Chapter 23

Rambling around the Milnor Fiber

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To Norbert, as a token of friendship and admiration

In this text we will present the pervading influence of Norbert's works on monodromy and the Milnor fiber in current research and their interplay with other topics like non-archimedean geometry, finite fields or symplectic geometry.

1 Foreword

Around 40 years ago, I was skiing on the main track of Les Grands Montets near Chamonix when by pure chance I ran into Bernard Teissier, my thesis advisor, who introduced me to his ski buddy: this was my first encounter with Norbert. During this first exchange with an unknown young student, Norbert was of course as welcoming and encouraging as I have always seen him since.

Amongst the many memories coming to my mind, there is one of an early breakfast in his kitchen (it was no later than 5am) where he was explaining to my half asleep self the ideas that came to his mind since we left late in the evening.

I am thus very happy to dedicate this small token of friendship and admiration to Norbert, a forever young[1] mathematician. Because of my own limitations, this text is unfortunately deprived of pictures and does not even mention Dehn twists, which is certainly a flaw in an homage to Norbert! I nevertheless hope that these ramblings amongst some of Norbert's favourite mathematical objects will convince the reader of the lasting influence of his vision and insight.

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[1]According to the foreword of [5], in his memories of Vladimir Rokhlin, Arnold quotes from Courant: "a mathematician should be considered young for as long as he is inclined to discuss math at the most inappropriate times".

2 Computing the Lefschetz numbers of the monodromy

2.1 Lefschetz numbers of the monodromy

Let us start by recalling the classical definition of the Milnor fiber. Let X be a smooth complex algebraic variety of dimension d, for instance $X = \mathbb{A}^d_{\mathbb{C}}$, and let $f\colon X \to \mathbb{A}^1_{\mathbb{C}}$ be a non-constant morphism to the affine line. Let x be a closed point of $f^{-1}(0)$.

Fix a distance function δ on some open neighborhood of x induced from a local embedding of this neighborhood in some complex affine space. For $\varepsilon > 0$ small enough, one may consider the corresponding closed ball $B(x,\varepsilon)$ of radius ε around x. For $\eta > 0$ we denote by D_η the closed disk of radius η around the origin in $\mathbb{C}$.

By Milnor's local fibration Theorem, there exists $\varepsilon_0 > 0$ such that, for every $0 < \varepsilon < \varepsilon_0$, there exists $0 < \eta < \varepsilon$ such that the morphism f restricts to a fibration, called the Milnor fibration,

$$B(x,\varepsilon) \cap f^{-1}(D_\eta \setminus \{0\}) \longrightarrow D_\eta \setminus \{0\}.$$

Set-theoretically the Milnor fiber at x,

$$F_{f,x} = f^{-1}(\eta) \cap B(x,\varepsilon),$$

depends on δ, η and ε, but its diffeomorphism type does not depend on these choices. The characteristic mapping of the fibration induces an automorphism on $F_{f,x}$, defined up to homotopy and called the local monodromy M_x at x. In particular the singular cohomology groups $H^i(F_{f,x},\mathbb{Q})$ are endowed with an automorphism M_x, and for any integer m one can consider the Lefschetz numbers

$$\Lambda(M_x^m) = \mathrm{Tr}(M_x^m; H^\bullet(F_{f,x},\mathbb{Q})) = \sum_{i\geq 0}(-1)^i\,\mathrm{Tr}(M_x^m; H^i(F_{f,x},\mathbb{Q})).$$

The first result of Norbert on the monodromy is the following:

Theorem 2.1 (A'Campo, [1]). *Assume x is a singular point of $f^{-1}(0)$, that is, $df(x) = 0$. Then*

$$\Lambda(M_x) = 0.$$

Norbert's proof used resolution of singularities, but some time later Lê Dũng Tráng constructed geometrically a representative of the monodromy without fixed points in [44], providing a proof without resolution of Theorem 2.1. Norbert's result was refined a bit later by Deligne in terms of the multiplicity of f at x.

Theorem 2.2 (Deligne, cf. [2]). *Let μ denote the multiplicity of f at x. Then*

$$\Lambda(M_x^m) = 0 \quad \textit{for} \quad 0 < m < \mu.$$

For $m \geq \mu$ it turns out that $\Lambda(M_x^m)$ can be expressed in terms of Euler characteristics of arc spaces. Indeed, for any integer $m \geq 0$, let $\mathcal{L}_m(X)$ denote the space of arcs modulo t^{m+1} on X: a $\mathbb{C}$-rational point of $\mathcal{L}_m(X)$ corresponds to a $\mathbb{C}[t]/t^{m+1}$-rational point of X, cf. [17]. Consider the locally closed subset $\mathcal{X}_{f,m,x}$ of $\mathcal{L}_m(X)$ defined as

$$\mathcal{X}_{f,m,x} = \{\varphi \in \mathcal{L}_m(X);\, f(\varphi) = t^m \mod t^{m+1}, \varphi(0) = x\}.$$

In [20], we proved the following remarkable formula:

Theorem 2.3 (Denef–Loeser, [20]). *For every $m \geq 1$,*

$$\chi_c(\mathcal{X}_{f,m,x}) = \Lambda(M_x^m), \tag{2.1}$$

with χ_c the Euler characteristic with compact supports.

Note that Theorem 2.2 follows as a corollary since $\mathcal{X}_{f,m,x}$ is empty for $0 < m < \mu$.

We arrived at this statement by an analogy between the use of arcs in motivic integration and that of symplectic disks in Floer theory that was suggested to us by Paul Seidel. In particular, as noted by Paul Seidel, there exists a remarkable analogy between Theorem 2.3 and the fact that, in symplectic Floer homology, the Lefschetz number of a symplectomorphism is equal to the Euler characteristic of the corresponding Floer homology groups, cf. [22]. We will return to this in Section 6.

2.2 A proof via Norbert's formula

The proof of Theorem 2.3 in [20] relies in a fundamental way on a result of Norbert (Theorem 1 in [2]) expressing $\Lambda(M_x^m)$ in terms of resolution data.

Let $h\colon Y \to X$ be a log-resolution of $(X, f^{-1}(0))$, that is, a proper morphism with Y smooth such that the restriction of $h\colon Y \setminus h^{-1}(f^{-1}(0)) \to X \setminus f^{-1}(0)$ is an isomorphism, and $h^{-1}(f^{-1}(0))$ is a divisor with simple normal crossings. We denote by E_i, i in A, the set of irreducible components of the divisor $h^{-1}(f^{-1}(0))$. Hence, by definition the E_i's are smooth and intersect transversally. We will assume that the reduced preimage $|h^{-1}(x)|$ is the union of components E_i, $i \in A_0$. For $I \subset A$, we set $E_I := \bigcap_{i\in I} E_i$ and $E_I^\circ := E_I \setminus \bigcup_{j\notin I} E_j$. We write E_i° for $E_{\{i\}}^\circ$. We denote by N_i the order of vanishing of $f \circ h$ along E_i and we define the log discrepancies ν_i by the equality of divisors $K_Y = h^* K_X + \sum_{i\in A}(\nu_i - 1)E_i$, with K denoting the canonical sheaf, so if ω is a top degree non-vanishing holomorphic form at x, $\nu_i - 1$ is the order of vanishing along E_i of $h^*(\omega)$.

Theorem 2.4 (A'Campo, [2]). *For $m \geq 1$, the Lefschetz number $\Lambda(M_x^m)$ is equal to $\sum_{N_i|m, i\in A_0} N_i \chi_c(E_i^\circ)$.*

To evaluate the left hand side of (2.1), we used ideas coming from motivic integration, using a cyclic Galois cover $\tilde{E}_I^\circ \to E_I^\circ$ that we describe now. For $I \subset A$, set $m_I := \gcd(N_i)_{i \in I}$. Let U be a Zariski open subset of Y, such that, on U $f \circ h = uv^{m_I}$ with u a unit. Then the restriction of $\tilde{E}_I^\circ \to E_I^\circ$ above $E_I^\circ \cap U$ is isomorphic to $\{(z, y) \in \mathbb{A}^1 \times E_I^\circ \cap U; z^{m_i} = u^{-1}\}$.

