.3.6. *Decomposition into combinatorial elementary complexes.* By the natural decomposition of the relative filtrations in the Key lemma, we have isomorphisms, functorial for the differentials of $C^K_r L$

$$Gr^{W^K}_{a_{K(J,r)}}(\cap_{K \underset{\neq}{\supset} s_\lambda \in s.} W^{s_\lambda}_{a_\lambda(J,r-1)})L \simeq \oplus_{m. \in X(J,s.,r)} Gr^{W^n}_{m_n} \cdots Gr^{W^i}_{m_i} \cdots Gr^{W^1}_{m_1} L,$$

where for all $(J, s.) \in K^+_{\cdot} \times S(K)$,

$$X(J, s., r) = \{m. \in \mathbb{Z}^n : \Sigma_{i \in K} m_i = a_K(J, r) \text{ and } \forall s_\lambda \in s., s_\lambda \neq K, \Sigma_{i \in s_\lambda} m_i \leq a_\lambda(J, r-1)\}$$

In particular, if we define for $J = \emptyset, X(s., r) = X(\emptyset, s., r)$ as

$$X(s., r) = \{m. \in \mathbb{Z}^n : \Sigma_{i \in K} m_i = |K| + r \text{ and } \forall s_\lambda \in s., s_\lambda \neq K, \Sigma_{i \in s_\lambda} m_i \leq |s_\lambda| + r - 1\}$$

the complex $C^K_r L(s.)$ splits as a direct sum of elementary complexes
$C^K_r L(s.) \simeq \oplus_{X(s.,r)} K(m_1, \ldots, m_n)L$.

A.3.7. *The combinatorial elementary complex.* . For each $(m_1, \ldots, m_n) \in \mathbb{Z}^n$ we define a complex with indices in $s. \in S(K)$ as follows:

$$K(m_1, \ldots, m_n; r)L(s.) = K(m_1, \ldots, m_n)L$$

if $(m_1, \ldots, m_n) \in X(s., r)$ and 0 otherwise.

Definition A.22. The combinatorial elementary complex is defined by summing over s.

$$\tilde{K}(m_1, \ldots, m_n; r)L = s[K(m_1, \ldots, m_n; r)L(s.)]_{s. \in S(K)}$$

Lemma A.23. *Define $X(r) = \{m. \in \mathbb{Z}^n : \Sigma_{i \in K} m_i = |K| + r\}$, then we have the decomposition:*

$$C^K_r L \simeq \oplus_{m. \in X(r)} \tilde{K}(m_1, \ldots, m_n; r). \tag{3.6}$$

Example A.24. For $K = \{1,2\}$, $s.$ is one of the three elements $s.' = \{1,2\} \supset \{1\}$, $s.'' = \{1,2\} \supset \{1\}$ or $K = \{1,2\}$. Then $\tilde{K}(m_1,\dots,m_n;r)L$ is defined for fixed r by the combinatorial complex
$K(m_1,m_2;r)L(s.') \simeq K(m_1,m_2)L$ if $m_1+m_2 = 2+r$ and $m_1 \leq r$ and 0 otherwise
$K(m_1,m_2;r)L(s.'') \simeq K(m_1,m_2)L$ if $m_1+m_2 = 2+r$ and $m_2 \leq r$ and 0 otherwise
$K(m_1,m_2;r)L(K) \simeq K(m_1,m_2)L$ if $m_1+m_2 = 2+r$ and 0 otherwise.

Notice $\tilde{K}(m_1,m_2;r)(W_1^1L) \cong 0$ for $r \geq 0$ that is the case $m_1 < 2$. In fact suppose $r > 0$ in the example and $m_1 < 2$, then $m_2 = 2+r-m_1 > r$, hence $K(m_1,m_2;r)L(s.'') \simeq 0$. Moreover $K(m_1,m_2;r)L(s.') \simeq K(m_1,m_2;r)L(K)$ so that $\tilde{K}(m_1,m_2;r)(W_1^1L) \cong 0$.
This is a main point that we prove in a more general setting in the coming basic lemma.

A.3.8. *The elementary sub-complexes supporting the cohomology.* For $r > 0$ we define
$T(r) = \{(m_1,\cdots,m_n) \in \mathbf{N}^n : \forall i \in K, m_i \geq 2 \text{ and } \Sigma_{i\in K} m_i = |K| + r\}$
(for $r = 0$, $T(0) = \emptyset$) so to introduce the complex

$$C(T(r))L \simeq \oplus_{(m_1,\cdots,m_n)\in T(r)} K(m_1,\dots,m_n)L$$

Dually, for $r < 0$ we define
$T'(r) = \{(m_1,\cdots,m_n) \in \mathbf{Z}^n : \forall i \in K, m_i \leq 0 \text{ and } \Sigma_{j\in K} m_j = |K| + r\}$
so to introduce the complex

$$C(T'(r))L \simeq \oplus_{(m_1,\cdots,m_n)\in T'(r)} K(m_1,\dots,m_n)L$$

Lemma A.25. *The complex $C(T(r))L$ embeds diagonally into the direct sum of $C_r(s.)L$ for all sequences $(s.)$ of maximal length $|K|$ so to define a morphism of complexes $C(T(r))L \to C_r^K L$.*
Dually, the complex $C(T'(r))$ embeds in $C_r(s.)L$ for $s. = K$ consisting of one subset K, so to define a morphism of complexes $C(T'(r))L[1-|K|] \to C_r^K L$.

Proof. We check the conditions defined by all $s_\lambda \in (s.)$, $s_\lambda \neq K$, namely $\Sigma_{i\in s_\lambda} m_i \leq |s_\lambda| + r - 1$ for all $(m.) \in T(r)$ by induction on $|s_\lambda|$. We start with the condition $\Sigma_{j\in K} m_j = |K| + r$ defined by K, which is satisfied by definition of $T(r)$. Let $k \in K$, then $\Sigma_{j\in K-\{k\}} m_j \leq |K - \{k\}| + r - 1$ by subtracting $m_k > 1$, which proves the assertion for $s_\lambda = K - \{k\}$, hence for all s_λ such $|s_\lambda| = |K| - 1$. Let $A \subset K, A \neq K$ and suppose $\Sigma_{j\in A} m_j \leq |A| + r - 1$ true, by induction on $|A|$, then for $k \in A, B = A - \{k\}$ we deduce $\Sigma_{j\in B} m_j \leq |B| + r - 2$ by subtracting $m_k > 1$.
Dually, for $(s.) = K$ there is no additional conditions, so the statement is clear. □

Proposition A.26. i) For $r > 0$, the canonical embedding of $C(T(r))L$ into $C_r^K L$ induces an isomorphism on the cohomology.
In particular the cohomology of $C_r^K L$, concentrated in degree $|K|$, is isomorphic to
$H^{|K|}(C_r^K L) \simeq Gr^{W^K}_{r-|K|}[L/(\Sigma_{i\in K} N_i L)] \simeq \oplus_{(m.)\in T(r)} Gr^{W^n}_{m_n-2} \cdots Gr^{W^1}_{m_1-2}[L/(\Sigma_{i\in K} N_i L)]$
it is a polarized HS of weight $r + m - |K|$ with the weight filtration induced by W^K shifted by m and Hodge filtration induced by F^K.
ii) If $r = 0$, the complex $C_r^K L$ is acyclic.
iii) Dually, for $r < 0$, the canonical embedding of $C(T'(r))L[1-|K|]$ into $C_r^K L$ induces an isomorphism on the cohomology.

In particular the cohomology of $C_r^K L$, concentrated in degree $|K|-1$, is isomorphic to

$$H^{|K|-1}(C_r^K L) \simeq Gr^{W^K}_{r+|K|}(\cap_{i\in K}(kerN_i : L \to L)) \simeq$$
$$\oplus_{(m.)\in T'(r)} Gr^{W^n}_{m_n} \cdots Gr^{W^1}_{m_1}[\cap_{i\in K}(kerN_i : L \to L)]$$

it is a polarized HS of weight $r+m+|K|$ with the weight filtration induced by W^K shifted by m and Hodge filtration induced by F^K.

Remark A.27. If $r \in [1, |K|-1]$, $T(r)$ is empty and $C_r^K L$ is acyclic. If $r \in [-|K|+1, 0]$, $T'(r)$ is empty and $C_r^K L$ is acyclic. In all cases $C_r^K L$ appears in $Gr^{\mathcal{W}}\Omega^* L$.

The principal ingredient in the proof is based on the

Lemma A.28. *i) For $r \geq 0$, the complex $\tilde{K}(m_1, \ldots, m_n; r)L$ is acyclic whenever at least one $m_i < 2$. Equivalently, for each $i \in K$, the complex $C_r^K(W_1^i L)$ is acyclic.*
ii) Dually, for $r \leq 0$, the complex $\tilde{K}(m_1, \ldots, m_n; r)L$ is acyclic whenever at least one $m_i \geq 2$. Equivalently, for each $i \in K$, the complex $C_r^K(L/W_0^i L)$ is acyclic.

The equivalences follow from the decompositions
$C_r^K(W_1^i L) \simeq \oplus_{m.\in X(r), m_i<2}\tilde{K}(m_1, \ldots, m_n)L$
$C_r^K(L/W_1^i L) \simeq \oplus_{m.\in X(r), m_i\geq 2}\tilde{K}(m_1, \ldots, m_n, r)L.$
We note that $\tilde{K}(m_1, \ldots, m_n, r)L \cong 0$ if at least one $m_i = 1$.

Proof. The lemma is based on the following elementary remark. Consider in $s.$ a sequence $s_{a+2} = s_{a+1} \cup \{i\} \supset s_{a+1}$ with $i \notin s_{a+1}$, then the condition on $m. \in X(s., r)$ associated to $s.$ defined by $s_{a+1} \cup \{i\}$ is $m_i + \Sigma_{j\in s_{a+1}} m_j \leq |s_{a+1} \cup \{i\}| + r - 1$ to compare with the condition $\Sigma_{j\in s_{a+1}} m_j \leq |s_{a+1}| + r - 1$ defined by s_{a+1}. Precisely when $m_i < 2$ (that is in $W_1^i L$) the condition for $s_{a+1} \cup \{i\}$ follows from the condition for s_{a+1}, hence the conditions defined by the subsequence $s_{a+1} \cup \{i\} \supset s_{a+1}$ in $s.$ is the same as the condition defined by s_{a+1} in $d_{s_{a+2}}(s.)$ where $d_{s_{a+2}}$ is the differential consisting in the removal of s_{a+2}, so that in the sum over all $s.$ this couple is quasi-isomorphic to zero. The following rigorous proof consists on filtering the complex by carefully choosing subsets in $S(K)$. All constructions below are compatible with the decompositions and apply to each complex $\tilde{K}(m_1, \ldots, m_n, r)L$.
i) We construct a filtration of $C_r^K(W_1^i L)$ with acyclic sub-complexes. We define the i-length $|s.|_i$ of a sequence $s.$ as the number of subset s_λ not containing i. Let S_a be the full subcategory whose objects satisfy $|s.|_i \leq a$, hence an object $s.$ in S_a is written as $s.'' \cup \{i\} \supset s.'$ with $|s.'| \leq a$ with no subset in $s.'$ containing i. Deleting a subset of $s.$ in S_a gives another object in S_a, hence $C(S_a)L\colon = s[C_r^K L(s.)]_{s.\in S_a}$ is a sub-complex of $C_r^K L$ where $C_r^K L(s.)$ is obtained by summing over over $J \subset K$ for a fixed $(s.)$.
1) To start we write $C(S_0)L$ as a cone over a morphism inducing a quasi-isomorphism on $C(S_0)(W_1^i L)$. We divide the objects of S_0 in two families : S_0' whose objects are defined by the sequences $(s.)$ starting with the subset $\{i\}$ and S_0'' whose objects are defined by the sequences $(s.)$ whose elements s_λ contain i but are different from $\{i\}$. Then we consider the two complexes $C(S_0')L\colon = s[C_r^K L(s.)]_{s.\in S_0'}$ and $C(S_0'')L\colon = s[C_r^K L(s.)]_{s.\in S_0''}$.
Let $(s.) = K \supset \ldots s_\lambda \supset \ldots \supset \{i\}$ in S_0'; deleting $s_\lambda \neq \{i\}$ is a morphism in S_0' but deleting $\{i\}$ gives an element in S_0'' . It is easy to check that deleting $\{i\}$ defines

a morphism of complexes $I_{\{i\}} : C(S'_0)L \to C(S''_0)L$ defined by embedding $C_r^K L(s.)$ into $C_r^K L(d_{\{i\}}(s.))$ where $d_{\{i\}}(s.) = (s.) - \{i\}$. The cone over this $I_{\{i\}}$ is isomorphic to $C(S_0)L[1]$.

We show now that if we reduce the construction to $W_1^i L$ instead of L, the morphism $I_{\{i\}} : C(S'_0)(W_1^i L) \to C(S''_0)(W_1^i L)$ is an isomorphism. Let $s. = (K \supset \ldots s_\lambda \supset \ldots \supset \{i\})$ in S'_0. It is enough to notice that the condition on $m. \in X(s., r)$ associated to $\{i\}$ defined by $m_i \leq |\{i\}| + r - 1 = r$ is irrelevant since $m_i \leq 1$ and $r > 0$ (since for $m_i = 1$ the elementary complexes are acyclic, we could also suppose $m_i < 1$ and $r = 0$) . It follows that $C(S_0)(W_1^i L)$ is acyclic.

2) *We extend the proof from S_0 to S_a.*

Suppose by induction that $C(S_a)(W_1^i L)$ is acyclic for $a \geq 0$, we prove $C(S_{a+1})(W_1^i L)$ is also acyclic. It is enough to prove that the quotient $G_a L\colon = C(S_{a+1})L/C(S_a)L$ is acyclic.

Let $s. = s.'' \cup \{i\} \supset s.'$ with $|s.'| \leq a + 1$ ($s.'$ not containing i). Deleting $s.'' \cup \{i\}$ is a morphism in S_{a+1} but deleting $s.'$ gives an element in S_a, hence defines a differential zero in $G_a L$.

We divide the objects of S_{a+1} in two families containing S_a according to the subsequence $s''_{a+2} \cup \{i\} \supset s'_{a+1}$, the family S'_{a+1} (respectively S''_{a+1}) whose objects are defined by the sequences satisfying $s''_{a+2} = s'_{a+1}$ (resp. $s''_{a+2} \underset{\neq}{\supset} s'_{a+1}$). Deleting $s''_\lambda \cup \{i\}$ for $s''_\lambda \neq s''_{a+2}$ is a morphism in S'_{a+1} but for $s''_\lambda = s''_{a+2}$ it defines a functor $d_{s_{a+2}} : S'_{a+1} \to S''_{a+1}$. If we consider the complexes $C(S'_{a+1})L/C(S_a)L$ and $C(S''_{a+1})L/C(S_a)L$, we deduce a morphism of complexes $I_{s_{a+2}} : C(S'_{a+1})L/C(S_a)L \to C(S''_{a+1})L/C(S_a)L$ which consists in an embedding of $C_r^K L(s.)$ into $C_r^K L(d_{s_{a+2}}(s.))$ where $d_{s_{a+2}}(s.) = (s.) - s_{a+2}$. It is easy to check that the cone over this $I_{s_{a+2}}$ is isomorphic to $G_a L[1]$.

