On the weight filtration of the homology of algebraic varieties, The generalized Leray cycles

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Abstract

Let Y be a normal crossing divisor in the smooth complex projective algebraic variety X and let U be a tubular neighbourhood of Y in X. Using geometrical properties of different intersections of the irreducible components of Y, and of the embedding $Y \subset X$, we construct homological cycles generating $H_*(A, B)$, where (A, B) is one of the following pairs $(Y, \varnothing)$, (X, Y), $(X, X-Y)$, $(X-Y, \varnothing)$ and $(\partial U, \varnothing)$. The construction is compatible with the weights in $H_*(A, B, \mathbb{Q})$ of Deligne's mixed Hodge structure.

The main technical part is to construct "the generalised Leray inverse image" of chains of the components of Y, giving rize to a chain situated in ∂U.

1§. Introduction

Let X be a smooth complex projective algebraic variety, Y a normal crossing divisor in X, and U a tubular neighbourhood of Y in X. For any $p \geq 1$ denote by $\tilde{Y}^p$ the normalization of the p-fold intersections of the different irreducible components of Y, and let $\tilde{Y}^0$ denote X.

The main goal of the present article is the construction of (geometric) cycles generating $H_*(A, B)$, where (A, B) is one of the following pairs $(Y, \varnothing)$, (X, Y), $(X, X - Y)$, $(X - Y, \varnothing)$ and $(\partial U, \varnothing)$. The construction has several additional features. First, the corresponding cycles are constructed from (geometric) cycles of the spaces $\{\tilde{Y}^p\}_{p \geq 0}$, emphasizing the geometrical connections between the smooth spaces $\tilde{Y}^p$ and the different pairs (A, B). Second, the construction is compatible with the weights in $H_*(A, B, \mathbb{Q})$ of Deligne's mixed Hodge structure. Having the first property of the construction, the second one is rather natural having in mind Deligne's construction of the mixed Hodge structure on the cohomology of the pairs (A, B). Third, the construction is compatible with the Poincaré duality. Here the point is that we can define a natural morphism at the level of the complexes of chains which will induce all the wanted dualities. (In general such morphisms do not exist; the construction of complexes which admit similar morphisms is rather important, see e.g. [?] for more details.)

Here we wish to stress the main point (and philosophy) of the paper: our final goal is not (only) to construct the (abstract) homology groups of the pairs (A, B), and to describe their weight filtration (for this one only needs to dualize Deligne's theory at (co)homolical level), but to provide a geometrical/topological method of identifying a set of closed cycles which generates the groups $Gr_*^W H_*(A, B)$. The beauty of the construction is that this program is established in a uniform way for all the pairs (A, B).

The construction has two important parts: first, for each pair (A, B) we define a double complex $A_{**}(A, B)$ together with a weight filtration, in such a way that its weight spectral sequence $\{E_{**}^k\}_k$ converges to $H_*(A, B)$, and the induced weights agree with Deligne's weights. The entries A_{st} of the double complex are (some kind of) geometrical chains of the spaces $\{\tilde{Y}^p\}_p$. In the second part, we construct a quasi-isomorphism $m_{A,B} : Tot_*(A_{**}(A, B)) \to C_*(A, B)$,

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where Tot_* denotes the total complex associated with a double complex and $C_*(A, B)$ is the group of geometric chains of the pair (A, B). Obviously, the map $m_{A,B}$ codifies those geometrical steps how one can construct cycles for the pair (A, B) from a compatible set of chains of the smooth spaces $\{\tilde{Y}_p\}_p$.

Both parts faces its own difficulties. First notice that if one wants to dualize Deligne's construction (which uses e.g. differential forms with logarithmic poles), then it is not hard to dualise the spectral sequence E_{**}^k starting from the term E_{**}^1, since all these entries can be described in terms of homology groups, the homological Gysin maps, and morphisms induced by some inclusions. But the construction of the E_{**}^0 term (i.e. of the double complex A_{**}) can be rather involved. The difficulty stands in the fact that if one tries to work with usual simplicial or singular chains, then there is no good Gysin map and no good intersection theory of these chains.

Therefore, excepting those cases when the double complex only involves maps induced by inclusions (these are exactly the cases $(A, B) = (Y, \varnothing)$ and $(A, B) = (X, Y)$), the construction also uses special chains compatible with intersections (i.e. with a geometrical Gysin map). It turns out that the zero-perversity chains of Goresky-MacPherson will do the job. (In this paper we call them "dimensionally transverse" chains.) In the case of ∂U, we even have to take a mixture of these intersection chains and the usual chains. The reader probably will be familiar with the double complexes A_{**} for the pairs $(Y, \varnothing)$ and (X, Y), but A_{**} for all the other pairs is a novelty of the present paper.

The chain morphism $m_{A,B}$ is a kind of generalization of the construction of the geometrical cycles living in a tubular neighbourhood of a smooth divisor Y in a smooth space X (the Leray cycles), but in the present case of a normal crossing divisor Y, it is more complex. It codifies the receipe how one can construct closed cycles for (A, B), with a given weight from E_{**}^*. So, in the final picture, via the maps $m_{A,B}$, instead of having only some groups − isomorphic to $Gr_*^W H_*(A, B)$ in some abstract way − we give precise geometric representatives of their elements. These kind of constructions (excepting the "easy cases" $(Y, \varnothing)$ and (X, Y)) were known only in sporadic low dimensional situations.

Even if the "easy cases" were known, since they appear in some (needed) exact sequences and Poincaré dualities involving the other cases, we decided to insert them in all our discussions. In fact, our final goal was the understanding of the case $(\partial U, \varnothing)$, which, in some sense involves all the other cases.

In a forthcoming paper we will provide duality properties between the present double complexes A_{**} and the cohomological double complexes used by Deligne. These dualities are established by some residues along the (normal crossing divisor) Y, generalizing Leray's theory (the Y smooth case).

In the present paper the proofs are not based on these (residue type) dualities. Nevertheless, at some point we will need to establish a duality at weighted (co)homological level with the Deligne's cohomological mixed Hodge structures. In fact, the degeneration of the spectral sequence E_{**}^k at rank two is proved in this way, and this property is absolutely crucial in the construction of the geometric cycles. More precisely: the degeneration of the spectral sequence imposes strong conditions about the cycles of $\tilde{Y}^p$. The very existence of "the receipe" how one constructs the cycles lies in these conditions.

Another feature of the present construction is that it provides an extremely suitable way to compare the weight filtration with the so called "support filtration" (or Zeeman filtration). For the pairs $(Y, \varnothing)$ and $(X, X-Y)$ we pove that Deligne's weight filtration and the support filtration

agree. (This can be extended to all the pairs showing that the weights depend only on the topological type of (X, Y).) The result for $(Y, \varnothing)$ was conjectured by Verdier and MacPherson, and proved by McCrory [?] (see also [?] and [?] for different proofs). Our new proof is rather elementary and geometrical, and automatically provides the corresponding result for $(X, X - Y)$ as well.

Finally, we mention the following (probably not absolutely neglectable) advantage of the pesent presentation: it gives a uniform way to describe the weight filtration of different homology groups appearing in the theory of complex algebraic variaties using only a topological language, not involving hypercohomologies, or Deligne's mixed Hodge complexes. (Of course, the price for this is that we can say anything about the corresponding Hodge filtrations.) Moreover, as a byproduct, one gets a geometrical interpretation of the rank two degenerations of the corresponding weight spectral sequences.

The second part of the introduction presents the main notations and provides a formal presentation of the main steps of the article. The rest of the paper is organized as follows. In the second section we analyze the complex of "dimensionally transverse" sub-analytic chains. This is in fact the complex of the sub-analytic "intersection chains" associated with the natural stratification of Y and zero perversity (in the sense of Goresky–MacPherson [?]). Moreover, here we also construct the (generalized Leray) operator L (which latter will be used in the definition of the morphisms $m_{A,B}$). Using the above chains, in section 3 we define the double complexes $A_{**}(A, B)$ and we list some properties of their spectral sequences. In section 4 we review the (needed) algebraic properties of a double complex whose spectral sequence degenerates at rank 2. Using the result of the previous two sections, in section 5 we construct cycles compatible with the weight filtration. The case of $(\partial U, \varnothing)$ is the most involved, so we separated it in section 6.

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In the next paragraphs, we summarize in more formal language the main steps of the article, in order to provide a precise guiding line for the reader.

1.1 The stratification of Y. Let X be a smooth projective algebraic variety containing a normal crossing divisor Y. Let $\{Y_\alpha\}_{\alpha \in I}$ be the decomposition of Y into irreducible components. Here the index set I is a totally ordered set. The divisor Y has a natural stratification by subspaces Y^p consisting of "points of multiplicity $\geq p$ in Y", i.e.:

$$Y^p = \bigcup_{\alpha_1 < \cdots < \alpha_p} Y_{\alpha_1} \cap \cdots \cap Y_{\alpha_p}, \ \ \text{with normalization } \ \tilde{Y}^p = \coprod_{\alpha_1 < \cdots < \alpha_p} Y_{\alpha_1} \cap \cdots \cap Y_{\alpha_p}.$$

It is convenient to write $Y_{\alpha_1, \ldots, \alpha_p} = Y_{\alpha_1} \cap \cdots \cap Y_{\alpha_p}$ and $\tilde{Y}^0 = Y^0 = X$.

1.2 The tubular neighbourhood of Y. For any variety X and divisor Y, X admits a triangulation compatible with Y, hence there exists a regular (tubular) neighbourhood in X which is a deformation retract of Y (see e.g. [?]). In the case of a normal crossing divisor, A' Campo [?] describes such a neighbourhood as follows. For any $\alpha \in I$, the real (oriented) blow-up $\Pi_\alpha : Z_\alpha \to X$ with center Y_α provides a differentiable manifold Z_α with boundary and

a projection Π_α inducing a diffeomorphism above the complement of Y_α. Moreover, $\Pi_\alpha^{-1}(Y_\alpha)$ is isomorphic to the S^1-bundle associated to the oriented normal bundle $N_{Y_\alpha/X}$. We can identify the fiber above a point $y \in Y_\alpha$ with the set of real oriented normal directions to Y_α. For example, if $Y = Y_\alpha$ is a point in $X = \mathbb{C}$, then Z_α is a half-cylinder whose boundary S^1 lies over the point Y. In general, the boundary $\Pi_\alpha^{-1}(Y_\alpha)$ of Z_α is diffeomorphic to the boundary of the complement of an open tubular neighbourhood of Y_α in X.

Next we consider the fibered product $\Pi : Z \to X$ of the projections $\{\Pi_\alpha\}_{\alpha \in I}$ over X. Then Z is a manifold with corners whose boundary ∂Z is exactly $\Pi^{-1}(Y)$. In fact, Z is homeomorphic to the complement of an open tubular neighbourhood V of Y in X, hence ∂Z is homeomorphic also to the boundary $\partial \bar{V}$ of the closure $\bar{V}$ of V. This shows that $\bar{V}$ is homeomorphic to the mapping cylinder of $\Pi|\partial Z : \partial Z \to Y$.

More precisely, fix a homeomorphism $\phi : \partial Z \times [0, \epsilon] \to Z_\epsilon$, where $Z_\epsilon \subset Z$ is some collar of ∂Z, and $\phi|\partial Z \times \{0\}$ is the inclusion $\partial Z \hookrightarrow Z_\epsilon$. There is a natural projection $p : Z_\epsilon \to \partial Z$ given by $pr_1 \circ \phi^{-1}$, where pr_1 is the projection on the first factor. (This retract can be extended to a strong deformation retract of the inclusion $\partial Z \hookrightarrow Z_\epsilon$.) Using this, one can define a tubular neighbourhood U_ϵ of Y in X by $U_\epsilon := \Pi(Z_\epsilon)$. Its boundary ∂U_ϵ is clearly $\Pi(\phi(\partial Z \times \{\epsilon\}))$.

This construction is completely satisfactory in the discussion of topological invariants. But, in fact, Z has a natural semi-analytic structure and the above homeomorphism ϕ can be chosen as a sub-analytic homeomorphism as well (in fact, even as a piecewise analytic isomorphism with respect to the natural stratification). For the proof of this last statement, see [?]. Therefore, all the homeomorphisms and maps discussed in the previous paragraphs can be considered in the sub-analytic category. Moreover, $\Pi : Z \to X$ is compatible with the semi-analytic structures of Z and X.

1.3 The construction of the double complexes $A_{**}(A, B)$. For any $p \geq 0$, we denote the complex of the usual (respectively the "dimensionally transverse") sub-analytic chains defined on $\tilde{Y}^p$ by $(C_*(\tilde{Y}^p), \partial)$ (respectively by $(C_*^{\pitchfork}(\tilde{Y}^p), \partial)$) (cf. 2.I and 2.II). One has the following chain morphisms:

$$\cdots \quad C_{*-4}^{\pitchfork}(\tilde{Y}^2) \quad \overset{\cap}{\longleftarrow} \quad C_{*-2}^{\pitchfork}(\tilde{Y}^1) \quad \overset{\cap}{\longleftarrow} \quad C_*^{\pitchfork}(\tilde{Y}^0)$$
$$\downarrow j$$
$$C_*(\tilde{Y}^0) \quad \overset{i}{\longleftarrow} \quad C_*(\tilde{Y}^1) \quad \overset{i}{\longleftarrow} \quad C_*(\tilde{Y}^2) \quad \cdots$$

Then define (modulo some shift of indexes): $A_{**}(X, X - Y) = (C_*^{\pitchfork}(\tilde{Y}^p), \partial, \cap)_{p \geq 1}$, $A_{**}(X - Y) = (C_*^{\pitchfork}(\tilde{Y}^p), \partial, \cap)_{p \geq 0}$, $A_{**}(Y) = (C_*(\tilde{Y}^p), \partial, i)_{p \geq 1}$, $A_{**}(X, Y) = (C_*(\tilde{Y}^p), \partial, i)_{p \geq 0}$. Finally, let $A_{**}(\partial U)$ be the cone of the morphism j (considered as a morphism of double complexes) after we replace in the above diagram $\tilde{Y}^0$ by U. The weight filtration W_* of a double complex A_{**} is defined by $W_s := \oplus_{p \leq s} A_{pq}$. $\{E_{st}^*\}$ denotes the corresponding weight spectral sequence of (A_{**}, W). The main property of these spectral sequences is that they degenerate at rank two.

1.4 The construction of the cycles. First, using the geometrical properties of the projection Π, for any pair (A, B) we construct a chain morphism $m_{A,B} : Tot_*(A_{**}(A, B)) \to C_*(A, B)$, and we prove that it is a quasi-isomorphism. Then, using the degeneration property of the spectral sequence, any representative c_{st} of an element $[c_{st}] \in ker(d^1|E_{st}^1)$ is completed to a chain $c_{st}^\infty = c_{st} + c_{s-1,t+1} + \cdots \in Z_{st}^\infty$ with $D(c_{st}^\infty) = 0$, where D is the differential of the associated

total complex $Tot_*(A_{**})$ (and $c_{pq} \in A_{pq}$). The final product of the construction is the $k = s + t$ dimensional closed cycle $m_{A,B}(c_{s,t}^\infty)$ in $C_k(A, B)$.

For the various pairs (A, B) considered above, we have the following result.