Using the key geometric statement (namely the fibration Lemma 3.4) underlying the change of variables formula for motivic integration proved in [17], we are able to compute the class of $\mathcal{X}_{f,m,x}$ in the Grothendieck ring of complex algebraic varieties (after localization by the class $\mathbb{L}$ of the affine line) in terms of the classes of the covers $\tilde{E}_I^\circ$

$$[\mathcal{X}_{f,m,x}] = \mathbb{L}^{md} \sum_{I \cap A_0 \neq \emptyset} (\mathbb{L} - 1)^{|I|-1} [\widetilde{E}_I^\circ] \left(\sum_{k_i \geq 1, i \in I, \sum_I k_i N_i = m} \mathbb{L}^{-\sum k_i \nu_i} \right). \tag{2.2}$$

After taking χ_c of both sides, all terms with $|I| \geq 2$ cancel out, and one gets

$$\chi_c(\mathcal{X}_{f,m,x}) = \sum_{N_i | m, i \in A_0} N_i \chi_c(E_i^0),$$

so that one can conclude by using Theorem 2.4.

Of course, such a proof cannot be considered as fully satisfactory, as it consists in computing explicitly both sides of (2.1) and checking both quantities are equal. In particular one misses a geometric explanation of why it should be true and whether it is connected with the Lefschetz fixed point formula. In Section 3 we shall present another approach, based on non-archimedean geometry, that is more conceptual, avoids explicit computations on resolutions and allows to see Theorem 2.3 as a consequence of an honest form of the Lefschetz fixed point formula.

3 Non-archimedean geometry enters the game

3.1 Fields of power series

We shall work over the valued field $K = \mathbb{C}((t))$ with ring of integers $R = \mathbb{C}[[t]]$. They should be seen as corresponding respectively to the punctured disk and the disk around the origin. The algebraic closure of $\mathbb{C}((t))$ is the field of Puiseux series $\mathbb{C}((t))^{\mathrm{alg}} = \cup_{m \geq 1} \mathbb{C}((t^{1/m}))$, we shall denote by $\widehat{\mathbb{C}((t))^{\mathrm{alg}}}$ its completion. The Galois group $G = \mathrm{Aut}(\widehat{\mathbb{C}((t))^{\mathrm{alg}}} / \mathbb{C}((t))) = \mathrm{Aut}(\mathbb{C}((t))^{\mathrm{alg}} / \mathbb{C}((t)))$ is canonically isomorphic to the group $\hat{\mu} = \varprojlim \mu_n$ of roots of unity, namely, $(\zeta_n)_{n \geq 1} \in \hat{\mu}$ sends the series $\sum a_m t^{i/m}$ to $\sum a_m \zeta_m^i t^{i/m}$. In other words we view $\hat{\mu}$ as the étale fundamental group of the punctured formal disk. As a profinite group it is topologically generated by $\varphi = (\exp 2\pi i/n)_{n\geq 1}$ which can therefore be viewed as an algebraic version of the monodromy.

3.2 Motivic integration according to Hrushovski–Kazhdan

We shall now present a quick review of some of the main results of [30], in a simplified form adapted to our needs. For this it is convenient to work in the setting of valued fields, that is of fields L endowed with a valuation val: $L^\times \to \Gamma(L)$ with $\Gamma(L)$ an ordered abelian group. Setting $\mathrm{val}(0) = \infty$ one extends val to val: $L \to \Gamma_\infty(L) := \Gamma(L) \cup \{\infty\}$. We will mostly consider $K = \mathbb{C}((t))$ and $\overline{K} = \mathbb{C}((t))^{\mathrm{alg}}$ with their standard valuation satisfying $\mathrm{val}(t^\gamma) = \gamma$, thus there is an inclusion $\Gamma(K) = \mathbb{Z} \subset \Gamma(\overline{K}) = \mathbb{Q}$.

We define semi-algebraic subsets of $\overline{K}^n$ as elements of the Boolean algebra generated by subsets of $\overline{K}^n$ defined by conditions of the form $\mathrm{val}(f) \geq \mathrm{val}(g)$ with f and g polynomials with coefficients in K. More generally, if X is a K-algebraic variety, a subset Z of $X(\overline{K})$ is called semi-algebraic if there is a cover by affine K-varieties U_i, such that $Z \cap U_i(\overline{K})$ is semi-algebraic for each i. We define a category VF_K whose objects are semi-algebraic subsets of some K-algebraic variety, morphisms being functions whose graph are semi-algebraic. In the terminology of [30], VF_K is (equivalent to) the category of K-definable sets in the VF-sort.

Let L be a valued field. Denote by M_L the maximal ideal of its valuation ring. The quotient $\mathrm{RV}(L) := L^\times/1 + M_L$ plays a central role in the Hrushovski–Kazhdan approach. It fits in a short exact sequence

$$1 \longrightarrow k(L)^\times \longrightarrow \mathrm{RV}(L) \longrightarrow \Gamma(L) \longrightarrow 0$$

with $k(L)$ the residue field of L. We denote by rv: $L^\times \to \mathrm{RV}(L)$, and more generally rv: $(L^\times)^n \to \mathrm{RV}(L)^n$, the quotient morphism and by val: $\mathrm{RV}(L) \to \Gamma(L)$ the morphism induced by val.

We will say a subset of $\Gamma(\overline{K})^n$ is semi-algebraic if it belongs to the Boolean algebra generated by subsets of $\Gamma(\overline{K})^n$ of the form $\sum_{i=1}^n a_i x_i + b \geq 0$ with a_i in $\mathbb{Z}$ and $b \in \Gamma(\overline{K})$. Semi-algebraic subsets of $\Gamma(\overline{K})^n$, for variable n, form a category that we denote by Γ_K. For $n \geq 0$, we note by $\Gamma_K[n]$ the subcategory of semi-algebraic subsets of $\Gamma(\overline{K})^n$.

Similarly, one may define a notion of semi-algebraic subsets of $\mathrm{RV}(\overline{K})^n$ (K-definable sets in the RV-sort in the terminology of [30]). We will not give a precise definition here, but here are some properties that should allow to get some feeling about them.

(a) If X is a semi-algebraic subset $\mathrm{RV}(\overline{K})^n$, its projection to $\Gamma(\overline{K})^n$ is semi-algebraic and its intersection with $(k(\overline{K})^\times)^n$ is the set of $\mathbb{C}$-points of a $\mathbb{C}$-constructible set.

(b) The image, resp. preimage, under $\mathrm{rv}\colon (L^\times)^n \to \mathrm{RV}(L)^n$ of a semi-algebraic set is semi-algebraic.

Semi-algebraic subsets of $\mathrm{RV}(\overline{K})^n$, for variable n, form a category that we denote by RV_K. For $n \geq 0$, we denote by $\mathrm{RV}_K[n]$ the category of morphisms $f\colon X \to \mathrm{RV}(\overline{K})^n$ in RV_K with finite fibers.

For each of the category VF_K, RV_K, ..., one denotes by $K(\mathrm{VF}_K)$, $K(\mathrm{RV}_K)$, ..., the corresponding Grothendieck ring. It is the free abelian group on isomorphism classes of objects modulo the cut and paste relation.