We show now that if we reduce the construction to $W_1^i L$ the morphism $I_{s_{a+2}}$ is an isomorphism. The condition on $m. \in X(s., r)$ associated to $s.$ defined by $s''_{a+2} \cup \{i\}$ when $s''_{a+2} = s'_{a+1}$ is $m_i + \Sigma_{j \in s'_{a+1}} m_j \leq |s'_{a+1} \cup \{i\}| + r - 1$ to compare with the condition $\Sigma_{j \in s'_{a+1}} m_j \leq |s'_{a+1}| + r - 1$ defined by s'_{a+1}. Precisely when $m_i < 2$ (that is in $W_1^i L$) the condition for $s'_{a+1} \cup \{i\}$ follows from the condition for s'_{a+1}, hence the conditions defined by the subsequence $s'_{a+1} \cup \{i\} \supset s'a + 1$ in $s.$ is the same as the condition defined by s'_{a+1} in $d_{s_{a+2}}(s.)$. This proves that $I_{s_{a+2}}$ is an isomorphism for $W_1^i L$, hence $G_a(W_1^i L)$ is acyclic and i) follows by induction.

ii) *Dual proof* (to be skipped). We construct a dual filtration of $C_r^K(L/W_0^i L)$ with acyclic sub-complexes. To simplify notations we denote a sequence by $s^\cdot$ and define its i-colength $|s^\cdot|^i$ as the number of subsets s_λ containing i. Let S^a be the full subcategory whose objects satisfy $|s^\cdot|^i \leq a$, hence an object $s^\cdot$ in S^a is written as $s.'' \cup \{i\} \supset s'^\cdot$ with $|s''^\cdot| \leq a$ and $s'^\cdot$ not containing i. Deleting a subset of $s^\cdot$ in S^a gives another object in S^a, hence $C(S^a)L\colon = s[C_r^K(s^\cdot)L]_{s^\cdot \in S^a}$ is a sub-complex of $C_r^K L$. To start with $C(S^0)L = 0$ since S^0 is empty. Suppose by induction that $C(S^a)(L/W_0^i L)$ is acyclic for $a \geq 0$, we prove $C(S^{a+1})(L/W_0^i L)$ is also acyclic. It is enough to prove that the quotient $G^a L\colon = C(S^{a+1})L/C(S^a)L$ is acyclic.

Let $s^\cdot = s''^\cdot \cup \{i\} \supset s'^\cdot$ with $|s''^\cdot| \leq a + 1$ ($s''^\cdot$ not containing i). Deleting $s'^\cdot$ is a morphism in S^{a+1} but deleting $s''^\cdot \cup \{i\}$ gives an element in S^a, hence defines a differential zero in $G^a L$.

We divide the objects of S^{a+1} in two families containing S^a according to the subsequence $s''^{a+1} \cup \{i\} \supset s'^{a+2}$, the family S_1^{a+1} (resp. S_2^{a+1}) whose objects are defined

by the sequences $s''^{a+1} = s'^{a+2}$ (resp. $s''^{a+1} \underset{\neq}{\supset} s'^{a+2}$). Deleting s'^{λ} for $s'^{\lambda} \neq s'^{a+2}$ is a morphism in S_1^{a+1} but for $s'^{\lambda} = s'^{a+2}$ it defines a functor $d_{s^{a+2}} : S_1^{a+1} \to S_2^{a+1}$. If we consider the complexes $C(S_1^{a+1})L/C(S^a)L$ and $C(S_2^{a+1})L/C(S^a)L$, we deduce a morphism of complexes $I_{s^{a+2}} : C(S_1^{a+1})L/C(S^a)L \to C(S_2^{a+1})L/C(S^a)L$ by embedding $C_r^K L(s^{\cdot})$ into $C_r^K L(d_{s^{a+2}}(s^{\cdot}))$. It is easy to check that the cone over this $I_{s^{a+2}}$ is isomorphic to $G^a L[1]$.

We show now that if we reduce the construction to $L/W_0^i L$ the morphism $I_{s^{a+2}}$ is an isomorphism. The condition on $m. \in X(s^{\cdot}, r)$ defined by $s''^{a+1} \cup \{i\}$ when $s''^{a+1} = s'^{a+2}$ (resp. by s'^{a+2}) is $m_i + \Sigma_{j \in s'^{a+2}} m_j \leq |s'^{a+2} \cup \{i\}| + r - 1$ (resp. $\Sigma_{j \in s'^{a+2}} m_j \leq |s'^{a+2}| + r - 1$), but precisely when $m_i > 0$ (that is in $L/W_1^i L$) the condition for s'^{a+2} follows from the condition for $s'^{a+2} \cup \{i\}$, hence the conditions defined by the subsequence $s'^{a+2} \cup \{i\} \supset s'^{a+2}$ in $s^{\cdot}$ is the same as the condition defined by $s'^{a+2} \cup \{i\}$ in $d_{s^{a+2}}(s^{\cdot})$. This proves that $I_{s^{a+2}}$ is an isomorphism for $L/W_0^i L$, hence $G^a(L/W_0^i L)$ is acyclic.

This shows by induction that $C(S^{|K|-1})(L/W_0^i)L$ is acyclic. At the last step, we show that $C_r^K(L/W_0^i L)$ is quasi-isomorphic to $C(S^{|K|})(L/W_0^i)L = C(S^{|K|-1})(L/W_0^i)L$. In colength $|K|$ there is only one sequence not in $S^{|K|-1}$ that is the full sequence starting with $\{i\}$ which imposes the condition $m_i \leq r$ which is impossible since $r \leq 0$ and $m_i \geq 1$. □

Then the proposition follows immediately from

Corollary A.29. i) For $r > 0$, the complex $C(T(r))$ is contained in each $C_r^K L(s.)$ that is $T(r) \subset X(s., r)$ and the complex $\tilde{C}(T(r)) = s(C(T(r))_{s. \in S(K)}$ is contained in and quasi-isomorphic to $C_r^K L$.

ii) Dually, the complex $C(T'(r))$ is contained only in $C_r^K L(s.)$ for $s. = K$ and is quasi-isomorphic to $C_r^K L[|K| - 1]$.

Proof. We did check that the complex $C(T(r))$ is contained in each $C_r^K(s.)$ for $s.$ of maximal length, hence for all $(s.)$. The lemma shows that $\tilde{K}(m_1, \ldots, m_n, r)L \cong 0$ whenever at least one $m_i < 2$, hence i) follows.

Dually, the condition for K, $\Sigma_{j \in K} m_j = |K| + r \Rightarrow \forall A = K - k \subset K, \Sigma_{j \in A} m_j > |A| + r - 1$ by subtracting $m_k < 1$. If this is true for all $A : |A| = a$ then $\forall B = (A - k) \subset A, \Sigma_{j \in B} m_j > |B| + r - 1$ as well. The shift in degree corresponds to the shift for $s. = K$ in the total complex $C_r^K L$. □

A.3.9. *Local decomposition.* Since the purity result is established, we can easily prove now the decomposition theorem after a careful study of the category of indices $S(I)$.

Theorem A.30. *(decomposition). For a nilpotent orbit $L, N_i, i \in M$, $|M| = n$ and for all subsets $K \subset M$ there exist canonical morphisms of $C_r^{KM} L$ in $Gr_r^{\mathcal{W}} \Omega^* L$ inducing a quasi-isomorphism (decomposition as a direct sum)*

$$Gr_r^{\mathcal{W}}(\Omega^* L) \cong \oplus_{K \subset M} C_r^{KM} L.$$

Moreover $Gr_0^{\mathcal{W}} \Omega^* L \cong 0$ is acyclic.

For $n = 1$, K and M reduces to one element 1 and the theorem reduces to

$$Gr_r^{\mathcal{W}}(\Omega^* L) \simeq C_r^1 L \text{ is the complex } Gr_{r+1}^{W^1} L \xrightarrow{N_1} Gr_{r-1}^{W^1} L$$

By the elementary properties of the weight filtration of N_1, it is quasi-isomorphic to $Gr^{W^1}_{r-1}(L/N_1L)[-1]$ if $r>0$, $Gr^{W^1}_{r+1}(kerN_1 : L \to L)$ if $r<0$ and $C^1_0L \simeq 0$.

The proof is by induction on n; we use only the property $Gr^{\mathcal{W}}_0(\Omega^*L) \simeq 0$ in $n-1$ variables to get the decomposition for n, then we use the fact that C^K_0L is acyclic for all K to get again $Gr^{\mathcal{W}}_0(\Omega^*L) \cong 0$ for n variables so to complete the induction step. The proof of the decomposition is carried in the three lemmas below.

For each $i \in \mathbf{N}$ we define a map into the subsets of M
$\varphi_i : S(M) \to \mathcal{P}(M) : \varphi_i(s.) = Sup\{s_\lambda : |s_\lambda| \leq i\}$. For each $(J, s.) \in M.^+ \times S(M)$, we consider the subspaces of L with indices i and t
$W_t(i, J, s.) := (\bigcap_{\varphi_i(s.) \underset{\neq}{\subset} s_\lambda \in s.} W^{s_\lambda}_{a_{s_\lambda}(J,r+t)}) \cap (\bigcap_{s_\lambda \subset \varphi_i(s.), s_\lambda \in s.} W^{s_\lambda}_{a_{s_\lambda}(J,r-1)}L)$
We define
$G_i(J, s.)(L) = Gr^{W(i,J,s.)}_0 L = W_0(i, J, s.)/W_{-1}(i, J, s.)$ and the complexes

$$G_i(s.)L = s(G_i(J, s.)L)_{J \subset M}, \quad G^{SM}_i L = s(G_i(s.)L)_{s. \in S(M)}$$

In particular, $\varphi_0(s.) = \emptyset$, and $\varphi_{|M|} = M$ so that

$$G_0(s.)L = Gr^{W(s.)}_r L,\ G^{SM}_0 L = Gr^{\mathcal{W}}_r \Omega^* L, \quad G^{SM}_{|M|} L = C^M_r L$$

The proof of the theorem by induction on i, starting with $i = 0$, is based on

Lemma A.31.

$$G^{SM}_i L \cong G^{SM}_{i+1} L \oplus (\oplus_{K \subset M, |K|=i+1} C^{KM}_r L)$$

To relate $G^{SM}_i L$ and $G^{SM}_{i+1} L$, we define $S^i(M) = \{s. \in S(M) : |\varphi_i(s.)| = i\}$ in $S(M)$ and consider the subcategory $S(M) - S^{i+1}(M)$. The restrictions of the simplicial vector spaces G_i and G_{i+1} to $S(M) - S^{i+1}(M)$, define two sub-complexes:
$G'_i L = s(G_i(s.)L)_{s. \in S(M) - S^{i+1}(M)}$ embedded in $G^{SM}_i L$ and
$G''_{i+1} L = s(G_{i+1}(s.)L)_{s. \in S(M) - S^{i+1}(M)}$ embedded in $G^{SM}_{i+1} L$
since deleting an object $s. \in S(M) - S^{i+1}(M)$ gives an object in the same subcategory. We have $G''_{i+1} L = G'_i L$ since $\varphi_i = \varphi_{i+1}$ on $S(M) - S^{i+1}(M)$; hence we are reduced to relate the the quotient complexes: $G^{SM}_i L/G'_i L$ and $G^{SM}_{i+1} L/G''_{i+1} L$ which are obtained by summing over $S^{i+1}(M)$.
We remark that $S^{i+1}(M) \simeq \cup_{|K|=i+1} S_K(M)$ is a disjoint union of $S_K(M)$ where $\varphi_{i+1}(s.) = K$ with $|K| = i + 1$.
Definition of $B^{KM}_r L$.
For K fixed in M, we introduce the subspaces with index t
$W_t(K, J, s.)L := (\bigcap_{K \underset{\neq}{\subset} s_\lambda \in s.} W^{s_\lambda}_{a_{s_\lambda}(J,r+t)}) \cap (\bigcap_{s_\lambda \subset K, s_\lambda \in s.} W^{s_\lambda}_{a_{s_\lambda}(J,r-1)}L)$
$Gr^{W(K,J,s.)}_0 L = W_0(K, J, s.)L/W_{-1}(K, J, s.)L$
$B^{KM}_r L = s[Gr^{W(K,J,s.)}_0 L]_{(J,s.) \in M.^+ \times S_K(M)}$
It follows by construction
$G^{S^{i+1}(M)}_{i+1} L := s(G_{i+1}(s.)L)_{s. \in S^{i+1}(M)} \simeq \oplus_{|K|=i+1} B^{KM}_r L$,
and a triangle

$$G''_{i+1} L \to G^{SM}_{i+1} L \to \oplus_{|K|=i+1} B^{KM}_r L$$

Definition of $A^{KM}_r L$.
$A^{KM}_r(J, s.)L = [(\bigcap_{K \subset s_\lambda \in s.} W^{s_\lambda}_{a_{s_\lambda}(J,r)}L) \cap (\bigcap_{s_\lambda \underset{\neq}{\subset} K, s_\lambda \in s.} W^{s_\lambda}_{a_{s_\lambda}(J,r-1)}L)]\ /\ [(\bigcap_{s_\lambda \in s.} W^{s_\lambda}_{a_{s_\lambda}(J,r-1)}L)]$
$A^{KM}_r L = s(A^{KM}_r(J, s.))_{(J,s.) \in M.^+ \times S_K(M)}$
We have by construction: $s(G_i(s.)L)_{s. \in S_K M} = A^{KM}_r L$. Suppose $|K| = i + 1$, then

$G_i^{S^{i+1}(M)}L = s(G_i(s.)L)_{s.\in S^{i+1}(M)} \simeq \oplus_{|K|=i+1}A_r^{KM}L.$
Considering the triangle

$$G_i'L \to G_i^{SM}L \to \oplus_{|K|=i+1}A_r^{KM}L$$

and the morphism of triangles: $G_{i+1}^{SM}L \to G_i^{SM}L$, $B_r^{KM}L \to A_r^{KM}L$, $G_{i+1}''L \simeq G_i'L$, the relation in the lemma follows from a relation between $C_r^{KM}L$ and $A_r^{KM}L$

Lemma A.32. *We have a quasi-isomorphism in the derived category:*

$$C_r^{KM}L \oplus B_r^{KM}L \cong A_r^{KM}L$$

The proof is based on the following elementary remark:

Remark A.33. Let W^i for $i = 1, 2$ be two increasing filtrations on an object V of an abelian category and a_i two integers, then we have an exact sequence:

$$0 \to W^2_{a_2-1}Gr^{W^1}_{a_1} \oplus W^1_{a_1-1}Gr^{W^2}_{a_2} \to W^1_{a_1} \cap W^2_{a_2}/W^1_{a_1-1} \cap W^2_{a_2-1} \to Gr^{W^1}_{a_1}Gr^{W^2}_{a_2} \to 0$$

We apply this remark to the space $V = V(J, s.) = \bigcap_{s_\lambda \subsetneq K, s_\lambda \in (s.)} W^{s_\lambda}_{a_{s_\lambda}(J,r-1)}L$,

filtered by:
$W^1_t(K, J, s.)L = W^K_{a_K(J,r+t)}L \cap V(J, s.))$,
$W^2_t(K, J, s.)L = (\bigcap_{K \subsetneq s_\lambda \in (s.)} W^{s_\lambda}_{a_{s_\lambda}(J,r+t)})L \cap V(J, s.)$ so that
$W^1_t(K, J, s.)L \cap W^2_t(K, J, s.)L = (\bigcap_{K \subset s_\lambda \in s.} W^{s_\lambda}_{a_{s_\lambda}(J,r+t)})L) \cap (\bigcap_{s_\lambda \subsetneq K, s_\lambda \in (s.)} W^{s_\lambda}_{a_{s_\lambda}(J,r-1)}L)$.
Let $a_1 = a_2 = t = 0$. , then we deduce from the above sequence an exact sequence of vector spaces

$$0 \to W^2_{-1}(J, s.)Gr^{W^K}_{a_K(J,r)}(V(J, s.)) \oplus W^K_{a_K(J,r-1)}Gr^{W^2}_0(V(J, s.)) \to$$
$$W^K_{a_K(J,r)} \cap W^2_0(V(J, s.))/W^K_{a_K(J,r-1)} \cap W^2_{-1}(V(J, s.)) \to Gr^{W^K}_{a_K(J,r)}Gr^{W^2}_0(V(J, s.)) \to 0$$

By summing over $(J, s.)$ in $M.^+ \times S_K(M)$ we get an exact sequence of complexes

$$0 \to C_r^{KM}L \oplus B_r^{KM}L \to A_r^{KM}L \to D_r^{KM}L \to 0$$

where by definition
$D_r^{KM}L = s[Gr_0^{W^2(J,s.)}Gr^{W^K}_{a_K(J,r)}(\bigcap_{s_\lambda \subsetneq K, s_\lambda \in s.} W^{s_\lambda}_{a_{s_\lambda}(J,r-1)}L)]_{(J,s.)\in M.^+ \times S_K(M)}$
The lemma follows if we prove

Lemma A.34. $D_r^{KM}L \cong 0$.