1.5 Theorem. *a)* (E_{st}^1, d^1) *of the weight spectral sequence can be explicitly determined from the homology of the spaces $\tilde{Y}^p$ and from the various normal bundles of the components of $\tilde{Y}^p$ in $\tilde{Y}^{p-1}$.*

b) $E_{st}^r \Longrightarrow H_{s+t}(A, B, \mathbb{Z})$, *and induces a weight filtration on the integer homology. Moreover $E_{st}^\infty \otimes \mathbb{Q} = Gr_{-t}^W H_{s+t}(A, B, \mathbb{Q})$ (the last considered in Deligne's MHS).*

c) $E_{**}^* \otimes \mathbb{Q}$ *degenerates at rank 2.*

d) The above construction provides all the cycles of the pair (A, B) (modulo boundary) according to their weights. More precisely, the homology class $[m_{A,B}(c_{s,t}^\infty)]$ of dimension $k = s + t$ is well-defined modulo $W_{-t-1}H_k(A, B, \mathbb{Q})$, and these type of classes generate $W_{-t}H_k(A, B, \mathbb{Q})$.

1.6 Example. Suppose $Y = Y_1 \cup Y_2$ has two components with smooth intersection $Y_{1,2} = Y_1 \cap Y_2$. One wishes to describe all the k dimensional cycles in ∂U. Let us exemplify the weight $-k - 1$ case ($s = -1$).

Two homology classes $[a_i] \in H_{k-1}(Y_i)$, $i = 1, 2$, satisfying $[a_1] \cap [Y_{1,2}] = [a_2] \cap [Y_{1,2}]$ in $H_{k-3}(Y_{1,2})$ give rise to such a homology class in ∂U. The corresponding representative can be constructed as follows. Assume that the representative a_i is transversal to $Y_{1,2}$ in Y_i ($i = 1, 2$). Due to the condition about $[a_i] \cap [Y_{1,2}]$, there exists a chain $a_{1,2}$ in $Y_{1,2}$ such that $\partial a_{1,2} = a_2 \cap Y_{1,2} - a_1 \cap Y_{1,2}$. Now, $\Pi^{-1}(a_i)$ is an "S^1-bundle over a_i" ($i = 1, 2$) — the precise meaning of $\Pi^{-1}(a_i)$ will be explained latter, cf. the operator L in section 2 — and $\Pi^{-1}(a_{1,2})$ is an "$S^1 \times S^1$-bundle over $a_{1,2}$". Therefore, $\dim \Pi^{-1} a_i = \dim \Pi^{-1} a_{1,2} = k$. Moreover, by the very construction, $\Pi^{-1}(a_1 + a_2 + a_{1,2})$ has no boundary. It is the wanted closed cycle in ∂U (via an identification of ∂Z with ∂U). The ambiguity provided by the choice of $a_{1,2}$ is modulo a cycle of weight $-k - 2$.

Now, consider the case when Y has three irreducible components Y_i ($i = 1, 2, 3$) with $Y_{1,2,3} \neq \varnothing$. Similarly as above, we want to lift some closed cycles a_i into ∂U. The first obstruction is $[a_i \cap Y_{i,j}] = [a_j \cap Y_{i,j}]$ in the homology of $Y_{i,j}$ (for any pair $i \neq j$). (This condition is codified in (E_{**}^1, d^1).) Using this, we create the new chains $a_{i,j}$ in $Y_{i,j}$ as above. Now, if we want to lift these new chains and glue them together, we face the second obstruction provided by the triple intersection $Y_{1,2,3}$ (codified in (E_{**}^2, d^2)). The main point is that *there exists a good choice of $a_{i,j}$ such that this new obstruction is trivial*; the vanishing of this second obstruction is equivalent with the vanishing of the second differential d^2 of the spectral sequence. Now the construction can be continued (i.e. in this case one gets an additional cycle $a_{1,2,3}$) which, together with the other pieces, lifted via Π, provides the wanted representative as above.

For a more complete example see **??** .

2§. Topological Preliminaries

I. Geometric chains.

2.1 Preliminary remarks. In some cases it is not absolutely evident how one can dualize a result established in cohomology. For example, if we want to compute the homology of Y, then Deligne spectral sequence in cohomology has a very natural analog in homology. On the other

hand, if we want to find the homological analogue of the cohomological theory of $X - Y$, then we have to realize that there is no obvious homological candidate. Actually, the E_1-term of the spectral sequence associated with the log complex $(\Omega_X^*(\log Y), W)$ can be easily dualized; but we want (and need to) dualize the whole spectral sequence, in particular the E_0 term as well.

Since the *Gysin differential*, in the cohomological E_1-term of $X - Y$, dualizes to the intersection of cycles, one needs to work with *chains with good intersection properties* with respect to the stratification defined by Y. Similarly as in the case of the intersection homology groups, one has several options to define the chain complex. In the original definition of the intersection homology groups, Goresky and MacPherson used geometric chains with some restrictions provided by the perversities [**?**]. On the other hand, H. King recovered these groups using singular chains. We will follow here the first option: we will use sub-analytic geometric chains. The sub-analytic assumption is also motivated by the fact that these chains are stable with respect to the real blowing up along Y (in contrast with the P.L. chains).

Moreover, in order to have a good intersection theory, one needs some kind of transversality property of the chains. Here again one has several possibilities. Our choice asks (only) a "dimensional transversality" of the chains. This has the big advantage that the complex of these chains coincides with the complex of zero perversity chains of Goresky–MacPherson; but has the disadvantage, that in the intersections of the cycles we have to handle an intersection multiplicity problem (see the second subsection II).

2.2 Definition. Sub-analytic geometric chains. Before we start the precise definition of our geometric chains, let us review briefly the general theory. We will follow the presentation from [**?**], pages 146-155.

Fix a manifold M. The group of *geometric chains* is defined in three steps. First, one defines *a good class $\mathcal{C}$ of subsets of M*. This should satisfy the following properties: (1) if a subset S of M is in $\mathcal{C}$, then M has a Whitney stratification such that S is a union of strata, and each stratum is in $\mathcal{C}$; (2) the class $\mathcal{C}$ is closed under unions, intersections and differences; (3) the closure of a subset in $\mathcal{C}$ is in $\mathcal{C}$.

In the second step, one defines the *geometric prechains* (relative to $\mathcal{C}$). Basically, they can be represented as $\sum_\alpha m_\alpha S_\alpha$, where each S_α is a closed subset from $\mathcal{C}$ (with a fixed orientation) and m_α is an integer (the coefficient or the "multiplicity" of S_α). A geometric chain is an equivalence class of geometric prechains with respect to a natural equivalence relation. (For details, see [**?**].)

In this paper we will use sub-analytic chains: if M is a complex analytic manifold, and we fix a real analytic structure on M, then the class of all sub–analytic subsets of M form the good class of subsets $\mathcal{C}$.

Since for any $p \geq 0$, $\tilde{Y}^p$ is a complex analytic manifold, the above definition can be applied. The corresponding chain complex is denoted by $C_*(\tilde{Y}^p)$. We emphasize (even if this is clear since X is compact) that we deal with chains with *compact* supports. (Cf. also with [**?**], I.)

2.3 Sub-analytic geometric chains via triangulation. In this paragraph we present a different realization of $C_*(\tilde{Y}^p)$. We closely follow the paper [**?**] of Goresky and MacPherson (where the case of the P.L. geometric chains is considered instead of sub-analytic ones).

We regard our manifold X as a pseudomanifold of dimension $2n$, where n is the complex dimension of X. We consider its natural stratification $X_{2n-2p+1} = X_{2n-2p} = Y^p$ for $p \geq 1$, hence $\Sigma = Y$. Similarly as above $\tilde{Y}^0 = Y^0 = X$.

By [**?**], X admits a (canonical) sub-analytic triangulation, compatible with the stratification. In fact, any two sub-analytic triangulations admit a common refinement (see [**?**], 2.4).

For the next definitions, we fix an integer $p \geq 0$. Then $\tilde{Y}^p$ is again a pseudomanifold. Its stratification is given by the intersection points of "multiplicity $\geq p$". If T is a sub-analytic triangulation of $\tilde{Y}^p$, compatible with its stratification, let $C_*^T(\tilde{Y}^p)$ be the chain complex of simplicial chains of $\tilde{Y}^p$ with respect to T. By definition, a chain of $\tilde{Y}^p$ is an element of $C_*^T(\tilde{Y}^p)$ for some sub-analytic triangulation T, however one identifies two chains $c \in C_*^T$ and $c' \in C_*^{T'}$ if their canonical images in $C_*^{T''}$ coincide, for some common refinement T'' of T and T'. Hence, the group of all chains is the inductive limit with respect to all the triangulations, and it is denoted by $\lim_{\to} C_*^T(\tilde{Y}^p)$. Obviously, there is a natural boundary operator ∂ which makes $\lim_{\to} C_*^T(\tilde{Y}^p)$ a complex.

If $\xi \in C_k^T$, then the support $|\xi|$ of ξ is the union of the closures of those k-simplices σ whose coefficient in ξ is non-zero. Actually, the support of ξ is independent on T, and it is a k-dimensional sub-analytic subset (in particular, it is well-defined for any element of $\lim_{\to} C_*^T(\tilde{Y}^p)$).

2.4 Identification and homological properties of the chains.

(a) $\lim_{\to} C_*^T(\tilde{Y}^p) = C_*(\tilde{Y}^p)$;

(b) The homology of the complex $C_*(\tilde{Y}^p)$ is the usual (i.e. singular) homology $H_*(\tilde{Y}^p)$ of $\tilde{Y}^p$;

(c) For any triangulation T, the natural morphism $C_*^T(\tilde{Y}^p) \to \lim_{\to} C_*^T(\tilde{Y}^p)$ is a quasi-isomorphism.

Proof. For (a) notice that for an arbitrary closed sub-analytic subset S of $\tilde{Y}^p$, by [**?**], there is a sub-analytic triangulation T which makes S an element of $C_*^T(\tilde{Y}^p)$ (with all multiplicities one). For (b) see [**?**], page 155. Finally, for (c) notice that the triangulation homeomorphism $t : |K| \to \tilde{Y}^p$ from the simplicial complex $|K|$ to $\tilde{Y}^p$ identifies the simplicial homology of $|K|$ and the usual homology of $\tilde{Y}^p$. $\square$

II. "Dimensionally transverse" chains.

For any $p \geq 0$, we introduce the following complex of *dimensionally transverse* chains in $\tilde{Y}^p$. We recall that the stratification of $\tilde{Y}^p$ is provided by the intersection points of multiplicity $\geq p$. It will be denoted by $\{\tilde{Y}^p_{2n-2p-2r}\}_{r \geq 0}$ (i.e. $2r$ denotes the real codimension in $\tilde{Y}^p$).

2.5 Definition. We say that a chain $\xi \in C_k(\tilde{Y}^p)$ is *dimensionally transverse*, if

a) $\dim(|\xi|) \cap \tilde{Y}^p_{2n-2p-2r} \leq k - 2r$, for any $r > 0$; and

b) $\dim(|\partial\xi|) \cap \tilde{Y}^p_{2n-2p-2r} \leq k - 1 - 2r$, for any $r > 0$.

The subgroup of $C_*(\tilde{Y}^p)$ consisting of the *dimensionally transverse* chains is denoted by $C_*^{\pitchfork}(\tilde{Y}^p)$. The boundary operator ∂ maps dimensionally transverse chains into dimensionally transverse chains, hence defines a complex with $\partial^2 = 0$.

In fact, the complex $C_*^{\pitchfork}(\tilde{Y}^p)$ is exactly the complex $IC_*^{\bar{0}}(\tilde{Y}^p)$ of intersection chains corresponding to the zero perversity (see [**?**]).

2.6 Lemma. *The natural inclusion $j : (C_*^{\pitchfork}(\tilde{Y}^p), \partial) \to (C_*(\tilde{Y}^p), \partial)$ is a quasi-isomorphism. In particular, the homology of $(C_*^{\pitchfork}(\tilde{Y}^p), \partial)$ is $H_*(\tilde{Y}^p)$.*

Proof. This follows from section 4.3 of [**?**] (because the Poincaré map $H^{2n-2p-k}(\tilde{Y}^p) \to H_k(\tilde{Y}^p)$ is an isomorphism, provided by the smoothness of $\tilde{Y}^p$). Actually, since $\tilde{Y}^p$ is smooth, and the intersection homology group is independent on the stratification (cf. [**?**], 3.2), all the intersection homology groups are the same to the usual homology groups. $\qquad\square$

2.7 The intersections with the strata. Now, for any chain $\xi = \xi^k_{\alpha_1,\dots,\alpha_p} \in C^{\pitchfork}_k(Y_{\alpha_1,\dots,\alpha_p})$ and $\alpha \notin \{\alpha_1,\dots,\alpha_p\}$ we define the intersection chain $\xi \cap Y_\alpha \in C^{\pitchfork}_{k-2}(Y_{\alpha_1,\dots,\alpha_p} \cap Y_\alpha)$. Here, similarly as above, if $p = 0$ then $Y_{\alpha_1,\dots,\alpha_p}$ denotes X. In the definition an "intersection multiplicity" plays a crucial role. It is clear that the intersection $|\xi| \cap Y_\alpha$ of the supports has dimension $\leq k-2$, and this intersection satisfies the transversality restrictions imposed by the $(k-2)$-dimensional supports. But, still we have to determine the coefficients of those simplices which support this intersection. For this, we will follow [**?**]. We start with the following fact.

2.8 Fact. ([**?**] page 138) *If $C \subset \tilde{Y}^{p+1}$ is a (k–2)–dimensional sub-analytic subset of $\tilde{Y}^{p+1}$ and if $D \subset C$ is a (k–3)–dimensional sub-analytic subset, then there is a one-to-one correspondence between chains $\beta \in C_{k-2}(\tilde{Y}^{p+1})$ such that $|\beta| \subset C$, $|\partial\beta| \subset D$, and between homology classes $[\beta] \in H_{k-2}(C, D)$. Furthermore, the homology class of $\partial\beta$ in $H_{k-3}(D)$ is exactly $\partial_*([\beta])$, where $\partial_* : H_{k-2}(C, D) \to H_{k-3}(D)$ is the natural connecting homomorphism.*

Using the above Fact, the *intersection* $\xi \mapsto \xi \cap Y_\alpha$ is completely determined by the following composition:

$$H_k(|\xi|, |\partial\xi|)$$
$$\approx \uparrow \quad \cap[Y_{\alpha_1,\dots,\alpha_p}]$$
$$H^{2n-2p-k}(Y_{\alpha_1,\dots,\alpha_p} - |\partial\xi|, Y_{\alpha_1,\dots,\alpha_p} - |\xi|)$$
$$\downarrow \quad i^*$$
$$H^{2n-2p-k}(Y_{\alpha_1,\dots,\alpha_p} \cap Y_\alpha - |\partial\xi|, Y_{\alpha_1,\dots,\alpha_p} \cap Y_\alpha - |\xi|)$$
$$\approx \downarrow \quad \cap[Y_{\alpha_1,\dots,\alpha_p} \cap Y_\alpha]$$
$$H_{k-2}(|\xi| \cap Y_\alpha, |\partial\xi| \cap Y_\alpha).$$

Above, the first map is the (inverse) of the cap product with the fundamental class of $Y_{\alpha_1,\dots,\alpha_p}$, as it is presented in the Appendix of [**?**], page 162; the second map is the restriction i^*, where i is the natural inclusion; and finally, the third map is again the cap product with the fundamental class of $Y_{\alpha_1,\dots,\alpha_p} \cap Y_\alpha$. The cap products are isomorphisms (cf. [*loc.cit.*]).