Given a semi-algebraic subset X of $\mathrm{RV}(\overline{K})^n$, one may consider its preimage $\mathrm{rv}^{-1}(X)$ in $\mathrm{VF}(\overline{K})^n$, which one denotes by $\mathbf{L}(X)$ ($\mathbf{L}$ stands for "lifting"). More generally, given $f\colon X \to \mathrm{RV}(\overline{K})^n$ in RV_K with finite fibers, one may consider the set $\mathbf{L}(X) = \{(x, y) \in X \times \mathrm{VF}(\overline{K})^n;\ f(x) = \mathrm{rv}(y)\}$. With some thought one may identify $\mathbf{L}(X)$ with an object of VF_K well defined up to isomorphism. The assignment $X \to \mathbf{L}(X)$ gives rise to a morphism

$$\mathbf{L}\colon K(\mathrm{RV}_K[n]) \longrightarrow K(\mathrm{VF}_K).$$

Setting $K(\mathrm{RV}_K[*]) = \oplus_n K(\mathrm{RV}_K[n])$, one gets a morphism

$$\mathbf{L}\colon K(\mathrm{RV}_K[*]) \longrightarrow K(\mathrm{VF}_K). \tag{3.1}$$

Hrushovski and Kazhdan proved the following remarkable result:

Theorem 3.1 (Hrushovski–Kazhdan, [30]). *The morphism (3.1) is surjective.*

What is the connexion with motivic integration? Motivic integration is supposed to assign to a object defined over a valued field a "volume" taking place in a Grothendieck ring of objects defined over the residue ring or the RV-sort. Hrushovski–Kazhdan strategy is to "invert" the morphism (3.1). This is not possible directly since the morphism (3.1) is not injective. Indeed, if $[1]_n$ stands for the class of a point embedded in $\mathrm{RV}(\overline{K})^n$, $\mathbf{L}([1]_0)$ is equal to the class of a point, while $\mathbf{L}([1]_1)$ is equal to the class of the open ball $1 + M_{\overline{K}}$. On the other hand, if $[\mathrm{RV}^{>0}]_1$ denotes the class of the subset defined by the condition $\mathrm{val}(x) > 0$ in $\mathrm{RV}(\overline{K})$, one notices that $\mathbf{L}([\mathrm{RV}^{>0}]_1)$ is the class of the punctured open ball $M_{\overline{K}} \setminus \{0\}$. Thus $[\mathrm{RV}^{>0}]_1 + [1]_0 - [1]_1$ belongs to the kernel of $\mathbf{L}$. An important result of Hrushovski and Kazhdan states that this is the only relation, namely:

Theorem 3.2 (Hrushovski–Kazhdan, [30]). *The kernel of the morphism (3.1) is exactly the ideal I generated by* $[\mathrm{RV}^{>0}]_1 + [1]_0 - [1]_1$.

Putting together Theorem 3.1 and Theorem 3.2 and inverting $\mathbf{L}$, one gets an isomorphism

$$\int : K(\mathrm{VF}_K) \longrightarrow K(\mathrm{RV}_K[*])/I. \tag{3.2}$$

3.3 An Euler characteristic

Using the isomorphism (3.2) together with some additional results of Hrushovski and Kazdhan, we construct in [31] a canonical morphism

$$\mathrm{EU}_\Gamma \colon K(\mathrm{VF}_K) \longrightarrow K^{\hat{\mu}}(\mathrm{Var}_\mathbb{C})/(\mathbb{L}-1)K^{\hat{\mu}}(\mathrm{Var}_\mathbb{C}),$$

assigning to a semi-algebraic set over $\mathbb{C}((t))$ the class of a constructible set over $\mathbb{C}$ endowed with an automorphism. This is done so to speak by throwing away the Γ-part in the RV-sort.

Here $K^{\hat{\mu}}(\mathrm{Var}_\mathbb{C})$ is an equivariant Grothendieck ring defined as follows. Let us say a $\hat{\mu}$-action on a complex quasi-projective variety is good if it factorizes through some μ_n-action, for some $n \geq 1$. We denote by $K^{\hat{\mu}}(\mathrm{Var}_\mathbb{C})$ the quotient of the abelian group generated by isomorphism classes of complex quasi-projective varieties with a good $\hat{\mu}$-action by the standard cut and paste relations and the following additional relations: for every complex quasi-projective variety X with good $\hat{\mu}$-action, for every finite dimensional complex vector space V endowed with two good linear actions ϱ and ϱ', the class of $X \times (V, \varrho)$ is equal to the class of $X \times (V, \varrho')$; we denote by $\mathbb{L}$ the class of the affine line with trivial $\hat{\mu}$-action.

3.4 Equivariant Euler characteristics

Let X be a K-algebraic variety of dimension d. We denote by X^{an} its Berkovich analytification. Now let U be a semi-algebraic subset of $X(\overline{K})$. It is defined Zariski locally by some finite Boolean combination of inequalities between valuations of functions, with data defined over K. We denote by U^{an} the subset of X^{an} defined by the same conditions. We set $\overline{X}^{\mathrm{an}} = X^{\mathrm{an}} \widehat{\otimes} \widehat{\mathbb{C}((t))^{\mathrm{alg}}}$ and we denote by $\overline{U}^{\mathrm{an}}$ the preimage of U^{an} in $\overline{X}^{\mathrm{an}}$ under the canonical morphism $\overline{X}^{\mathrm{an}} \to X^{\mathrm{an}}$.

When U^{an} is locally closed in X^{an}, the theory of germs in [9] allows to define cohomology groups $H^i_c(\overline{U}^{\mathrm{an}}, \mathbb{Q}_\ell)$ endowed with an action of the Galois group $\mathrm{Aut}(\widehat{\mathbb{C}((t))^{\mathrm{alg}}}/\mathbb{C}((t))) = \hat{\mu}$. Furthermore, Florent Martin proved in [35] that they are finite dimension $\mathbb{Q}_\ell$-vector spaces and that they are zero for $i > 2d$.

Let $K(\hat{\mu}\text{-Mod})$ be the Grothendieck ring of the category of $\mathbb{Q}_\ell[\hat{\mu}]$-modules that are finite dimensional as $\mathbb{Q}_\ell$-vector spaces. When U^{an} is locally closed in X^{an}, one defines $\mathrm{EU}_{\text{ét}}(U)$ as the class of

$$\sum_i (-1)^i [H_c^i(\overline{U}^{\mathrm{an}}, \mathbb{Q}_\ell)]$$

in $K(\hat{\mu}\text{-Mod})$.

Using further results from [35] one proves the existence of a unique morphism

$$\mathrm{EU}_{\text{ét}}\colon K(\mathrm{VF}_K) \longrightarrow K(\hat{\mu}\text{-Mod})$$

extending the previous construction.

Let now Y be a complex quasi-projective variety endowed with a $\hat{\mu}$-action factoring for some n through a μ_n-action. The ℓ-adic étale cohomology groups $H_c^i(Y, \mathbb{Q}_\ell)$ are endowed with a $\hat{\mu}$-action, and we may consider the element

$$\mathrm{eu}_{\text{ét}}(Y) := \sum_i (-1)^i [H_c^i(Y, \mathbb{Q}_\ell)]$$

in $K(\hat{\mu}\text{-Mod})$. Note that $\mathrm{eu}_{\text{ét}}([V, \varrho]) = 1$ for any finite dimensional $\mathbb{C}$-vector space V endowed with a $\hat{\mu}$-action factoring for some n through a linear μ_n-action. Thus, $\mathrm{eu}_{\text{ét}}$ factors to give rise to a morphism

$$\mathrm{eu}_{\text{ét}}\colon K^{\hat{\mu}}(\mathrm{Var}_{\mathbb{C}})/(\mathbb{L}-1)K^{\hat{\mu}}(\mathrm{Var}_{\mathbb{C}}) \longrightarrow K(\hat{\mu}\text{-Mod}).$$

We have the following fundamental compatibility property between $\mathrm{EU}_{\text{ét}}$ and $\mathrm{eu}_{\text{ét}}$.