Proof. The idea of the proof is to consider $C_r^K L$ as the fibre of a local system on Y_K^* and form the filtered complex $(\Omega^*(C_r^K L), \mathcal{W})$ for the polarized nilpotent orbit $(C_r^K L, N_i, i \in M - K$ to which we can apply the theorem on the lower dimensional space Y_K^*. It happens that D_r^{KM} is quasi-isomorphic to $Gr_0^{\mathcal{W}}(\Omega^*(C_r^K L))$ hence acyclic by induction on dimension. We can either use that $C_r^K L$ is reduced to its unique non zero cohomology or as well prove the acyclicity for each term in $C_r^K L$, as we do now.
We consider the filtration induced by W^2 on $Gr^{W^K}_{a_K(J,r)}(\bigcap_{s_\lambda \subsetneq K, s_\lambda \in s.} W^{s_\lambda}_{a_{s_\lambda}(J,r-1)}L)$
$W^2_t(K, J, s.) := \bigcap_{K \subsetneq s_\lambda \in s.} W^{s_\lambda}_{a_{s_\lambda}(J,r+t)}Gr^{W^K}_{a_K(J,r)}(\bigcap_{s_\lambda \subsetneq K, s_\lambda \in s.} W^{s_\lambda}_{a_{s_\lambda}(J,r-1)}L)$.
We want to rewrite the complex $D_r^{KM}L$ as a sum in two times over $(J, s.) \in K.^+ \times S(K)$ and $(J', s.') \in (M - K).^+ \times S(M - K))$ corresponding to $((J \cap J'), (K \cup$

$s.', s.)) \in M.^+ \times S(M)$.
Using the expression of sequences in $S_K(M)$ as $s. = (K \cup s.', s.)$ and the result on relative weight filtration with respect to $Gr^{W^K}_{a_K(J,r)}$ we rewrite W^2_t as
$W^2_t(K,(J,J'),(K\cup s.',s.)) =$
$\bigcap_{s'_\alpha \in s.'} W^{K\cup s'_\alpha}_{a_{s'_\alpha}(J',a_K(J,r)+t)} Gr^{W^K}_{a_K(J,r)}(\bigcap_{s_\lambda \subsetneq K, s_\lambda \in s.} W^{s_\lambda}_{a_{s_\lambda}((J,J'),r-1)} L) =$
$\bigcap_{s'_\alpha \in s.'} W^{s'_\alpha}_{a_{s'_\alpha}(J',t)} Gr^{W^K}_{a_K(J,r)}(\bigcap_{s_\lambda \subsetneq K, s_\lambda \in s.} W^{s_\lambda}_{a_\lambda(J,r-1)} L)$

For a fixed $(J,s.) \in K.^+ \times S(K)$ we introduce the filtration $W'^2_t = \bigcap_{s'_\alpha \in s.'} W^{s'_\alpha}_{a_{s'_\alpha}(J',t)}$ on the space $L(r,J,s.) := Gr^{W^K}_{a_K(J,r)}(\bigcap_{s_\lambda \subsetneq K, s_\lambda \in s.} W^{s_\lambda}_{a_{s_\lambda}(J,r-1)} L)$
and the complex
$D(M-K)(L(r,J,s.) := s[Gr^{W'^2}_0(L(r,J,s.))]_{(J',s.')\in(M-K).^+\times S(M-K)}$
We have by construction

$$D^{KM}_r L = s[D(M-K)(L(r,J,s.)]_{(J,s.)\in K.^+\times S(K)}$$

We prove by induction on n that each complex
$D(M-K)(L(r,J,s.) :=$
$s[Gr^{W'^2}_0(Gr^{W^K}_{a_K(J,r)}(\bigcap_{s_\lambda \subsetneq K, s_\lambda \in s.} W^{s_\lambda}_{a_{s_\lambda}(J,r-1)} L)]_{(J',s.')\in(M-K).^+\times S(M-K)} \cong 0$
is acyclic. Fixing $(J,s.)$, we decompose $L(r,J,s.)$ into a direct sum of
$L(i.) = Gr^{W^{i_s}}_{m_{i_s}-2|i_s\cap J|} \cdots Gr^{W^{i_1}}_{m_{i_1}-2|i_1\cap J|} L$ for $\{i_1,\dots,i_s\} = K$
and $\sum_{i_p\in K} m_{i_p} = a_K(J,r)$.
We reduce the proof to $D(M-K)(L(i.)) \cong 0$, then we introduce the weight filtration $\mathcal{W}$ on the combinatorial DeRham complex $\Omega^*(L(i.))$ for the nilpotent orbit $L(i.)$ of dimension strictly less then n and weight $a_K(J,r)$ and we notice that $D(M-K)(L(i.)) \simeq Gr^{\mathcal{W}}_0(\Omega^*(Gr^{W^K}_{a_K(J,r)}(L(i.)))$ is acyclic by the inductive hypothesis in dimension $< n$. This ends the proof of the lemma. □

A.3.10. *The relation between $C^K_r L$ and $C^{KM}_r L$.* The following result describes $C^{KM}_r L$ as the fibre of an intersection complex of the local system defined by $C^K_r L$.

Proposition A.35. Let $H = H^*(C^K_r L)$ be considered as a nilpotent orbit with indices $i \in M-K$, then we have a quasi-isomorphism: $C^{KM}_r L \cong \mathcal{W}_{-1}\Omega^*(H)$.

Proof. It is based on a decomposition as above: $(K^+_. \times S(K) \times (M-K)^+_. \times S(M-K)) \simeq M^+_. \times S_K(M)$, with the correspondence $(J,s.,J',s.') \to ((J,J'),(s'_. \cup K, s.))$ then using the relations:
$a_{s_\lambda}((J,J'),r-1) = a_{s_\lambda}(J,r-1)$ when $s_\lambda \subset K$
$a_{s'_\lambda}((J,J'),r-1) = a_{s'_\lambda}(J',-1) + a_K(J,r)$ and
$W^{s'_\lambda\cup K}_{a_{s'_\lambda\cup K}((J,J'),r-1)}(Gr^{W^K}_{a_K(J,r)}) = W^{s'_\lambda}_{a_{s'_\lambda}(J',-1)}(Gr^{W^K}_{a_K(J,r)})$
since $W^{s'_\lambda\cup K}$ is relative to W^K we write

$C^{KM}_r L := s(\cap_{K\neq s_\lambda\in(s.'\cup K,s.)} W^{s_\lambda}_{a_{s_\lambda}((J,J'),r-1)} Gr^{W^K}_{a_K((J,J'),r)} L)_{((J,J'),(s'_.\cup K,s.))\in M^+_.\times S_K M}$
as a double sum, first as
$C_r(J',s') = (\cap_{s'_\lambda\in s.'} W^{K\cup s'_\lambda}_{a_{s'_\lambda}(J',a_K(J,r-1)}(s[Gr^{W^K}_{a_K(J,r)}(\cap_{K\supsetneq s_\lambda\in s.} W^{s_\lambda}_{a_{s_\lambda}(J,r-1)} L]_{(J,s.)\in K^+_.\times S(K)})$

then as $s(C_r(J',s'))_{(J',s.')\in(M-K)^+_{\cdot}\times S(M-K)}$

that is $C_r^{KM}L \simeq s[\cap_{s'_\lambda\in s.'}W^{s'_\lambda}_{a_{s'_\lambda}(J',-1)}(C_r^KL)]_{(J',s.')\in(M-K)^+_{\cdot}\times S(M-K)}$

where we used $\cap_{s'_\lambda\in s.'}W^{s'_\lambda}_{a_{s'_\lambda}(J',-1)}(C_r^KL) \simeq C_r^K(\cap_{s'_\lambda\in s.'}W^{K\cup s'_\lambda}_{a_{s'_\lambda}(J',-1)}L)$ as defined above for each subset of L.

This formula is precisely the weight $\mathcal{W}_{-1}$ of the terms of C_r^KL. It shows that $C_r^{KM}L$ is constructed in two times, once as C^K over $K^+_{\cdot}\times S(K)$ (that is a space normal to Y_K) and once as a weight filtration over $(M-K)^+_{\cdot}\times S(M-K)$ (that is the space Y_K). □

A.4. Global construction of the weight filtration. In this section we construct a global bi-filtered combinatorial logarithmic complex and prove a global decomposition of the graded weight into intermediate extensions of polarized VHS on the various intersections of components of Y. We use a formula of the intersection complex announced by Kashiwara and Kawai [20] that we prove since we have no reference for its proof.

Let Y be a NCD in X with smooth irreducible components $Y_{i\in I}$ with indices in the set I. The derived direct image of the complex local system $Rj_*\mathcal{L}$ is computed globally via the logarithmic complex with coefficients in Deligne's analytic extension $\Omega^*_X(LogY)\otimes\mathcal{L}_X$. It is on a quasi-isomorphic constant combinatorial complex with indices $s.\in S(I)$

$$\Omega^*\mathcal{L} = s(\Omega^*_{X_{s.}}(LogY)\otimes\mathcal{L}_{X_{s.}})_{s.\in S(I)}$$

that we can define the two filtrations $\mathcal{W}$ and F.

Lemma A.36. *Let $M\subset I$, $y\in Y^*_M$ and $L\simeq\mathcal{L}_X(y)$ the space of multivalued sections of $\mathcal{L}$ at y, then the correspondence from $v\in L$ to $\tilde{v}\in\mathcal{L}_{X,y}$ extends to a quasi-isomorphism*

$$\Omega^*L\cong(\Omega^*\mathcal{L})_y$$

The quasi-isomorphism $\Omega(L,N_j,j\in M)\cong(\Omega^*_X(LogY)\otimes\mathcal{L}_X)_y$ (A(2.2)) is compatible with the differentials when defined with indices $s.\in S(I)$.

A.4.1. *The weight $\mathcal{W}$.* For each $(s.)\in S(I)$ we deduce, from the weight filtration by sub-complexes $W_r(s.)L = s(W_r(J,s.)L)_{J\subset M}$ (A.3.2) of the locally defined DeRham complex $\Omega(L,N.)$, a corresponding global filtration by sub-complexes $\mathcal{W}_r(X,\mathcal{L})(s.)$ of $\Omega^*_X(LogY)\otimes\mathcal{L}_X$.

Definition A.37. The filtration $\mathcal{W}_*(X,\mathcal{L})(s.)$ of $\Omega^*_X(LogY)\otimes\mathcal{L}_X$ is defined locally in $(\Omega^*_X(LogY)\otimes\mathcal{L}_X)_y$, in terms of a set of coordinates $z_i, i\in M$, equations of Y_M in a neighbourhood of $y\in Y^*_M$, as follows

$\mathcal{W}_r(X,\mathcal{L})_y(s.)$ is generated as an $\Omega^*_{X,y}-$ sub-module by the germs of the sections $\wedge_{j\in J}\frac{dz_j}{z_j}\otimes\tilde{v}$ for $v\in W_r(J,s.)L$.

This formula is independent of the choice of coordinates, since if we choose a different coordinate $z'_i = fz_i$ instead of z_i with f invertible holomorphic at y, the difference $\frac{dz'_i}{z'_i}-\frac{dz_i}{z_i}=\frac{df}{f}$ is holomorphic at y, hence the difference of the sections $\wedge_{j\in J}\frac{dz'_j}{z'_j}\otimes\tilde{v}-\wedge_{j\in J}\frac{dz_j}{z_j}\otimes\tilde{v}$ is still a section of the sub-complex $\mathcal{W}_r(X,\mathcal{L})(s.)$.

Moreover the restriction of the section is still defined in the sub-complex near y, since $W_r(J,s.)L\subset W_r(J-i,s.)L$ for all $i\in J$, then we have a quasi-isomorphism

LEMMA A.38. *We have an induced quasi-isomorphism* $W_r(s.)L \cong \mathcal{W}_r(X,\mathcal{L})_y(s.)$, *functorial in* $(s.)$.

DEFINITION A.39. (weight and Hodge filtrations). The weight filtration is defined on the combinatorial logarithmic complex with indices $s. \in S(I)$

$$\Omega^*\mathcal{L} = s(\Omega^*_{X_{s.}}(LogY) \otimes \mathcal{L}_{X_{s.}})_{s.\in S(I)} \tag{4.1}$$

as follows:

$$\mathcal{W}_r(X,\mathcal{L}) := s(\mathcal{W}_r(X,\mathcal{L})(s.))_{s.\in S(I)}$$

The Hodge filtration F is constant in $(s.)$ and deduced from Schmid's extension to $\mathcal{L}_X$

$F^p(s.) = 0 \to F^p\mathcal{L}_X \ldots \to \Omega^i_{X_{s.}}(LogY) \otimes F^{p-i}\mathcal{L}_{X_{s.}} \to \ldots; \quad F^p = s(F^p(s.))_{s.\in S}$

The fibre of Deligne's bundle $\mathcal{L}_X(y)$ at the point y is identified with the space of multivalued sections L.

DEFINITION A.40. (weight). With the same notations, let $M \subset I, |M| = p$ and $y \in Y^*_M$, then in terms of a set of n coordinates $z_i, i \in [1,n]$ where we identify M with $[1,p]$, we write a section
$f = (f^{s.})_{s.\in S}$ with $f^{s.} = \Sigma_{J\subset M, J'\cap M=\emptyset} \frac{dz_J}{z_J} \wedge dz_{J'} \otimes f^{s.}_{J,J'}$ s.t. $f^{s.}_{J,J'}$ is not divisible by $y_j, j \in J$, then f is in $(\mathcal{W}_r\Omega^*\mathcal{L})_y$ if and only if $f^{s.}_{J,J'}(y)$ in $\mathcal{L}_X(y) = L$ satisfy

$$\forall J \subset M,\ f^{s.}_{J,J'}(y) \in \bigcap_{s_\lambda \in s., s_\lambda \subset M} W^{s_\lambda}_{a_\lambda(J,r)} L$$

A.4.2. *Definition via residue.* Let W^Y denotes the weight along Y on $\Omega^*_X(LogY)$; choose an order on I and an integer m, then the residue morphism Res_m of order m is defined on $W^Y_m(\Omega^p_X(LogY)) \otimes \mathcal{L}_X$ with value in $\Omega^{p-m}_{Y^m} \otimes \mathcal{L}_{Y^m}$ on the disjoint union of intersections of m components of Y (the residue does not commute with differentials). For $M \subset I$ s.t. $Y_M \neq \emptyset$ and $|M| = m$ we deduce the residue $Res_M : W^Y_m(\Omega^p_X(LogY)) \otimes \mathcal{L}_X \to \Omega^{p-m}_{Y_M} \otimes \mathcal{L}_{Y_M}$ by composition of the residue morphism Res_m with the obvious projection. At a point $y \in Y_M$, the morphism induced on the fibre with value in $(\Omega^{p-m}_{Y^m} \otimes \mathcal{L}_{Y^m})(y)$ is denoted by $Res_{M,y}$.