Then the map $\xi \mapsto \xi \cap Y_\alpha$ is defined as follows: ξ gives an element in $H_k(|\xi|, |\partial\xi|)$ via **??** , and the image of that element by the above composition determines $\xi \cap Y_\alpha$, via the same **??** .

2.7 Remark. Obviously, the supports satisfy the inclusion $|\xi \cap Y_\alpha| \subset |\xi| \cap Y_\alpha$. Nevertheless, it is important to notice that $|\xi \cap Y_\alpha|$ is not necessarily equal to $|\xi| \cap Y_\alpha$ (see e.g. **??**).

Next, we define an operator $\cap : C^{\pitchfork}_k(\tilde{Y}^p) \to C^{\pitchfork}_{k-2}(\tilde{Y}^{p+1})$. If $\xi^k \in C^{\pitchfork}_k(\tilde{Y}^p)$, we write $\xi^k = \oplus_{\alpha_1 < \dots < \alpha_p} \xi^k_{\alpha_1,\dots,\alpha_p}$ for the corresponding components of ξ^k corresponding to the decomposition of $\tilde{Y}^p$ (the index k emphasizes the dimension, and sometimes it is omitted). Similarly, the components of $\cap\xi$ are $\oplus_{\alpha_1 < \dots < \alpha_{p+1}} (\cap\xi)_{\alpha_1,\dots,\alpha_{p+1}}$. Then, by definition:

$$(\cap\xi)_{\alpha_1,\dots,\alpha_{p+1}} = \sum_{i=0}^{p} (-1)^{k+i} \xi^k_{\alpha_1,\dots,\alpha_i,\widehat{\alpha_{i+1}},\alpha_{i+2},\dots,\alpha_{p+1}} \cap Y_{\alpha_{i+1}}$$

(where $\widehat{\alpha_{i+1}}$ means that α_{i+1} is omitted).

2.7 Lemma. *a)* $\cap^2 = 0$ *and* *b)* $\partial \cap + \cap \partial = 0$.

Proof. The proof follows from a standard index manipulation and from the identity $(\xi_{\alpha_1,...,\alpha_p} \cap Y_\alpha) \cap Y_\beta = (\xi_{\alpha_1,...,\alpha_p} \cap Y_\beta) \cap Y_\alpha$ for any $\alpha < \beta$ and $\alpha, \beta \notin \{\alpha_1, \ldots, \alpha_p\}$ (which follows from [?], page 144). For b) also notice that the sign $(-1)^k$ in the definition of $\cap$ provides the wanted identity instead of $\cap \partial = \partial \cap$. $\square$

The fact that the homology of $(C_*^{\text{ch}}(\tilde{Y}^p), \partial)$ is exactly $H_*(\tilde{Y}^p)$ (cf. Lemma **??**) and part (b) of **??** show that $\cap$ induces an operator $H_*(\tilde{Y}^p) \to H_{*-2}(\tilde{Y}^{p+1})$, still denoted by $\cap$.

2.6 Lemma. *For any $p \geq 0$, the operator $\cap : H_k(\tilde{Y}^p) \to H_{k-2}(\tilde{Y}^{p+1})$, $[\xi] \mapsto \cap[\xi]$, induced by* $\cap : C_k^{\text{ch}}(\tilde{Y}^p) \to C_{k-2}^{\text{ch}}(\tilde{Y}^{p+1})$ *is*

$$(\cap[\xi])_{\alpha_1,...,\alpha_{p+1}} = \sum_{i=0}^{p}(-1)^{k+i}\, [\xi]_{\alpha_1,...,\alpha_i,\widehat{\alpha_{i+1}},\alpha_{i+2},...,\alpha_{p+1}} \cap [Y_{\alpha_{i+1}}]$$

where $\cap[Y_\alpha]$ denotes the homological Gysin map, or transfer map $i_!$, where $i : Y_{\alpha_1,...,\alpha_p} \cap Y_\alpha \hookrightarrow Y_{\alpha_1,...,\alpha_p}$ is the natural inclusion. (I.e. $i_! = PD \circ i^* \circ PD$, where PD denotes the Poincaré dualities in the corresponding spaces; cf. e.g. [?], page 368.)

Proof. The result follows from the definition of the intersection $\xi \cap Y_\alpha$; cf. also with the next discussion about Poincaré dualities. $\square$

2.7 The Poincaré duality map. We will need later the Poincaré isomorphism between homology of Y and cohomology of X with support in Y. Therefore, we review the construction of [?], pages 139-140, inspired by the classical construction which provides the Poincaré duality for manifolds (see e.g. [?], page 338).

Similarly as above, for any triangulation T of X, compatible with the stratification, one can consider the chain complex of simplicial cochains $(C_T^*(\tilde{Y}^p), \delta)$ of $\tilde{Y}^p$. Here $C_T^i(\tilde{Y}^p) = Hom(C_i^T(\tilde{Y}^p), \mathbb{Z})$. Let T' be the first barycentric subdivision of T, and let $\hat{\sigma}$ denote the barycentre of the simplex $\sigma \in T$. Let T_i be the i-skeleton of T, thought of as a subcomplex of T'. It is spanned by all vertices $\hat{\sigma}$ such that $\dim \sigma \leq i$. Let D_i be the i-coskeleton spanned, as a subcomplex of T', by all the vertices $\hat{\sigma}$ such that $\dim \sigma \geq i$. There are canonical simplex preserving deformation retracts:

2.8 $$X - |T_i| \to |D_{i+1}| \quad \text{and} \quad X - |D_{i+1}| \to |T_i|.$$

Now, identify $C_T^i(X)$ with $\oplus_{\dim \sigma = i} H^i(\sigma, \partial\sigma) = H^i(|T_i|, |T_{i-1}|)$, and define $pd : C_T^i(X) \to C_{2n-i}^{T'}(X)$ by the following composition:

$$H^i(|T_i|, |T_{i-1}|)$$

$$\downarrow \cap[X]$$

$$H_{2n-i}(X - |T_{i-1}|, X - |T_i|)$$

$$\approx \downarrow \text{(deformation retract)}$$

$$H_{2n-i}(|D_i|, |D_{i+1}|)$$

$$\downarrow$$
$$H_{2n-i}(|T'_{2n-i}|, |T'_{2n-i-1}|)$$

Actually, $H_{2n-i}(|T'_{2n-i}|, |T'_{2n-i-1}|) = C^{T'}_{2n-i}(X)$, but we have even something more. Since the image of any chain by this composition is supported by union of $|D_i|$'s, and any $|D_i|$ is dimensionally transverse, one obtains a homomorphism $pd : C^i_T(X) \to C^{T', \pitchfork}_{2n-i}(X)$. Similarly, for any $Y_{\alpha_1,\ldots,\alpha_p}$, one can define a homomorphism:

$$pd : C^i_T(Y_{\alpha_1,\ldots,\alpha_p}) \to C^{T', \pitchfork}_{2n-2p-i}(Y_{\alpha_1,\ldots,\alpha_p}).$$

By [?] (7.2), this is a chain map:

2.9 $$\partial \circ pd = pd \circ \delta.$$

2.10 Lemma. *a) Fix $\alpha \notin \{\alpha_1, \ldots, \alpha_p\}$. Then the following diagram is commutative:*

$$
\begin{array}{ccc}
C^i_T(Y_{\alpha_1,\ldots,\alpha_p}) & \xrightarrow{\ pd\ } & C^{T',\pitchfork}_{2n-2p-i}(Y_{\alpha_1,\ldots,\alpha_p}) \\
\downarrow{\scriptstyle i^*} & & \downarrow{\scriptstyle \cap} \\
C^i_T(Y_{\alpha_1,\ldots,\alpha_p} \cap Y_\alpha) & \xrightarrow{\ pd\ } & C^{T',\pitchfork}_{2n-2p-i-2}(Y_{\alpha_1,\ldots,\alpha_p} \cap Y_\alpha)
\end{array}
$$

b) *The above diagram (and **??**) provides at homology level a commutative diagram:*

$$
\begin{array}{ccc}
H^i(Y_{\alpha_1,\ldots,\alpha_p}) & \xrightarrow{\ PD\ } & H_{2n-2p-i}(Y_{\alpha_1,\ldots,\alpha_p}) \\
\downarrow{\scriptstyle i^*} & & \downarrow{\scriptstyle \cap} \\
H^i(Y_{\alpha_1,\ldots,\alpha_p} \cap Y_\alpha) & \xrightarrow{\ PD\ } & H_{2n-2p-i-2}(Y_{\alpha_1,\ldots,\alpha_p} \cap Y_\alpha)
\end{array}
$$

*where the horizontal maps are the Poincaré duality isomorphisms (cf. also with **??**).*

Proof. Use the definitions of $\cap$ and pd and **??** . $\square$

III. The chain correspondence L.

2.8   In this subsection we present our *main topological tool*: the generalized Leray correspondence L which provides chains in Z, respectively in its boundary ∂Z, as "strict inverse" of chains on $\tilde{Y}^p$. But first we have to fix some orientation conventions regarding the fibers $\Pi^{-1}(y^o)$ for different points $y^o \in Y$.

2.9 Orientation convention. We denote the oriented boundary of a disc in the complex plane by $S^1 = \partial D$ (where we consider the orientation of a boundary in the usual way). For any point $y^o \in Y^1 - Y^2$, the circle $\Pi^{-1}(y^o)$ appears as the boundary of the complement of a disc in $\mathbb{C}$, hence (as an oriented 1-manifold) it is $-S^1$. If $y^o \in Y^p - Y^{p+1}$ then the situation is similar. Y, in a local model, is defined by $\{y_1 \ldots y_p = 0\} \subset \mathbb{C}^p$ where y^o stays for the origin. Let U be defined by $\{y \mid \min|y_i| \le 1\} \subset \mathbb{C}^p$ and the component Y_{α_i} by $y_i = 0$. Then $\Pi^{-1}(y^o)$, set-theoretically, is the tori $S^1_1 \times \ldots \times S^1_p$, where S^1_i is defined by $\{y_i : |y_i| = 1\}$. Then, we fix the orientation of $\Pi^{-1}(y^o)$ as given by the product orientation $(-S^1_1) \times \cdots \times (-S^1_p)$. Moreover, if σ is a contractible C^∞-submanifold in $Y_{\alpha_1,\ldots,\alpha_p} - Y^{p+1}$ ($\alpha_1 < \cdots < \alpha_p$), then on the set-theoretical inverse image $\Pi^{-1}(\sigma)$ we define the product orientation $\sigma \times (-S^1_{\alpha_1}) \times \cdots \times (-S^1_{\alpha_p})$.

With this notations, the following holds.

2.10 Lemma. *Fix indices $\alpha_1 < \cdots < \alpha_i < \alpha < \alpha_{i+1} < \cdots < \alpha_p$ and let $\sigma \subset Y_{\alpha_1,\dots,\alpha_p}$ be an oriented $\mathbb{C}^\infty$ sub-manifold with boundaries such that σ and $\partial\sigma$ intersect Y_α transversally, but $\sigma \cap Y_\beta = \varnothing$ for any $\beta \notin \{\alpha, \alpha_1, \dots, \alpha_p\}$ (here the transversality is considered in $Y_{\alpha_1,\dots,\alpha_p}$). Then the oriented boundary $\partial(\Pi^{-1}\sigma)$ of $\Pi^{-1}\sigma$ is the union of $\Pi^{-1}(\partial\sigma)$ and $(-1)^{\dim\sigma+i}\Pi^{-1}(\sigma\cap Y_\alpha)$.*

Proof. In a neighbourhood of an intersection point $p \in \sigma \cap Y_\alpha$ (respectively $(\partial\sigma) \cap Y_\alpha$), σ has a product structure $D \times T$ where T is a ball (respectively a half ball) in $Y_\alpha \cap Y_{\alpha_1,\dots,\alpha_p}$, and D is a real 2-disc transversal to Y_α. We denote the boundary of D by ∂D. Let $D_\eta = D - \{$small open disc of radius η and origin $0\}$; hence $\partial D_\eta = \partial D - S^1_\eta$. Then we have the diffeomorphism $\Pi^{-1}(D \times T) \approx \Pi^{-1}(D_\eta \times T)$. Since $\Pi^{-1}(D_\eta \times T) = D_\eta \times T \times \Omega$, where $\Omega = (-1)^p S^1_{\alpha_1} \times \cdots \times S^1_{\alpha_p}$, we get: $\partial\Pi^{-1}(D_\eta \times T) = \partial D \times T \times \Omega - S^1_\eta \times T \times \Omega + D_\eta \times \partial T \times \Omega$. Hence $\partial\Pi^{-1}(D \times T) = (-1)^{\dim T + i} T \times (-1)^{p+1} S^1_{\alpha_1} \times \cdots \times S^1_\alpha \times \cdots \times S^1_{\alpha_p} + \partial(D \times T) \times \Omega = (-1)^{\dim\sigma + i} \Pi^{-1}(\sigma \cap Y_\alpha) + \Pi^{-1}\partial\sigma$ (with the obvious notations). $\square$

2.11 Definition of $L(\xi)$. Next, we lift an arbitrary sub-analytic chain via Π. Let $n : \tilde{Y}^p \to Y$ be the natural map. Then notice that for each sub-analytic subset $S \subset \tilde{Y}^r$, the strict transform $\tilde{S} := cl(\Pi^{-1}(n(S) - Y^{r+1}))$ (i.e. the closure of the inverse image of the complement of Y^{r+1} in $n(S)$) is a sub-analytic subset of Z. Indeed, the strict transform can be constructed using a real analytic isomorphism and the permitted operations listed in **??** (2-3).

Fix again the integer $p \geq 0$. For any chain $\xi \in C_k^{\pitchfork}(\tilde{Y}^p)$ we construct a chain $L_p(\xi)$ in $C_{k+p}(\partial Z)$ if $p \geq 1$, respectively in $C_k(Z)$ if $p = 0$.

In a simple way, the chain $L_p(\xi)$ can be defined as follows. Write ξ as a finite sum $\sum m_\sigma \sigma$, where σ are the simplices in the support of ξ with non-zero coefficients. Let σ^0 be $n(\sigma) - Y^{p+1}$. Then the closure $cl(\Pi^{-1}(\sigma^0))$ of $\Pi^{-1}(\sigma^0)$ is a $(k+p)$–dimensional sub-analytic set. Then define $L_p(\xi)$ by $\sum m_\sigma cl(\Pi^{-1}(\sigma^0))$.

In the next discussions we will use another (equivalent) definition based on Fact **??** . Again notice that the inverse image $\Pi^{-1}(n(|\xi|))$ of the support of ξ is a $(k+p)$–dimensional sub-analytic subset of Z. For simplicity, sometimes we will omit the map n, e.g. we will write $\Pi^{-1}(|\xi|)$.

First, we consider the morphism

$$i : H_k(|\xi|, |\partial\xi|) \longrightarrow H_k(|\xi|, |\partial\xi| \cup (|\xi| \cap Y^{p+1}))$$

induced by the inclusion $(|\xi|, |\partial\xi|) \longrightarrow (|\xi|, |\partial\xi| \cup (|\xi| \cap Y^{p+1}))$. (In fact i is an isomorphism since $\mathrm{codim}|\xi| \cap Y^{p+1}$ is 2 in $|\xi|$ and in the long homology exact sequence of the above pair one has $H_i(|\partial\xi| \cup (|\xi| \cap Y^{p+1}), |\partial\xi|) = H_i(|\xi| \cap Y^{p+1}, |\partial\xi| \cap Y^{p+1}) = 0$ if $i = k$ or $k - 1$.)