Theorem 3.3 (Hrushovski–Loeser, [31]). *The diagram*

$$\begin{tikzcd} K(\mathrm{VF}) \arrow[rr, "\mathrm{EU}_\Gamma"] \arrow[dr, "\mathrm{EU}_{\acute{e}t}"'] & & K^{\hat{\mu}}(\mathrm{Var}_{\mathbb{C}})/(\mathbb{L}-1)K^{\hat{\mu}}(\mathrm{Var}_{\mathbb{C}}) \arrow[dl, "\mathrm{eu}_{\acute{e}t}"] \\ & K(\hat{\mu}\text{-Mod}) & \end{tikzcd}$$

is commutative.

In other words "étale Euler characteristics commute with motivic integration".

This result was extended by A. Forey in [26] to "motivic Euler characteristics" within the framework of Ayoub's rigid motives. Recall that in [6], Ayoub constructs a category Rig SH of rigid motives and shows it is equivalent to the category QUSH($\mathbb{C}$) of quasi-unipotent motives over $\mathbb{C}$. Forey constructs Euler characteristics morphisms $\chi_{\mathrm{Rig}}\colon K(\mathrm{VF}) \to K(\mathrm{Rig\,SH})$ and $\chi_{\hat{\mu}}\colon K^{\hat{\mu}}(\mathrm{Var}_{\mathbb{C}}) \to K(\mathrm{QUSH}(\mathbb{C}))$ and shows they are compatible with motivic integration in the sense that the diagram

$$\begin{tikzcd}
K(\mathrm{VF}) \arrow[r, "\int"] \arrow[d, "\chi_{\mathrm{Rig}}"'] & K(\mathrm{RV}_K[*])/I \arrow[r, "\Theta\circ\mathcal{E}_c"] & K^{\hat{\mu}}(\mathrm{Var}_{\mathbb{C}}) \arrow[d, "\chi_{\hat{\mu}}"] \\
K(\mathrm{Rig\,SH}) \arrow[rr, "\simeq"] & & K(\mathrm{QUSH}(\mathbb{C}))
\end{tikzcd}$$

is commutative, with $\Theta \circ \mathcal{E}_c$ the composition of two canonical morphisms, and the bottom isomorphism being induced by Ayoub's equivalence.

3.5 A non-archimedean Milnor fiber

Let X be a smooth complex algebraic variety of dimension d and let $f : X \to \mathbb{A}^1_{\mathbb{C}} = \mathrm{Spec}\,\mathbb{C}[t]$ be a non-constant morphism to the affine line. Via f, we can endow X with an $\mathbb{C}[t]$-scheme structure. Let $\mathcal{Y}$ be its t-adic completion and Y the corresponding rigid space. For any closed point x of $f^{-1}(0)$, we can consider the tube $]x[$ in Y consisting of those points in Y mapping to x under the specialization map $Y \to X$. This set was first considered by Nicaise and Sebag in [41], where they call it the analytic Milnor fiber at x and denote it by $\mathcal{F}_{f,x}$. Let us now explain why it deserves such a naming. Write $\overline{\mathcal{F}_{f,x}}^{\mathrm{an}}$ for $\mathcal{F}^{\mathrm{an}}_{f,x}\widehat{\otimes}\widehat{\mathbb{C}((t))^{\mathrm{alg}}}$ and $H^\bullet(\overline{\mathcal{F}_{f,x}}^{\mathrm{an}}, \mathbb{Q}_\ell)$ for the corresponding ℓ-adic étale cohomology groups in the Berkovich sense. Note that the Galois element φ acts on $H^\bullet(\overline{\mathcal{F}_{f,x}}^{\mathrm{an}}, \mathbb{Q}_\ell)$.

Using a general comparison theorem due to Berkovich, Nicaise and Sebag indeed show in [41] that, for every $i \geq 0$, there is an isomorphism

$$H^i(\overline{\mathcal{F}_{f,x}}^{\mathrm{an}}, \mathbb{Q}_\ell) \simeq H^i(F_{f,x}, \mathbb{Q}) \otimes_{\mathbb{Q}} \mathbb{Q}_\ell. \tag{3.3}$$

such that the action of φ on the left hand side corresponds to the action of the monodromy M_x on the right hand side.

3.6 A fixed point formula

We shall use the following version of the Lefschetz fixed point theorem. It is classical and follows in particular from [12, Theorem 3.2]

Proposition 3.4. *Let Y be a quasi-projective variety over an algebraically closed field of characteristic zero. Let T be a finite order automorphism of Y. Let Y^T be the fixed point set of T and denote by $\chi_c(Y^T, \mathbb{Q}_\ell)$ its ℓ-adic Euler characteristic with compact supports. Then*

$$\chi_c(Y^T, \mathbb{Q}_\ell) = \mathrm{Tr}(T; H^\bullet_c(Y, \mathbb{Q}_\ell)).$$

Remark 3.5. In general one cannot expect to have a fixed point theorem for non proper varieties without a good control of the behaviour of the automorphism at infinity. Thus, in the above the statement the condition that T is of finite order is crucial.

3.7 A proof of Theorem 2.3 using non-archimedean geometry

We are now in position to explain the proof of Theorem 2.3 given in [31].

Fix $m \geq 1$. By (3.3), one may write

$$\Lambda(M_x^m) = \mathrm{Tr}(\varphi^m; H^\bullet(\overline{\mathcal{F}_{f,x}}^{\mathrm{an}}, \mathbb{Q}_\ell)).$$

One deduces easily from Poincaré Duality that

$$\mathrm{Tr}(\varphi^m; H^\bullet(\overline{\mathcal{F}_{f,x}}^{\mathrm{an}}, \mathbb{Q}_\ell)) = \mathrm{Tr}(\varphi^m; H_c^\bullet(\overline{\mathcal{F}_{f,x}}^{\mathrm{an}}, \mathbb{Q}_\ell)).$$

Let $X_{f,t,x}$ the semi-algebraic subset of $X(\mathcal{O}_K)$ defined by $f(\varphi) = t$ and $\varphi(0) = x$. By definition, we have

$$\mathrm{Tr}(\varphi^m; H_c^\bullet(\overline{\mathcal{F}_{f,x}}^{\mathrm{an}}, \mathbb{Q}_\ell)) = \mathrm{Tr}(\varphi^m; \mathrm{EU}_{\text{ét}}([X_{f,t,x}])).$$

On the other hand, by Theorem 3.3,

$$\mathrm{Tr}(\varphi^m; \mathrm{EU}_{\text{ét}}([X_{f,t,x}])) = \mathrm{Tr}(\varphi^m; \mathrm{eu}_{\text{ét}} \circ \mathrm{EU}_\Gamma([X_{f,t,x}])).$$

Using the Lefschetz fixed point formula provided by Proposition 3.4, we get

$$\mathrm{Tr}(\varphi^m; \mathrm{eu}_{\text{ét}} \circ \mathrm{EU}_\Gamma([X_{f,t,x}])) = \chi_c(\mathrm{EU}_\Gamma([X_{f,t,x}])^{\varphi^m}),$$

where $\mathrm{EU}_\Gamma([X_{f,t,x}])^{\varphi^m}$ denotes the fixed point set of φ^m acting on the virtual object $\mathrm{EU}_\Gamma([X_{f,t,x}])$. Finally, one proves that $\mathrm{EU}_\Gamma([X_{f,t,x}])^{\varphi^m}$ and $\mathcal{X}_{f,m,x}$ have the same class in $K_0(\mathrm{Var}_\mathbb{C})/(\mathbb{L}-1)K_0(\mathrm{Var}_\mathbb{C})$. In particular,

$$\chi_c(\mathrm{EU}_\Gamma([X_{f,t,x}])^{\varphi^m}) = \chi_c(\mathcal{X}_{f,m,x}),$$

which finishes the proof of Theorem 2.3.