DEFINITION A.41. With the same notations, let $M \subset I, |M| = p$ and $y \in Y^*_M$, then the fibre of the sub-analytic sheaf $(\mathcal{W}_r\Omega^*\mathcal{L})_y$ at y is defined by induction on its W^Y weight i, that is its intersection with $(W^Y_i(\Omega^p_{X_{s.}}(LogY) \otimes \mathcal{L}_{X_{s.}})_y$ by the following formula:
setting $\mathcal{L}_{Y^m}(y) = L$, a section $f \in (W^Y_i\Omega^p_X(LogY) \otimes \mathcal{L}_X)_y$ is in $(\mathcal{W}_r\Omega^*\mathcal{L})_y$ if and only if

$\forall J, s_\lambda \subset M, s_\lambda \in s. : |J| = i,\ Res_{J,y}(f) \in \Omega^{p-m}(y) \otimes W^{s_\lambda}_{a_\lambda(J,r)}\mathcal{L}_{Y^m}(y)$
and $f - \sum_{|J|=i} \frac{dz_J}{z_J} \otimes \widetilde{Res_{J,y}(f)} \in (\mathcal{W}_r\Omega^*\mathcal{L})_y \cap (W^Y_{i-1})_y$

REMARK A.42. By construction, for all integers r, $\mathcal{W}_r/X-Y = \Omega^*(\mathcal{L})/X-Y$, so that $\mathcal{W}_r$ is exhaustive for r big enough, and equal to the extension by zero for r small enough. It is a filtration by sub-complexes of analytic sub-sheaves globally defined on X.

PROPOSITION A.43. (Comparison with the local definition). Let $L = \mathcal{L}_X(y)$ denotes the space of multi-valued sections of $\mathcal{L}$ at a point $y \in Y^*_M$, then we have a bi-filtered quasi-isomorphism

$$(\Omega^* L, \mathcal{W}, F) \cong (\Omega^* \mathcal{L}, \mathcal{W}, F)_y$$

Proof. At left we sum on $S(M)$ and at right on S(I). The statement asserts that on Y_M^*, we still have a quasi-isomorphism. For a fixed subset M in I, the definition of the weight filtration near a point in Y_M^* involves only the subsets $s_\lambda \subset M$, hence if we consider the correspondence $s.' \in S(M)$ with the family $(M \cup s.'', s.') \in S_M(I)$ where $s.'' \in S(I-M)$, the diagonal embedding of the restrictions to Y_M^*, $\Omega^*_{X_{(s.')}}(LogY) \otimes \mathcal{L}_{X_{(s.')}}, \mathcal{W}, F)_{|Y_M^*}$ into

$$s((\Omega^*_{X_{(M\cup(s.''),(s.'))}}(LogY) \otimes \mathcal{L}_{X_{(M\cup(s.''),(s.'))}}, \mathcal{W}, F)_{|Y_M^*})_{(s.'')\in S(I-M)}$$

is a bi-filtered quasi-isomorphism, hence the local study at points of Y_M^* of $(\Omega^* L, \mathcal{W}, F)$ reduces to $s(\Omega^*_{X_{(s.')}}(LogY) \otimes \mathcal{L}_{X_{(s.')}}, \mathcal{W}, F)_{|Y_M^*})_{s.'\in S(M)}$. □

A.4.3. *The variation of Hodge structures* $(\mathcal{L}_r^K, F)$. Let $i_K : Y_K^* \to X$ and recall the definitions in the introduction:
$\mathcal{L}^K = i_K^* R^{|K|} R j_* \mathcal{L}, \quad \mathcal{L}'^K = i_K^* R^0 R j_* \mathcal{L}$ $\mathcal{L}_{Y_K}^K = \mathcal{L}_{Y_K}/(\Sigma_{i\in K}\mathcal{N}_i \mathcal{L}_{Y_K})$,
$\mathcal{L}'^K_{Y_K} = \cap_{i\in K}(ker\mathcal{N}_i : \mathcal{L}_{Y_K} \to \mathcal{L}_{Y_K})$ and $\mathcal{W}_{Y_K}^K = \mathcal{W}(\Sigma_{i\in K}\mathcal{N}_i)$
for the filtration by sub-bundles defined on $\mathcal{L}_{Y_K}$ by $\Sigma_{i\in K}\mathcal{N}_i$.

Proposition A.44. i) $\mathcal{L}_{Y_K}^K$ (resp. $\mathcal{L}'^K_{Y_K}$) induces a flat bundle on Y_K^*, with flat sections isomorphic to the local system $\mathcal{L}^K$ (resp. $\mathcal{L}'^K$); precisely they are respectively Deligne's extension of the corresponding complex local system.

ii) The filtration $\mathcal{W}_{Y_K}^K$ induces a filtration by flat sub-bundles of $\mathcal{L}_{Y_K}^K$ (resp $\mathcal{L}'^K_{Y_K}$) on Y_K^*, hence induces a filtration by complex sub-local systems $\mathcal{W}^K$ of $\mathcal{L}^K$ (resp. $\mathcal{L}'^K$).

iii) The filtration $\mathcal{W}^K$ is rationally defined on the rational local system $\mathcal{L}^K$ (resp. $\mathcal{L}'^K$).

iv) Let $\mathcal{L}_r^K := Gr^{\mathcal{W}^K}_{r-|K|}\mathcal{L}^K$ and $\mathcal{L}^K_{Y_K,r} := Gr^{\mathcal{W}^K_{Y_K}}_{r-|K|}\mathcal{L}^K_{Y_K}$ for $r \geq |K|$, resp.
$\mathcal{L}_r^K := Gr^{\mathcal{W}^K}_{r+|K|}\mathcal{L}'^K$ and $\mathcal{L}^K_{Y_K,r} := Gr^{\mathcal{W}^K_{Y_K}}_{r+|K|}\mathcal{L}'^K_{Y_K}$ for $r \leq |K|$,
then the system defined on Y_K^* by $(\mathcal{L}_r^K, (\mathcal{L}^K_{Y_K,r}), F)$ where F is the Hodge filtration induced from $\mathcal{L}_X$, is a polarized variation of Hodge structures VHS of weights $r - |K| + m$ for $r \geq |K|$ and $r + |K| + m$ for $r \leq |K|$.

Proof. We deduce from the comparison propositions with local definitions at each point $y \in Y_K^*$ the following complexes quasi-isomorphic to $(Rj_*\mathcal{L})_y$

$(\Omega_X^*(LogY) \otimes \mathcal{L}_X)_y \cong \Omega(L, N_j, j \in K) \cong s(L(J), N.)_{J\subset K}$,
$(\Omega^* L, \mathcal{W}, F) \cong (\Omega^* \mathcal{L}, \mathcal{W}, F)_y$

hence near each point $y \in Y_K^*$, $\mathcal{L}^K$ and $\mathcal{L}'^K$ are (locally) constant. The local system $\mathcal{L}^K$ (resp. $\mathcal{L}'^K$) is defined by the flat sections of the bundles $\mathcal{L}_{Y_K}^K$ (resp. $\mathcal{L}'^K_{Y_K}$) whose connection has logarithmic singularity since it is induced by the connection on $\mathcal{L}_X$ which proves (i).
The same argument apply to the filtration $\mathcal{W}^K$, which proves ii).

iii) Let us denote by $\mathcal{L}^{K,rat}$ the rational local system underlying the complex $\mathcal{L}^K$. The intersection $\mathcal{W}^K \cap \mathcal{L}^{K,rat}$ defines a rational filtration underlying the complex one. This can be checked locally as the graded vector space $Gr_r^{W^K} L$ has a rational structure at each point y.

iv) The sheaf $\mathcal{L}_r^K$ is locally constant and isomorphic to the cohomology of the complex $C_r^K L$ for $L = \mathcal{L}_X(y)$ (A.3.6) which shows that the local system $\mathcal{L}_r^K$ is defined by the flat sections of the bundle $\mathcal{L}_{Y_K,r}^K$, then (iv) follows. □

Remark A.45. Given the VHS $\mathcal{L}_r^K$, we can construct a corresponding complex $(\Omega^*\mathcal{L}_r^K, \mathcal{W}, F)$, then for each point $y \in Y_M^*$, $K \subset M$, we have a quasi-isomorphism $C_r^{KM}L \cong \mathcal{W}_{-1}(\mathcal{L}_r^K[-|K|])_y$ for $r \geq |K|$ and $C_r^{KM}L \cong \mathcal{W}_{-1}(\mathcal{L}_r^K[1-|K|])_y$ for $r \leq |K|$ (recall $C_0^{KM}L \cong 0$). That is the cohomology of the various complexes $C_r^{KM}L$ is globally defined on Y_M^*. In fact we will see in the next result it is the restriction of the intermediate extension of $\mathcal{L}_r^K$ to Y_M^*.

It will follow from the next proof that we could define $\mathcal{L}_r^K$ as
$\mathcal{L}_r^K := i_K^*\mathcal{H}^{|K|}(Gr_r^{\mathcal{W}}\Omega^*\mathcal{L}), r > |K|$ (*resp.* $\mathcal{L}_r'^K := i_K^*\mathcal{H}^{|K|-1}(Gr_r^{\mathcal{W}}\Omega^*\mathcal{L}), r < |K|$)

Theorem A.46. *Let $\mathcal{L}$ be a local system with locally unipotent monodromy, underlying a variation of polarized Hodge structures of weight m on $X - Y$ of dim n and let $j^K : Y_K^* \to Y_K$, $i_K : Y_K^* \to X$, then the bi-filtered complex*

$$(\Omega^*\mathcal{L}[n], \mathcal{W}, F)$$

is filtered quasi-isomorphic to $(\Omega_X^*(LogY) \otimes \mathcal{L}_X[n], F)$.
i) The restriction to Y_K^*, $i_K^*\mathcal{H}^{|K|}(Gr_r^{\mathcal{W}}\Omega^*\mathcal{L})$ for $r \geq |K|$ (resp. $i_K^*\mathcal{H}^{|K|-1}(Gr_r^{\mathcal{W}}\Omega^*\mathcal{L})$ for $r \leq |K|$) is a complex local system isomorphic to $\mathcal{L}_r^K$, moreover the following decomposition property into intermediate extensions is satisfied

$$(Gr_r^{\mathcal{W}}\Omega^*\mathcal{L}[n], F) \cong \oplus_{K\subset I} j_{!*}^K\mathcal{L}_r^K[n-|K|], F[-|K|]), \text{ for } r \geq |K|$$
$$(Gr_0^{\mathcal{W}}\Omega^*\mathcal{L}[n], F) \cong 0$$
$$(Gr_r^{\mathcal{W}}\Omega^*\mathcal{L}[n], F) \cong \oplus_{K\subset I} j_{!*}^K\mathcal{L}_r^K[n+1-|K|], F), \text{ for } r \leq |K|$$

ii) The sub-complex $\mathcal{W}_{-1}\Omega^*(\mathcal{L}[2n])$ is quasi-isomorphic to the intermediate extension $j_{!*}\mathcal{L}[n]$ of $\mathcal{L}[n]$ and $(\mathcal{W}_{-1}\Omega^*(\mathcal{L}[n]), F)$ is a Hodge complex of weight $n+m$ (Kashiwara and Kawai's formula [20]*).*

Proof. i) The decomposition of $(Gr_r^{\mathcal{W}}\Omega^*\mathcal{L}, F)$ reduces near a point $y \in Y_M^*$ to the local decomposition for the nilpotent orbit L defined at the point y by the local system $Gr_r^{\mathcal{W}}\Omega^*L \cong \oplus_{K\subset M} C_r^{KM}L$. The global decomposition that follows is $Gr_r^{\mathcal{W}}\Omega^*\mathcal{L} \cong \oplus_{K\subset I}\mathcal{W}_{-1}\Omega^*(\mathcal{L}_r^K)$ that is the complex Ω^* considered for the polarized local system $\mathcal{L}_r^K$, as it follows from the local formula $C_r^{KM}L \cong \mathcal{W}_{-1}L_r^K)$ where $L_r^K \simeq H^*(C_r^K L)$ which has been checked. The fact that $C_r^{KM}L$ is precisely the fibre of $j_{!*}^K\mathcal{L}_r^K[-|K|]$ for $r > 0$ (resp. $j_{!*}^K\mathcal{L}_r^K[1-|K|]$ for $r < 0$ will follow from (ii) by induction on the dimension.
The count of weight and the shift in F take into account for $r > 0$ the residue in the isomorphism with L that shifts W and F but also the shift in degrees, while for $r < 0$ there is no residue (since the cohomology is in degree 0 of the logarithmic complex with index $s. = K \in S(K)$ but only a shift in degrees $|K| - 1$ in the combinatorial complex, the rule being as follows:
Let (K, W, F) be a mixed Hodge complex then for all $m, h \in \mathbb{Z}$, $(K', W', F') = (K[m], W[m-2h], F[h])$ is also a mixed Hodge complex.
The same proof apply for $r = 0$.

ii) The proof is based on the decomposition of $S_K M$ as a product in (A.3.3) and follows by induction on the dimension n, from the local decomposition of the graded parts of the weight filtration above in i).
The proof is true in dimension 1 and if we suppose the result true in dimension

strictly less than n, we can apply the result for $Gr_r^{\mathcal{W}}\Omega^*\mathcal{L}$ that is for local systems defined on open subsets of the closed sets Y_K, namely the local system $\mathcal{L}_r^K[-|K|]$ for $r > 0$ (resp.$\mathcal{L}_r^K[1-|K|]$ for $r < 0$) whose fibre at each point $y \in Y_K^*$ is quasi-isomorphic to $C_r^K L$. Let $j^K : Y_K^* \to Y_K$ be the open embedding in Y_K and consider the associated DeRham complex $\Omega^*(\mathcal{L}_r^K)$ on Y_K whose weight filtration will be denoted locally near a point in Y_M^* by $\mathcal{W}^{M-K}$ for $K \subset M$; then by the inductive hypothesis we have at the point y: $\mathcal{W}_{-1}^{M-K}\Omega^* L_r^K \simeq \mathcal{W}_0^{M-K}\Omega^* L_r^K$ is also quasi-isomorphic to the fibre of the intermediate extension of $\mathcal{L}$, that is

$$\forall r > 0, C_r^{KM}(L) \simeq (j_{!*}^K \mathcal{L}_r^K[-|K|])_y \simeq \mathcal{W}_{-1}^{M-K}\Omega^* L_r^K$$

and similarly for $r < 0$.