Then, like in the case of Leray's cohomological residue, we have the following morphism of relative homology:

$$\Pi^{-1}_{rel} : H_k(|\xi|, |\partial\xi| \cup (|\xi| \cap Y^{p+1})) \longrightarrow H_{k+p}(\Pi^{-1}(|\xi|), \Pi^{-1}(|\partial\xi|) \cup \Pi^{-1}(|\xi| \cap Y^{p+1})).$$

Since $\Pi^{-1}(|\xi|) \to |\xi|$ is an oriented fiber bundle over $|\xi| - Y^{p+1}$, with fibers $(S^1)^p$, the classical spectral sequence argument shows that it is an isomorphism (use deformation retract tubular neighbourhoods to thicken $|\xi| \cap Y^{p+1}$ and its inverse image $\Pi^{-1}(|\xi| \cap Y^{p+1})$, then the excision theorem to reduce the situation to the fiber bundle case).

Now, the image of ξ, via the composed map $\Pi^{-1}_{rel} \circ i$, determines completely $L_p(\xi)$ (via Fact). By definition, this is the application $\xi \mapsto L_p(\xi)$. Sometimes the index p will be omitted. (The fact that i and Π^{-1}_{rel} are isomorphisms shows that the two definitions of L agree.)

The next result generalizes the above lemma **??** .

2.9 Proposition. *Fix the integer $p \geq 0$. If $p \geq 1$ then fix the indices $\alpha_1 < \cdots < \alpha_p$ as well. For any $\alpha \notin \{\alpha_1, \ldots, \alpha_p\}$, set $i(\alpha) = i$ if $\alpha_1 < \cdots < \alpha_i < \alpha < \alpha_{i+1} < \cdots < \alpha_p$. Then for any $\xi \in C_k^{\pitchfork}(Y_{\alpha_1, \ldots, \alpha_p})$:*

$$\partial(L_p \xi) = L_p(\partial \xi) + \sum_\alpha (-1)^{k+i(\alpha)} \, L_{p+1}(\xi \cap Y_\alpha).$$

Above, the sum is over all α with $\alpha \notin \{\alpha_1, \ldots, \alpha_p\}$. The equality is considered in $C_*(\partial Z)$ if $p \geq 1$, respectively in $C_*(Z)$ if $p = 0$.

Proof. The contribution $L_p(\partial \xi)$ is clear. Next we want to determine the coefficients of the simplices which lie in $\Pi^{-1}(|\xi| \cap Y_\alpha)$. It is enough to work modulo $Y_\alpha \cap (\cup_\beta Y_\beta)$, where the union is over all $\beta \notin \{\alpha, \alpha_1, \ldots, \alpha_p\}$, since it intersects $|\xi|$ in codimension 4 and its inverse image intersects $\Pi^{-1}(|\xi|)$ in codimension 2.

Consider the composition:

$$H_{k+p}(\Pi^{-1}(|\xi|), \Pi^{-1}(|\partial \xi|) \cup \Pi^{-1}(|\xi| \cap Y^{p+1}))$$

$$\downarrow \partial$$

$$H_{k+p-1}(\Pi^{-1}(|\partial \xi| \cup (|\xi| \cap Y^{p+1})), \Pi^{-1}(|\partial \xi| \cup (|\xi| \cap \cup_\beta Y_\beta)))$$

$$\approx \downarrow e$$

$$H_{k+p-1}(\Pi^{-1}(|\xi| \cap Y_\alpha), \Pi^{-1}(|\partial \xi| \cap Y_\alpha) \cup \Pi^{-1}(|\xi| \cap Y_\alpha \cap \cup_\beta Y_\beta))).$$

Again, the union $\cup_\beta$ is over all $\beta \notin \{\alpha, \alpha_1, \ldots, \alpha_p\}$. If $A = \Pi^{-1}(|\xi|)$, $B = \Pi^{-1}(|\partial \xi| \cup (|\xi| \cap \cup_\beta Y_\beta)))$, and $C = \Pi^{-1}(|\xi| \cap Y_\alpha)$, then the first map is the boundary operator $H_{k+p}(A, B \cup C) \to H_{k+p-1}(B \cup C, B)$, and the second is the excision isomorphism $H_{k+p-1}(B \cup C, B) \to H_{k+p-1}(C, B \cap C)$. The composed map $e \circ \partial$ and the map $\xi \mapsto \xi \cap Y_\alpha$ are connected by the following diagram, which commutes up to sign:

$$H_k(|\xi|, |\partial \xi|) \xrightarrow{\Pi_{rel}^{-1}} H_{k+p}(\Pi^{-1}(|\xi|), \Pi^{-1}((|\partial \xi|) \cup (|\xi| \cap Y^{p+1})))$$

$$\downarrow \cap Y_\alpha \qquad\qquad\qquad \downarrow e \circ \partial$$

$$H_{k-2}(|\xi| \cap Y_\alpha, |\partial \xi| \cap Y_\alpha) \xrightarrow{\Pi_{rel}^{-1}} H_{k+p-1}(\Pi^{-1}(|\xi| \cap Y_\alpha), \Pi^{-1}((|\partial \xi| \cap Y_\alpha) \cup (|\xi| \cap Y_\alpha \cap \cup_\beta Y_\beta)))$$

The commutativity of the diagram (up to a sign), basically comes from the fact that for a manifold with boundary, the Lefschetz duality identifies the boundary operator in homology with the restriction map (to the boundary) in cohomology (see e.g. [?], page 357). The corresponding sign is universal, depends only on the orientation conventions. Therefore, it can be determined using C^∞-transversal chains, as in lemma **??** . $\qquad\square$

The above proposition and the definition of the operator $\cap : C_k^{\pitchfork}(\tilde{Y}^p) \to C_{k-2}^{\pitchfork}(\tilde{Y}^{p+1})$ have the following corollary.

2.9 Corollary. $\partial L = L(\partial + \cap)$.

2.10 Example. Let $\xi \in C_*^{\pitchfork}(X)$ with $\partial \xi = 0$. In general, the three sets $\Pi^{-1}(|\xi| \cap Y)$, $|L(\xi)| \cap \partial Z$ and $|\partial(L(\xi))|$ are all distinct. To see this, set $X = \mathbb{C}^2$, $Y = \{0\} \times \mathbb{C}$ $(Y^2 = \varnothing)$. Fix coordinates $z_j = x_j + iy_j$ $(j = 1, 2)$ in $\mathbb{C}^2$. Let ξ be the closed cycle with support $(x_1 - 1)^2 + y_1^2 + x_2^2 - 1 = y_2 = 0$ and coefficient one.

Then $|\xi|$ is a 2-dimensional real sphere with $|\xi| \cap Y = (0,0)$, hence $\Pi^{-1}(|\xi| \cap Y) = S^1$. On the other hand, the intersection chain $\xi \cap Y = 0$ (even if $|\xi| \cap Y \neq \varnothing$). Indeed, $\xi \cap Y$ is the chain supported by $|\xi| \cap Y$ with coefficient the intersection multiplicity of $|\xi|$ and Y, which is zero. Therefore, by **??** , $\partial L(\xi) = 0$, hence $|\partial(L(\xi))| = \varnothing$. Finally, the reader is invited to verify that $|L(\xi)| \cap \partial Z$ is a *half circle* in $S^1 = \Pi^{-1}(0,0)$.

2.10 Remark. If $p = 0$, then in the above discussion one can replace the space X by the (compact) tubular neighbourhood $U = U_\epsilon = \Pi(Z_\epsilon)$ (cf. **??**). This means that the complex $C_*(X)$ is replaced by $C_*(U)$, and $C_*^{\pitchfork}(X)$ by $C_*^{\pitchfork}(U)$. Notice that $\cap : C_*^{\pitchfork}(U) \to C_{*-2}^{\pitchfork}(\tilde{Y}^1)$ is well-defined and still satisfies $\cap \partial + \partial \cap = 0$. Moreover, one has a map $L : C_*^{\pitchfork}(U) \to C_*(Z_\epsilon)$, defined similarly as $L : C_*^{\pitchfork}(X) \to C_*(Z)$, which satisfies $\partial L = L(\partial + \cap)$.

3§. The double complexes and their spectral sequences

3.1 Preliminary remarks and notations. In this section we construct the homological double complexes giving rise to the homology groups mentioned in the introduction, together with their weight filtration. The "easy" cases $(Y, \varnothing)$ and (X, Y) are well-known, nevertheless we decided to include them since their properties are useful in the study of the other pairs as well. The construction of the complexes for the pairs $(X - Y, \varnothing)$, $(X, X - Y)$ and $(\partial U, \varnothing)$ is new (to the best of the author's knowledge). These constructions use the dimensionally transverse cycles. The most involved case is ∂U, which is separated in section 6.

For any double complex $(A_{**}, \partial, \delta)$, we denote its total complex by $(Tot_*(A_{**}), D)$, where $Tot_k(A_{**}) = \oplus_{s+t=k} A_{st}$ and $D = \partial + \delta$. Here the degree of ∂ is $(0, -1)$, of δ is $(-1, 0)$. The weight filtration of A_{**} is defined by $W(A_{**})_s := \oplus_{p \leq s} A_{pq}$. The homological spectral sequence associated with the weight filtration W is denoted by E_{**}^r. Recall that $E_{st}^1 = H_t(A_{s*}, \partial)$ and d^1 is induced by δ. However, we will violate the weight notation on the ∞-term, and we will use Deligne's convention: on E_{st}^∞ the weight is $-t$ (instead of s); i.e. image$\{H_k(Tot_*(W_s) \to H_k(Tot_*(A))\}$ is $W_{s-k} H_k(Tot_*(A))$.

The *dual double complex* of A_{**} is $B_{st} = Hom_{\mathbb{Z}}(A_{st}, \mathbb{Z})$ with the corresponding dual maps. Its weight filtration is $W(B)_s = \{\varphi \in A^* : \varphi(W(A)_{-s-1}) = 0\}$, or equivalently, $W(B)_{-s} = \oplus_{p \geq s} B_{pq}$.

In some of the proofs we use some properties of the "cohomological mixed Hodge complexes" (for the definition and properties see [**?**]) and "mixed cones" (they correspond to the mapping cones in the category of mixed Hodge complexes; for details, see [**?**], page 21, or [**?**], page 49).

I. The homological double complex of Y.

Consider the double complex $A_{s,t}(Y) := C_t(\tilde{Y}^{s+1})$ (with $s \geq 0$ and $t \geq 0$) together with the natural operators:

$$\partial : C_k(\tilde{Y}^p) \to C_{k-1}(\tilde{Y}^p) \text{ and } i : C_k(\tilde{Y}^p) \to C_k(\tilde{Y}^{p-1}).$$

Here i is defined as follows. If $\oplus_{\alpha_1 < \cdots < \alpha_p} c_{\alpha_1,\ldots,\alpha_p} \in C_k(\tilde{Y}^p)$, then $i(\oplus_\alpha c_\alpha) = \oplus_\beta d_\beta$ if:

$$d_{\alpha_1,\ldots,\alpha_{p-1}} = \sum_{\alpha_1 < \cdots < \alpha_i < \alpha < \cdots < \alpha_{p-1}} (-1)^{k+i} \, i_{\alpha_1,\ldots,\alpha_{p-1};\alpha}(c_{\alpha_1,\ldots,\alpha_i,\alpha,\ldots,\alpha_{p-1}}),$$

where $i_{\alpha_1,\ldots,\alpha_{p-1};\alpha}$ is the natural inclusion $Y_{\alpha_1,\ldots,\alpha_i,\alpha,\ldots,\alpha_{p-1}} \hookrightarrow Y_{\alpha_1,\ldots,\alpha_{p-1}}$.

3.2 Lemma. *a) $i^2 = 0$, and b) $i\partial + \partial i = 0$.*

In particular, $D := i + \partial$ is a differential of the total complex $Tot_*(A_{**}(Y))$. The weight filtration $\{W(A)_s\}_s$ provides a spectral sequence over $\mathbb{Z}$. Here are some of its properties.

3.3 Proposition. $E_{st}^1 = H_t(\tilde{Y}^{s+1})$ *(s $\geq$ 0), $d^1 = i_*$, and $\{E_{st}^r\}_r$ satisfies* **??** *.b-c.*

Proof. Step 1. Before we start the proof of the proposition, we make the following discussion. The goal is to sfeafify the dual complex, and to obtain a quasi-isomorphism with $\mathbb{Z}_{\tilde{Y}^p}$.

(In fact, $C_*(\tilde{Y}^p)$ has a natural sheafification $\mathcal{C}_{\tilde{Y}^p}^*$ which is fine and is quasi-isomorphic to the dualizing complex $\mathcal{D}_{\tilde{Y}^p}^*$ (cf. [?] page 97, or [?] page 33). In the present paper we will use a different construction.)

For any open set $V \subset \tilde{Y}^p$, let $C_k(V, \tilde{Y}^p)$ be the subgroup of chains $\xi \in C_k(\tilde{Y}^p)$ with compact support $|\xi|$ in V. Obviously, for any open pair $V \subset W \subset \tilde{Y}^p$, there is a natural inclusion $C_k(V, \tilde{Y}^p) \to C_k(W, \tilde{Y}^p)$. Now, define the dual $C^k(V, \tilde{Y}^p)$ by $Hom_\mathbb{Z}(C_k(V, \tilde{Y}^p), \mathbb{Z})$. Then for any $V \subset W$ as above, the "restriction" $C^k(W, \tilde{Y}^p) \to C^k(V, \tilde{Y}^p)$ defines a presheaf $\mathcal{C}^k(\tilde{Y}^p)$ on $\tilde{Y}^p$, satisfying the condition (S2) (i.e. it is "conjuctive" in the terminology of [?]). Let $\bar{\mathcal{C}}^k(\tilde{Y}^p)$ be the associated sheaf whose space global sections is denoted by $\bar{C}^k(\tilde{Y}^p)$. Let $C_0^k(\tilde{Y}^p)$ be the subgroups of elements of $C^k(\tilde{Y}^p)$ with empty support. Then

$$0 \to C_0^k(\tilde{Y}^p) \to C^k(\tilde{Y}^p) \to \bar{C}^k(\tilde{Y}^p) \to 0$$

is exact (cf. [?] page 22). Moreover, $C^*(\tilde{Y}^p)$ and $C_0^*(\tilde{Y}^p)$ form complexes, and $H^*(C_0^*(\tilde{Y}^p)) = 0$ (by a subdivision argument, see [?], page 26 in the case of singular chains).

Therefore, the complexes $C^*(\tilde{Y}^p)$ and $\bar{C}^*(\tilde{Y}^p)$ are quasi-isomorphic. On the other hand, for any open V, there is a natural augmentation map $C_0(V, \tilde{Y}^p) \to \mathbb{Z}$ which give rise to a resolution $0 \to \mathbb{Z}_{\tilde{Y}^p} \to \bar{C}^*(\tilde{Y}^p)$. The sheaf $\bar{\mathcal{C}}^k(\tilde{Y}^p)$ is a module over the ring of $\mathbb{Z}$-constructible functions on $\tilde{Y}^p$. Indeed, for any constructible function f and $\varphi \in C^k(\tilde{Y}^p)$ one can define $f \cdot \varphi \in C^k(\tilde{Y}^p)$ by $(f \cdot \varphi)(\sigma) = f(\hat{\sigma})\varphi(\sigma)$, where $\hat{\sigma}$ is the barycenter of σ. This shows that the above resolution is a resolution of fine sheaves.