4 The motivic Milnor fiber

4.1 The motivic zeta function

Since $\mathcal{X}_{f,m,x}$ is endowed with a natural μ_m-action, we can consider its class $[\mathcal{X}_{f,m,x}]$ in the equivariant Grothendieck ring $K^{\hat{\mu}}(\mathrm{Var}_\mathbb{C})$ defined in 3.3 and also in its localization $K^{\hat{\mu}}(\mathrm{Var}_\mathbb{C})[\mathbb{L}^{-1}]$. The motivic zeta function (cf. [16, 18]) is defined as the formal series

$$Z_{f,x}^{\mathrm{mot}}(T) := \sum_{n\geq 1}[\mathcal{X}_{f,m,x}]\mathbb{L}^{-nd}T^n$$

in $K^{\hat{\mu}}(\mathrm{Var}_{\mathbb{C}})[\mathbb{L}^{-1}][\![T]\!]$. It follows from the formula (2.2) expressing $[\mathcal{X}_{f,m,x}]$ in terms of a log-resolution, that the series $Z_{f,x}^{\mathrm{mot}}(T)$ is in fact rational.

More precisely, it belongs to the ring $K^{\hat{\mu}}(\mathrm{Var}_{\mathbb{C}})[\mathbb{L}^{-1}][T]_{\dagger}$, where the subscript $\dagger$ stands for localization by the multiplicative family $(1-\mathbb{L}^aT^b)$, $a\in\mathbb{Z}$, $b\in\mathbb{N}_+$. It follows in particular that one can consider its formal limit as $T\to\infty$. Guided by the analogy with Denef's work in [13], where the limit as $s\to-\infty$ of p-adic Igusa local zeta functions was expressed in terms of trace of liftings of the Frobenius acting on the cohomology of Milnor fibers, the motivic Milnor fiber was defined in [16, 18] as

$$\mathcal{S}_{f,x} := -\lim_{T\to\infty} Z_{f,x}^{\mathrm{mot}}(T)$$

in $K^{\hat{\mu}}(\mathrm{Var}_{\mathbb{C}})[\mathbb{L}^{-1}]$. This denomination was justified by the fact that invariants of the topological Milnor fiber $F_{f,x}$ such as the monodromy zeta function and the Hodge spectrum can be recovered from $\mathcal{S}_{f,x}$. However the proofs in [16] are not very enlightening since they rely on explicit computations on log-resolutions.

4.2 A more conceptual framework

In section 8 of our work with E. Hrushovski [31] we provided a more conceptual framework to explain how the motivic Milnor fiber $\mathcal{S}_{f,x}$ can be recovered from the non-archimedean Milnor fiber $\mathcal{F}_{f,x}$ using Hrushovski–Kazhdan motivic integration. In fact, instead of working with $\mathcal{F}_{f,x}$, it is more convenient to consider its "tubular neighborhood"

$$\mathcal{X}_{f,x} := \left\{y\in x+M_K^d \mid \mathrm{rv}(f(y))=\mathrm{rv}(t)\right\},$$

with the notation of 3.2. Unfortunately, the arguments given in section 8 of [31] are not fully complete as they overlooked some technical issues. These issues have now been resolved by Forey and Yin in [27]. To achieve this Forey and Yin construct an integration theory interpolating between the Hrushovski–Kazhdan integral $\int\colon K(\mathrm{VF}_K)\to K(\mathrm{RV}_K[*])/I$ outlined in 3.2 and its version with volume forms constructed in [30]. More precisely objects of the categories VF and RV[$*$] can be equipped with Γ-volume forms, and the resulting categories are denoted by μVF and μRV[$*$]. It follows from [30] that $\int$ admits a volume form version in the form of an isomorphism

$$\int^{\mu} : K(\mu\mathrm{VF}_K[*]) \longrightarrow K(\mu\mathrm{RV}_K[*])/\mu I$$

with μI the homogeneous ideal generated by $[\mathrm{RV}^{>0}]_1 - [1]_1$. The main construction of [27] is that of a so-called bounded integral

$$\int^{\diamond} : K(\mu\mathrm{VF}_K^{\diamond}[*]) \longrightarrow K(\mu\mathrm{RV}_K^{\mathrm{db}}[*])/I_{\Gamma}$$

interpolating between $\int$ and $\int^{\mu}$ in the sense that it fits into a commutative diagram

$$\begin{tikzcd}
K(\mathrm{VF}_K) \arrow[d, "\int"] & K(\mu\mathrm{VF}_K^{\diamond}[*]) \arrow[l] \arrow[r] \arrow[d, "\int^{\diamond}"] & K(\mu\mathrm{VF}_K[*]) \arrow[d, "\int^{\mu}"] \\
K(\mathrm{RV}_K[*])/I & K(\mu\mathrm{RV}_K^{\mathrm{db}}[*])/I_{\Gamma} \arrow[l] \arrow[r] & K(\mu\mathrm{RV}_K[*])/\mu I.
\end{tikzcd}$$

Forey and Yin show that $\int^{\diamond}$ restricts to a morphism

$$\int^{\diamond} : K^{\natural}(\mu\mathrm{VF}_K^{\diamond}[*]) \longrightarrow K^{\natural}(\mu\mathrm{RV}_K^{\mathrm{db}}[*])/I_{\Gamma},$$

where the exponent $\natural$ refers to a certain integrability condition. To each $[U]$ in

$$K^{\natural}(\mu\mathrm{RV}_K^{\mathrm{db}}[*])/I_{\Gamma}$$

they assign a power series

$$Z([U])(T) := \sum_{m\geq 1} \mathbf{H}_m([U])T^m$$

in $K^{\hat{\mu}}(\mathrm{Var}_{\mathbb{C}})[\mathbb{L}^{-1}][T]_{\dagger}$. Here $\mathbf{H}_m$ is lifting a morphism H_m that was originally considered in [31].

Theorem 4.1 (Forey–Yin). *Let $[U]$ be the class of $\int^{\diamond}([\mathcal{X}_{f,x}]$ in $K^{\natural}(\mu\mathrm{RV}_K^{\mathrm{db}}[*])/I_{\Gamma}$. Then the equality*

$$-\lim_{T\to\infty} Z([U])(T) = \mathcal{S}_{f,x}$$

holds in $K^{\hat{\mu}}(\mathrm{Var}_{\mathbb{C}})[\mathbb{L}^{-1}]$.

Remark 4.2. The fact that within this framework Z is defined on the whole Grothendieck ring $K^{\natural}(\mu\mathrm{RV}_K^{\mathrm{db}}[*])/I_{\Gamma}$ (and that it is a ring morphism with respect to the Hadamard product) makes it a quite powerful tool. See also [40] for a related comparison result (using explicit computations on a log-resolution).

Remark 4.3. In [24] Fichou and Shiota study a version of the analytic Milnor fiber over the reals and in [25] Fichou and Yin show its relation with the real version of the motivic Milnor fiber.

5 Character sums and the monodromy zeta function of discriminants of Coxeter groups

5.1 The monodromy zeta function

In [2] Norbert derives from Theorem 2.4 his famous formula for the monodromy zeta function

$$Z^{\mathrm{mono}}_{f,x}(T) := \prod_q \det(\mathrm{Id} - TM_x, H^q(F_{f,x}, \mathbb{C}))^{(-1)^{q+1}}.$$

Theorem 5.1 (A'Campo, [2]). *A log-resolution being given,*

$$Z^{\mathrm{mono}}_{f,x}(T) = \prod_{m\geq 1} (1 - T^m)^{-\chi(S_m)},$$

with S_m the disjoint union of all the sets E_i° with $N_i = m$.