In order to check the result for $\mathcal{W}_0\Omega^*(\mathcal{L}[2n])$, we use the following criteria characterizing intermediate extension [13] where the degree shift is by $2n$:
Consider the stratification defined by Y on X and the middle perversity $p(2k) = k-1$ associated to the closed subset $Y^{2k} = \cup_{|K|=k} Y_K$ of real codimension $2k$. We let $Y^{2k-1} = Y^{2k}$ and $p(2k-1) = k-1$. For any complex of sheaves $S.$ on X which is constructible with respect to the stratification, let $S.^{2k} = S.^{2k-1} = S.|X - Y^{2k}$ and consider the four properties:

a) Normalization: $S.|X - Y^2 \cong \mathcal{L}[2n]$

b) Lower bound: $\underline{H}^i(S.) = 0$ for all $i < -2n$

c) vanishing condition: $\underline{H}^m(S.^{2(k+1)}) = \underline{H}^m(S.^{2k+1}) = 0$ for all $m > k - 2n$

d) dual condition: $\underline{H}^m(j_{2k}^! S.^{2(k+1)}) = 0$ for all $k \geq 1$ and all $m > k - 2n$ where $j_{2k} : (Y^{2k} - Y^{2(k+1)}) \to (X - Y^{2(k+1)})$ is the closed embedding,

then we can conclude that $S.$ is the intermediate extension of $\mathcal{L}[2n]$.
In order to prove the result in dim. n we check the above four properties for $\mathcal{W}_0\Omega^*(\mathcal{L}[2n])$. The first two are clear and we use the exact sequences

$$0 \to \mathcal{W}_{r-1}\Omega^*(\mathcal{L}[2n]) \to \mathcal{W}_r\Omega^*(\mathcal{L}[2n]) \to Gr_r^{\mathcal{W}}\Omega^*(\mathcal{L}[2n]) \to 0$$

to prove d)(resp. c)) by descending (resp. ascending)indices from $\mathcal{W}_r$ to $\mathcal{W}_{r-1}$ for $r \geq 0$ (resp. $r-1$ to r for $r < 0$) applying at each step the inductive hypothesis to $Gr_r^{\mathcal{W}}$.

Proof of d). For r big enough $\mathcal{W}_r\Omega^*(\mathcal{L}[2n])$ coincides with the whole complex $Rj_*\mathcal{L}[2n]$, then the dual condition is true for r big enough. Now to check d) for $Gr_r^{\mathcal{W}}\Omega^*(\mathcal{L}[2n])$, we apply d) to a component with support $Y_{K'}$ with $|K'| = k'$. We choose $k > k'$ and consider $j'_{2k} : (Y^{2k} \cap Y_{K'} - Y^{2(k+1)} \cap Y_{K'}) \to (Y_{K'} - Y^{2(k+1)} \cap Y_{K'})$ (notice that $Y^{2k} \cap Y_{K'} = (Y \cap Y_{K'})^{2(k-k')}$ is of codim. $2(k-k')$ in $Y_{K'}$), then for $S.'$ equal to the intermediate extension of $\mathcal{L}_r^{K'}[2n-2k']$ on $Y_{K'}$ we have on $Y_{K'}$ the property $\underline{H}^m(Rj'^!_{2k} S.'^{2(k-k')+1)}) = 0$ for all $(k-k') \geq 1$ and all $m > k-k'-2(n-k') = k+k'-2n$ which gives for $S.'[k']$ on X: $\underline{H}^m(Rj_{2k}^! S.'^{2(k+1)}[k']) = 0$ for all $k > k'$ and all $m > k - 2n$, hence d) is true.
If $k = k'$, then $Y^{2k} \cap Y_{K'} = Y_{K'}$ and we have a local system in degree $k' - 2n$ on $Y_{K'} - Y^{2(k+1)} \cap Y_{K'}$ hence d) is still true, and for $k < k'$, the support $Y_{K'} - Y^{2(k+1)} \cap Y_{K'}$ of $S.'$ is empty. From the decomposition theorem and the induction, this argument apply to $Gr_r^{\mathcal{W}}$ and hence apply by induction on $r \geq 0$ to $\mathcal{W}_0$ and also to $\mathcal{W}_{-1}$.

Proof of c). Dually, the vanishing condition is true for r small enough since then $\mathcal{W}_r$ coincides with the extension by zero of $\mathcal{L}[2n]$ on $X - Y$.

Now we use the filtration for $r < 0$, for $S.'$ equal to the intermediate extension of $\mathcal{L}_r^{K'}[2n-2k']$ on $Y^{K'}$ we have for $k > k'$: $\underline{H}^m(S.'^{2(k-k')+1}) = 0$ for all $m > k+k'-2n$, which gives for $S.'[k'+1]$, $(r < 0)$ on X: $\underline{H}^m(S.^{2(k+1)}) = \underline{H}^m(S.^{2k+1}) = 0$ for all $m > k-1-2n$. If $k = k'$, then $S.'[k'+1]$ is a local system in degree $-2n+k-1$ on $Y_{K'} - Y^{k+1}$ and for $k < k'$, $Y_{K'} - Y^{k+1}$ is empty. □

Corollary A.47. The weight filtration $\mathcal{W}$ of $\Omega^*\mathcal{L}[n]$ is defined over $\mathbb{Q}$.

The proof is based on the following lemma applied to $Rj_*\mathcal{L} \simeq \Omega^*\mathcal{L}$ with its filtration $\mathcal{W}$.

Lemma A.48. *Let K be a $\mathbb{Q}-$perverse sheaf such that $K^c = K \otimes \mathbb{C}$ is filtered by a finite filtration W^c of complex perverse sub-sheaves s.t.$Gr_r^{W^c}K^c$ is rationally defined and the rational filtration $W^r = W^c \cap K$ induces the rational structure on $Gr_r^{W^c}K^c$, then W^r is a rational filtration by perverse sub-sheaves of K such that $W^r \otimes \mathbb{C} \simeq W^c$.*

The proof is similar to the case of local systems and is by induction on the weight i since by hypothesis it applies to the lowest weight. Considering the extension $0 \to W_i^c K^c \to W_{i+1}^c K^c \xrightarrow{\pi} Gr_{i+1}^{W^c}K^c \to 0$, then $W_{i+1}^r K \otimes \mathbb{C} \simeq W_{i+1}^c K$ follows from the hypothesis $(Gr_{i+1}^{W^r}K) \otimes \mathbb{C} \simeq (Gr_{i+1}^{W^c}K)$ and the inductive isomorphism for $W_i^r K$.

Corollary A.49. If X is proper and if we forget the negative weights in the filtration $\mathcal{W}$ that is we consider $\mathcal{W}''$ with $\mathcal{W}''_i = \mathcal{W}_i$ for $i \geq 0$ and $\mathcal{W}''_{-1} = 0$, then the bi-filtered complex

$$(\Omega^*\mathcal{L}[n], \mathcal{W}''[m+n], F)$$

is a mixed Hodge complex.

A.5. The complex of nearby cycles $\Psi_f(\mathcal{L})$. Let $f\colon X \to D$ and suppose $Y = f^{-1}(0)$ a NCD, the complex of sheaves $\Psi_f\mathcal{L}$ of nearby co-cycles on Y has been introduced in [10]; its cohomology fibre $H^i((\Psi_f\mathcal{L})_y) \simeq H^i(F_y, \mathcal{L})$ at a point y in Y is isomorphic to the cohomology of the Milnor fibre F_y at y. The monodromy induces an action $\mathcal{T}$ on the complex itself. If $dim\, X = n$, $\Psi_f\mathcal{L}[n-1]$ is perverse on Y. Since the local system $\mathcal{L}$ is defined over $\mathbb{Q}$, the monodromy decomposes in the abelian category of $\mathbb{Q}-$perverse sheaves as the product $\mathcal{T} = \mathcal{T}^s\mathcal{T}^u$ of simple and unipotent endomorphisms. Let $\mathcal{N} = Log\mathcal{T}^u$, then Deligne's filtration $\mathcal{W}(\mathcal{N})$ is defined over $\mathbb{Q}$. The aim of this section is to describe the structure of a mixed Hodge complex (MHC) on $\Psi_f\mathcal{L}$ with weight filtration $\mathcal{W}(\mathcal{N})$. This problem is closely related to the weight filtration in the open case since we have the following *relation between $\Psi_f^u\mathcal{L}$, the direct image $Rj_*\mathcal{L}[n]$ and $j_{!*}\mathcal{L}[n]$*

$$i_Y^*(Coker(j_{!*}\mathcal{L}[n] \to Rj_*\mathcal{L}[n])) \quad \simeq \quad Coker(\mathcal{N}\colon \Psi_f^u\mathcal{L}[n-1] \to \Psi_f^u\mathcal{L}[n-1])$$

Notice that we realized $Coker(j_{!*}\mathcal{L}[n] \to Rj_*\mathcal{L}[n])$ as a quotient that we denote abusively $Rj_*\mathcal{L}[n]/j_{!*}\mathcal{L}[n]$, so that the filtration $W(\mathcal{N})$ on $\Psi_f^u(\mathcal{L})[n-1]$ induces a filtration $\mathcal{W}$ on $Coker(\mathcal{N}/\Psi_f^u\mathcal{L})[n-1]$, hence it induces one on our model $Rj_*\mathcal{L}[n]/j_{!*}\mathcal{L}[n]$.
This induced filtration is independent of the choice of f. For a rigorous proof one should use the result of Verdier [25]. A path in the space of functions between two local equations f and f' of Y gives rise to an isomorphism between $\Psi_f\mathcal{L}$ and $\Psi_{f'}\mathcal{L}$; it is only modulo $Coker\mathcal{N}$ that this isomorphism is independent of the path. We do check here that the weight filtration $\mathcal{W}$ on $\Omega^*\mathcal{L}$ is induced locally by $W(\mathcal{N})$.

A.5.1. *The weight filtration on the nearby co-cycles* $\Psi_f\mathcal{L}$. When we consider the coefficients in the complex local system $\mathcal{L}\otimes\mathbb{C}$ (denoted also $\mathcal{L}$), the method to compute Ψ_f as explained in [11] uses the restriction $i_Y^*\mathrm{R}\,\mathrm{j}_*\mathcal{L}$ of the derived direct image of $\mathcal{L}$ to Y and the cup-product $H^i(X^*,\mathcal{L})\otimes H^1(X^*,\mathbb{Q})\overset{\smile\eta}{\longrightarrow} H^{i+1}(X^*,\mathcal{L})$ by the inverse image $\eta = f^*c \in H^1(X^*,\mathbb{Q})$ of a generator c of the cohomology $H^1(D^*,\mathbb{Q})$. We construct effectively, using Deligne's bundle extension, a bi-filtered complex on which η is defined as a morphism (of degree 1), $\eta\colon\ i_Y^*(\Omega_X^*(LogY)\otimes\mathcal{L}_X)\to i_Y^*(\Omega_X^*(LogY)\otimes\mathcal{L}_X)[1]$ satisfying $\eta^2=0$ so to get a double complex whose simple associated complex is quasi-isomorphic to $\Psi_f^u(\mathcal{L})$.

A.5.2. *The global weighted complex* $(\Psi_f^u(\mathcal{L})[n-1],\mathcal{W},F)$. Let t denotes a coordinate on the disc D and $\eta\colon\ = f^*(\frac{dt}{t})$, then $\wedge\eta$ defines a morphism of degree one on $i_Y^*\Omega_X^*(LogY)\otimes\mathcal{L}_X$. We consider the simple complex

$$(\Psi_f^u\mathcal{L})_X\colon\ = s(i_Y^*(\Omega_X^*(LogY)\otimes\mathcal{L}_X)[p],\eta)_{p\le 0} \tag{5.1}$$

defined by the double logarithmic complex (that is the direct sum $\oplus_{p\le 0} i_Y^*(\Omega_X^i(LogY)\otimes\mathcal{L}_X)$ is in degree i). To define as previously a constant combinatorial resolution of $(\Psi_f^u\mathcal{L})_X$, we put $(\Psi_f^u\mathcal{L})_X(s.) = (\Psi_f^u\mathcal{L})_X$ for each $s.\in S(I)$ and let

$$\Psi_f^u\mathcal{L}\colon\ = s((\Psi_f^u\mathcal{L})_X(s.))_{s.\in S(I)} \simeq s(i_Y^*\Omega^*\mathcal{L}[p],\eta)_{p\le 0} \tag{5.2}$$

where the isomorphism shows that we can sum first over $s.\in S(I)$ or over p in which case we define the weight filtration and the Hodge filtration by

$$\mathcal{W}_r(\Psi_f^u\mathcal{L}) = s(i_Y^*\mathcal{W}_{r+2p-1}\Omega^*\mathcal{L}[p],\eta)_{p\le 0},\ F^r(\Psi_f^u\mathcal{L}) = s(i_Y^*F^{r+p}\Omega^*\mathcal{L}[p],\eta)_{p\le 0} \tag{5.3}$$

The logarithm of the monodromy $\mathcal{N}$ is defined on this complex and we want to show that the filtration $\mathcal{W}$ above coincides with $W(\mathcal{N})$. Notice that the sum could be for $p\le 0$ or dually for $p\ge 0$ as in the work of Steenbrink (see A.5.6) below).

THEOREM A.50. *Suppose that $\mathcal{L}$ underlies a unipotent variation of polarized Hodge structures of weight m, then $W(\mathcal{N})=\mathcal{W}$.*

With this result we can conclude that the weight filtration in the open case is induced locally by the weight filtration defined by the monodromy on the nearby co-cycles.

The proof of this theorem is based on the results in the local case, that is for the nilpotent orbit defined by $\mathcal{L}$ at a point $y\in Y_M^*$, $M\subset I$.

A.5.3. *Local description of the weight and Hodge filtrations.* Near a point $y\in Y_M^*, M\subset I$, we can find coordinates z_i for $i\in M$ defining Y_M^* locally and non zero integers n_i s.t. $f = \overset{n}{\underset{i=1}{\prod}} z_i^{n_i}$ where we do suppose $i\in[1,n]$, where $|M|=n$ (n is less or equal to the *dim* of X), then in DeRham cohomology $\eta = f^*(\frac{dt}{t}) = \sum_{i=1}^n n_i\frac{dz_i}{z_i}$. Thus η defines a morphism of degree one on the DeRham complex $\Omega(L,N_i)_{i\in[1,n]}$ satisfying $\eta^2=0$. We define

$$\Psi^0(L) = s(\Omega(L,N_i)_{i\in[1,n]})[p],\eta)_{p\le 0}$$

as the simple complex defined by the double complex for $p\le 0$.

Remark: In order to take into account the action of $N = LogT^u$ we may write $L[N^p]$ for $L[p]$ and $L[N^{-1}]$ for the direct sum over $p\le 0$, so that the action of N

is just multiplication by N, then

$$\Psi^0(L) \simeq \Omega(L[N^{-1}], N_i - n_i N)_{i\in[1,n]}.$$

is the Koszul complex on $N_i - n_i N$ acting on $L[N^{-1}]$.
The complex $\Psi^0_M L$. To describe the weight in terms of the filtrations $(\Omega^* L, \mathcal{W}, F)$ associated to L, we need to use the constant complex with index $s. \in S(M)$, $\Psi^0 L(s.) = \Psi^0 L$ and introduce the complex

$$\Psi^0_M L\colon = s(\Psi^0 L(s.))_{s.\in S(M)} \tag{5.4}$$

which can be viewed also as $s(\Omega^* L[p], \eta)_{p\leq 0}$, then we define on it the *weight and Hodge filtrations*

$$\mathcal{W}_r(\Psi^0_M L) = s(\mathcal{W}_{r+2p-1}\Omega^* L[p], \eta)_{p\leq 0}, \quad F^r(\Psi^0_M L) = s(F^{r+p}\Omega^* L[p], \eta)_{p\leq 0}. \tag{5.5}$$

Monodromy.

The logarithm $\mathcal{N}$ of the monodromy is defined by an endomorphism ν of the complex $\Psi^0_M L$, given by the formula

$$\forall a. = \Sigma_{p\leq 0} a_p \in \Psi^0_M L : \quad (\nu(a.))_p = a_{p-1}$$

such that $\nu(\mathcal{W}_r) \subset \mathcal{W}_{r-2}$ and $\nu(F^r) \subset F^{r-1}$.