Step 2. Let $n : \tilde{Y}^p \to Y$ be the natural map. Recall that the $K_\mathbb{Q}^\cdot$ term in Deligne's cohomological mixed Hodge complex associated with the space Y is the "Mayer-Vietoris resolution" $n_*\mathbb{Q}_{\tilde{Y}^\cdot}$:

$$0 \to n_*\mathbb{Q}_{\tilde{Y}^1} \to n_*\mathbb{Q}_{\tilde{Y}^2} \to \cdots.$$

This is considered with its "bête" filtration $W_{-s}(n_*\mathbb{Q}_{\tilde{Y}^\cdot}) = \sigma_{\geq s}(n_*\mathbb{Q}_{\tilde{Y}^\cdot})$.

Consider now the dual double complex B_{**} of A_{**}. Then $(n_*\mathbb{Q}_{\tilde{Y}^\cdot}, W)$ and $(B_{**}, W) \otimes 1_\mathbb{Q}$ are quasi-isomorphic. This follows from the above discussion (Step 1). Therefore, their spectral sequence (for $r > 0$) are isomorphic. This gives $a)$ and $b)$. Finally notice that by a result of Deligne, the weight spectral sequence (over $\mathbb{Q}$) of a mixed Hodge complex degenerates at rank two, which provides $c)$. $\qquad\square$

II. The homological double complex of (X, Y).

Starting with $\tilde{Y}^0 = X$, we consider now the double complex $A_{s,t}(X,Y) := C_t(\tilde{Y}^s)$ (with $s \geq 0$ and $t \geq 0$) together with the natural operators: $\partial : C_k(\tilde{Y}^p) \to C_{k-1}(\tilde{Y}^p)$ and $i : C_k(\tilde{Y}^p) \to C_k(\tilde{Y}^{p-1})$ as above. Again, $D := i + \partial$ is a differential of the total complex $Tot_*(A_{**}(X,Y))$.

In fact, the double complex $A_{**}(X,Y)$ is a cone. To see this, first introduce the double complex $A_{**}(X)$ of X defined by $A_{s*}(X) = C_*(X)$ if $s = 0$ and $= 0$ otherwise. Then define $\tilde{i} : A_{**}(Y) \to A_{**}(X)$ so that $\tilde{i}|A_{s*}(Y) = 0$ if $s \neq 0$, and $\tilde{i}|A_{0*}(Y)$ is exactly $i : C_*(\tilde{Y}^1) \to C_*(X)$. Then $A_{s*}(X,Y) = A_{s*}(X) \oplus A_{s-1,*}(Y)$, and $A_{**}(X,Y)$ can be interpreted as the $Cone(\tilde{i})$ of $\tilde{i}$.

3.3 Proposition. $E^1_{st} = H_t(\tilde{Y}^s)$ $(s \geq 0)$, $d^1 = i_*$, and $\{E^r_{st}\}_r$ satisfies **??** .b-c.

Proof. The proof is similar as in the case of **??** . In the present case, $K^{\cdot}_{\mathbb{Q}}$ is $0 \to n_*\mathbb{Q}_{\tilde{Y}^0} \to n_*\mathbb{Q}_{\tilde{Y}^1} \to \cdots$. In fact this (and the whole cohomologically mixed Hodge complex of (X,Y)) can be constructed as a mixed cone of the complexes of Y and X. This is compatible with the construction of $A_{**}(X,Y)$. $\square$

III. The homological double complex of $(X, X-Y)$ or $(U, \partial U)$.

We define
$$A_{s,t}(X, X-Y) := C^{\pitchfork}_{t+2(s-1)}(\tilde{Y}^{-(s-1)}),$$
with $s \leq 0$ and $t + 2(s-1) \geq 0$. Then ∂ and $\cap$ act as $\partial : A_{s,t} \to A_{s,t-1}$ and $\cap : A_{s,t} \to A_{s-1,t}$, hence $D = \partial + \cap$ is the differential of the total complex $Tot_*(A_{**}(X, X-Y))$. Corollary **??** reads as:

3.1 Corollary. $L : (Tot_*(A_{**}(X, X-Y)), D) \to (C_{*-1}(\partial Z), \partial)$ *is a morphism of complexes i.e.* $\partial L = LD$.

Moreover, **??** , **??** , **??** and Poincaré duality imply the following result.

3.-1 Proposition.
a) $E^1_{s+1,t} = H_{t+2s}(\tilde{Y}^{-s})$ $(s+1 \leq 0)$, d_1 *is the transfer map* $i_!$, *and* $\{E^r_{st}\}_r$ *satisfies* **??** .b-c.
b) The Poincaré duality map pd (cf. **??** *) induces an isomorphism of spectral sequences (for any* $r \geq 1$*) between the cohomological spectral sequence of* Y *and the above homological spectral sequence of* $(X, X-Y)$*, which provides exactly the ismorphism* $\cap[X] : Gr^W_{2n-t}H^{2n-s-t}(Y) \to Gr^W_{-t}H_{s+t}(X, X-Y)$.
c) L from **??** *induces a morphism* $H_*(X, X-Y, \mathbb{Z}) \to H_{*-1}(\partial Z, \mathbb{Z})$.

Proof. The result follows via duality *d)* and the results of the subsection I (case Y). The proof of the duality is as follows.

Fix a triangulation T and let T' be its first barycentric subdivision. Then the Poincaré duality map (cf. **??**) can be organized in the following morphism of double complexes.

Set $A^T_{st}(Y) = C^T_t(\tilde{Y}^{s+1})$, which form a double complex with ∂ and i, similarly as in 3.I. Set the double complex $B^T_{st}(Y) = Hom(A^T_{st}(Y), \mathbb{Z})$ with dual morphisms δ and i^*. Consider $A^{T'}_{s+1,t}(X, X-Y) = C^{\pitchfork,T'}_{t+2s}(\tilde{Y}^{-s})$ with boundary morphisms ∂ and $\cap$ (similarly as $A_{**}(X, X-Y)$ defined above). Then $pd : B^T_{-s,2n-t}(Y) \to A^{T'}_{st}(X, X-Y)$ satisfies $\partial \circ pd = pd \circ \delta$ (cf. **??**) and $\cap \circ pd = pd \circ i^*$ (cf. **??**).

Now, $A_{**}(Y)$ is quasi-isomorphic to $A_{**}^T(Y)$ (i.e. their spectral sequences are the same for $r \geq 1$), and the later is dual to $B_{**}^T(Y)$. Using **??** , pd induces an isomorphism at the level of the E^1 term, hence it is an quasi-isomorphism. In particular, it induces isomorphism at the level of any E^r ($r \geq 2$). On the other hand, $E^r(A_{**}^{T'}(X, X-Y)) = E^r(A_{**}(X, X-Y))$ for $r \geq 1$. Hence the result follows. $\qquad\square$

3.-7 Remark. Let $U = U_\epsilon$ be a "tubular neighbourhood" of Y in X (cf. **??**). Then obviously $H_*(X, X-Y, \mathbb{Z}) = H_*(U, U-Y, \mathbb{Z}) = H_*(U, \partial U, \mathbb{Z})$. Hence the above double complex $A_{**}(U, \partial U) := A_{**}(X, X-Y)$ also computes the homology of $H_*(U, \partial)$ with its weight filtration and with all the properties listed in the above proposition. In fact, if one identifies ∂Z and ∂U, the operator induced by L (cf. part c) is exactly the boundary operator $H_*(U, \partial U, \mathbb{Z}) \to H_{*-1}(\partial U, \mathbb{Z})$.

IV. The homological double complex of $X - Y$.

Define $A_{s,t}(X-Y) := C_{t+2s}^{\pitchfork}(\tilde{Y}^{-s})$, with $s \leq 0$ and $t + 2s \geq 0$. Then $D = \partial + \cap$ is a differential of the total complex $Tot_*(A_{**}(X-Y))$.

Similarly to the case of the pair (X, Y), we can define $A_{**}^{\pitchfork}(X)$ by $A_{s*}^{\pitchfork}(X) = C_*^{\pitchfork}(X)$ if $s = 0$ and $= 0$ otherwise. Then $A_{s*}(X-Y) = A_{s*}^{\pitchfork}(X) \oplus A_{s+1,*}(X, X-Y)$. Actually, $A_{*-1,*}(X-Y)$ is the cone of the morphism $\cap : A_{**}^{\pitchfork}(X) \to A_{**}(X, X-Y)$, where $\cap$ is the intersection for $s = 0$ and zero otherwise.

Hence the Poincaré duality from (the proof of) **??** extends to:

3.-8 Proposition.
*a) $E_{s,t}^1 = H_{t+2s}(\tilde{Y}^{-s})$ ($s \leq 0$), d_1 is the transfer map $i_!$, and $\{E_{st}^r\}_r$ satisfies **??** .b-c.*
*b) The Poincaré duality map pd (cf. **??**) induces an isomorphism of spectral sequences (for any $r \geq 1$) between the cohomological spectral sequnence of (X, Y) and the above homological spectral sequence of $X - Y$, which provides exactly the ismorphism $\cap[X] : Gr_{2n-t}^W H^{2n-s-t}(X, Y) \to Gr_{-t}^W H_{s+t}(X-Y)$.*

4§. The spectral sequence associated with a double complex

4.1 The main result of this section is proposition **??** . In order to state the result, we need to review the basic notations.

Let $A = \oplus_{s,t} A_{s,t}$ be a finite homological double complex with operators $\delta : A_{s,t} \to A_{s-1,t}$ and $\partial : A_{s,t} \to A_{s,t-1}$ with $\delta^2 = 0$, $\partial^2 = 0$ and $\delta\partial + \partial\delta = 0$. Similarly as above, define the associated total complex by $Tot_k A = \oplus_{s+t=k} A_{s,t}$ and $D = d + \delta$; and the weight filtration by $W_\ell A = \oplus_{s \leq \ell} A_{s,t}$. By general theory, there is a spectral sequence $(E_{s,t}^r, d_r)_{r \geq 0}$, converging to the graded homology of A. Here $E_{s,t}^0 = A_{s,t}$; $E_{s,t}^1 = Ker(\partial : A_{s,t} \to A_{s,t-1})/Im(\partial : A_{s,t+1} \to A_{s,t})$, and d^1 is induced by δ. Set

$$Z_{s,t}^r = \{c \in W_s Tot_{s+t} A : Dc \in W_{s-r} A\} \qquad Z_s^r = \{c \in W_s A : Dc \in W_{s-r} A\}$$

$$Z_{s,t}^\infty = \{c \in W_s Tot_{s+t} A : Dc = 0\} \qquad Z_s^\infty = \{c \in W_s A : Dc = 0\}.$$

Then

$$E_{s,t}^r = Z_{s,t}^r/(Z_{s-1}^{r-1} + DZ_{s+r-1}^{r-1}) \qquad E_{s,t}^\infty = Z_{s,t}^\infty/(Z_{s-1}^\infty + DA \cap W_s A).$$

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In particular, $E_{s,t}^r$ is generated by the class of elements

$$c_{st} + c_{s-1,t+1} + \cdots + c_{s-r+1,t+r-1} \in Z_{st}^r \quad (c_{pq} \in A_{pq})$$

which satisfy

$(*_r)$ $\qquad \partial c_{st} = 0; \quad \partial c_{s-1,t+1} + \delta c_{st} = 0; \quad \ldots; \quad \partial c_{s-r+1,t+r-1} + \delta c_{s-r+2,t+r-2} = 0.$

For $r = 1$, this means that $c_{st} \in A_{pq}$ with $\partial c_{st} = 0$. For $r = 2$, Z_{st}^2 is generated by elements of type $c_{st} + c_{s-1,t+1}$ such that $\partial c_{s,t} = 0, \partial c_{s-1,t+1} + \delta c_{s,t} = 0$. These type of elements generate $Ker(d^1) \subset E^1$. For $r = \infty$, E_{st}^∞ is generated by class of cycles of type

$$c_{st} + c_{s-1,t+1} + \cdots + c_{s-r,t+r} + \cdots$$

satisfying

$(*_\infty)$ $\qquad\qquad \partial c_{st} = 0; \quad \partial c_{s-r,t+r} + \delta c_{s-r+1,t+r-1} = 0 \ \text{ for any } \ r \geq 1.$

In general, for any $r \geq 1$, the map d^r can be defined as:

$$d_{s,t}^r[c_{st} + c_{s-1,t+1} + \cdots + c_{s-r+1,t+r-1}] = [\delta c_{s-r+1,t+r-1}].$$

4.1 Now, assume that $d^2 = d^3 = \cdots = 0$ which means $E_{st}^2 \equiv E_{st}^\infty$. Then consider

$$Ker(d_1 : E_{st}^1 \to E_{s-1,t}^1) \to E_{st}^\infty.$$

This shows that for any $c^2 = c_{st} + c_{s-1,t+1}$ satisfying $(*_2)$, there is a class $c^\infty = c'_{st} + c'_{s-1,t+1} + \cdots$ with $(*_\infty)$, such that $[c^2] = [c^\infty]$ in E_{st}^∞. The next proposition shows that we can choose the cycle c^∞ with $c'_{st} = c_{st}$.

4.2 Proposition. *For any $r \geq 2$, consider an element $c^r = \sum_{i=0}^{r-1} c_{s-i,t+i} \in Z_{s,t}^r$. If $d^r = 0$ then c^r can be replaced by $\tilde{c}^r = c_{st} + \sum_{i=1}^{r-1} \tilde{c}_{s-i,t+i} \in Z_{s,t}^r$ with $[c^r] = [\tilde{c}^r] \in E_{s,t}^r$ such that $\tilde{c}^r$ can be completed to $\tilde{c}^{r+1} = \tilde{c}^r + \tilde{c}_{s-r,t+r} \in Z_{st}^{r+1}$. The fact that $d^r = 0$ is equivalent to the fact that this can be done for any cycle c^r.*

In particular, if $d^2 = d^3 = \cdots = 0$, then
(i) any representative c_{st} (with $\partial c_{st} = 0$ and $\delta c_{st} \in \partial A_{s-1,t+1}$) of a class in $Ker(d_1|E_{st}^1)$ can be completed to a cycle
$$c_{s,t}^\infty = c_{s,t} + \sum_{i \geq 1} c_{s-i,t-i} \in Z_{st}^\infty;$$

(ii) if c_{st}^∞ and $c_{st}'^\infty$ are two liftings of c_{st} as in (i), then $c_{st}^\infty - c_{st}'^\infty \in Z_{s-1}^\infty$;
(iii) if c_{st} and c'_{st} are representatives of the same class of $Ker(d_1|E_{st}^1)$ (i.e. $c_{st} - c'_{st} \in \partial B_{s,t+1}$) and we complete c_{st} and c'_{st} to c_{st}^∞ and $c_{st}'^\infty$, respectively, as in (i), then

$$c_{st}^\infty - c_{st}'^\infty \in DA \cap W_s + Z_{s-1}^\infty.$$

The proof is standard linear algebra (using the above definitions) and it is left to the reader.