Indeed, this follows from Theorem 2.4 together with the classical linear algebra formula

$$\det(\mathrm{Id} - TA) = \exp\Big(-\sum_{i\geq 1} \frac{T^i}{i} \operatorname{Tr} A^i\Big).$$

5.2 Discriminants of finite Coxeter groups

I will now discuss the computation of the monodromy zeta function for discriminants of finite Coxeter groups, which may not be too inappropriate in view of Norbert's work on Coxeter systems [3].

Let V be a complex vector space of finite dimension n and let G be a finite subgroup of $\mathrm{GL}(V)$ generated by pseudo-reflections, that is, endomorphisms of finite order fixing pointwise a hyperplane. Such a group is called a finite complex reflection group and pseudo-reflections of order 2 are called reflections. By a theorem of Chevalley the ring of polynomial invariants $\mathbb{C}[V]^G$ is a free algebra on n homogeneous invariant polynomials, whose degrees $d_1, \ldots, d_n$ only depend on G and are called the degrees of the group G. For every pseudo-reflection in G with corresponding hyperplane H, choose a linear form ℓ_H defining H and denote by $e(H)$ the order of the subgroup of elements of G fixing H pointwise. When G is a finite Coxeter group (that is, $V = \mathbb{C}^n$ and G is a subgroup of $\mathrm{GL}(\mathbb{R}^n)$), the integers e_H are all equal to 2. Set $\Delta := \prod \ell_H^{e(H)}$. The induced function $V/G \to \mathbb{C}$ is the discriminant of G and we denote by $Z(T, G)$ its monodromy zeta function at the origin.

In joint work with Denef, we found the following remarkable recursion formula for $Z(T, G)$:

Theorem 5.2. *Let G be a finite Coxeter group. We have*

$$\prod_{\Gamma} Z(-T, G(\Gamma))^{(-1)^{|\Gamma|}} = \prod_{1 \leq i \leq n} \frac{1 - T^{d_i}}{1 - T},$$

where the product on the left-hand side runs over all connected subgraphs Γ *of the Coxeter diagram of* G*,* $G(\Gamma)$ *denotes the Coxeter group with diagram* Γ*, and* $|\Gamma|$ *the number of vertices of* Γ*.*

Our initial proof in [14] was based on some new properties of Springer's regular elements in finite complex reflection groups. In fact, we computed in [14] the zeta function of the local monodromy of the discriminant for all irreducible finite complex reflection groups. This involved a case by case analysis, already for finite Coxeter groups.

The motivation for computing the monodromy zeta function of discriminants of Coxeter groups arose from our study of a finite field analogue of Macdonald's conjecture. Let us recall the statement of Macdonald's conjecture. Let G be a finite subgroup of $\mathrm{GL}(\mathbb{R}^n)$ which is generated by reflections and let q be a positive definite quadratic form which is invariant under G. Macdonald's conjecture, proved by Opdam in [42], is the following remarkable identity

$$\int_{\mathbb{R}^n} \Delta(x)^s e^{-q(x)} dx = (\operatorname{discr} q)^{-1/2} \pi^{n/2} \kappa^s \prod_{i=1}^{n} \frac{\Gamma(d_i s + 1)}{\Gamma(s + 1)} \tag{5.1}$$

with $\kappa = \prod_H \frac{q(\ell_H)}{4}$.

Let us now state its finite field analogue. Let F be a finite field of characteristic p different from 2. We consider a finite-dimensional F-vector space V, a finite subgroup G of $\mathrm{GL}(V)$ generated by reflections, and q a G-invariant and non degenerate symmetric bilinear form on V. If p does not divide $|G|$, one may define the degrees of G, $d_1, \ldots, d_n$, as in the complex case. One also defines Δ and $\kappa \in F$ similarly. Fix a non trivial additive character $\psi : F \to \mathbb{C}$. The analogue of the integral in (5.1) will be the character sum

$$S_G(\chi) := \sum_{x \in (U/G)(F)} \chi(\Delta(x)) \psi(q(x))$$

where χ is a multiplicative character and U denotes the complement of the hypersurface $\Delta = 0$ in V. In [15], we proved the following finite field analogue of Macdonald's conjecture. Assume that p does not divide $|G|$. Then, for every multiplicative character $\chi\colon F^\times \to \mathbb{C}^\times$,

$$S_G(\chi) = (-1)^n \phi(\operatorname{discr} q) g(\phi)^n \phi(\kappa)\chi(\kappa) \prod_{i=1}^{n} \frac{g(\phi\chi)^{d_i}}{g(\phi\chi)}. \tag{5.2}$$

Here the Gauss sum $-g(\chi) := \sum_{x\in F^\times} \chi(x)\psi(x)$ is the finite field analogue of the Gamma function and ϕ denotes the unique multiplicative character of order 2. In the special case when G is the symmetric group S_n, this identity was proved by Evans in [23].

Our proof of (5.2) is based on the cohomological interpretation of character sums, using the Grothendieck–Lefschetz trace formula. In this specific situation, the cohomology is concentrated in middle dimension and has rank 1, so that we have only to calculate the determinant of the Frobenius action on the cohomology. To compute this determinant we use Laumon's product formula [32]. It is here that the formula for the monodromy zeta function of the discriminant in Theorem 5.2 plays an essential role, as a local factor in Laumon's formula.

Conversely, closing up the loop, we explain in [19] how, using work of Anderson [4] and Loeser and Sabbah [33] on determinants of Aomoto complexes and determinants of integrals, one can derive Theorem 5.2 from Macdonald's formula (in fact Theorem 5.2 is equivalent to knowing the precise form of the gamma factors in Macdonald's formula).

6 Connections with symplectic geometry

6.1 Symplectic monodromy and Floer cohomology

We assume from now on that $X = \mathbb{A}^d_{\mathbb{C}}$. Considering the standard (exact) symplectic form on $\mathbb{A}^d_{\mathbb{C}}$, X is endowed with a symplectic structure. In particular we can view the Milnor fiber $F_{f,x} = f^{-1}(\eta) \cap B(x,\varepsilon)$ as an exact symplectic manifold with boundary, the boundary being endowed with a contact structure. One may choose a representative M_x of the monodromy which respects this structure. As explained in Section 4 of [43] and section 4 of [36], it is possible to define in this setting Floer cohomology groups $HF^*(M_x,+)$, the symbol $+$ referring to the way of treating the fixed points near the boundary.

We consider a log-resolution $h\colon Y \to X$ as in 2.2, and keep the notation from therein. Fix an integer $m \geq 1$. We shall say that h is m-separating if $N_i + N_j > m$ whenever $E_i \cap E_j \neq \emptyset$ with $i, j \in A_0$. Set $S_m = \{i \in A_0; N_I | m\}$. Consider a relatively ample divisor $W = -\sum_{i\in A} w_i E_i$ and set $S_{m,p} := \{i \in S_m; m w_i + pN_i = 0\}$. Under the assumption f has an isolated singularity at x, McLean constructed in [36] a spectral sequence

$$'E_1^{p,q} = \bigoplus_{i \in S_{m,p}} H_{d-1-2m\nu_i/N_i-1-(p+q)}(\tilde{E}_i^0, \mathbb{Z}) \implies HF^*(M_x^m, +)$$

converging to the Floer cohomology of the m-th iterate of the monodromy on the Milnor fiber $F_{f,x}$.