Lemma A.51. *The local quasi-isomorphism on the stalk at a point $y \in Y^*_M$ of the logarithmic complex with coefficients in Deligne's extension $(\Omega^*_X(LogY) \otimes \mathcal{L}_X)_y$ extends to a quasi-isomorphism from $(\Psi^u_f \mathcal{L})_{X,y}$ (A(5.1))) to $\Psi^0 L$ (resp. from $(\Psi^u_f \mathcal{L})_y$ (A(5.2)) to $\Psi^0_M L$ (A(5.4))) respecting the weight and Hodge filtrations*

$$((\Psi^u_f \mathcal{L}_X)_y, W, F) \simeq (\Psi^0 L, W, F) \quad , \quad ((\Psi^u_f \mathcal{L})_y, \mathcal{W}, F) \simeq (\Psi^0_M L, \mathcal{W}, F)$$

This lemma is the needed link between the global and local cases.

Lemma A.52. *We have a triangle in the derived category represented by the exact sequence*
$0 \to i^*_Y(\Omega^*_X(LogY) \otimes \mathcal{L}_X \to \Psi^u_f \mathcal{L}_X \xrightarrow{\nu} \Psi^u_f \mathcal{L}_X \to 0$

A.5.4. *Main local results.* The proof of the theorem can be reduced to the local case at a point $y \in Y^*_M$. We introduce first the following complexes.
The morphism η induces a morphism denoted also by $\eta : C^{KM}_r L \to C^{KM}_{r+2} L[1]$ so that we can define a double complex and the associated simple complex

$$\Psi^{KM}_r L = s(C^{KM}_{r+2p-1} L[p], \eta)_{p\leq 0}, \ \Psi^K_r L\colon = \Psi^{KK}_r L \tag{5.6}$$

i) *Decomposition of $Gr^{\mathcal{W}}_r(\Psi^0_M L)$. There exist natural injections of $\Psi^{KM}_r L$ into $Gr^{\mathcal{W}}_r \Psi^0_M L$ and a decomposition*

$$Gr^{\mathcal{W}}_r \Psi^0_M L \simeq \oplus_{K\subset M} \Psi^{KM}_r L \tag{5.7}$$

Proof. The result follows from the decomposition of $Gr^{\mathcal{W}}_r \Omega^* L$ in the previous open case, applied to the spectral sequence with respect to p in the double complex above. □

ii) We introduce now the the complex A^K_i and prove

Lemma A.53. *(Basic lemma). For all $i \geq 1$, the complex*
$A^K_i\colon = s[C^K_{i+2p-1} L[p], \eta]_{1-i\leq p\leq 0}$ *is acyclic.*

Proof. We view A_i^K as a double complex where η of degree 1 is a differential of the direct sum of complexes $C^K_{i+2p-1}L$ without shift in degrees :

$$A_i^K := C^K_{-i+1} \xrightarrow{\eta} \cdots \xrightarrow{\eta} C^K_{i+2p-1}L \cdots \xrightarrow{\eta} C^K_{i-1}L, \quad A_i^K(J) = \oplus_{1-i\leq p\leq 0} C^K_{i+2p-1}L(J), J\subset K$$

We may filter A_i^K by sub-complexes U_i. One way is to take $U_0 = C^K_{i-1}L$, $U_1 = s[C^K_{i+2p-1}L[p],\eta]_{1-i<p\leq 0}$ and $U_2 = A_i^K$, that is we write A_i^K as:

$$s[C^K_{-(i-1)}L[-2], s[C^K_{i+2p-1}L[p],\eta]_{0<p<i-1}[-1], C^K_{i-1}L,\eta]$$

so to use by induction $Gr_1^U = A^K_{i-2} \cong 0$.

However we will use the technique of the spectral sequence defined by the increasing filtration

$U_r = s[C^K_{i+2p-1}L[p],\eta]_{-r\leq p\leq 0}$ for $0\leq r\leq i-1$

with Deligne's notations

$$E_1^{a,b} = H^{a+b}(Gr^U_{-a}) = H^{a+b}(C^K_{i+2a-1}L)$$

Since $H^u(C^K_jL) = 0$ if $j<0$ and $u\neq |K|-1$ or $j>0$ and $u\neq |K|$, we get

1) if $2a > 1-i$, $E_1^{a,|K|-1-a} = H^{|K|-1}(C^K_{i+2a-1}L) = Gr^{W^K}_{|K|+i+2a-1}[(\cap_{i\in K} kerN_i : L\to L]$

and 0 otherwise

2) if $2a < 1-i$, $E_1^{a,|K|-a} = H^{|K|}(C^K_{i+2a-1}L) = Gr^{W^K}_{-|K|+i+2a-1}[L/(\Sigma_{i\in K}N_iL)]$

and 0 otherwise

3) if $2a = i-1$, $E_1^{a,b} = 0$.

Starting at the level 1, the term $E_1^{-(i-1)-d,|K|-1+(i-1+d)}$ remains unchanged until we reach the level $r = i-1+2d$ where the only non trivial differential appears and we want to show it is an isomorphism. The proof is based on the following study of this differential.

The cohomology space of C^K_rL for will be identified with the following polarized subspace of $Gr^{W^K}_{r-|K|}L$. For each $r>0$, and $(m_1,\dots,m_n)\in T(r)$ (that is $m_i>1, \Sigma m_i = r+|K|$), let

$P(m.)L = \cap_{i\in[1,n]}(kerN_i^{m_i-1} : Gr^{W^n}_{m_n-2}\cdots Gr^{W^1}_{m_1-2}L \to Gr^{W^n}_{-m_n}\cdots Gr^{W^1}_{-m_1}L) \subset Gr^{W^K}_{r-|K|}L$,

be the primitive polarized subset, then we have the isomorphism

$P(m.)L \simeq Gr^{W^n}_{m_n-2}\cdots Gr^{W^1}_{m_1-2}(L/(\Sigma_iN_iL))$.

A.5.5. *Sub-lemma. Let $N_K = \Sigma_{i\in K}n_iN_i$ and for each $(m.)\in T(i+2d-1)$, let $P(m.)L$ denotes the primitive sub-HS as above. The differential at the level $r = i-1+2d$ of the spectral sequence*

$$E_r^{-(i-1)-d,|K|-2+i+d} \to E_r^{d,|K|-d}, \quad r = i-1+2d$$

is given by the inverse of the isomorphism up to a constant

$$Gr^{W^K}_{-|K|+r}[L/(\Sigma_{i\in K}N_iL)] \simeq \oplus_{m.\in T(r)}P(m.)L \xrightarrow{N_K^{r-|K|}}$$
$$\oplus_{m.\in T'(r)}P(m.)L \simeq Gr^{W^K}_{|K|-r}(\cap_{i\in K}kerN_i).$$

inducing for each $(m.)$ precisely the inverse of

$$(-1/(n_1^{(m_1-1)}\dots n_n^{(m_n-1)})N_1^{(m_1-2)}\dots N_n^{(m_n-2)} :$$
$$P((m_1,\dots,m_n)) \longrightarrow Gr^{W^n}_{-m_n+2}\cdots Gr^{W^1}_{-m_1+2}(\cap_{i\in K}kerN_i).$$

1) We start the proof for $n=1$, that is one dimensional nilpotent orbit (L,N). Then we need to prove that the differential is the inverse of

$$-\frac{1}{n^r}N^{r-1} : Gr^W_{r-1}L/NL \xrightarrow{\sim} P(r-1)L \to Gr^W_{-(r-1)}kerN$$

which can be checked on the diagram

$$\begin{array}{ccccccc} Gr^W_{-i+2}L & & Gr^W_{-i+4}L & \cdots & Gr^W_{i-2}L & & Gr^W_iL \\ N\downarrow & \eta\searrow & N\downarrow & \cdots & N\downarrow & \eta\searrow & N\downarrow \\ Gr^W_{-i}L & & Gr^W_{-i+2}L & \cdots & Gr^W_{i-4}L & & Gr^W_{i-2}L \end{array}$$

where $\eta = -nId$. For $a \in Gr^W_{i-2}L$ primitive, the element

$$\sum_{0\le j\le i-2}(1/n)^{i-1-j}N^{i-2-j}(a) \in \oplus Gr^W_{i-2+2j}L$$

is a cohomology class modulo the complex $C^{\{0\}}_{i-1}L$ inducing the cohomology class $((1/n)^{i-1}N^{i-2}(a)$ in $C^{\{0\}}_{1-i}L$ whose image by η is $-a$ the original primitive element up to sign.

2) In general we notice that $(N_K)^{r-2|K|}$ decomposes on $P(m.)L$ as the product $(n_nN_n)^{m_n-2}\circ\cdots\circ(n_1N_1)^{m_1-2}$.

We use this relation to give an inductive proof on the number of endomorphisms N_i.

The cohomology of an elementary complex $K(m_1,\cdots,m_n)$ is isomorphic to the cohomology of a complex with the unique endomorphism N_n acting on $L_n = \cap_{i\notin J(m.),i\neq n}kerN_i : L/(\Sigma_{i\in J(m.)-\{n\}}N_iL)$ with L_n depending on m_i for $i\neq n$, then the diagram is similar to the case of one variable until we reach $-m_n+2$ for which the morphism $N_n^{m_n-2}$ is needed. The morphisms $N_i^{m_i-2}$ will appear inductively with the variable i. So we can deduce in general:

$N_1^{m_1-2}\cdots N_n^{m_n-2} : Gr^{W^n}_{m_n}\cdots Gr^{W^1}_{m_1-2}P(m.)L \simeq Gr^{W^n}_{-m_n+2}\cdots Gr^{W^1}_{-m_1+2}(\cap_{i\in K}kerN_i)$

the sum over $\{(m_1\ge 2,\cdots,m_n\ge 2) : \Sigma_{i\in K}m_i = i+2d-1+|K|\ \}$ induces an isomorphism:

$$\gamma : Gr^{W^K}_{i+2d-1-|K|}[L/(\Sigma_{i\in K}N_iL)] \to Gr^{W^K}_{-(i-1)-2d+|K|}[(\cap_{i\in K}(kerN_i : L\to L].$$

□

Example A.54. Consider L, N_1, N_2 in dimension 2, $K=\{1,2\}$ the origin in $\mathbb{C}^2$, A^K_5, $a\in Gr^{W^K}_2L$, $(m.) = (m_2=4, m_1=2)$ with the following conventions for differentials : the restriction from $s. = K\supset\{1\}$ to K is $-I$ (I is Identity), the restriction from $s. = K\supset\{2\}$ to K is I, the differentials on C^K_rL are ($-N_1$ on dz_2,N_i otherwise and η is n_1 on dz_2, $-n_i$ otherwise.

An element a of A^K_5 is written as the sum of various components of the underlying groups in $C^K_{-4}L\oplus C^K_{-2}L\oplus C^K_0L\oplus C^K_2L\oplus C^K_4L$, that is $a=\sum a(dz_J,s.,r)$ where dz_J stands for $\wedge_{i\in J}dz_i$, $s.$ is $\{1,2\}\supset\{1\}$, or $\{1,2\}\supset\{2\}$ or K for $\{1,2\}$ and r for $a(dz_J,s.,r)\in C^K_rL$ ($r=-4,-2,0,2,4$). Still we need to specify for an element $b\in Gr^{W^K}_rL\simeq\oplus_{m_1+m_2=r}Gr^{W^1}_{m_1}Gr^{W^2}_{m_2}L$ its components $b=\Sigma_{m_1+m_2=r}b(m_1,m_2)$.

A bi-primitive element $a\in P(2,4)\subset Gr^{W^1}_0Gr^{W^2}_2L\subset Gr^{W^K}_2L, a=a(dz_K,(s.=K\supset\{1\},4)$ in C^K_4L (hence $N_1a=0, N_2^3a=0$) defines the following cohomology class $\beta(a)\in A^K_5/C^K_4L$ modulo C^K_4L:

$[N_2^2a(m_2=-2,m_1=0)(dz_\emptyset,s.=K,-4)\in Gr^{W^K}_{-2}L$ in $C^K_{-4}L$, $(n_2N_2a((0,0)(\emptyset,K,-2)\in Gr^{W^K}_0L$ in $C^K_{-2}L$,

$n_1N_2^2(a)(0,-2)(dz_1,s.=K\supset\{2\},-2)\in Gr^{W^K}_{-2}L$ in $C^K_{-2}L$, $n_1n_2N_2(a)(0,0)(dz_1,(s.=K\supset\{1\},0)\in Gr^{W^K}_2L$ in $C^K_0L(s.=K\supset\{1\})$, $(n_2)^2a((2,0)(dz_\emptyset,s.=K,0)\in Gr^{W^K}_2L$ in C^K_0L, $(n_2)^2n_1a(2,0)(dz_1,s.=(K\supset\{2\}),2)$ in $C^K_2L(s.=K\supset\{2\})$,

$-(n_2)^3a(2,0)(dz_2,s.=(K\supset\{2\}),2)$ in $C^K_2L(s.=K\supset\{2\})]$ which ends the components of $\beta(a)$,

whose image by η has two components $-(n_2)^3 n_1 a((2,0)(dz_K, s. = (K \supset \{2\}), 4)$ in $C_4^K L(s. = K, 2)$ and $-(n_2)^3 n_1 a((2,0)(dz_K, s. = (K \supset \{1\}), 4)$ in $C_4^K L(s. = (K \supset \{1\})$ which represents the cycle $-a$ in the cohomology as represented by the complex defined by $T(r)$ with $r = 4$.
Notice that the conditions for $s. = (K \supset \{1\}), r = 2$ are , $W_1(L) \leq 2, W_1(Ldz_1) \leq 0, W_1(Ldz_2) \leq 2$ are satisfied by $a(dz_2)$ while the conditions for $s. = (K \supset \{2\}), r = 2), r = 2$ are $W_2 \leq 2, W_2(dz_1) \leq 2, W_2(dz_2) \leq 0$ are not satisfied since $W_2(adz_2) = 2$ which forces the lifting in $C_2^K L(s. = (K \supset \{1\})$.

Corollary A.55. For all $i > 0$, the complex $A_i^{KM} := s[C_{i+2p-1}^{KM} L[p], \eta]_{-i<p\leq 0}$ is acyclic.

Proof. We can easily check, as in the previous open case, that the cohomology of A_i^{KM} is quasi-isomorphic to the stalk at y of the intermediate extension of the local system on Y_K^* defined by the cohomology of A_i^K, hence it is quasi-isomorphic to zero since $A_i^K \cong 0$.

The iterated monodromy morphism defines an exact sequence

$0 \to ker\, \nu_X^i \to \Psi_f^u \mathcal{L}_X \xrightarrow{\nu_X^i} \Psi_f^u \mathcal{L}_X \to 0$ where $ker\, \nu_X^i = s(\Omega_X^*(LogY)[p], \eta)_{-i<p\leq 0}$,

$0 \to ker\, \nu^i \to \Psi_f^u \mathcal{L} \xrightarrow{\nu^i} \Psi_f^u \mathcal{L} \to 0$ where $ker\, \nu^i = s(\Omega^* \mathcal{L}[p], \eta)_{-i<p\leq 0}$. □

Corollary A.56. For all $i \geq 1$, $Gr_i^{\mathcal{W}} ker\, \nu^i = s(Gr_{i+2p-1}^{\mathcal{W}} \Omega^* L[p], \eta)_{-i<p\leq 0}$ is acyclic.

Proof: By the decomposition A(5.4) we have: $Gr_i^{\mathcal{W}} ker\, \nu^i \cong \oplus_{K\subset M} A_i^{KM}$.

Corollary A.57. For all $i \geq 1$, $\nu^i : Gr_i^{\mathcal{W}}(\Psi_M^0 L) \simeq Gr_{-i}^{\mathcal{W}}(\Psi_M^0 L)$.