5§. The cycles and topological characterizations of the weights

In this section we construct closed *rational* cycles which generate $W_{-t}H_k(A, B)$, where $(A, B) = (Y, \varnothing)$, (X, Y), $(X, X - Y)$, $(X - Y, \varnothing)$. As we already summarized in the introduction (cf. e.g **??** , part d), all the cycles have the form $m_{A,B}(c_{st}^\infty)$ for some quasi-isomorphisms $m_{A,B}$, and for some closed cycles c_{st}^∞ provided by the spectral-sequence argument of section 4. Nevertheless, we prefer to stress more the particular form of the cycles instead of the properties of the morphisms $m_{A,B}$. Therefore, at each case, we will describe in details the corresponding cyles and only at the very end (for the interested reader) we will indicate briefly what the morphism $m_{A,B}$ are.

The construction depends on a choice of c_{st}^∞. By proposition **??** , different choices modify the class of the cycle by elements in $W_{-t-1}H_k(A, B)$. (This general fact will be not repeated at each case again.) All the homology groups are considered with rational coefficients.

5.-1 Cycles in Y. Consider $(A_{**}(Y), \partial, i)$ and fix a pair (s, t) with $s + t = k$. If one wants to construct a closed cycle in Y, it is natural to start with a chain $c_{st} \in C_t(\tilde{Y}^{s+1})$ (with $\partial c_{st} = 0$) and tries to extent it. The first obstruction is $i(c_{st}) \in im\partial$, i.e. the existence of $c_{s-1,t+1}$ with $i(c_{st}) + \partial c_{s-1,t+1} = 0$. Then the second obstruction is $i(c_{s-1,t+1}) \in im\partial$, and so one. The remarkable fact is that once the first obstruction is satisfied then all the others are automatically satisfied (for the precise formulation of this fact, see section 4). This follows from the degeneration of the spectral sequence (**??**), and it is a consequence of the algebraicity of Y; in a simple topological context it is not true.

The above completion procedure is formalized in section 4 in the language of the spectral sequence, and it allows us to construct closed cycles in Y. More precisely, if $c_{st} \in A_{st}(Y)$ satisfies $\partial c_{st} = 0$ and $i(c_{st}) \in im\partial$, then it can be completed to a cycle $c_{st}^\infty = c_{st} + c_{s-1,t+1} + \cdots + c_{0,k}$ with $Dc_{st}^\infty = 0$. For any $p \geq 1$, let $n : \tilde{Y}^p \to Y$ be the natural projection. Then we claim that $n_*(c_{0,k}) \in C_*(Y)$ is a closed chain. Indeed, $\partial n_*(c_{0,k}) = n_*(\partial c_{0,k}) = n_*(ic_{1,k-1}) = n_*c_{1,k-1} - n_*c_{1,k-1} = 0$. $n_*(c_{0,k})$ is our wanted closed cycle.

Apparently, considering $n_*c_{0,k}$ we lose the other cycles $\{c_{s-i,t+i}\}_i$, but this is not exactly so. E.g., the dimensions of their supports can be recovered from $n_*c_{0,k}$. Indeed, first notice that n is a finite map. Moreover, for any i with $s \geq i \geq 0$, if one writes $p = s - i + 1$, then:

$$|n_*c_{0,k}| \cap Y^p = n(|c_{p-1,k-p+1}|), \text{ whose dimension is } k - p + 1.$$

On the other hand, if $p \geq s + 2$, then in the homology class $[c_{st}]$ one can take a representative c_{st} which is in a general position with respect to Y^p (e.g. it is transversal to all the components of Y^p). Therefore (with a good choise of c_{st}):

$$|n_*c_{0,k}| \cap Y^p = |n_*c_{st}| \cap Y^p \text{ whose dimension is } \leq t - 2(p - s - 1) = 2k - t - 2p + 2.$$

This motivates the following definition:

5.-1 Definition. The "support filtration". Define in $H_k(Y)$ the incresing filtration V_* by

$$V_{-t}H_k(Y) := \{ [c] \in H_k(Y) \text{ so that if } 1 \leq p \leq k - t + 1, \text{then } \dim |c| \cap Y^p \leq k - p + 1;$$

$$\text{if } p \geq \min(1, k - t + 2), \text{then } \dim |c| \cap Y^p \leq 2k - t - 2p + 2. \}$$

This can be rewritten in the language of intersection homology as follows. For any integer $s \geq 0$, consider the perversity $\underline{s}$ defined by $\underline{s}(2i) = i$ for $0 \leq i \leq s$, and $\underline{s}(2i) = s$ for $i \geq s$. (Since we

have no stratum with odd codimension, $\underline{s}(2i+1)$ is unimportant.) Notice that $\underline{s}$ does not satisfy $\underline{s}(2) = 0$ (as any perversity in [**?**]); but it is a *generalized perversity* in the sense of [**?**] (see also [**?**]). Then by the definition of the intersection homology groups one has:

$$V_{-t}H_k(Y) = im\left(IH_k^s(Y) \to H_k(Y)\right) \text{ (with } k = t+s).$$

This filtration is called "support filtration" (or Zeeman filtration) (for some more details see e.g. [**?**]; in fact, in [loc. cit.] the definition is slightly different, but one can show that they are equivalent). One of its main properties is that it depends only on the topological type of the space Y.

The results from the previous sections can be summarized in the following statement.

5.0 Proposition. *Using the above notations, one has:*
a) The homology classes $[n_*c_{0,k}] \in H_k(Y)$ (associated with c_{st} as above) generate $W_{-t}H_k(Y)$.
b) $W_{-t}H_k(Y) \subset V_{-t}H_k(Y)$ for any k and t.

In fact, Verdier and MacPherson conjectured that in part *b)* one has equality. This fact was verified by C. McCrory in [**?**] (see also [**?**] and [**?**]). In the sequel we will provide a new proof of this identity. The present proof is partially based on the duality between Y and $(X, X - Y)$, therefore, before the proof, we need to treat the case $(X, X - Y)$ as well.

5.1 Cycles in $(X, X - Y)$ (or in $(U, \partial U)$). In the next paragraphs we will construct relative cycles in $(U, \partial U)$, i.e chains ξ supported in U with $|\partial \xi| \subset \partial U$. Since the natural inclusion induces a weighted isomorphism $H_*(U, \partial U) \to H_*(X, X - Y)$, this settles the $(X, X - Y)$ case as well.

For any (s, t) with $s + t = k$, fix $c_{st} \in A_{st}(X, X - Y)$ with $\partial c_{st} = 0$ and $\cap c_{st} \in im \partial$. Then it can be completed to a cycle $c_{s,t}^\infty = c_{st} + c_{s-1,t+1} + \cdots$ with $Dc_{s,t}^\infty = 0$ (cf. section 4). Consider $L(c_{s,t}^\infty) \in C_{k-1}(\partial Z)$. Then by **??** one has $\partial(L(c_{s,t}^\infty)) = 0$. Fix a sub-analytic homeomorphism $\phi : \partial Z \times [0, \epsilon] \to Z_\epsilon$, where Z_ϵ is a collar of $\partial Z \subset Z$ and set $U = \Pi(Z_\epsilon)$ (cf. **??**). For any chain $\xi \in C_*(\partial Z)$, one can associate in a natural way a new chain $\xi \times [0, \epsilon] \in C_{*+1}(\partial Z \times [0, \epsilon])$. Indeed, if $\xi = \sum m_S S$, then $\xi \times [0, \epsilon] := \sum m_S S \times [0, \epsilon]$. Then $\Pi_* \circ \phi_*(L(c_{s,t}^\infty) \times [0, \epsilon]) \in C_k(U)$ has boundary supported in ∂U. For simplicity, we will denote this relatice cycle by $c_{s,t}^{rel}$.

Next, similarly as in the case of Y, we analyse the intersection properties of the cycle $c_{s,t}^{rel}$ with Y^p. Notice that for $p \geq 1$, $|c_{s,t}^{rel}| \cap Y^p = n|c_{s',k-s'}|$, where $s' := \min\{s, -p+1\}$. More precisely, if $p \geq 1 - s$, the intersection $|c_{s,t}^{rel}| \cap Y^p$ is $n|c_{-p+1,k+p-1}|$, which has dimension $k - p - 1$. For $p \leq -s$, the intersection $|c_{s,t}^{rel}| \cap Y^p$ is $n|c_{s,t}|$, which has dimension $2k - t - 2$.

Then similarly as for Y, we introduce the topological "support filtration" as follows:

5.0 Definition. Define in $H_k(X, X - Y)$ the increasing filtration V_* by

$$V_{-t}H_k(X, X-Y) := \{\, [c] \in H_k(X, X-Y) \text{ so that if } 1 \leq p \leq -k+t, \text{then } \dim|c|\cap Y^p \leq 2k-t-2;$$

$$\text{if } p \geq \min(1, -k+t+1), \text{then } \dim|c| \cap Y^p \leq k - p - 1. \,\}.$$

Then clearly one has:

5.1 Proposition.
a) The relative cycles $c_{s,t}^{rel}$ generate $W_{-t}H_k(U, \partial U) = W_{-t}H_k(X, X - Y)$.
b) $W_{-t}H_k(X, X - Y) \subset V_{-t}H_k(X, X - Y)$ for any k and t.

Our next goal is to prove that in **??** .b and **??** .b one has identities: i.e. Deligne's weight filtration and the support filtration agree.

5.0 Theorem. *If $(A, B) = (Y, \varnothing)$ or $(A, B) = (X, X - Y)$ then $W_* H_k(A, B) = V_* H_k(A, B)$ for any k. In particular, the weight filtration of $H_k(Y)$ (resp. of $H_k(X, X - Y)$) depends only on the homeomorphism type of Y (resp. of $(X, X - Y)$).*

Proof. First consider the non-degenerate intersection pair $\cap : H_k(Y) \otimes H_{2n-k}(X, X - Y) \to \mathbb{Q}$. By the general theory (or by **??** .b of the present paper) $\cap$ induces a duality at the level of weight filtrations as well. In other words, for any t:

$$(\perp_1) \qquad W_{-2n-1+t} H_{2n-k}(X, X - Y)^{\perp} = W_{-t} H_k(Y).$$

Next, we verify the following property for the support filtrations: for any t,

$$(\perp_2) \qquad \text{the restriction of } \cap \text{ to } V_{-t} H_k(Y) \otimes V_{-2n-1+t} H_{2n-k}(X, X - Y) \text{ is trivial.}$$

Indeed, let $[c] \in V_{-t} H_k(Y)$ and $[c'] \in V_{-2n-1+t} H_{2n-k}(X, X - Y)$ so that the representatives c and c' satisfy the imposed restrictions. We show that the intersection $|c| \cap |c'|$ is empty, after a possible small deformation of c.

Since $|c| \subset Y$, it is clear that $|c| \cap |c'| \subset Y$. Assume that for some $p \geq 1$ one has $|c| \cap |c'| \cap Y^{p+1} = \varnothing$, and we analyze the dimensions $d := \dim |c| \cap Y^p$ and $d' := \dim |c'| \cap Y^p$ of the intersections with Y^p. For this, set $t' := 2n+1-t$ and $k' := 2n-k$. Notice that $-k'+t' = k-t+1$.

If $p \leq k - t + 1$ $(*)$, then $p \leq -k' + t'$ as well, hence $d \leq k - p + 1$ and $d' \leq 2k' - t' - 2$. Therefore, using $(*)$, one gets: $d + d' \leq 2n - 2p - 1$. Similarly, if $p \geq k - t + 2$ $(**)$, or equivalently $p \geq -k' + t' + 1$, one has $d \leq 2k - t - 2p + 2$ and $d' \leq k' - p - 1$. Hence, again by $(**)$, $d + d' \leq 2k - 2p - 1$.

The point is that if one deforms slightly and generically the cycle c, the dimensions of the intersections $\{|c| \cap Y^r\}_r$ will not increase. Moreover, the property $|c| \cap |c'| \cap Y^{p+1} = \varnothing$ will also be preserved under a small deformation. Since $Y^p - Y^{p+1}$ is smooth of dimension $2n - 2p$, for a generic deformation, the supports $|c|$ and $|c'|$ will have empty intersection in Y^p. Therefore $|c| \cap |c'| = \varnothing$ by induction, hence $(\perp_2)$ follows.

Now, we finish the proof of theorem. Consider the inclusions **??** .b and **??** .b. We show that the dual of the second inclusion is the opposite inclusion of the first one. Indeed, by $(\perp_2)$, duality and $(\perp_1)$:

$$V_{-t} H_k(Y) \subset V_{-t'} H_{k'}(X, X - Y)^{\perp} \subset W_{-t'} H_{k'}(X, X - Y)^{\perp} = W_{-t} H_k(Y).$$

Hence everywhere one has equality. $\qquad\qquad\square$

5.-2 Remark. (The "neighbourhood filtration") Let $U^p \subset X$ be a small regular neighbourhood of Y^p in X (cf. [?], ch. 3). Then for U sufficiently small (with respect to U^p), and for any $p \geq 1$, one can consider the groups

$$N_{-p} H_k(U, \partial U) := im \left(H_k(U^p, U^p \cap \partial U) \to H_k(U, \partial U) \right).$$

Then the filtration $\{N_{-p} H_k(U, \partial U)\}_p$ is independent on the choice of neighbourhoods U^p. By our construction, $|c^{rel}_{s,t}| \subset U^p$ provided that $p = 1 - s$ and $s + t = k$. Therefore:

$$(*_p) \qquad W_{-k-p+1} H_k(U, \partial U) \subset N_{-p} H_k(U, \partial U).$$

Nevertheless, the filtration N_* does not characterize topologically the weight filtration, since the inclusion in $(*_p)$, in general, is strict (take e.g. $n = 1$, $k = 2$ and $p = 2$).

5.-1 Cycles in (X, Y) and their topological characterization. The construction is similar to the case Y. Take $c_{st} \in A_{st}(X, Y)$ with $\partial c_{st} = 0$ and $i(c_{st}) \in im\partial$. Then complete to $c_{st}^\infty = c_{st} + \cdots + \tilde{c}_{0,k}$, where $k = s + t$. Here $\tilde{c}_{0,k} \in C_k(X)$ with $\partial\tilde{c}_{0,k} = -i(c_{1,k-1})$, hence $|\partial\tilde{c}_{0,k}| \subset Y$. Then, one has:

- The relative cycles $\tilde{c}_{0,k}$ (associated with c_{st} as above) generate $W_{-t}H_k(X, Y)$.
- The weight filtration of $H_k(X, Y)$ is completely characterized by the boundary operator $\partial : H_k(X, Y) \to H_{k-1}(Y)$ and the topological characterization of the weight filtration of $H_k(Y)$. Indeed, $W_{-t}H_k(X, Y) = \partial^{-1}W_{-t}H_{k-1}(Y)$. Hence $W_*H_k(X, Y)$ depends only on the homeomorphism type of the pair (X, Y).

5.0 Cycles in $X - Y$ and their topological characterization. Here we will use the identification $H_*(X - Y) = H_*(Z)$ (via Π). Similarly as above, for any (s, t) with $s + t = k$ fix $c_{st} \in A_{st}(X - Y)$ with $\partial c_{st} = 0$ and $\cap c_{st} \in im\partial$. Then it can be completed to a cycle $c_{s,t}^\infty = c_{st} + c_{s-1,t+1} + \cdots$ with $Dc_{s,t}^\infty = 0$. Consider $Lc_{s,t}^\infty \in C_k(Z)$. Then one has:

- The cycles $Lc_{s,t}^\infty$ generate $W_{-t}H_k(Z) = W_{-t}H_k(X - Y)$.
- The weight filtration of $H_k(X - Y)$ is completely characterized by the boundary operator $\partial : H_{k+1}(X, X-Y) \to H_k(X-Y)$ and the topological characterization of the weight filtration of $H_{k+1}(X, X-Y)$. Indeed, $W_{-k}H_k(X-Y) = H_k(X-Y)$; and for $t < -k$ one has $W_{-t}H_k(X-Y) = \partial W_{-t}H_{k+1}(X, X-Y)$. Therefore, $W_*H_k(X-Y)$ depends only on the homeomorphism type of the pair (X, Y).