From this spectral sequence McLean deduces the following theorem:

Theorem 6.1 (McLean, [36]). *For each $m > 0$, set $\alpha_m = \sup\{i : HF^i(M_x^m, +) \neq 0\}$ and $\mu_m = \inf\{m\nu_i/N_i : i \in S_m\}$. Then, $\alpha_m = d - 1 - 2\mu_m$. In particular $HF^*(M_x^m, +)$ vanishes if and only if $\mu_m = \infty$ and the numbers μ_m are invariants of the link up to embedded contactomorphism.*

This theorem has the following remarkable corollary:

Corollary 6.2. *The multiplicity $m(f, x)$ of f at x is the smallest $m > 0$ such that $HF^*(M_x^m, +) \neq 0$. The log canonical threshold $\mathrm{lct}_x(f) := \min_{i \in A_0} \nu_i/N_i$ is equal to*

$$\liminf_{m\to\infty} \inf_i \left\{-i/2m : HF^i(M_x^m, +) \neq 0 \textit{ or } -i/2m = 1\right\}.$$

In particular both the multiplicity and log canonical threshold of f at x are invariants of the link of f at x up to embedded contactomorphism.

Note that this is quite striking in terms of the analogy between the use of arcs in motivic integration and that of symplectic disks in Floer theory which was already mentioned, since both the multiplicity and the log canonical threshold admit an expression in terms of arcs: $m(f, x)$ as the smallest $m > 0$ such that $\mathcal{X}_{f,m,x}$ is non-empty and $\mathrm{lct}_x(f)$ can be expressed in terms of dimensions of jet schemes by a remarkable result due to Mustaţă [39].

Furthermore, one should emphasize that (a generalization of) the McLean spectral sequence for symplectic monodromy has been used by J. Fernández de Bobadilla and T. Pełka in their spectacular proof of Zariski's conjecture that families of isolated hypersurface singularities with constant Milnor number have constant multiplicity [10].

6.2 Connections with arc spaces

In [36] McLean asks what is the relationship between the $\mathcal{X}_{f,m,x}$ considered in Section 2 and the groups $HF^*(M_x^m, +)$ and in particular whether there exists a spectral sequence similar to $'E_1^{p,q}$ converging to the cohomology of $\mathcal{X}_{f,m,x}$. This was answered recently in [11] by N. Budur, J. Fernández de Bobadilla, Q. T. Lê, and H. D. Nguyen as follows.

For any integer n denote by $\pi_n\colon \mathcal{L}(X) \to \mathcal{L}_n(X)$ the morphism sending an arc $\varphi(t) \in X(\mathbb{C}[\![t]\!])$ to its truncation modulo t^{n+1}. Because of the assumption of h being m-separating, $\mathcal{X}^{\infty}_{f,m,x} := \pi_m^{-1}(\mathcal{X}_{f,m,x})$ splits as a disjoint union

$$\mathcal{X}^{\infty}_{f,m,x} = \bigsqcup_{i\in S_m} \mathcal{X}^{\infty}_{f,m,i}$$

where

$$\mathcal{X}^{\infty}_{f,m,i} = \{\gamma \in \mathcal{X}^{\infty}_{f,m,x};\, \tilde{\gamma}(0) \in E_i^{\circ}\},$$

with $\tilde{\gamma}$ the lift of γ to Y. In fact, for $\ell \gg 0$, each $\mathcal{X}^{\infty}_{f,m,i} = \pi_\ell^{-1}(\mathcal{X}^{\ell}_{f,m,i})$ with $\mathcal{X}^{\ell}_{f,m,i}$ constructible in $\mathcal{L}_\ell(X)$, and we have a decomposition

$$\mathcal{X}^{\ell}_{f,m,x} = \bigsqcup_{i\in S_m} \mathcal{X}^{\ell}_{f,m,i}$$

of $\mathcal{X}^{\ell}_{f,m,x} = \pi_\ell(\mathcal{X}^{\infty}_{f,m,x})$, cf. [11].

The authors of [11] consider the filtration $F_p\mathcal{X}^{\ell}_{f,m,x} = \bigsqcup_{i\in S_m, mw_i+pN_i\geq 0} \mathcal{X}^{\infty}_{f,m,i}$ and they show that for $\ell \gg 0$, $\mathcal{X}^{\ell}_{f,m,i}$ is homotopy equivalent to $\tilde{E}_i^{\circ}$. This allows them to prove the existence of a spectral sequence

$$E_1^{p,q} = \bigoplus_{i\in S_{m,p}} H_{2(d(m+1)-m\nu_i/N_i-1)-(p+q)}(\tilde{E}_i^0, \mathbb{Z}) \implies H_c^*(\mathcal{X}_{f,m,x}, \mathbb{Z})$$

converging to the cohomology with compact supports of $\mathcal{X}_{f,m,x}$.

This is extremely suggestive in view of McLean's result. Indeed, E_1 differs from $'E_1$ by a $(2dm+d-1)$-shift in the total degree $p+q$, hence up to relabelling, the two pages are the same. This led the authors of [11] to conjecture that for $X = \mathbb{A}^d_{\mathbb{C}}$ and f having an isolated singularity at x, the two spectral sequences (E_r, d_r) and $('E_r, 'd_r)$ are isomorphic, and that for $m \geq 1$, $HF^*(M_x^m, +)$ is isomorphic to

$$H_c^{*+2dm+d-1}(\mathcal{X}_{f,m,x}, \mathbb{Z}).$$

6.3 Some speculation

This provides more substance to the still mysterious analogy between formal arcs and motivic integration on one side and symplectic disks and symplectic homology on the other side. The story does not stop here in view of some further recent results by McLean and collaborators.

Indeed, McLean proves in [37] that birational projective Calabi–Yau manifolds have isomorphic small quantum cohomology algebras after a certain change of Novikov rings. The key tool used is a version of an algebra called symplectic cohomology, which is constructed using Hamiltonian Floer cohomology. In particular it follows that they have the same Betti numbers. This is quite striking if one keeps in mind that Batyrev's proof of birational invariance of Betti numbers for birational Calabi–Yau manifolds [7] was the starting point for the introduction of motivic integration by Kontsevich!

Recently McLean and Ritter proved a version of McKay correspondence for isolated singularities using Floer theory [38]. Recall McKay correspondence was first proved by Batyrev in [8] and then, using a different method, by Denef and me in [21]. Both proofs relied on motivic integration. By the work of Yasuda [45, 46] we know it is indeed an instance of motivic integration on Deligne–Mumford stacks and it plays an important role in recent developments regarding Mirror symmetry for moduli spaces of Higgs bundles and the Fundamental Lemma [28, 29, 34], but that is another story!

All these results suggest that there might exist some kind of non-archimedean analogues of McLean's constructions in symplectic geometry. This would be a way to start to understand the mysterious analogy between formal arcs and symplectic disks. A possible approach could be provided by the work of Tony Yue Yu [47, 48] who developed a theory of counting non-archimedean holomorphic cylinders in log Calabi–Yau surfaces in the Berkovich setting, in which heuristically, non-archimedean holomorphic cylinders are to be understood as "limits" of complex holomorphic disks in SYZ fibrations. It is unclear to us whether developing a variant of Yu's constructions relatively to maps $f : \mathbb{A}^2_{\mathbb{C}} \to \mathbb{A}^1_{\mathbb{C}}$ (curve singularities) could help to shed some light in this case. A key issue is that in the Berkovich setting we don't have a geometric monodromy acting on the Milnor fiber at our disposal, we only have a Galois action on cohomology, which is much weaker.