Proof. The equivalence with the above corollary follows from the exact sequence

$0 \to Gr_i^{\mathcal{W}} ker\, \nu^i \to Gr_{-i}^{\mathcal{W}}(\Psi_M^0 L) \xrightarrow{\nu^i} Gr_{-i}^{\mathcal{W}}(\Psi_M^0 L) \to 0.$ □

The theorem follows from the corollaries.

A.5.6. *The global weighted complex* $(\Psi_f^u \mathcal{L}[n-1], \mathcal{W}, F)$. Returning to the global situation, we need to define the Hodge filtration on $\Psi_f^u(\mathcal{L}_X)$. We could define

$$(\Psi_f^u \mathcal{L})_X \cong s(i_Y^*(\Omega_X^*(LogY) \otimes \mathcal{L}_X)[i+1], \eta)_{i\geq 0},\ (\Psi_f^u \mathcal{L}) := s(\Psi_f^u \mathcal{L}(s.))_{s.\in S(M)}$$

First F extends to the logarithmic complex by the formula under (A(4.1)) in (definition A.34), then F extends to $(\Psi_f^u(\mathcal{L})_X$ via the formula

$$F^r(s(i_Y^*(\Omega_X^*(LogY) \otimes \mathcal{L}_X)[i+1])_{i\geq 0}) = s(F^{r+i+1}(i_Y^*(\Omega_X^*(LogY) \otimes \mathcal{L}_X)[i+1], \eta)_{i\geq 0}$$

The definition of the global weight filtration reduces to the local construction at a point $y \in Y_M^*$, using the quasi-isomorphism $(\Psi_f^u\, \mathcal{L})_y \simeq \Psi^0 L$.
We suppose again $\mathcal{L}$ unipotent and define as above $\Psi_f^u \mathcal{L}$ which can be viewed also as $s(i_Y^* \Omega^* \mathcal{L}[i+1], \eta)_{i\geq 0}$, then we define the weight and Hodge filtrations

$$\mathcal{W}_r(\Psi_f^u \mathcal{L}) = s(i_Y^*(\mathcal{W}_{r+2i+1} \Omega^* \mathcal{L})[i+1], \eta)_{i\geq 0},\ F^r(\Psi_f^u \mathcal{L}) = s(i_Y^* F^{r+i+1} \Omega^* \mathcal{L}[i+1], \eta)_{i\geq 0}$$

The logarithm of the monodromy $\mathcal{N}$ is defined on this complex as in the local case. The filtration $W(\mathcal{N})$ is defined on $\Psi_f^u \mathcal{L}$ in the abelian category of perverse sheaves.

THEOREM A.58. *Suppose $\mathcal{L}$ underlies a unipotent variation of polarized Hodge structures of weight m, then the graded part of the weight filtration of the complex*
$$(\Psi_f^u\mathcal{L}[n-1], \mathcal{W}[m+n-1], F)$$
decomposes into a direct sum of intermediate extension of polarized VHS. Moreover it is a MHC for X proper.
We have $W(\mathcal{N}) = \mathcal{W}$; $\mathcal{W}_{r+1}\Omega^*\mathcal{L}[n]$ is induced by $\mathcal{W}_r\Psi_f^u\mathcal{L}[n-1]$ on $Coker\mathcal{N}$ and $Gr_{r+1}^{\mathcal{W}} i_Y^*\Omega^*\mathcal{L}[n]$ is the primitive part of $Gr_r^{\mathcal{W}}\Psi_f^u\mathcal{L}[n-1]$.

The proof of this theorem reduces by definition to show that $(Gr_r^{\mathcal{W}}(\Psi_f^u\mathcal{L})_X, F)$ decomposes which result can be reduced to the local case where it follows from

LEMMA A.59. *Let i_0 be a positive integer large enough to have $Gr_j^{\mathcal{W}}\Omega^*\mathcal{L} = 0$ for all $|j| > i_0$ and $I(r) = \{p \geq 0 : |r| + 1 \leq r + 2p + 1 \leq i_0\}$, then*
$$Gr_r^{\mathcal{W}}(\Psi^0 L)_M = s(Gr_{r+2p-1}^{\mathcal{W}}\Omega^* L[p], \eta)_{p\in I(r)};\ Gr_r^{\mathcal{W}}(\Psi^0 L)_M \overset{\nu^r}{\underset{\cong}{\to}} Gr_{-r}^{\mathcal{W}}(\Psi^0 L)_M, r \geq 0$$

PROOF. Suppose $r > 0$, then $Gr_r^{\mathcal{W}}(\Psi^0 L)_M \cong s(Gr_{r+2p+1}^{\mathcal{W}}\Omega^* L[p], \eta)_{p\in I(r)}$ since $Gr_{r+2p+1}^{\mathcal{W}}\Omega^* L \cong 0$ for $r \notin I(r)$, while for $r < 0$, the sum $s(Gr_{r+2p+1}^{\mathcal{W}}\Omega^* L[p], \eta)_{-1<p<-r} \cong s(Gr_i^{\mathcal{W}}\Omega^* L[p], \eta)_{i\in[r+1,-(r+1)]}$ is acyclic.
The quasi-isomorphism defined by ν^r follows since the formula is symmetric in r and $-r$. □

EXAMPLE A.60. $Gr_0^{\mathcal{W}}(\Psi^0 L)_M \cong s(Gr_1^{\mathcal{W}}\Omega^* L[1] \xrightarrow{\eta} Gr_3^{\mathcal{W}}\Omega^* L[2] \ldots)$
$Gr_1^{\mathcal{W}}(\Psi^0 L)_M \cong s(Gr_2^{\mathcal{W}}\Omega^* L[1] \xrightarrow{\eta} Gr_4^{\mathcal{W}}\Omega^* L[2] \ldots)$
$Gr_2^{\mathcal{W}}(\Psi^0 L)_M \cong s(Gr_3^{\mathcal{W}}\Omega^* L[1] \xrightarrow{\eta} Gr_5^{\mathcal{W}}\Omega^* L[2] \ldots)$
For $r = i_0 - 1$ and $r = i_0 - 2$, $Gr_r^{\mathcal{W}}(\Psi^0 L)_M \cong Gr_{i_0}^{\mathcal{W}}\Omega^* L[1]$

COROLLARY A.61. The graded part of $(\Psi^0 L)_M$ is non zero for only a finite number of indices which decomposes into direct sum of of intermediate extension of polarized VHS.

PROOF. We can introduce the notion of primitive parts starting with $Gr_r^{\mathcal{W}}(\Psi^0 L)_M$ for $r = i_0 - 1$ and $r = i_0 - 2$ which are direct sum of of intermediate extension of polarized VHS, then using the decomposition into direct sum of primitive parts we prove by induction that such decomposition is valid for all indices r (there is no extension by η). □

Then the theorem follows easily.

REMARK A.62. (*dual statement*). Let us return to $\Psi_f^u\mathcal{L}\colon = s(\Psi_f^u\mathcal{L}(s.))_{s.\in S(I)}$ defined with indices $p \leq 0$, which can be viewed locally at a point $y \in Y$ as $s(\Omega^*\mathcal{L}[p], \eta)_{p\leq 0}$, the weight and Hodge filtrations defined by the formula (A(5.3)).

THEOREM A.63 (dual statement). *Suppose $\mathcal{L}$ underlies a unipotent variation of polarized Hodge structures of weight m, then the graded part of the weight filtration of the complex*
$$(\Psi_f^u\mathcal{L}[n-1], \mathcal{W}[m+n-1], F)$$
decomposes into a direct sum of intermediate extension of polarized VHS. Moreover it is a MHC for X proper.
We have $W(\mathcal{N}) = \mathcal{W}$; $\mathcal{W}_{r-1} i_Y^*\Omega^*\mathcal{L}$ is induced by $\mathcal{W}_r\Psi_f^u\mathcal{L}$ on $Ker\mathcal{N}$ for $r \leq 0$ and $Gr_{r-1}^{\mathcal{W}} i_Y^*\Omega^*\mathcal{L}$ is isomorphic to the primitive part of $Gr_{-r}^{\mathcal{W}}\Psi_f^u\mathcal{L}$.

The proof of this theorem reduces by definition to show that $(Gr_r^{\mathcal{W}}\Psi_f^u\mathcal{L}, F)$ decomposes; such result can be reduced to the local case where it follows from

LEMMA A.64. *Let i_0 be a positive integer large enough to have $Gr_j^{\mathcal{W}}\Omega^*\mathcal{L} = 0$ for all $|j| > i_0$ and $I(r) = \{p \leq 0, -i_0 + 1 \leq r + 2p - 1 \leq -|r| - 1\}$, then*

$$Gr_r^{\mathcal{W}}(\Psi^0 L)_M = s(Gr_{r+2p-1}^{\mathcal{W}}\Omega^* L[p], \eta)_{p\in I(r)},\ Gr_r^{\mathcal{W}}(\Psi^0 L)_M \overset{\nu^r}{\underset{\cong}{\to}} Gr_{-r}^{\mathcal{W}}(\Psi^0 L)_M, r \geq 0$$

PROOF. Suppose $r > 0$, then $Gr_r^{\mathcal{W}}(\Psi^0 L)_M$ is the sum of $s(Gr_{r+2p-1}^{\mathcal{W}}\Omega^* L[p], \eta)_{p\leq -r}$ and $s(Gr_{r+2p-1}^{\mathcal{W}}\Omega^* L[p], \eta)_{-r<p\leq 0} \cong 0$. In particular, $Gr_r^{\mathcal{W}}(\Psi^0 L)_M \cong 0$ for all r such that $|r| \geq i_0$. Using A(5.6) and A(5.7) we get a decomposition into $\Psi_r^{KM} L = s(C_{r+2p-1}^{KM} L[p], \eta)_{p\in I(r)}$ where $C_{r+2p-1}^{KM} L$ is the fibre of an intermediate extension of a local system defined by $C_{r+2p-1}^{K} L$ for $r + 2p - 1 < 0$ on Y_K^*. Then the proof is similar to the previous case in the dual definition of $\Psi_f^u\mathcal{L}$. □

A.6. The weight filtration after M. Saito. *local situation.* We give in this subsection a brief view on Kashiwara and Saito's constructions and proof of the decomposition and the purity results for $\Psi_f^u(\mathcal{L})$ [22]. This result is embedded in the theory of Hodge modules and uses its language, a theory adapted for a general setting but not necessary at this stage.

We consider the polynomial ring $\mathbb{C}[N]$ in one variable (resp. the field $\mathbb{C}[N, N^{-1}]$) and the module $L[N] = L\otimes_{\mathbb{C}}\mathbb{C}[N]$ (resp. $L[N, N^{-1}]$) endowed with commuting endomorphisms $N_i \otimes Id$, denoted also by N_i, where N_i is nilpotent and multiplication by N denoted also by N. For each family of integers $n_i > 0$, for $i \in [1, n]$, we consider the endomorphisms $A_i = N_i - n_i N$ on $L[N]$ (resp. $L[N, N^{-1}]$). The inverse of the endomorphisms A_i are defined on $L[N, N^{-1}]$ and equal to

$$A_i^{-1} = -\Sigma_{j\geq 0}(N_i)^j/(n_i N)^{j+1}$$

where the sum is finite since N_i is nilpotent for all i. In particular A_i and $A_J = \Pi_{i\in J} A_i$, $J \subset [1, n]$, are injective on $L[N]$. Given $(L, N_i)_{i\in[1,n]}$, we introduce the Koszul complex $\Omega(L[N], A.)$ to compute the fibre in the nearby cycle case.

LEMMA A.65. *i) The Koszul complex $\Omega(L[N, N^{-1}],\ A_i = N_i - n_i N, i \in [1, n])$ is acyclic.*
ii) The following complexes are isomorphic

$$\Omega(L[N^{-1}], A.) \quad \xrightarrow{\sim} \quad \Omega(L[N], A.)[1]$$

iii) The complex

$$IC(L[N], A.) = s(ImA_J)_{J\subset[1,n]} \simeq 0 \tag{6.1}$$

is an acyclic sub-complex of the Koszul complex $\Omega(L[N], A.)$.
iv) Let $\Psi_J(L) := L[N]/ImA_J$ and denote by $\Psi^0 L = s(\Psi_J(L)_{J\subset[1,n]}, (A_i)_{i\in[1,n]})[1]$, the complex associated to the simplicial complex with differential induced by

$$A_i : \Psi_{J-i}(L) \to \Psi_J(L)$$

then we have the following isomorphism
(6.2)

$$\Omega(L[N], A.) \overset{\Pi}{\underset{\sim}{\to}} \Psi^0 L = s(\Psi_J(L)_{J\subset[1,n]}, A.)[1] = s(L[N]/Im(A_J), A.)_{J\subset[1,n]}[1]$$

Remark A.66. i) The importance of the introduction of $\Psi_J(L)$ is that they are canonically associated to the perverse sheaf $\Psi^0(L)$, so that the construction of the weight filtration reduce to its construction on these vector spaces. These vector spaces characterize the perverse sheaf (for example on the opposite the relation between a perverse sheaf and its cohomology is less precise to work with then the above spaces).

ii) The isomorphism A(1.4)
$Coker(j_{!*}\mathcal{L}[n] \to Rj_*\mathcal{L}[n]) \simeq CokerN\colon \Psi^u_f(\mathcal{L})[n-1] \to \Psi^u_f(\mathcal{L})[n-1]$
is viewed locally at a point $y \in Y^*_M$ as follows.
Notice that $Coker(N/\Psi_J L) \simeq Coker(N_J/L)$ since $A_i = N_i - n_i N$ is equal to N_i modulo N, so that we have locally the isomorphisms

$$\begin{aligned}(Rj_*\mathcal{L}/j_{!*}\mathcal{L})_y \simeq s(CokerN_J/L)_{J\subset[1,n]} &\simeq s(CokerN/\Psi_J L)_{J\subset[1,n]}\\ &\simeq Coker(\mathcal{N}/\Psi^u_f\mathcal{L})_y[-1]\end{aligned}$$

where we suppose abusively in the first term $j_{!*}\mathcal{L}$ realized by a subcomplex.

iii) In general, the graded part of the cokernel is isomorphic to the primitive part $P_k(N)$ for all $k \geq 0$:

$$Gr_k^{W(N)}(CokerN/\Psi_J L) \simeq P_k(N)$$

The filtration $W(N)$ on $\Psi_J L$ defines a filtration by sub-complexes of $\Psi^0 L$ and corresponds to the filtration $W(\mathcal{N})$ of $\Psi^u_f(\mathcal{L})$.

The above facts are special features of basic facts that we recall now since they are needed to understand the properties of the weight filtration.

The *category of perverse sheaves* $\mathcal{K}$ on X with respect to the natural stratification $Y^\star_M$ defined by Y (such that for each $M \subset I$, the cohomology of $\mathcal{K}/Y^*_M$ is locally constant), is described locally at a point y considered as the center of a polydisc $(D^*)^{|M|}$ by the following combinatorial construction in [19, p 996].