On the other hand, it is well known that one cannot recover the weight filtration from the homeomorphism type (or even from the analytic type) of $X - Y$. For a counterexample, see e.g. [**?**], (2.12).

For the space $X - Y$, similarly as for the pair $(X, X - Y)$, one can define the "neighbourhood filtration" (cf. **??**). Then our construction provides an inclusion (like in **??**) which, in general, is strict. (The details are left for the interested reader.)

5.-1 Remark. The homology of Y^p and $(X, X - Y^p)$. Using the double complexes A_{**}, one can easily recover the weight filtration from the homology of the spaces Y^p and $(X, X - Y^p)$. The case $(A, B) = (Y, \varnothing)$ is classical, see e.g. [**?**].

If A_{**} is a double complex, we denote by $\sigma_{s \leq i}A_{**}$ the subcomplex of A_{**} defined by $(\sigma_{s \leq i}A_{**})_{st} = A_{st}$ for $s \leq i$ and zero otherwise. $\sigma_{s \geq i+1}A_{**}$ is the quotient double complex $A_{**}/\sigma_{s \leq i}A_{**}$.

If we start with the double complex of Y, then for any $p \geq 1$: $H_k(Tot_*(\sigma_{s \geq p-1}A_{**}(Y))) = H_{k-p+1}(Y^p)$. Moreover, the exact sequence $0 \to \sigma_{s \leq p-2}A_{**} \to A_{**} \to \sigma_{s \geq p-1}A_{**} \to 0$ provides the identity:

$$W_{p-2-k}H_k(Y) = Ker\,(\,H_k(Y) \xrightarrow{b} H_{k-p+1}(Y^p)\,),$$

where b is the ("Mayer-Vietoris") boundary map (associated with the closed covering $Y = \cup_i Y_i$). Similar identities are valid for $A_{**}(X, Y)$, in particular, for $p \geq 1$:

$$W_{p-1-k}H_k(X, Y) = Ker\,(\,H_k(X, Y) \xrightarrow{b'} H_{k-p}(Y^p)\,),$$

where b' is the composite of the boundary operator and b. In the case of $(X, X - Y)$, for $p \geq 1$, one has $H_k(Tot_*(\sigma_{s \leq -p+1}A_{**}(X, X - Y))) = H_{k+p-1}(X, X - Y^p)$. Hence one gets:

$$W_{-p+1-k}H_k(X, X - Y) = im\,(\,H_{k+p-1}(X, X - Y^p) \xrightarrow{b} H_k(X, X - Y)\,),$$

$$W_{-p-k}H_k(X-Y) = im\left(H_{k+p}(X, X-Y^p) \xrightarrow{b'} H_k(X-Y)\right),$$

The above properties of the pairs $(Y, \varnothing)$ and $(X, X-Y)$, respectively of (X, Y) and $(X-Y, \varnothing)$, correspond by Poincaré Duality.

5.0 The morphisms $m_{A,B}$**.** The reader can easily realize that in the above constructions all the cycles have the form $m_{A,B}(c_{st}^\infty)$. Now, we will indicate briefly these morphisms $m_{A,B}$.

1. $(A, B) = (Y, \varnothing)$. $m_Y := m_{Y,\varnothing} : Tot_*(A_{**}(Y)) \to C_*(Y)$ defined as follows. If $c_{st} \in A_{st}(Y)$ then $m_Y(c_{s,t}) = n_*(c_{s,t})$ if $s = 0$ and equals zero otherwise. Since $n_* ic_{1,t} = 0$, one gets $\partial \nu = \nu D$.

2. $(A, B) = (X, Y)$. Consider $i_* : C_*(Y) \to C_*(X)$ induced by the inclusion of Y into X. Let $Cone_*(i_*)$ denote its cone. By the cone construction of $A_{**}(X, Y)$ there is a natural morphism $m_{X,Y} := id \oplus m_Y : Tot_*(A_{**}(X, Y)) \to Cone_*(i_*)$. If one prefers $C_*(X)/C_*(Y)$ instead of $Cone_*(i_*)$, then one can consider the natural quasi-isomorphism $\theta : Cone_*(i_*) \to C_*(X)/C_*(Y)$ given by $(x_q, y_{q-1}) \mapsto \widehat{x}_q$. Hence one can take $m_{X,Y} := \theta \circ (id \oplus m_Y)$ instead of $id \oplus m_Y$.

3. $(A, B) = (X-Y, \varnothing)$. Notice that by **??** and **??** , $L : Tot_*(A_{**}(X-Y)) \to C_*(Z)$ is a quasi-isomorphism of complexes (cf. also with the isomorphisms of **??**). On the other hand, $C_*(Z - \partial Z) \equiv C_*(X-Y)$ is quasi-isomorphic to $C_*(Z)$, hence one can take $m_{X-Y,\varnothing} := L$.

4. $(A, B) = (X, X-Y)$. Consider the complex (Ker_{*+1}, ∂) defined as the kernel of the morphism $\Pi'_* : (C_*(\partial Z), \partial) \to (C_*(Y), \partial)$ induced by Π. Since $\partial Z \approx \partial U$, and $U \sim Y$, (Ker_{*+1}, ∂) is quasi-isomorphic to $C_{*+1}(U, \partial U)$ (for a quasi-isomorphism, see below). The complex (Ker_{*+1}, ∂) coincide (by a natural identification via the morphism $C_*(\partial Z) \to C_*(Z)$) with the kernel of the morphism $\Pi_* : (C_*(Z), \partial) \to (C_*(X), \partial)$ since Π induces an isomorphism over $X - Y$.

We claim that $L : A_{s,t}(X-Y) \to C_{s+t}(Z)$ maps $A_{s,t}(X-Y)$ into Ker_{s+t+1} provided that $s \leq -1$. Indeed, for any $p \geq 1$, the image of any cycle $\xi \in C_k^{\pitchfork}(\tilde{Y}^p)$ via the composite map $C_k^{\pitchfork}(\tilde{Y}^p) \xrightarrow{L} C_{k+p}(\partial Z) \xrightarrow{\Pi_*} C_{k+p}(Y)$ has k-dimensional support.

Now we can define $m_{X,X-Y}$. First for any $s \leq -1$ identify $A_{s+1,t}(X, X-Y)$ with $A_{s,t}(X-Y)$. Then the restriction of L gives a morphism $Tot_*(A_{*+1,*}(X, X-Y)) \to Ker_{*+1}$. Now, if one prefers the "relative cycle realization" of $C_{*+1}(U, \partial U)$ instead of Ker_{*+1}, one can consider additionally the following morphism: $\theta' : Ker_{*+1} \to C_{*+1}(U, \partial U)$. Fix a sub-analytic homeomorphism $\phi : \partial Z \times [0, \epsilon] \to Z_\epsilon$, where Z_ϵ is a collar of $\partial Z \subset Z$ and set $U = \Pi(Z_\epsilon)$ (cf. **??**). For any chain $\xi \in C_*(\partial Z)$, one can associate in a natural way a new chain $\xi \times [0, \epsilon] \in C_{*+1}(\partial Z \times [0, \epsilon])$. Then $\theta'(\xi) := \Pi_* \circ \phi_*(\xi \times [0, \epsilon])$. Then $m_{X,X-Y} := \theta' \circ L$ (or simply the restriction L).

6§. The homology of ∂U.

I. The homological double complex of ∂U.

6.1 Preliminary remarks. The space ∂U appears in a natural way in two homological exact sequences. One of them is the pair $(U, \partial U)$. The homological exact sequence of this pair, and the isomorphism $H_*(U) = H_*(Y)$ suggests that a possible double complex for ∂U should satisfy

$$0 \to A_{*+1,*}(U, \partial U) \to A_{**}(\partial U) \to A_{**}(Y) \to 0.$$

Hence, we can try to define the double complex of ∂U by

$$A_{s,t}(\partial U) = \begin{cases} C_{t+2s}^{\pitchfork}(\tilde{Y}^{-s}) & \text{for } s \leq -1 \\ C_t(\tilde{Y}^{s+1}) & \text{for } s \geq 0. \end{cases}$$

But now we are confronted with the definition of the arrows of the double complex. Actually, we have all the vertical arrows, and all the horizontal arrows corresponding to $s \leq -1$ and $s \geq 0$. But we need to define a map $C_*(\tilde{Y}^1) \to C_{*-2}^{\pitchfork}(\tilde{Y}^1)$ with some nice properties. First of all, this map should be compatible with the other arrows, in the sense that the whole complex should form a double complex. On the other hand, we expect (since we know the cohomological E_1 term) that this map should induce at the homology level the "intersection matrix" I (see **??**). Therefore, the wanted map $C_*(\tilde{Y}^1) \to C_{*-2}^{\pitchfork}(\tilde{Y}^1)$ should be some kind of intersection. But we realize immediately that we face serious obstructions: we have to intersect cycles which are not "transversal", and the result of the intersection should be "transversal". Even if we try to modify our complexes, similar type of obstruction will survive. This can be explained as follows. The cap products $\{c_\alpha \cap [Y_\alpha]\}_\alpha$, sitting on the diagonal of I, cannot be determined only from the spaces $\{\tilde{Y}^p\}_{p\geq 1}$, we need the Poincaré dual of the spaces Y_α in X, hence we need also to consider the space X (or at least U) in our double complex.

Therefore, we have to think about ∂U as the boundary of Z, and we have to consider the pair of spaces $(Z, \partial Z)$. Notice that $H_*(Z, \partial Z) = H_*(X, Y)$ and $H_*(Z) = H_*(X - Y)$, so it is natural to ask for for a double complex with $0 \to A_{*+1,*}(X, Y) \to A_{**}(\partial U) \to A_{**}(X - Y) \to 0$. The construction is done in the next subsection.

6.1 The double complex $A'_{**}(\partial U)$.

Consider the following diagram of complexes:

$$\cdots \quad C_{*-4}^{\pitchfork}(\tilde{Y}^2) \quad \overset{\pitchfork}{\leftarrow} \quad C_{*-2}^{\pitchfork}(\tilde{Y}^1) \quad \overset{\pitchfork}{\leftarrow} \quad C_*^{\pitchfork}(\tilde{Y}^0)$$
$$\downarrow j$$
$$C_*(\tilde{Y}^0) \quad \overset{i}{\leftarrow} \quad C_*(\tilde{Y}^1) \quad \overset{i}{\leftarrow} \quad C_*(\tilde{Y}^2) \quad \cdots$$

The first line corresponds to the double complex $A_{**}(X - Y)$ (where the column $s = 0$ is $C_*^{\pitchfork}(\tilde{Y}^0) = C_*^{\pitchfork}(X)$), and the second line is the double complex $A_{**}(X, Y)$. Notice that j can be considered as a morphism of double complexes, hence the usual cone construction provides the double complex $A'_{**}(\partial U) = \{A'_{s*}(\partial U)\}_s$:

$$\cdots \quad C_{*-4}^{\pitchfork}(\tilde{Y}^2) \quad \overset{\pitchfork}{\leftarrow} \quad C_{*-2}^{\pitchfork}(\tilde{Y}^1) \quad \overset{\pitchfork}{\leftarrow} \quad C_*^{\pitchfork}(\tilde{Y}^0)$$
$$\oplus \qquad \overset{j}{\swarrow} \qquad \oplus$$
$$C_*(\tilde{Y}^0) \quad \overset{i}{\leftarrow} \quad C_*(\tilde{Y}^1) \quad \overset{i}{\leftarrow} \quad C_*(\tilde{Y}^2) \quad \cdots$$

$$s = -2 \qquad\qquad s = -1 \qquad\qquad s = 0 \qquad\qquad s = 1$$

6.2 Proposition.

a) The E^1 term of the spectral sequence associated with $(A'_{**}(\partial U), W)$ is:

$$\cdots \quad H_{*-4}(\tilde{Y}^2) \quad \overset{\pitchfork}{\leftarrow} \quad H_{*-2}(\tilde{Y}^1) \quad \overset{\pitchfork}{\leftarrow} \quad H_*(\tilde{Y}^0)$$
$$\oplus \qquad \overset{id}{\swarrow} \qquad \oplus$$
$$H_*(\tilde{Y}^0) \quad \overset{i_*}{\leftarrow} \quad H_*(\tilde{Y}^1) \quad \overset{i_*}{\leftarrow} \quad H_*(\tilde{Y}^2) \quad \cdots$$

$$s = -2 \qquad\qquad s = -1 \qquad\qquad s = 0 \qquad\qquad s = 1$$

Here $\cap$ is given in ?? , e.g. for $c \in H_k(X)$ one has $\cap(c) = (-1)^k \oplus_\beta c \cap [Y_\beta] \in \oplus_\beta H_{k-2}(Y_\beta) = H_{k-2}(\tilde{Y}^1)$; i_* is induced by i (cf. 3.I), e.g. for $\oplus_\alpha c_\alpha \in \oplus_\alpha H_k(Y_\alpha) = H_k(\tilde{Y}^1)$ one has $i_*(\oplus_\alpha c_\alpha) = (-1)^k \sum_\alpha i_\alpha(c_\alpha) \in H_k(X)$. The E^1 term is quasi-isomorphic to the complex:

$$\cdots H_{*-4}(\tilde{Y}^2) \overset{\cap}{\leftarrow} H_{*-2}(\tilde{Y}^1) \overset{I}{\leftarrow} H_*(\tilde{Y}^1) \overset{i_*}{\leftarrow} H_*(\tilde{Y}^2) \cdots,$$

where I denotes the "intersection matrix" $\cap \circ i_*$, i.e.

$$I(\oplus_\alpha c_\alpha) = \oplus_\beta \left(\sum_\alpha i_\alpha(c_\alpha) \cap [Y_\beta] \right).$$

b) $E^r_{st} \Longrightarrow H_{s+t}(\partial U, \mathbb{Z})$ and $E^\infty_{st} \otimes \mathbb{Q} = Gr^W_{-t} H_{s+t}(\partial U, \mathbb{Q})$.

c) $E^*_{**} \otimes \mathbb{Q}$ degenerates at level two, i.e. $d^r \otimes 1_\mathbb{Q} = 0$ for $r \geq 2$.