References

[1] N. A'Campo, Le nombre de Lefschetz d'une monodromie. *Nederl. Akad. Wetensch. Proc. Ser. A* ***76*** *= Indag. Math.* **35** (1973), 113–118

[2] N. A'Campo, La fonction zêta d'une monodromie. *Comment. Math. Helv.* **50** (1975), 233–248

[3] N. A'Campo, Sur les valeurs propres de la transformation de Coxeter. *Invent. Math.* **33** (1976), no. 1, 61–67

[4] G. W. Anderson, Local factorization of determinants of twisted DR cohomology groups. *Compositio Math.* **83** (1992), no. 1, 69–105

[5] V. I. Arnold, *Mathematical understanding of nature*. American Mathematical Society, Providence, RI, 2014

[6] J. Ayoub, Motifs des variétés analytiques rigides. *Mém. Soc. Math. Fr. (N.S.)* (2015), no. 140-141, vi+386

[7] V. V. Batyrev, Birational Calabi–Yau n-folds have equal Betti numbers. In *New trends in algebraic geometry (Warwick, 1996)*, pp. 1–11, London Math. Soc. Lecture Note Ser. 264, Cambridge Univ. Press, Cambridge, 1999

[8] V. V. Batyrev, Non-Archimedean integrals and stringy Euler numbers of log-terminal pairs. *J. Eur. Math. Soc. (JEMS)* **1** (1999), no. 1, 5–33

[9] V. G. Berkovich, Étale cohomology for non-Archimedean analytic spaces. *Inst. Hautes Études Sci. Publ. Math.* (1993), no. 78, 5–161 (1994)

[10] J. F. de Bobadilla and T. Pełka, Symplectic monodromy at radius zero and equimultiplicity of μ-constant families, 2022, arXiv:2204.07007.

[11] N. Budur, J. F. de Bobadilla, Q. T. Lê, and H. D. Nguyen, Cohomology of contact loci. *J. Differential Geom.* **120** (2022), no. 3, 389–409

[12] P. Deligne and G. Lusztig, Representations of reductive groups over finite fields. *Ann. of Math. (2)* **103** (1976), no. 1, 103–161

[13] J. Denef, Degree of local zeta functions and monodromy. *Compositio Math.* **89** (1993), no. 2, 207–216

[14] J. Denef and F. Loeser, Regular elements and monodromy of discriminants of finite reflection groups. *Indag. Math. (N.S.)* **6** (1995), no. 2, 129–143

[15] J. Denef and F. Loeser, Character sums associated to finite Coxeter groups. *Trans. Amer. Math. Soc.* **350** (1998), no. 12, 5047–5066

[16] J. Denef and F. Loeser, Motivic Igusa zeta functions. *J. Algebraic Geom.* **7** (1998), no. 3, 505–537

[17] J. Denef and F. Loeser, Germs of arcs on singular algebraic varieties and motivic integration. *Invent. Math.* **135** (1999), no. 1, 201–232

[18] J. Denef and F. Loeser, Geometry on arc spaces of algebraic varieties. In *European Congress of Mathematics, Vol. I (Barcelona, 2000)*, pp. 327–348, Progr. Math. 201, Birkhäuser, Basel, 2001

[19] J. Denef and F. Loeser, Macdonald integrals and monodromy. *Internat. J. Math.* **12** (2001), no. 9, 987–1004

[20] J. Denef and F. Loeser, Lefschetz numbers of iterates of the monodromy and truncated arcs. *Topology* **41** (2002), no. 5, 1031–1040

[21] J. Denef and F. Loeser, Motivic integration, quotient singularities and the McKay correspondence. *Compositio Math.* **131** (2002), no. 3, 267–290

[22] S. Dostoglou and D. A. Salamon, Self-dual instantons and holomorphic curves. *Ann. of Math. (2)* **139** (1994), no. 3, 581–640

[23] R. J. Evans, The evaluation of Selberg character sums. *Enseign. Math. (2)* **37** (1991), no. 3-4, 235–248

[24] G. Fichou and M. Shiota, Real Milnor fibres and Puiseux series. *Ann. Sci. Éc. Norm. Supér. (4)* **50** (2017), no. 5, 1205–1240

[25] G. Fichou and Y. Yin, Motivic integration and Milnor fiber. *J. Eur. Math. Soc. (JEMS)* **24** (2022), no. 5, 1617–1678

[26] A. Forey, Virtual rigid motives of semi-algebraic sets. *Selecta Math. (N.S.)* **25** (2019), no. 1, Paper No. 6, 43

[27] A. Forey and Y. Yin, Bounded integral and motivic Milnor fiber, 2019, arXiv:1910.12764.

[28] M. Groechenig, D. Wyss, and P. Ziegler, Geometric stabilisation via p-adic integration. *J. Amer. Math. Soc.* **33** (2020), no. 3, 807–873

[29] M. Groechenig, D. Wyss, and P. Ziegler, Mirror symmetry for moduli spaces of Higgs bundles via p-adic integration. *Invent. Math.* **221** (2020), no. 2, 505–596

[30] E. Hrushovski and D. Kazhdan, Integration in valued fields. In *Algebraic geometry and number theory*, pp. 261–405, Progr. Math. 253, Birkhäuser Boston, Boston, MA, 2006

[31] E. Hrushovski and F. Loeser, Monodromy and the Lefschetz fixed point formula. *Ann. Sci. Éc. Norm. Supér. (4)* **48** (2015), no. 2, 313–349

[32] G. Laumon, Transformation de Fourier, constantes d'équations fonctionnelles et conjecture de Weil. *Inst. Hautes Études Sci. Publ. Math.* (1987), no. 65, 131–210

[33] F. Loeser and C. Sabbah, Équations aux différences finies et déterminants d'intégrales de fonctions multiformes. *Comment. Math. Helv.* **66** (1991), no. 3, 458–503

[34] F. Loeser and D. Wyss, Motivic integration on the Hitchin fibration. *Algebr. Geom.* **8** (2021), no. 2, 196–230

[35] F. Martin, Cohomology of locally closed semi-algebraic subsets. *Manuscripta Math.* **144** (2014), no. 3-4, 373–400

[36] M. McLean, Floer cohomology, multiplicity and the log canonical threshold. *Geom. Topol.* **23** (2019), no. 2, 957–1056

[37] M. McLean, Birational Calabi–Yau manifolds have the same small quantum products. *Ann. of Math. (2)* **191** (2020), no. 2, 439–579

[38] M. McLean and A. Ritter, The McKay correspondence via Floer theory, 2018, arXiv:1802.01534.

[39] M. Mustaţă, Singularities of pairs via jet schemes. *J. Amer. Math. Soc.* **15** (2002), no. 3, 599–615

[40] J. Nicaise and S. Payne, A tropical motivic Fubini theorem with applications to Donaldson-Thomas theory. *Duke Math. J.* **168** (2019), no. 10, 1843–1886

[41] J. Nicaise and J. Sebag, Motivic Serre invariants, ramification, and the analytic Milnor fiber. *Invent. Math.* **168** (2007), no. 1, 133–173

[42] E. M. Opdam, Some applications of hypergeometric shift operators. *Invent. Math.* **98** (1989), no. 1, 1–18

[43] P. Seidel, More about vanishing cycles and mutation. In *Symplectic geometry and mirror symmetry (Seoul, 2000)*, pp. 429–465, World Sci. Publ., River Edge, NJ, 2001

[44] L. D. Tráng, The geometry of the monodromy theorem. In *C. P. Ramanujam—a tribute*, pp. 157–173, Tata Inst. Fund. Res. Studies in Math. 8, Springer, Berlin-New York, 1978

[45] T. Yasuda, Twisted jets, motivic measures and orbifold cohomology. *Compos. Math.* **140** (2004), no. 2, 396–422

[46] T. Yasuda, Motivic integration over Deligne-Mumford stacks. *Adv. Math.* **207** (2006), no. 2, 707–761

[47] T. Y. Yu, Enumeration of holomorphic cylinders in log Calabi–Yau surfaces. I. *Math. Ann.* **366** (2016), no. 3-4, 1649–1675

[48] T. Y. Yu, Enumeration of holomorphic cylinders in log Calabi–Yau surfaces. II: positivity, integrality and the gluing formula. *Geom. Topol.* **25** (2021), no. 1, 1–46