The category $\mathcal{P}$ of perverse sheaves $\mathcal{K}$ on $(D^*)^{|M|}$, with respect to its NCD stratification is equivalent to the abelian category defined as follows :

i) A family of vector spaces L_A for $A \subset M$,
ii) A family of morphisms
$f_{AB}\colon L_B \to L_A$ *and* $g_{BA}\colon L_A \to L_B$ *for* $B \subset A \subset I$ *such that :*
$f_{AB} \circ f_{BC} = f_{AC}$, $g_{CB} \circ g_{BA} = g_{CA}$ *for* $C \subset B \subset A$
$f_{AA} = g_{AA} = id$, $g_{A,A\cup B} \circ f_{A\cup B,B} = f_{A,A\cap B} \circ g_{A\cap B,B}$ *for all* A, B
and if $A \supset B, \mid A \mid = \mid B \mid +1$, *then* $1 - g_{BA}f_{AB}$ *is invertible.*

Minimal extensions of the category $\mathcal{M}_A$
We will need the following description for $A \subset M$ of the minimal extension $\mathcal{F}$ of a locally constant sheaf $\mathcal{L}$ on $X^\star_A$: in terms of the family of the family of vector spaces L_B for $B \subset M$ defined by $\mathcal{F}$; it is equivalent to $L_B = 0$ for $A \underset{\neq}{\subset} B$, and for $B \supset A$, $f_{BA}\colon L_A \to L_B$ is surjective and $g_{AB}\colon L_B \to L_A$ is injective. We denote by $\mathcal{M}$ the objects isomorphic to a direct sum of objects in $\cup_A \mathcal{M}_A$.

The category $\mathcal{M}$ of sums of minimal extensions A result of Kashiwara states [19, p 997]
A perverse sheaf $\mathcal{K} \in \mathcal{P}$ is a direct sum of minimal extensions ($\mathcal{K}$ in $\mathcal{M}$) if and only if

$$\forall A, B \subset M,\ B \subset A, \quad L_A \simeq Imf_{AB} \oplus Kerg_{BA} \tag{6.3}$$

Moreover, it is enough to consider $|A| = |B| + 1$.
The above condition is equivalent to the isomorphism:

$$\oplus_{B\subset A} f_{AB}(P_B(\mathcal{K})) \quad \xrightarrow{\sim} \quad L_A \tag{6.4}$$

where $P_B(\mathcal{L}^{\cdot}) = \cap_{C\subsetneq B} Ker\, g_{CB}$, *then moreover* $g_{BA} : f_{AB}(P_B(\mathcal{K})) \to P_B(\mathcal{K})$ *is injective for* $B \subset A$.

Description of the weight filtraion in the category of perverse sheaves.
The family $\Psi_J(L) = L[N]/ImA_J$ for $J \subset M, J \neq \emptyset$ gives precisely the description of $\Psi^0(L)$ as a perverse sheaf, where for $i \in J$, the morphisms mentioned above are $f_{J(J-i)} = A_i : \Psi_{J-i}(L) \to \Psi_J(L)$ and $g_{J(J-i)} = p_i : \Psi_J(L) \to \Psi_{J-i}(L)$ the canonical projection. The product by N induces on each $\Psi_J(L)$ a nilpotent endomorphism denoted also by N which commutes with A_i and p_i, hence these morphisms are compatible with $W(N)$; they send $W_{r-1}(N)$ into itself (it is enough to show that for $b \in \Psi_{J-i}(L)$, if $N^s(b) = 0$ for $s \geq r$, $N^s(A_i(b)) = A_i(N^s(b) = 0$ (resp.for p_i)).
For each integer r, let $Gr_r^{W(N)}\Psi_J(L), p_i^r, A_i^r$ denote the corresponding perverse graded objects, then we define

$$K_i^r = Ker\ p_i^r : Gr_r^{W(N)}\Psi_J(L) \to Gr_r^{W(N)}\Psi_{J-i}(L), \quad K_J^r : \cap_{i\in J} K_i^r \tag{6.5}$$

in particular $\forall i \in J, K_J^r \subset KerN_i \subset Gr_r^{W(N)}\Psi_J(L)$. The aim of the next part is to deduce the decomposition property (6.4) via the proof of (6.3) in presence of a polarized Hodge filtration.

THEOREM A.67 (decomposition). *(Kashiwara - Saito). For each integer* a *and point* $y \in Y$

$$Gr_a^{W(\mathcal{N})}(\Psi_f(\mathcal{L}))_y \simeq Gr_a^{W(N)}\Psi^0(L)$$

is isomorphic to a direct sum of fibres at y *of various intermediate extension of variations of polarized Hodge structures. Precisely*

$$Gr_a^{W(N)}\Psi(L) \simeq \oplus_{J\subset M} IC((K_J^a(L), N_i, i \in M - J) \tag{6.6)}$$

where K_J^a, *defined by* (6.5), *is a pure Hodge structure of weight* $a + m$ *with the induced Hogde filtration* F.

Elements of Kashiwara's proof [22(2), prop.3.19, and Appendix]. We associate to $(L, F, P, N_i, i \in M = [1, n])$ the module $L[N]$ where N is a polynomial variable, endowed with two filtrations as follows. Consider $W(L) = W(\Sigma_{i\in M} N_i)[m]$ and F on L, then define

$$W_k(L[N]) = \Sigma_j W_{k+2j} L \otimes N^j \ , \ F^p(L[N]) = \Sigma_j F^{p+j} L \otimes N^j \tag{6.7}$$

Since the endomorphisms $A_i = N_i - n_i N$ shift W by -2 and F by -1, the two filtrations induce a MHS on the cokernel $\Psi_J = L[N]/ImA_J$ for $J \subset M$. We have an isomorphism compatible with the filtrations

$$(\oplus_{j\leq l-1} L \otimes N^j, W, F) \quad \simeq \quad (\Psi_J L, W, F) \tag{6.8}$$

obtained via the composition of the natural embedding in $L[N]$ with the projection on $\Psi_J L$, where W and F are defined on the left term as in the formula (6.7) above. In fact the relation : $N^l = \Sigma_{j\in[1,l]}(-1)^{j+1}\sigma_j(\frac{N_i}{n_i}, i \in J) \otimes N^{l-j}$ where $l =| J |$ and σ_j is the jth elementary symmetric function of $\frac{N_i}{n_i}, i \in J$, on the quotient of the right term leads to the definition of the action of N on the left term by the formula

$$N(a \otimes N^{l-1}) = \Sigma_{1 \leq j \leq l}(-1)^{j+1}\sigma_j((N_i/n_i), i \in J)(a) \otimes N^{l-j}.$$

In order to define a polarization we introduce a product P_J on $\Psi_J(L)$ as follows

$$P_J(aN^i, bN^j) = P(a, (-1)^i \quad Res\ (A_J^{-1}(b \otimes N^{i+j})))$$

where A_J^{-1} is defined on $L[N, N^{-1}]$, N is considered as a variable x and the residue Res is equal to the coefficient of $1/N$ in the fraction in N. This formula shows directly that the product is well defined on $Coker A_J$; in fact, $P_J(aN^i, A_J(c) = P(a, (-1)^i \quad \mathrm{Res}(c \otimes N^i) = 0$. Using an explicit expression of A_i^{-1}, we find $P_J(aN^p, bN^q) = (-1)^p P(a, \Sigma_{a_i}(\Pi_i(N_i^{a_i}(b)/n_i^{a_i+1}))$ where $a_i \geq 0$ and $\Sigma_i a_i = i+j-l+1$. In particular

$$P_J(a, bN^r) = (1/\Pi_i n_i)P(a, b) \text{ if } r = l-1, \text{ and zero otherwise}$$

$$P_J(aN^i, bN^j) = (-1)^i P_J(a, bN^{i+j}).$$

In [22(2)], the following result is attributed to Kashiwara

Theorem A.68. *With the previous notations, namely W and F*

$$\Psi_J(L) = (L[N]/ImA_J, N_1, \ldots, N_n, N; W, F, P_J) \tag{6.9}$$

underlies a polarized nilpotent orbit of weight $m+1-|J|$, that is: the weight filtration $W(N + \Sigma_{i \in M} N_i)[m+1-|J|] = W$ underlies the weight of a MHS on $\Psi_J(L)$ with the Hodge filtration F.

(see M. Saito [22(1),5.2.15, 5.2.14], [22(2),3.20.4]). The induced morphisms N, N_i and A_i shift W by -2 and F by -1. Since $W(N + \Sigma_{i \in J} N_i)$ is the weight filtration of the endomorphism $\Sigma_{i \in J} N_i$ relative to $W(N)$ that is for all $\Psi_J(L)$:

$$Gr_{r+j}^{W(N+\Sigma_{i \in J} N_i)} Gr_r^{W(N)} \Psi_J(L) \xrightarrow{(\Sigma_{i \in J} N_i)^j} Gr_{r-j}^{W(N+\Sigma_{i \in J} N_i)} Gr_r^{W(N)} \Psi_J(L)$$

we have for $\Psi_J(L)$:

$$Gr_{j+r}^{W(N+\Sigma_{i \in J} N_i)} Gr_r^{W(N)} \simeq Gr_j^{W(\Sigma_{i \in J} N_i)} Gr_r^{W(N)} \simeq Gr_j^{W(\Sigma_{i \in J} A_i)} Gr_r^{W(N)}$$

Now we may consider the orbit with only two endomorphisms $(\Psi_J(L), N_i, N, F = F(N_i))$ (F is the limit along the axis Y_i), then we deduce commutative diagrams for j varying in an interval of $\mathbf{Z}$ symmetric with center 0 with at left HS of weight $n+j$ where $n = m + a - |J|$ and $n + j - 1$ at right

(6.10)

$$\begin{array}{ccc} Gr_j^{W(N_i)} Gr_a^{W(N)} \Psi_{J-i} L & \xrightarrow{A_i} & Gr_{j-1}^{W(N_i)} Gr_a^{W(N)} \Psi_J L(-1) \\ \downarrow N_i & \swarrow p_i & \downarrow N_i \\ Gr_{j-2}^{W(N_i)} Gr_a^{W(N)} \Psi_{J-i} L(-1) & \xrightarrow{A_i} & Gr_{j-3}^{W(N_i)} Gr_a^{W(N)} \Psi_J L(-2) \end{array}$$

moreover we have: $P_J(A_i u, v) = P_{J-i}(u, p_i v)$ for all $u \in \Psi_{J-i} L$ and $v \in \Psi_J L$
In this situation, a result of M. Saito [22(1),5.2.15] applies and shows

Proposition A.69. For all $J \subset I$ and $i \in J$, consider the morphisms

$$Gr_a^{W(N)} \Psi_{J-i} L \xrightarrow{\mathrm{A_i}} Gr_a^{W(N)} \Psi_J L(-1) \xrightarrow{\mathrm{p_i}} Gr_a^{W(N)} \Psi_{J-i} L(-1)$$

then we have a decomposition

$$Gr_a^{W(N)} \Psi_J L \quad \simeq \quad ImA_i \oplus ker\, p_i \tag{6.11}$$

compatible with the primitive decomposition . In particular, p_i induces an isomorphism of ImA_i in $Gr_a^{W(N)} \Psi_J L$ onto ImA_i in $Gr_a^{W(N)} \Psi_{J-i} L$.

The result is deduced from the sequence in the proposition by taking its graded version $Gr_{j-1}^{W(N_i)}$ for various j as in (6.10) and using the polarization of HS induced on, to prove for each j,

$$ImA_i \oplus Kerp_i \simeq Gr_{j-1}^{W(N_i)} Gr_a^{W(N)} \Psi_J L(-1)$$

hence, since A_i and p_i are compatible with the MHS of weight $W(N_i)$, we get: $ImA_i \oplus Kerp_i \simeq Gr_a^{W(N)} \Psi_J L(-1)$.

Now, to prove the decomposition theorem, it remains to show that K_J^a is pure and polarized, which is deuced from the

LEMMA A.70. *i)*$K_J^a \subset W_0(\Sigma_{i\in J} N_i) Gr_a^{W(N)} \Psi_J L$.
ii) $K_J^a \cap (W_{-1}(\Sigma_{i\in J} N_i) Gr_a^{W(N)} \Psi_J L) = 0$.

PROOF. i) the assertion (i) follows from the relation:

$$Ker(\Sigma_{i\in J} N_i) \subset W_0(\Sigma_{i\in J} N_i) Gr_a^{W(N)} \Psi_J L.$$

ii) Suppose $x \in W_{-s}(\Sigma_{i\in J} N_i) Gr_a^{W(N)} \Psi_J L \cap K_J^a$ where $-s \leq -1$, then there exists $y \in W_s(\Sigma_{i\in J} N_i) Gr_a^{W(N)} \Psi_J L \ = W_s(\Sigma_{i\in J} A_i) Gr_a^{W(N)} \Psi_J L$ such that $x = (\Sigma_{i\in J} A_i)^s(y)$ (by surjectivity of $\Sigma_{i\in J} A_i$ on negative weights) then for each i, we have $(N_i)^s(y) \in ImA_i \ mod\ W_{-s-1}$, hence $x = \sum_i N_i^s(y)$ is in $(\cap_i kerp_i) \cap \sum_i ImA_i = 0 \ mod\ W_{s-1}$, that is $x \in W_{-s-1}(\Sigma_{i\in J} N_i) Gr_a^{W(N)} \Psi_J L$. We deduce (ii) by a descending inductive argument on $-s$. □

We deduce from the lemma that K_J^a is pure of weight a which ends the proof of the theorem.

A.6.1. *The global weighted logarithmic complex of nearby cycles* $(\Psi_f^u(\mathcal{L}), W(\mathcal{N}))$. We define the weight filtration abstractly without going back to an explicit formula as on the combinatorial logarithmic complex. The filtration $W(N)$ on each $\Psi^0(L)$ defines a filtration by sub-complexes and corresponds to the fibre at a point y of the filtration $W(\mathcal{N})$ of $\Psi_f^u(\mathcal{L})$ in the abelian category of perverse sheaves where $\mathcal{N} = -\frac{1}{2i\pi} Log\mathcal{T}^u$.
Consider Deligne's extension $\mathcal{L}_X$ and the associated logarithmic complex

$$i_Y^\star s(\Omega_X^*(LogY) \otimes \mathcal{L}_X[p], \eta)_{p\geq 0}.$$

As in (A.2.2) and (lemma A.10) there exists a global acyclic sub-complex $IC(\mathcal{L}_X[N])$ inducing at each fiber at the point y the complex $IC(L[N], A_i = N_i - n_i N)$. We define the weight filtration $W(\mathcal{N})$ on the quotient complex

$$\Psi_f^u(\mathcal{L}_X) : \ = (s(\Omega_X^*(LogY) \otimes (\mathcal{L})_X[p], \eta)_{p\geq 0} / IC(\mathcal{L}_X[N]))[1]$$

as the filtration inducing $W(N)$ at each fiber $\Psi^0(L)$ at points of Y. The filtration F extends to $(\Psi_f(\mathcal{L}_X)$ via its extension to $\Omega_X^*(LogY) \otimes \mathcal{L}_X[N][i]$ by the formula

$$F^p(i_Y^\star s(\Omega_X^*(LogY) \otimes \mathcal{L}_X[i])_{i\geq 0})[1] = i_Y^\star s(F^{p+i+1}(\Omega_X^*(LogY) \otimes \mathcal{L}_X)[i], \eta)_{i\geq 0}[1]$$

$$= i_Y^\star s[(s(\Omega_X^q(LogY) \otimes F^{p+i+1-q}\mathcal{L}_X)_{q\geq 0})[i], \eta)]_{i\geq 0}[1].$$

With the above local study, one can deduce the following result

Suppose $\mathcal{L}$ underlies a polarized VHS *of weight m, then the graded part of the weight on the above complex with the filtration F*

$$(\Psi_f(\mathcal{L}_X)[n-1], W(\mathcal{N})[m+n-1], F)$$

decomposes into intermediate extensions of polarized VHS.
Moreover $Coker(\mathcal{N})/\Psi_f^u(\mathcal{L}_X[n-1])$ is isomorphic to the quotient $(\Omega_X^*(LogY) \otimes \mathcal{L}_X[n])/IC(X, \mathcal{L}[n])$ with weight filtration induced by $W(\mathcal{N})$.

The proof reduces by definition to show that $(Gr_r^{W(\mathcal{N})[m]}, F)$ decomposes.

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