Proof. For *a*) and *b*) use ?? and ?? and the corresponding definitions. *c*) follows from the results of the previous subsections and from the construction of the mixed cone. $\square$

6.0 The double complex $A_{**}(\partial U)$. In the construction of the homological cycles for ∂U it is natural to combine chains situated in the neighbourhood U. Therefore, the above complex is not convenient because of the presence of the global terms $C^\pitchfork_*(\tilde{Y}^0)$ and $C_*(\tilde{Y}^0)$. In the next construction, we replace these complexes by the complexes of chains supported by the close neighbourhood U. More precisely, we define $A_{**}(\partial U)$:

$$
\begin{array}{ccccc}
\cdots & C^\pitchfork_{*-4}(\tilde{Y}^2) & \overset{\cap}{\leftarrow} & C^\pitchfork_{*-2}(\tilde{Y}^1) & \overset{\cap}{\leftarrow} & C^\pitchfork_*(U) \\
& & & & {\scriptstyle j} \swarrow & \\
& & \oplus & & & \oplus \\
& & C_*(U) & \overset{i}{\leftarrow} & C_*(\tilde{Y}^1) & \overset{i}{\leftarrow} & C_*(\tilde{Y}^2) \cdots
\end{array}
$$

Then $E^1(A_{**})$ is:

$$
\begin{array}{ccccc}
\cdots & H_{*-4}(\tilde{Y}^2) & \overset{\cap}{\leftarrow} & H_{*-2}(\tilde{Y}^1) & \overset{\cap}{\leftarrow} & H_*(U) \\
& & & & {\scriptstyle id} \swarrow & \\
& & \oplus & & & \oplus \\
& & H_*(U) & \overset{i_*}{\leftarrow} & H_*(\tilde{Y}^1) & \overset{i_*}{\leftarrow} & H_*(\tilde{Y}^2) \cdots
\end{array}
$$

which is quasi-isomorphic to $E^1(A'_{**})$, hence $(E^r(A_{**}), d^r) = (E^r(A'_{**}), d^r)$ for any $r \geq 2$. Therefore, a posteriori, one gets the degeneration of the spectral sequence of $A_{**}(\partial U)$ as well.

6.1 Topological characterization of the weights. By construction, $\sigma_{s \leq -1} A_{**}(X-Y) = A_{*+1,*}(X, X-Y)$ is a subcomplex of $A_{**}(\partial U)$. The corresponding pair provides the exact sequence:

$$\to H_{k+1}(X, X-Y) \overset{\partial}{\longrightarrow} H_k(\partial U) \overset{\alpha}{\longrightarrow} H_k(Y) \overset{\delta}{\longrightarrow} H_k(X, X-Y) \to .$$

Since the weights of $H_{k+1}(X, X-Y)$ are $\leq -k-1$, and the weights of $H_k(Y)$ are $\geq -k$, the weight filtration of $H_k(\partial U)$ is completely determined from this exact sequence and from the weight filtrations of $(X, X-Y)$ and Y. Indeed, $W_l H_k(\partial U) = \partial(W_l H_{k+1}(X, X-Y))$ for $l \leq -k-1$, and $W_l H_k(\partial U) = \alpha^{-1} W_l H_k(Y)$ for $l \geq -k$.

In particular, the weight filtration of $H_k(\partial U)$ is completely determined from the topology of the pair (X, Y). On the other hand, it cannot be determined from the diffeomorphism type of ∂U (cf. [?]; here one has to identify the link of on isolated singularity (S, s) with the boundary ∂U of a resolution $U \to S$ over a Stein representative S of (S, s).)

6.2 The morphism $m_{\partial U, \varnothing}$. In the next subsection we will construct our closed cycles. The morphism $m_{\partial U, \varnothing}$ what we will use is the following. Let $A_{**}(Z_\epsilon)$ be the double complex:

$$\cdots \quad C^{\pitchfork}_{*-4}(\tilde{Y}^2) \quad \overset{\Omega}{\leftarrow} \quad C^{\pitchfork}_{*-2}(\tilde{Y}^1) \quad \overset{\Omega}{\leftarrow} \quad C^{\pitchfork}_*(U) \quad \leftarrow \quad 0 \quad \cdots$$

$$s = -2 \qquad\qquad s = -1 \qquad\qquad s = 0 \qquad\quad s = 1$$

Then, there are two natural morphisms. First a projection $q : A_{**}(\partial U) \to A_{**}(Z_\epsilon)$, and then $L : Tot_*(A_{**}(Z_\alpha)) \to C_*(Z_\epsilon)$ (cf. **??**). Then $m_{\partial U} := L \circ Tot(q)$ (modulo the identifications $Z_\epsilon = \partial Z \times [0, \epsilon]$ and $\partial Z = \partial U$).

II. Rational cycles in ∂U.

In the sequel we assume that the homology groups have rational coefficients. Corresponding to the above discussion of the weights (cf. **??**), in the construction of the cycles in ∂U we also distinguish two different cases.

6.1 The case $s \leq -1$. Fix a pair (s, t) with $s + t = k$ and $s \leq -1$. Consider $c_{st} \in A_{s+1,t}(X, X - Y) \subset A_{s,t}(\partial U)$ with $\partial c_{st} = 0$ and $\cap c_{st} \in im\partial$. Complete to $c_{s,t}^\infty$ with $Dc_{s,t}^\infty = 0$. Then $Lc_{s,t}^\infty \in C_k(\partial Z)$ is closed (cf. **??**). Recall that we have a natural identification of ∂Z and ∂U (cf. **??**). Then one has:

- For $s \leq -1$, the cycles $Lc_{s,t}^\infty$ generate $W_{-t}H_k(\partial U)$.

Notice also that $[Lc_{s,t}^\infty] = \partial[c_{s,t}^{rel}]$, where $c_{s,t}^{rel}$ is constructed in **??** . In particular, these homology classes inherit all the properties of $W_* H_{k+1}(X, X - Y)$ (including their relationship with the "support" filtration). The details are left to the reader.

6.-1 The case $s \geq 0$. Assume that $s \geq 0$. In this case the construction of the cycles is more involved: we have to lift some cycles from Y to ∂U.

Assume that $c_{st}' \in A_{st}(\partial U)$ ($s \geq 0$) satisfies $\partial c_{st}' = 0$ and $d_1[c_{st}'] = 0$. If $s = 0$ this means that $c_{st}' = c_{0,k}^\perp + c_{0,k}$, where $c_{0,k}^\perp \in C_k^{\pitchfork}(U)$ and $c_{0,k} \in C_k(\tilde{Y}^1)$ such that $\partial c_{0,k}^\perp = \partial c_{0,k} = 0$ and $i_*(c_{0,k}) + c_{0,k}^\perp + \partial\gamma = 0$ for some $\gamma \in C_{k+1}(U)$.

If $s > 0$, then $c_{st}' \in A_{st}'(\partial U)$ has "only one component": $c_{st}' = c_{st} \in C_t(\tilde{Y}^{s+1})$ with $\partial c_{st} = 0$ and $i_* c_{st} \in im\partial$.

In both cases c_{st}' can be completed to a cycle

$$c_{s,t}^\infty = c_{s,t} + c_{s-1,t+1} + \cdots + c_{1,k-1} + (c_{0,k} + c_{0,k}^\perp) + (\gamma_{-1,k+1} + c_{-1,k+1}) + c_{-2,k+2} + \cdots$$

where $c_{s,t} \in C_t(\tilde{Y}^{s+1})$ for $s \geq 0$, $c_{s,t} \in C_{t+2s}^{\pitchfork}(\tilde{Y}^{-s})$ for $s \leq -1$, $c_{0,k}^\perp \in C_k^{\pitchfork}(U)$, and $\gamma_{-1,k+1} \in C_{k+1}(U)$. These chains satisfy the following relations:

$\partial c_{s,t} = 0$;

$i_* c_{s-l,t+l} + \partial c_{s-l-1,t+l+1} = 0$ for $0 \leq l \leq s - 1$;

$\partial c_{0,k}^\perp = 0$;

$i_* c_{0,k} + j c_{0,k}^\perp + \partial \gamma_{-1,k+1} = 0$;

$\cap c_{0,k}^\perp + \partial c_{-1,k+1} = 0$;

$\cap c_{l,k-l} + \partial c_{l-1,k-l+1} = 0$ for $l \leq -1$.

Now, we will separate the chain (for the notations, see **??**):

$$\bar{c}_{st} := c_{0,k}^{\perp} + c_{-1,k+1} + c_{-2,k+2} + \cdots \ \in A_{**}(Z_\epsilon).$$

Then by the above relations, $D(\bar{c}_{st}) = 0$, where here D is the differential in $A_{**}(Z_\epsilon)$. Therefore, $L\bar{c}_{st} \in C_k(Z_\epsilon)$ is closed. Obviously, the projection $pr : Z_\epsilon = \partial Z \times [0,\epsilon] \to \partial Z$ provides a closed cycle $pr_* L\bar{c}_{st} \in C_k(\partial Z)$.

- *For any $s \geq 0$ and $t + s = k$, the cycles $pr_* L\bar{c}_{st}$ generate $W_{-t} H_k(\partial U)$.*

Notice that in the cycles $\bar{c}_{st}$ we do not see the chains $c_{l,k-l}$ for $l \geq 0$, in particular neither $c_{s,t}$, the chain which generates $\bar{c}_{st}$. In fact, the above algorithm goes as follows. The chain c_{st} is completed to a sequence $c_{st} + c_{s-1,t+1} + \cdots + c_{0,k}$ with $c_{0,k} \in C_k(\tilde{Y}^1)$. The chain $i_* c_{0,k}$ actually is closed (in U) and supported by Y. Now, this is replaced by the transversal chain $c_{0,k}^{\perp} \in C_k^{\pitchfork}(U)$ such that $i_* c_{0,k} + c_{0,k}^{\perp} + \partial\gamma = 0$ for some γ, and finally $c_{0,k}^{\perp}$ is completed to $\bar{c}_{st}$. It is really remarkable that the above algebraic construction plays the role of a very geometric operation: it replaces a closed chain supported in Y by another closed chain supported in U, homologous with the original one in U, and dimensionally transversal to the stratification given by Y (i.e. it eliminates the obstruction mentioned in **??** by this "algebraic deformation".)

6.-2 Example. Assume that $n = 2$, and Y is a connected set of curves $\{Y_i\}_i$ in X. Set $g = \sum_i \mathrm{genus}(Y_i)$. If Γ is the dual graph of the curves then let $c_\Gamma = rank H_1(|\Gamma|)$ be the number of independent cycles in $|\Gamma|$. Let I be the intersection matrix of the irreducible curves Y_i. Then it is well-known that $rank H_1(\partial U) = rank\, Ker I + 2g + c_\Gamma$. These three contributions correspond exactly to the weight of $H_1(\partial U)$.

Indeed, for $s = -1$, take $c_{-1,2} \in C_0^{\pitchfork}(\tilde{Y}^1)$ as above. A possible chain $c_{-1,2}$ is an arbitrary point P in $Y^1 - Y^2$ with coefficient one. Then $c_{-1,2}^{\infty} = c_{-1,2}$ and $\Pi^{-1}(c_{-1,2})$ is a circle S^1, the loop around Y in a transversal slice at P. These loops γ_P generate $W_{-2} H_1$ (isomorphic to $coker\, I$). If $s = 0$, consider a closed 1-cycle $c_{0,1}$ in one of the components of Y. This can be changed by a homologous cycle $c_{0,1}^{\perp}$ in U, which has no intersection with Y. Hence it can be contracted to ∂U. These cycles generate $W_{-1} H_1$ (so that $\dim Gr_{-1}^{W} H_1 = 2g$). Notice that the lifting $c_{0,1} \mapsto c_{0,1}^{\perp}$ is defined modulo the cycles of type γ_P. Finally, consider $c_{1,0} \in C_0(Y^2)$ such that $d_1[c_{1,0}] = 0$. Here $d_1 : H_0(Y^2) \to H_0(\tilde{Y}^1)$ and $Ker\, d_1 \approx H_1(|\Gamma|)$. Take $c_{0,1} \in C_1(\tilde{Y}^1)$ such that $i c_{1,0} + \partial c_{0,1} = 0$. Then $i(c_{0,1})$ is a cycle in U supported by Y. We replace it by $c_{0,1}^{\perp}$ which has no intersection with Y. They provide the remaining cycles in $W_0 H_1$ so that $\dim Gr_0^{W} H_1 = c_\Gamma$.

Now, we discuss the case $H_2(\partial U)$ as well. Take a point $P \in Y^2$ (with coefficient one) corresponding to $c_{-2,4}$. Then $\Pi^{-1} c_{-2,4}$ is a torus in ∂U. They generate $W_{-4} H_2$. Notice that $W_{-4} H_2 \approx coker \cap : H_2(\tilde{Y}^1) \to H_0(Y^2)$ has dimension c_Γ. Next, take a generic closed 1-cycle in $\tilde{Y}^1$ whose image c in Y has no intersection with Y^2. The 2-cycle $\Pi^{-1}(c)$ in ∂U is an S^1 bundle over c; they generate $W_{-3} H_2$. Finally, consider $c_{02} = \sum_i m_i Y_i \in C_2(\tilde{Y}^1)$ such that $\cap[c_{02}] = 0$. This means that $[c_{02}] \cdot [Y_j] = 0$ for all j. The dimension of the space generated by these classes $[c_{02}]$ is $coker\, I$. Now, c_{02} is replaced by a transversal 2-cycle $c_{02}^{\perp}$. Transversality implies that $c_{02}^{\perp} \cap Y_i$ is a 0-cycle in Y_i, which by the above assumption is zero-homologous. In particular, $c_{02}^{\perp} \cap Y_i = \partial c_{-1,3}$. Now, take a very small tubular neighbourhood U' of $Y^1 - Y^2$ with projection $pr : U' \to Y^1 - Y^2$. (Here $U - U'$ stays for Z_ϵ.) Then the boundaries of the chains $c_{02}^{\perp} - U'$ and the S^1-bundle $(pr|\partial U')^{-1}(c_{-1,3})$ can be identified (modulo sign) so they can be glued. They generate the remaining classes in $W_{-2} H_2$.

6.-1 Remark. (Purity results) The above example shows that in general all the possible weights (permitted by the spectral sequence) can appear. For example, if ∂U is the boundary (or link) of a 1-parameter family of projective curves over a small disc, then the intersection matrix I has 1-dimensional kernel, hence $H_1(\partial U)$ can have weight $-2, -1$ and 0. On the other hand, if ∂U is the boundary of the resolution of a normal surface singularity, then I is non-degenerate, hence the non-trivial weight of $H_1(\partial U)$ are -1 and 0.

More generally, if S is a projective algebraic variety with unique singular point $s \in S$, and $\phi : X \to S$ is a resolution of this isolated singularity with normal crossing exceptional divisor $Y = \phi^{-1}(s)$, then the following additional restrictions hold. For details see e.g. [?].

1) If $k \leq n-1$ then $Gr_l^W H_k(\partial U) = 0$ for l not in $[-k, 0]$;
2) If $k \geq n$ then $Gr_l^W H_k(\partial U) = 0$ for l not in $[-2n, -k-1]$;
3) If $k \geq n$ then $H_k(Y)$ is pure of weight $-k$ and $H_k(U) \to H_k(U, \partial U)$ is injective;
4) If $k \leq n$ then $H_k(U, \partial U)$ is pure of weight $-k$ and $H_k(U) \to H_k(U, \partial U)$ is surjective;
5) Consider the exact sequence (cf. the E^1 term in **??**):

$$\cdots H_{k-4}(\tilde{Y}^2) \xleftarrow{\cap} H_{k-2}(\tilde{Y}^1) \xleftarrow{I_k} H_k(\tilde{Y}^1) \xleftarrow{i_*} H_k(\tilde{Y}^2) \cdots.$$

If $k \geq n$ then $Ker I_k = im\, i_*$, if $k \leq n$ then $Ker\cap = im\, I_k$.

Actually, between the above exact sequence and the the exact sequence **??** , there is the following connection: δ is non-trivial only for weight $-k$ and $Gr_{-k}^W \delta$ can be identified with $\hat{I}_k : H_k(\tilde{Y}^1)/im\, i_* \to Ker\cap$.

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